Dynamic Optimization of a Rimless Wheel with an Actuated Pendulum

Benjamin Thomas Mihevc

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Dynamic Optimization of a Rimless Wheel with an Actuated Pendulum

by

Benjamin Thomas Mihevc

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Science in Computer Engineering

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Abstract

Dynamic Optimization of a Rimless Wheel with an Actuated Pendulum

Benjamin Thomas Mihevc

As the demand for mobile robots that work alongside humans increases, the amount of energy that these co-robotic systems consume will become a critical limiting factor in their deployment. This need is clearly captured in one of the fifteen main goals of the 2009 Roadmap for US Robotics which is to create a robot that can walk with half the energy consumption of a human being. At this point, the most energy-efficient walking robot is about as energy efficient as a human.

Energy efficient bipedal motion is an active area of research. It has been proven that it is theoretically possible to design a robot with intermittent support, one of the most fundamental attributes of legged locomotion, to have a zero-energy cost collisionless gait.

Optimal control has been used by a number of researchers to study the generation of periodic gaits for walking robots. However, little research exists demonstrating walkers with energy efficient collisionless motion.
the energy lost to the system when walking is from losses due to step collisions.

In this work energy efficient locomotion of a prototype actuated rimless wheel on level ground is explored using numerical optimal control. The actuated rimless wheel has an internal pendulum driven by a DC motor. The locomotion problem is posed as an optimal control problem. Different cost functions and initial configurations are investigated and the corresponding gait trajectories analyzed and assessed based on their use of energy and the potential for collisionless motion.

The results of this work will provide the foundation for the design and implementation of more energy efficient actuated rimless wheel prototype with near collisionless motion.
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# List of Variables

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<th>Description</th>
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<tr>
<td>$\phi$</td>
<td>Angle between the stance leg and the ground</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Angle between the pendulum and the vertical</td>
</tr>
<tr>
<td>$L_1$</td>
<td>Length of a wheel leg</td>
</tr>
<tr>
<td>$L_2$</td>
<td>Half the length of the pendulum</td>
</tr>
<tr>
<td>$I_1$</td>
<td>Moment of inertia of the rimless wheel in respect to the center of mass</td>
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<tr>
<td>$I_2$</td>
<td>Moment of inertia of the pendulum in respect to the center of mass</td>
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<tr>
<td>$m_1$</td>
<td>Mass of the rimless wheel</td>
</tr>
<tr>
<td>$m_2$</td>
<td>Mass of the pendulum</td>
</tr>
<tr>
<td>$g$</td>
<td>Gravitational acceleration</td>
</tr>
<tr>
<td>$n$</td>
<td>Number of legs</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Angle between two legs</td>
</tr>
<tr>
<td>$K_m$</td>
<td>Motor constant</td>
</tr>
<tr>
<td>$N$</td>
<td>Motor gear ratio</td>
</tr>
<tr>
<td>$R_m$</td>
<td>Motor internal resistance</td>
</tr>
<tr>
<td>$\hat{u}_{1-4}$</td>
<td>System basis vectors</td>
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Chapter 1

Introduction

Numerical optimization is a tool that allows for the determination of an ideal solution based on a cost criteria. This cost must be minimized to find this optimal solution. To minimize this cost, function inputs are altered based on the first and second derivatives of the system until an optimal solution is found. This is the fundamental idea that optimal control is built upon.

Optimal control is an extension of numerical optimization. With optimal control, the cost is often a function of functions and the input is a set of trajectories instead of a constant. The system is then subject to a number of dynamic constraints. Then, by varying the input trajectories, an optimal set of state trajectories can be determined.

Optimal control can be, and often is, applied to energy minimization problems. By using several of the components of the energy in the system as a cost function, inputs can be found that minimize energy use. However, it must be noted that optimization is based on a cost function. If the cost function does not accurately reward the system for the desired dynamic,
an actual optimal solution may not be found.

Why is energy efficient optimal control important? The answer to that question changes constantly. In the 1950s, it was important to enable the efficient use of rockets for space travel. In the 1960s, it was important for developing optimal climb trajectories for high performance military aircraft. Today, it is important for efficient robotic motion.

Historically, robotic walkers started out focused on statically stable motion. Though practical for early research, maintaining constant stability is not energy efficient. Proceeding from stable gaits to half stable and periodic gaits was the focus of much of the early 1990s through the early 2000s. During this research, the rimless wheel was developed. This wheel, seen in Figure 1.1, can be likened to a wagon wheel with the wheel part removed. This leaves only the spokes.

![Diagram of a Rimless Wheel](image)

The primary motivation for this thesis was Ahlin's thesis[1]. Ahlin's work
focused on the creation and simulation of an energy efficient walker capable of level ground transport. The goal was to reduce the energy of transport by generating collision free motion. For his research, a five legged walker with an internal inertial device was created. The inertial device consisted of a torsional spring attached to a rotating mass.

In simulation, frictional losses prevented the passive system from achieving collisionless motion. Attempts to apply simple actuation and feedback control to the system did improve the cost of transport, but did not reintroduce collisionless motion [1].

In this thesis this problem will be reexamined using optimal control. To apply optimal control to the system, an understanding of the dynamics of the rimless wheel must obtained. First studied in detail by McGeer in the late 1980s and early 1990s, the rimless wheel is considered the simplest walking system [2]. It has been shown to be a half stable, 1-period system when walking down an incline. When walking on level ground the system is asymptotically stable [3].

Rimless wheels with actuation have been proposed and implemented to further gain insight into collisionless motion. Among the many actuation methods the most common are telescoping legs, spinning inertial devices, and wobbling pendulums. Most of these methods of actuation sought a stable gait frequency.

In our work, we will seek the most energy efficient step possible for a
walking rimless wheel. We will consider the minimization of the control effort as a surrogate to input energy, but we will also seek to determine the optimal trajectory that leads to a collisionless step.

We will begin by deriving the dynamics of the system and the equations of motion. From there, a simulation of the system will be developed. Following that, the optimal control problem will be formulated and solved for various wheel configurations with various costs functions. We will experiment with penalizing state movement and control input. Finally we will analyze our results and discuss physical implementation of the system.
Chapter 2

Literature Review

In the late 1980s and early 1990s, McGeer pioneered passive dynamic walking by introducing the rimless wheel [2]. The dynamics of the rimless wheel were studied significantly. Since the introduction of the rimless wheel it's passive stability on a gentle incline was determined to be 1-period half stable. With frictional losses, the system is asymptotically stable [4][5][3][6].

Gamus and Or [6] examined the dynamic legged locomotion of a robotic walker that undergoes a constant falling motion followed by foot placement. Their work involved a rimless walking wheel without any kind of secondary oscillation or actuated swinging body besides the wheel itself. In the paper, they examine the slipping dynamics of the wheel. Though beyond the scope of my research, the dynamics of their walker are similar to the simplified dynamics of the walker we are describing.

Passive walking wheels provide a simple way to model stepping motions, but do not have any inherent methods of actuation. Generating a walking motion on level ground requires input energy. Methods of adding energy
to these systems are numerous. Some of the more popular methods include telescoping legs [7], external actuated torsos [5][8], virtual slopes generated by an actuated heel [9], wobbling pendulums [10], small physical slopes [1], and many more.

Fumihiko Asano created a passive dynamic walking rimless wheel with a two degree of freedom wobbling mass on the center of mass of the wheel [10]. This walker was shown to demonstrate the effect of a passive wobbling mass on a walker moving down an incline. The wobbling mass swings on a pin joint at the center of the wheel’s mass and telescopes along the pendulum arm’s length. If the telescoping arm’s length is held constant, the walker is essentially the one in this thesis.

Asano used this walker to demonstrate the asymptotically stable gait of this type of walker. In addition, the swinging of the inner mass was used as a controller to achieve a reference frequency. Asano was able to demonstrate that this type of walker is capable of achieving a controlled gait for high efficiency movement using simple PID control while walking down a negative slope [10].

Much of the research involving rimless wheels is foundational work that leads directly to bipedal walking [6] [3] [7] [5]. These bipedal walkers almost exclusively focus on stable periodic walking gaits.

Walkers that can achieve stable periodic bipedal motion need to be actuated. Honjo, Nagano, and Lou [11] developed a bipedal robot that altered
the center of mass of the robot to allow for dynamic motion. In their work, they model their robot as an inverted double pendulum that has an upper mass and a swinging foot. By altering the center of mass of their system they were able to achieve a more efficient method of robotic locomotion.

A Lagrangian approach was used to determine the equations of motion of the rimless wheel. General background on this approach can be found in [12] and [13].

Efficient robotic locomotion yields directly to optimal control. Numerous authors have used optimal control for bipedal and rimless wheel walkers [14] [15] [16] [17] [18] [19] [20] [21] [22]. Where most of the work involving optimal control and walking robots differ are the actuation methods. In general, the cost function of the optimal control problem is either gait frequency or energy, or both. In some cases, electrical input energy and storing the systems kinetic energy prior to a step is considered [19], but the energy lost in the collision with the ground is rarely studied.

As was discovered by Ahlin [1] and previously discussed, a significant amount of energy is lost in foot collisions. Determining if a trajectory exists that can yield actuated collisionless motion is of significant interest.

The locomotion problem was formulated as an optimal control problem. For background on optimal control and dynamic optimization see [23] [24] [25] [26]. The optimal control problem was then solved numerically using the matlab toolbox GPOPS-II. This MATLAB package that solves
multiple-phase optimal control problems using Gaussian pseudospectral methods is based on research by Rao et al. [27]. This method discretizes the problem as approximations of the states and control using high order polynomials [27] [28] [29]. The coefficients of the polynomials can then be solved for using traditional optimization methods. This "collocation" of the system is dynamic in nature and does not require a constant sampling rate. This allows for high accuracy representation of the system [15].
Chapter 3

The Actuated Rimless Wheel

This section describes the dynamics of a rimless wheel system. In it the equations of motion, energy equations, and control torque equations will be presented and explained. Additionally, the derivations will be presented in detail. The relevant free body diagrams will be presented and their significance will be explained.

3.1 Newton-Euler Equations

The rimless wheel dynamics will be described as a double pendulum. This system is a system with two pin joints, one being the motionless pivot joint on the ground, and the other being the moving pivot at the center of gravity of the rimless wheel. This can be seen in Figure 3.1 where $\phi$ is the angle between the spoke and the ground and $\theta$ is the angle between the vertical axes and the inner pendulum's arm.

This system can be split into two separate free body diagrams. These free body diagrams can be seen in Figure 3.2. System 1 describes the forces
acting on the outer wheel of this system. In the simulation, this system acts as the first arm of the double pendulum. System 2 is describes the forces acting on the inner arm of the pendulum. System 1 in Figure 3.2 introduces two coordinate axis. The first axis is the standard $i$, $j$, and $k$ coordinate plane. The second coordinate plane is the system consisting of $\hat{u}_3$ and $\hat{u}_4$ system which moves with the outer wheel. System 2 in Figure 3.2 introduces a third coordinate axis, $\hat{u}_1$ and $\hat{u}_2$ which moves with
the actuated inner pendulum. The coordinate transform from \( \hat{u}_3 \) and \( \hat{u}_4 \) into \( \hat{i}, \hat{j}, \) and \( \hat{k} \) is summarized in Equations (3.1) and (3.2). The coordinate transform from \( \hat{u}_1 \) and \( \hat{u}_2 \) into \( \hat{i}, \hat{j}, \) and \( \hat{k} \) is summarized in Equations (3.3) and (3.4).

\[
\hat{u}_3 = \cos(\phi) \hat{i} - \sin(\phi) \hat{j} \tag{3.1} \\
\hat{u}_4 = \sin(\phi) \hat{i} + \cos(\phi) \hat{j} \tag{3.2} \\
\hat{u}_1 = \sin(\theta) \hat{i} - \cos(\theta) \hat{j} \tag{3.3} \\
\hat{u}_2 = \cos(\theta) \hat{i} + \sin(\theta) \hat{j} \tag{3.4}
\]

For context the following force equations were determined. Though ultimately, unused, they provide an understanding of how the system moves. To begin, System 1 in Figure 3.2 was analyzed.

\[
\sum \vec{F} = m_1 \vec{a}_1 \\
F_y \hat{j} + F_x \hat{i} + R_y \hat{j} + R_x \hat{i} - g m_1 \hat{j} = m_1 \vec{a}_1 \tag{3.5}
\]

To determine the acceleration of System 1, the position of its center of mass must first be determined. Let point ‘o’ be the pivot in System 1 in
Figure 3.2, where \( \dot{u}_3 = -\phi \dot{u}_4 \) and \( \dot{u}_4 = \phi \dot{u}_3 \)

\[ \tilde{r}_{cm/0} = -L_1 \dot{u}_3 \]
\[ \tilde{v}_1 = \tilde{r}_{cm/0} = -L_1 \dot{u}_3 = L_1 \phi \dot{u}_4 \]
\[ \tilde{\alpha}_1 = \tilde{v}_1 = L_1 (\phi \dot{u}_4 + \dot{\phi} \dot{u}_4) = L_1 (\phi \dot{u}_4 + \phi^2 \ddot{u}_3) \]
\[ \tilde{\alpha}_1 = L_1 (\phi^2 \dot{u}_3 + \dot{\phi} \dot{u}_4) \] (3.6)

Equations (3.1) and (3.2) were substituted into Equation (3.6).

\[ \tilde{\alpha}_1 = L_1 (\phi^2 (\cos(\phi) \dot{i} - \sin(\phi) \dot{j}) + \ddot{\phi} (\sin(\phi) \dot{i} + \cos(\phi) \dot{j})) \]

Then Equation (3.6) was substituted into Equation (3.5).

\[ F_y \dot{j} + F_x \dot{i} + R_y \dot{j} + R_x \dot{i} - g m_1 \dot{j} = \]
\[ = m_1 \left( L_1 (\phi^2 (\cos(\phi) \dot{i} - \sin(\phi) \dot{j}) + \ddot{\phi} (\sin(\phi) \dot{i} + \cos(\phi) \dot{j})) \right) \]
\[ = m_1 L_1 \phi^2 \cos(\phi) \dot{i} - m_1 L_1 \phi^2 \sin(\phi) \dot{j} + m_1 L_1 \ddot{\phi} \sin(\phi) \dot{i} + m_1 L_1 \dddot{\phi} \cos(\phi) \dot{j} \]

The above vector equation is equivalent to the following two scalar equations

\[ F_y + R_y = L_1 m_1 \left( \frac{g}{L_1} + \phi^2 \sin(\phi) + \ddot{\phi} \cos(\phi) \right) \] (3.7)
\[ F_x + R_x = L_1 m_1 (\phi^2 \cos(\phi) - \dddot{\phi} \sin(\phi)) \] (3.8)

Next, the rotational dynamics of System 1 were analyzed. This analysis can be seen in the derivation of Equation (3.9), where \( r_{cm/0} = L_1 \dot{u}_3 \) and
\[ \ddot{\alpha}_1 = \dot{\phi} \hat{k} \]

\[
\sum M_{cm_1} = I_1 \ddot{\alpha}_1
\]

\[
I_1 \ddot{\alpha}_1 = r_{cm_1/o} \times F_y \hat{j} + r_{cm_1/o} \times F_x \hat{i} +
\]

\[
r_{cm_1/cm_1} \times R_y \hat{j} + r_{cm_1/cm_1} \times R_x \hat{i} - r_{cm_1/cm_1} \times g m_1 \hat{j}
\]

Next, System 2 in Figure 3.2 was analyzed. To determine the acceleration of System 2’s center of mass, the position of the center of mass must first be determined. Point ‘o’ is the pivot in System 1 in Figure 3.2, where \( \dot{u}_1 = \)
\( \dot{\theta} \hat{u}_2 = -\dot{\theta} \hat{u}_1, \dot{\theta} \hat{u}_3 = -\phi \hat{u}_4, \) and \( \dot{\phi} \hat{u}_4 = \phi \hat{u}_3. \)

\[
\ddot{r}_{cm2/o} = \ddot{r}_{cm2/cm1} + \ddot{r}_{cm1/o} = L_2 \hat{u}_1 - L_1 \hat{u}_3
\]

\[
\ddot{v}_2 = L_2 \hat{u}_1 - L_1 \hat{u}_3 = L_2 \dot{\theta} \hat{u}_2 + L_1 \dot{\phi} \hat{u}_4
\]

\[
\ddot{\alpha}_2 = L_2 (\dot{\theta} \hat{u}_2 + \dot{\phi} \hat{u}_2) + L_1 (\dot{\phi} \hat{u}_4 + \dot{\phi} \hat{u}_4)
\]

\[
= L_2 (\dot{\theta} \hat{u}_2 - \dot{\phi} \hat{u}_1) + L_1 (\dot{\phi} \hat{u}_4 + \dot{\phi} \hat{u}_3)
\]

\[
= L_2 \ddot{\theta} \hat{u}_2 - L_2 \dot{\phi} ^2 \hat{u}_1 + L_1 \ddot{\phi} \hat{u}_4 + L_1 \dot{\phi} ^2 \hat{u}_3
\]

\[
\ddot{\alpha}_2 = L_1 \ddot{\phi} (\sin (\phi) \hat{i} + \cos (\phi) \hat{j}) + L_1 \dot{\phi} ^2 (\cos (\phi) \hat{i} - \sin (\phi) \hat{j})
\]

\[
+ L_2 \ddot{\theta} (\cos (\theta) \hat{i} + \sin (\theta) \hat{j}) - L_2 \dot{\theta} ^2 (\sin (\theta) \hat{i} - \cos (\theta) \hat{j})
\]

From Newton's 2nd law

\[
\sum \ddot{F} = m_2 \ddot{\alpha}_2
\]

\[
-R_y \hat{j} - R_x \hat{i} - g m_2 \hat{j} = m_2 \ddot{\alpha}_2
\]

The above vector equation leads to the following two scalar equations

\[
-R_x = m_2 (L_1 \ddot{\phi} \sin \phi + L_1 \dot{\phi} ^2 \cos \phi + L_2 \ddot{\theta} \cos \theta - L_2 \dot{\theta} ^2 \sin \theta)
\]

\[
-R_y - g m_2 = m_2 (L_1 \ddot{\phi} \cos \phi - L_1 \dot{\phi} ^2 \sin \phi + L_2 \ddot{\theta} \sin \theta + L_2 \dot{\theta} ^2 \cos \theta)
\]

Applying Newton's 2nd law to the rotational dynamics of System 2 and
noting that \( r_{cm1/cm2} = -L_2 \dot{u}_1 \) and \( \ddot{a}_2 = \ddot{\theta} \hat{k} \),

\[
\sum \ddot{H}_{cm2} = I_2 \ddot{a}_{cm2} \\
I_2 \ddot{a}_2 = r_{cm1/cm2} \times (-R_y \hat{j}) + r_{cm1/cm2} \times (-R_x \hat{i}) + r_{cm2/cm2} \times (-g m_1 \hat{j}) \\
I_2 \ddot{a}_2 = r_{cm1/cm2} \times (-R_y \hat{j}) + r_{cm1/cm2} \times (-R_x \hat{i}) \\
I_2 \ddot{a}_2 = L_2 \dot{u}_1 \times R_y \hat{j} + L_2 \dot{u}_1 \times R_x \hat{i} \\
I_2 \ddot{a}_2 = L_2 (\sin(\theta) \hat{i} - \cos(\theta) \hat{j}) \times R_y \hat{j} + L_2 (\sin(\theta) \hat{i} - \cos(\theta) \hat{j}) \times R_x \hat{i} \\
I_2 \ddot{a}_2 = L_2 R_y \sin(\theta) \hat{i} \times \hat{j} - L_2 R_x \cos(\theta) \hat{j} \times \hat{i} \\
I_2 \ddot{a}_2 = L_2 R_y \sin(\theta) \hat{k} + L_2 R_x \cos(\theta) \hat{k} \\
I_2 \ddot{a}_2 = L_2 (R_y \sin(\theta) \hat{k} + R_x \cos(\theta) \hat{k})
\] (3.13)

These equations were not used for the dynamics of the system. Solving for the equations of motion directly using the Newtonian equations of motion proved to become complicated quite quickly. They are included to provide context and reference for future work. The equations of motion are determined with the alternative Lagrangian method in the next section.

### 3.2 Lagrangian Dynamics

To derive the dynamic equations of the actuated rimless wheel we will use the Lagrangian approach. Let \( q = (\phi, \theta) \) denote the generalized coordinates of the system, \( \mathcal{T} \) kinetic energy, \( \mathcal{V} \) potential energy and \( \mathcal{L} = \mathcal{T} - \mathcal{V} \)
the Lagrangian. Then, Lagrange’s equations of motion can be written as

$$\frac{d}{dt} \left( \frac{\partial (L + D)}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = f$$  \hspace{1cm} (3.14)$$

where $D$ is the dissipation function and $f$ a vector of external generalized forces. In our model we assume that there is no dissipation, so $D = 0$. To find the Lagrangian one must first find the kinetic and potential energy of the system.

The potential energy of the system is

$$V = g L_1 (m_1 + m_2) \sin(\phi) - g m_2 L_2 \cos(\theta)$$  \hspace{1cm} (3.15)$$

The kinetic energy is the sum of the kinetic energies of the wheel $T_w$ and the pendulum $T_p$ where

$$T_w = \frac{1}{2} m_1 |\vec{v}_1|^2 + \frac{1}{2} I_1 \dot{\phi}^2$$  \hspace{1cm} (3.16)$$

$$T_p = \frac{1}{2} m_2 |\vec{v}_2|^2 + \frac{1}{2} I_2 \dot{\theta}^2$$  \hspace{1cm} (3.17)$$

Since the velocities of the center of mass with respect to the point ‘o’ is

$$\vec{v}_1 = L_1 \dot{\phi} \left( \sin(\phi) \hat{i} + \cos(\phi) \hat{j} \right)$$

$$\vec{v}_2 = L_2 \dot{\theta} \left( \cos(\theta) \hat{i} + \sin(\theta) \hat{j} \right) + L_1 \dot{\phi} \left( \sin(\phi) \hat{i} + \cos(\phi) \hat{j} \right)$$  \hspace{1cm} (3.18)$$
It follows that

\[ T_w = \frac{1}{2} m_1 |\vec{v}_{cm1/o}|^2 + \frac{1}{2} I_1 \dot{\phi}^2 \]
\[ = \frac{1}{2} m_1 |L_1 \dot{\phi} (\sin(\phi) \hat{i} + \cos(\phi) \hat{j})|^2 + \frac{1}{2} I_1 \dot{\phi}^2 \]
\[ = \frac{1}{2} m_1 |L_1 \dot{\phi} \sin(\phi) \hat{i} + L_1 \dot{\phi} \cos(\phi) \hat{j}|^2 + \frac{1}{2} I_1 \dot{\phi}^2 \]
\[ = \frac{1}{2} m_1 (L_1 \dot{\phi} \sin(\phi))^2 + \frac{1}{2} m_1 (L_1 \dot{\phi} \cos(\phi))^2 + \frac{1}{2} I_1 \dot{\phi}^2 \]
\[ T_w = \frac{(L_1^2 m_1 + I_1) \dot{\phi}^2}{2} \tag{3.19} \]

\[ T_p = \frac{1}{2} m_2 |L_2 \dot{\theta} (\cos(\theta) \hat{i} + \sin(\theta) \hat{j}) + L_1 \dot{\phi} (\sin(\phi) \hat{i} + \cos(\phi) \hat{j})|^2 + \frac{1}{2} I_2 \dot{\theta}^2 \]
\[ = \frac{1}{2} m_2 |(L_2 \dot{\theta} \cos(\theta) + L_1 \dot{\phi} \sin(\phi)) \hat{i} + (L_2 \dot{\theta} \sin(\theta) + L_1 \dot{\phi} \cos(\phi)) \hat{j}|^2 + \frac{1}{2} I_2 \dot{\theta}^2 \]
\[ = \frac{1}{2} m_2 (L_2 \dot{\theta} \cos(\theta))^2 + \frac{1}{2} m_2 2L_2 \dot{\theta} \cos(\theta) L_1 \dot{\phi} \sin(\phi) \]
\[ + \frac{1}{2} m_2 (L_1 \dot{\phi} \sin(\phi))^2 + \frac{1}{2} m_2 (L_2 \dot{\theta} \sin(\theta))^2 \]
\[ + \frac{1}{2} m_2 2L_2 \dot{\theta} \sin(\theta) L_1 \dot{\phi} \cos(\phi) + \frac{1}{2} m_2 (L_1 \dot{\phi} \cos(\phi))^2 + \frac{1}{2} I_2 \dot{\theta}^2 \]
\[ T_p = \frac{(L_2^2 m_2 + I_2) \dot{\theta}^2}{2} + \frac{L_1^2 m_2 \dot{\phi}^2}{2} + m_2 L_1 L_2 \dot{\theta} \dot{\phi} \sin(\theta + \phi) \tag{3.20} \]

Finally the total kinetic energy is

\[ T = T_w + T_p \]
\[ = \frac{(L_1^2 m_1 + L_1^2 m_2 + I_1) \dot{\phi}^2}{2} + \frac{(L_2^2 m_2 + I_2) \dot{\theta}^2}{2} + m_2 L_1 L_2 \dot{\theta} \dot{\phi} \sin(\theta + \phi) \tag{3.21} \]
To express these equations compactly, define the following variables.

\[ J_{e0} = m_1 L_1^2 + I_1 \]  \hfill (3.22)
\[ J_{e1} = (m_1 + m_2) L_1^2 + I_1 \]  \hfill (3.23)
\[ J_{e2} = m_2 L_2^2 + I_2 \]  \hfill (3.24)
\[ J_{e3} = L_1 L_2 m_2 \]  \hfill (3.25)

Substituting in the above equations

\[ V = g L_1 (m_1 + m_2) \sin(\phi) - g \cos(\theta) L_2 m_2 \]  \hfill (3.26)
\[ T = \frac{1}{2} J_{e1} \dot{\phi}^2 + \frac{1}{2} J_{e2} \dot{\theta}^2 + J_{e3} \sin(\phi + \theta) \dot{\phi} \dot{\theta} \]  \hfill (3.27)

The Lagrangian of the system is

\[ \mathcal{L} = (L_2 m_2 \cos \theta - L_1 (m_1 + m_2) \sin \phi) g 
+ \frac{J_{e1} \dot{\phi}^2 + J_{e2} \dot{\theta}^2}{2} + J_{e3} \sin(\phi + \theta) \dot{\phi} \dot{\theta} \]  \hfill (3.28)

**Remark 1** Note that the kinetic energy has the general form

\[ \mathcal{T}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} \dot{\mathbf{q}}^T M(\mathbf{q}) \dot{\mathbf{q}} \]  \hfill (3.29)
\[ M(\mathbf{q}) = \begin{bmatrix}
J_{e1} & J_{e3} \sin(\phi + \theta) \\
J_{e3} \sin(\phi + \theta) & J_{e2}
\end{bmatrix} \]  \hfill (3.30)

Since \( V \) does not depend on \( \dot{\mathbf{q}} \), then

\[ \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} = M(\mathbf{q}) \dot{\mathbf{q}} \]  \hfill (3.31)
\[
\frac{\partial \mathcal{L}}{\partial \theta} = J_{e_3} \dot{\phi} \dot{\theta} \cos (\phi + \theta) - g \sin (\theta) L_2 m_2 \\
\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \dot{\theta} J_{e_2} + J_{e_3} \dot{\phi} \sin (\phi + \theta) \\
\frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) = \ddot{\theta} J_{e_2} + \ddot{\phi} \sin (\phi + \theta) J_{e_3} + \dot{\phi} \cos (\phi + \theta) J_{e_3} (\dot{\phi} + \dot{\theta}) \\
f_1 = T \\
\frac{\partial \mathcal{L}}{\partial \phi} = J_{e_3} \dot{\phi} \dot{\theta} \cos (\phi + \theta) - g \alpha \cos (\phi) L_1 (m_1 + m_2) \\
\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} J_{e_1} + J_{e_3} \dot{\phi} \sin (\phi + \theta) \\
\frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) = \ddot{\phi} J_{e_1} + \ddot{\phi} \sin (\phi + \theta) J_{e_3} + \dot{\phi} \cos (\phi + \theta) J_{e_3} (\dot{\phi} + \dot{\theta}) \\
f_2 = 0
\]

The Euler-Lagrange equations are

\[
J_{e_3} \sin (\phi + \theta) \ddot{\theta} + J_{e_1} \ddot{\theta} + J_{e_3} \cos (\phi + \theta) \dot{\theta}^2 + L_1 (m_1 + m_2) \cos \phi g = 0 \\
J_{e_2} \ddot{\theta} + J_{e_1} \sin (\phi + \theta) \ddot{\phi} + J_{e_3} \cos (\phi + \theta) \dot{\phi}^2 + L_2 m_2 \sin \theta g = T
\]

**Remark 2** The Euler-Lagrange equations have the general form

\[
M(q) \ddot{q} + C(q, \dot{q}) \dot{q} + K(q) = f
\]
where \( f = (0; T) \), \( M(q) \) is given by (3.30) and

\[
C(q, \dot{q}) = \begin{bmatrix}
0 & J_{e3} \cos(\phi + \theta) \dot{\theta} \\
J_{e3} \cos(\phi + \theta) \dot{\phi} & 0
\end{bmatrix} \\
K(q) = \begin{bmatrix}
L_1 (m_1 + m_2) \cos \phi \\
L_2 m_2 \sin \theta
\end{bmatrix} g
\]

(3.41)

(3.42)

Solving for \( \ddot{q} \)

\[
\ddot{q} = -M(q)^{-1} C(q, \dot{q}) \dot{q} - M(q)^{-1} K(q) + M(q)^{-1} f
\]

(3.43)

where

\[
M(q)^{-1} = \frac{1}{\gamma} \begin{bmatrix}
J_{e2} & -J_{e3} \sin(\phi + \theta) \\
-J_{e3} \sin(\phi + \theta) & J_{e1}
\end{bmatrix}
\]

(3.44)

\[
\gamma = J_{e1} J_{e2} - J_{e3}^2 \sin(\phi + \theta)^2
\]

(3.45)

Expanding (3.43) and using the fact that \( x = (q; \dot{q}) = (\phi; \theta; \dot{\phi}; \dot{\theta}) \) the equations of motion of the actuated rimless wheel in state-space form are:

\[
\frac{d}{dt} \phi = \dot{\phi}
\]

\[
\frac{d}{dt} \theta = \dot{\theta}
\]

\[
\gamma \frac{d}{dt} \dot{\phi} = -g \left( \cos \phi J_{e2} L_1 m_1 + \cos \phi J_{e2} L_1 m_2 - \sin(\phi + \theta) \sin \theta J_{e3} L_2 m_2 \right)
\]

\[
- J_{e3}^2 \cos(\phi + \theta) \sin(\phi + \theta) \dot{\phi}^2 - J_{e2} J_{e3} \cos(\phi + \theta) \dot{\theta}^2
\]

\[
- \sin(\phi + \theta) J_{e3} T
\]

\[
\gamma \frac{d}{dt} \dot{\theta} = g \left( \sin(\phi + \theta) \cos(\phi) J_{e3} L_1 m_1 - \sin(\theta) J_{e1} L_2 m_2 \right)
\]

\[
+ g \sin(\phi + \theta) \cos(\phi) J_{e3} L_1 m_2 - J_{e1} J_{e3} \cos(\phi + \theta) \dot{\phi}^2
\]

\[
+ J_{e3}^2 \cos(\phi + \theta) \sin(\phi + \theta) \dot{\theta}^2 + J_{e1} T
\]

(3.46)
Note that the dynamic equations are affine in the control input $u = T$, that is the torque applied to the internal pendulum.

### 3.3 Actuator Dynamics

When implementing a prototype of this system, the torque input is provided by an actuator, in this case a permanent magnet DC motor. Therefore, the kinematics of the actuator must be taken into account. A linear model of the actuator can be obtained assuming the motor has negligible armature inductance and negligible mechanical losses. The motor can then be represented by the generalized circuit model of Fig. 3.3, where $V_m$ is the input armature voltage, $i_m$ is the motor armature current, $R_m$ is the motor armature resistance, $K_m$ is the motor constant, $\omega_m$ is the motor angular velocity, $T_m$ is the electromechanical torque of the motor. The dependent sources model the electromechanical conversion. The gearbox with gear ratio $N$ is modeled as an ideal transformer. The inertia of the rotor $\frac{J_m}{N^2}$ has been reflected to the output of the transformer and appears in parallel to $I_2$, the pendulum inertia. Finally, the torque delivered to the pendulum is $T$ and $\dot{\theta}$ is the angular velocity of the pendulum.

Since the motor is mounted on the wheel frame (see Fig. 3.1)

\[
\psi = \theta - \left(\frac{\pi}{2} - \phi\right) \quad (3.47)
\]
\[
\dot{\psi} = \dot{\phi} + \dot{\theta} \quad (3.48)
\]
The dynamic equations can be obtained applying Kirchhoff’s Laws to the
generalized circuit. Using Kirchhoff’s voltage law

\[ V_m(t) = R_m i_m(t) + K_m \omega_m(t) \]  

(3.49)

The equations for the ideal transformer yield a relation of motor velocity,
\( \omega_m \), to pendulum angular velocity, \( \dot{\theta} \)

\[ \omega_m(t) = N \dot{\psi}(t) = N \left( \dot{\phi}(t) + \dot{\theta}(t) \right) \]  

(3.50)

Substituting equation (3.50) into (3.49) yields

\[ V_m(t) = R_m i_m(t) + K_m N \left( \dot{\phi}(t) + \dot{\theta}(t) \right) \]  

(3.51)

The current dependent source gives the relation between the electromechanical torque and the armature current which can be related to the output torque via the ideal transformer as

\[ T_m(t) = K_m i_m(t) = \frac{T(t)}{N} \Rightarrow T(t) = NK_m i_m(t) \]  

(3.52)
Solving equation (3.51) for $i_m(t)$ and substituting in (3.52) yields

$$T(t) = -\frac{(K_mN)^2}{R_m}\dot{\phi}(t) - \frac{(K_mN)^2}{R_m}\dot{\theta}(t) + \frac{K_mN}{R_m}V(t) \quad (3.53)$$

This equation models the actuator. Replacing the torque $T$ in the equations of motion we obtain a new set of equations with state $x^T = (\phi, \dot{\phi}, \theta, \dot{\theta})$ and voltage input $u(t) = V(t)$ given by,

$$\dot{x} = p(x) + n(x)u$$

where $\gamma = J_{e_1}J_{e_2} - J_{e_3}^2 \sin(\phi + \theta)^2$ as in (3.45) and

$$p_1(x) = \dot{\phi} \quad (3.54)$$

$$p_2(x) = \dot{\theta} \quad (3.55)$$

$$\gamma p_3(x) = J_{e_3}^2 \sin(\phi + \theta) \cos(\phi + \theta) \dot{\phi}^2 - J_{e_2}J_{e_3} \cos(\phi + \theta) \dot{\theta}^2$$

$$+ J_{e_3}K_m^2N^2 \sin(\phi + \theta) \dot{\theta} + J_{e_3}K_m^2N^2 \sin(\phi + \theta) \dot{\phi}$$

$$- g J_{e_2}L_1 (m_1 + m_2) \cos(\phi)$$

$$+ g J_{e_3}L_2 m_2 \sin(\phi + \theta) \sin(\theta) \quad (3.56)$$

$$\gamma p_4(x) = J_{e_3}^2 \sin(\phi + \theta) \cos(\phi + \theta) \dot{\theta}^2 - J_{e_1}J_{e_3} \cos(\phi + \theta) \dot{\phi}^2$$

$$- J_{e_1}K_m^2N^2 \dot{\theta} - J_{e_1}K_m^2N^2 \dot{\phi}$$

$$g \left( J_{e_3}L_1 m_1 \sin(\phi + \theta) \cos(\phi) - J_{e_1}L_2, m_2 \sin(\theta) \right)$$

$$+ gJ_{e_3}L_1 m_2 \sin(\phi + \theta) \cos(\phi) \quad (3.57)$$
\[ n(x) = \frac{1}{\gamma R_m} \begin{bmatrix} 0 \\ 0 \\ -J_e K_m N^2 \sin(\phi + \theta) \\ J_e K_m N^2 \end{bmatrix} \] (3.58)

In the equations of the system the inertia of the motor reflected to the load is either assumed to be negligible or lumped into the inertia of the pendulum.

### 3.4 Collision Dynamics

When a foot of the wheel impacts the ground, a collision occurs. This collision is the main source of energy losses during locomotion. For simplicity it is assumed that the collisions are perfectly inelastic.
Collisions can occur when
\[
\phi = \begin{cases} 
\frac{\pi - \alpha}{2}, & \text{(backward step)} \\
\frac{\pi + \alpha}{2}, & \text{(forward step)} 
\end{cases}
\]
where \(\alpha = 2\pi / n\) is the angle subtended between legs and \(n\) is the number of legs. Here after we will assume that the wheel takes only forward steps.

Momentum, not energy, is conserved through a collision. This concept is the basis of the derivation of the collision equations. Let \(\vec{H}_s^+\) and \(\vec{H}_s^-\) denote the system's angular momentum just after and before collision, respectively. Similarly \(\vec{H}_p^+\) and \(\vec{H}_p^-\) will denote the momentum of the pendulum.

\[
\vec{H}_p^+ = I_2 \dot{\theta}^+ \hat{k} + \vec{r}_{cm2/a}^+ \times m_2 \vec{v}_2^+ \\
\vec{H}_p^- = I_2 \dot{\theta}^- \hat{k} + \vec{r}_{cm2/a}^- \times m_2 \vec{v}_2^-
\]

\[
\vec{H}_s^+ = -I_1 \dot{\phi}^+ \hat{k} + \vec{r}_{cm1/b}^+ \times m_1 \vec{v}_1^+ + I_2 \dot{\theta}^+ \hat{k} + \vec{r}_{cm2/b}^+ \times m_2 \vec{v}_2^+ \\
\vec{H}_s^- = -I_1 \dot{\phi}^- \hat{k} + \vec{r}_{cm1/b}^- \times m_1 \vec{v}_1^- + I_2 \dot{\theta}^- \hat{k} + \vec{r}_{cm2/b}^- \times m_2 \vec{v}_2^-
\]
where

\[
\begin{align*}
\vec{r}_{cm2/a}^+ &= L_2 \dot{u}_1^+ \\
\vec{v}_2^+ &= L_1 \dot{\phi}^+ \dot{u}_4^+ + L_2 \dot{\theta}^+ \dot{u}_2^+ \\
\vec{r}_{cm2/a}^- &= L_2 \dot{u}_1^- \\
\vec{v}_2^- &= L_1 \dot{\phi}^- \dot{u}_4^- + L_2 \dot{\theta}^- \dot{u}_2^- \\
\vec{r}_{cm1/b}^+ &= -L_1 \dot{u}_3^+ \\
\vec{v}_1^+ &= L_1 \dot{\phi}^+ \dot{u}_4^+ \\
\vec{r}_{cm2/b}^+ &= -L_1 \dot{u}_3^- + L_2 \dot{u}_1^- \\
\vec{r}_{cm1/b}^- &= -L_1 \dot{u}_4^- \\
\vec{v}_1^- &= L_1 \dot{\phi}^- \dot{u}_4^- \\
\vec{r}_{cm2/b}^- &= -L_1 \dot{u}_4^- + L_2 \dot{u}_1^-
\end{align*}
\]

Substituting the above expressions in the momentum equations leads to the following scalar equations:

\[
\begin{align*}
H_p^+ &= L_1 L_2 m_2 \sin(\phi^+ + \theta^+) \dot{\phi}^+ + \left( m_2 L_2^2 + I_2 \right) \dot{\theta}^+ \quad (3.63) \\
H_p^- &= L_1 L_2 m_2 \sin(\phi^- + \theta^-) \dot{\phi}^- + \left( m_2 L_2^2 + I_2 \right) \dot{\theta}^- \quad (3.64)
\end{align*}
\]
\[ H_s^+ = \left( L_1 L_2 m_2 \sin(\phi^+ + \theta^+) - L_1^2 (m_1 + m_2) - I_1 \right) \dot{\phi}^+ \]
\[ + \left( I_2 + L_2^2 m_2 \left( \sin(\phi^+ + \theta^+) - 1 \right) \right) \dot{\theta}^+ \]
\[ H_s^- = \left( L_1 L_2 m_2 \sin(\phi^- + \theta^-) - I_1 \right) \dot{\phi}^- \]
\[ + \left( L_1 L_2 m_2 \cos(\phi^- + \theta^-) + I_2 + L_2^2 m_2 \right) \dot{\theta}^- \]  

Using the variables defined in (3.22)-(3.25) the above equations become

\[ H_p^+ = J_{e_3} \sin(\phi^+ + \theta^+) \dot{\phi}^+ + J_{e_2} \dot{\theta}^+ \]  
\[ H_p^- = J_{e_3} \sin(\phi^- + \theta^-) \dot{\phi}^- + J_{e_2} \dot{\theta}^- \]  
\[ H_s^+ = \left( J_{e_3} \sin(\phi^+ + \theta^+) - J_{e_1} - I_1 \right) \dot{\phi}^+ \]
\[ + \left( I_2 + L_2^2 m_2 \left( \sin(\phi^+ + \theta^+) - 1 \right) \right) \dot{\theta}^+ \]
\[ H_s^- = \left( J_{e_3} \sin(\phi^- + \theta^-) - I_1 \right) \dot{\phi}^- + \left( J_{e_3} \cos(\phi^- + \theta^-) + J_{e_2} \right) \dot{\theta}^- \]  

The collision equations for forward stepping are given by the following four algebraic equations:

\[ \phi^+ = \phi^- - \alpha \]  
\[ \theta^+ = \theta^- \]  
\[ H_p^+ = H_p^- \]  
\[ H_s^+ = H_s^- \]
Since the first two equations are linear they can be solved first and substituted in the last two nonlinear equations. To do this first write the equations in matrix using the fact that \( x = (\phi, \dot{\phi}, \theta, \dot{\theta}) \) is the state vector. Note that \( H_p^+ = H_p(x^+) \) and \( H_p^- = H_p(x^-) \) where

\[
H_p(x) = h_1(\phi, \theta) \dot{\phi} + h_2(\phi, \theta) \dot{\theta} \tag{3.75}
\]

where

\[
h_1(\phi, \theta) = J_{e_3} \sin(\phi + \theta) \tag{3.76}
\]

\[
h_2(\phi, \theta) = J_{e_2} \tag{3.77}
\]

Following a similarly approach for \( H_s^+ \) and \( H_s^- \) define

\[
h_3(\phi, \theta) = J_{e_3} \sin(\phi + \theta) - J_{e_1} - I_1 \tag{3.78}
\]

\[
h_4(\phi, \theta) = I_2 + L_2^2 m_2 (\sin(\phi + \theta) - 1) \tag{3.79}
\]

\[
h_5(\phi, \theta) = J_{e_3} \sin(\phi + \theta) - I_1 \tag{3.80}
\]

\[
h_6(\phi, \theta) = J_{e_3} \cos(\phi + \theta) + J_{e_2} \tag{3.81}
\]

Therefore, the collision equations are:

\[
\begin{bmatrix}
  h_1(\phi^+, \theta^+) & h_2(\phi^+, \theta^+)
  \\
  h_3(\phi^+, \theta^+) & h_4(\phi^+, \theta^+)
\end{bmatrix}
\begin{bmatrix}
  \dot{\phi}^+
  \\
  \dot{\theta}^+
\end{bmatrix}
= \begin{bmatrix}
  h_1(\phi^-, \theta^-) & h_2(\phi^-, \theta^-)
  \\
  h_5(\phi^-, \theta^-) & h_6(\phi^-, \theta^-)
\end{bmatrix}
\begin{bmatrix}
  \dot{\phi}^-
  \\
  \dot{\theta}^-
\end{bmatrix} \tag{3.82}
\]

\[
M_L(\phi^+, \theta^+) \begin{bmatrix}
  \dot{\phi}^+
  \\
  \dot{\theta}^+
\end{bmatrix} = M_R(\phi^-, \theta^-) \begin{bmatrix}
  \dot{\phi}^-
  \\
  \dot{\theta}^-
\end{bmatrix}
\]
Note that, the left hand side matrix depends on the post-collision positions. It is easy to see that from (3.71), (3.72) $M_L(\phi^+, \theta^+) = M_L(\phi^-, \alpha, \theta^-)$

Solving for the post collision velocities:

$$\begin{bmatrix} \dot{\phi}^+ \\ \dot{\theta}^+ \end{bmatrix} = [M_L(\phi^+, \theta^+)]^{-1} M_R(\phi^-, \theta^-) \begin{bmatrix} \dot{\phi}^- \\ \dot{\theta}^- \end{bmatrix}$$

$$= \frac{1}{h_1^+ h_4^- - h_2^+ h_3^+} \begin{bmatrix} h_4^- & -h_2^+ \\ -h_3^+ & h_1^+ \end{bmatrix} \begin{bmatrix} h_1^- & h_2^- \\ h_5^- & h_6^- \end{bmatrix} \begin{bmatrix} \dot{\phi}^- \\ \dot{\theta}^- \end{bmatrix}$$

(3.83)

(3.84)

where we have dropped the explicit dependence of $h_i(\cdot)$ on $\theta$ and $\phi$ and used $h^+$ to denote $h(\phi^+, \theta^+) = h(\phi^- - \alpha, \theta^-)$. A closed form expression for the above equations can be obtained replacing the expressions in equations (3.76)-(3.81) for $h_i$ and is given below

$$\eta \dot{\phi}^+ = \dot{\theta}^- \left( J_2 e_2^2 - J_2 e_2 + \cos(\phi^- + \theta^-) J_2 e_3 \right) + \dot{\theta}^- \left( J_2 e_2^2 m_2 + \sin(\alpha - \phi^- - \theta^-) J_2 e_2^2 m_2 \right) + \dot{\phi}^- \left( \sin(\phi^- + \theta^-) J_2 e_3 - \sin(\phi^- + \theta^-) J_2 e_3 \right) + \dot{\phi}^- \left( \sin(\phi^- + \theta^-) J_3 e_2^2 m_2 - J_1 e_2 \right) + \dot{\phi}^- \left( \sin(\phi^- + \theta^-) J_3 e_2^2 m_2 - J_1 e_2 \right)$$

(3.85)

$$\eta \dot{\theta}^+ = -\dot{\phi}^- \left( \sin(\alpha - \phi^- - \theta^-) J_1 e_3 + \sin(\phi^- + \theta^-) J_1 e_3 \right) - \dot{\theta}^- \left( J_1 e_2 - \sin(\alpha - \phi^- - \theta^-) \cos(\phi^- + \theta^-) J_3 e_2^2 \right)$$

(3.86)
where

\[
\eta = J_{e1} J_{e2} + \sin(\alpha - \phi^- - \theta^-) J_{e3} \left( m_2 L_2^2 - J_2 + J_{e2} \right) \\
+ \sin(\alpha - \phi^- - \theta^-)^2 J_{e3} L_2^2 m_2
\] (3.87)

### 3.5 Overview of Optimal Control Theory

Optimal control theory is the study of dynamic optimization problems where the user seeks to obtain a control strategy, \( u \), that minimizes a cost functional \( J(u) \) subject to the dynamics of the system to be controlled and constraints on the allowable set of control and states. The functional \( J \) captures the "cost of control" and it has the general form of equation (3.88). It has two distinct components: the integral part, or Lagrangian, component and the scalar \( M \), or Mayer, component. The form of the cost functional below is known as the Bolza form[26][30][23][24].

\[
J(u) = M(x(t_0), t_0, x(t_f), t_f) + \int_{t_0}^{t_f} \mathcal{L}(x(t), u(t), t) \, dt 
\] (3.88)

In its simplest form the objective of an optimal control problem is to find

\[
\min_u J(u)
\] (3.89)

subject to the dynamic constraints, (3.90), and the initial conditions, (3.91).

\[
\dot{x}(t) = f(x(t), u(t), t) \\
x(t_0) = x_0(t)
\] (3.90)
The input to the system, $u(t)$ is a time function that must itself satisfy the dynamics of the system it is affecting. This identifies an optimal control problem.

Adjoining the dynamic constraints into the cost function yields

$$J(u) = M(x(t_0), t_0, x(t_f), t_f)$$

$$+ \int_{t_0}^{t_f} \left\{ \mathcal{L}(x(t), u(t), t) + \lambda^T(t) \left[ f(x(t), u(t), t) - \dot{x} \right] \right\} dt$$

(3.92)

where $\lambda(t)$ is the co-state vector. Each equality constraint on the system is integrated into the cost in this manner.

After obtaining the adjoined integrated performance index the critical, or stationary, points must be determined. Two conditions for a point to be stationary are a positive or negative definite Hessian matrix and a zero gradient matrix. The gradient matrix being equal to zero indicates that the system is not moving when at that point. A positive definite Hessian indicates a minimum and a negative definite Hessian indicates a maximum [25].

The necessary conditions for the optimality of a solution of an optimal control problem can be derived using the calculus of variations. These conditions for optimality are summarized in "Pontryagin's Minimum Principle". It is important to note, that these conditions are only necessary. This implies that a solution satisfying the "Pontryagin's Minimum Principle" is a candidate optimal solution, but not necessarily the only one.
The determination of a control trajectory that obtains an optimization requires solving a two point boundary value problem (TPBVP). The system dynamics in equation (3.90) and the co-state dynamics

\[
\dot{\lambda} = -H_x^T \equiv -\mathcal{L}_x^T - f_x^T \lambda
\]

subject to the boundary conditions at the initial conditions \( \dot{x}(t_0) \) and

\[
\lambda(t_f) = M_x^T
\]

where the subscripts denote partial derivatives.

The solution of an optimal control problem does not have, in general, a closed form. Therefore, most practical problems must be solved numerically. A number of packages and programs exist for solving optimal control problems numerically such as ACADO [31], GPOPS-II [32], ICLOCS [33] and BOCOP [34].

In this thesis we will use GPOPS-II to solve a number of optimal control problems that arise in the study of efficient locomotion of the experimental RIT rimless Wheel. GPOPS-II was chosen over other packages because of its ability to solve multiple phase problems and its ease of use.
Chapter 4

Experiments and Results

In this chapter we outline the numerical approach to solution of the optimal control problem that generates control trajectories. An overview of the problem setup used by the GPOP-II toolbox to solve single and multi-phase optimal control problems is given. Additionally, the methods of verification are also outlined. All code and design documents are available upon request.

4.1 Overview of Implementation

Upon determining the equations governing the states of the system, a working simulation was required. Creating programs that could both generate optimal trajectories and accurately verify these trajectories was imperative to the success of this thesis. GPOPS-II’s results must be verified independently. For this reason, two separate systems needed to be created.

The optimal control program and the system validation program were
created independently of one another. The only way they are similar is in that the equations of motion and collision equations are identical.

To begin, the validation simulation for the dynamics of the system was created. The solver ODE45, which is a MATLAB implementation of Dormand Prince’s explicit Runge-Kutta (4,5) formula, was used to simulate the system. The simulations were provided with the initial conditions of an execution of GPOPS-II and a similarly generated control trajectory.

GPOPS-II creates control trajectories with a dynamic time step. To provide input to the ODE45 simulation, a constant time re-sampled signal was required. To create this control signal, a linear interpolation was performed on the GPOPS-II control solution. This solution was then provided to ODE45 as the input trajectory for the system. The sampling rate of the linear interpolation was set equal to 1 kHz. Unfortunately if the signal has changes faster than 500 Hz, aliasing will create information loss.

In the event that a multi-step validation is required, ODE45 is provided with a event detection function. A collision occurs when equation (4.1) is satisfied.

\[
0 = \sin(\pi - \phi) - \sin\left(\frac{\pi - \alpha}{2}\right) \tag{4.1}
\]

This event is terminal and will allow the simulation to reinitialize the problem with the next steps initial conditions. This condition is only terminal
in the negative direction however as the system should never be underground.

Upon the completion of the simulation, the energy in the system was analyzed. This input is compared to expectations and goals. When the motor losses are removed, the energy in the system is constant. These results can then be recorded.

After creating the equations of motion and validating them with the verification model, an optimal control trajectory was required. To do this, a new folder was created. GPOPS-II requires a minimum of two external functions to run. These functions are the \texttt{rimlessWheelEndpoint.m} and \texttt{rimlessWheelContinuous.m} functions and must be declared. In addition to these two functions, they must be called by a main script that sets up the system state.

The main files contain a call to \texttt{gpops2}. This call can be seen in line 20 of Listing 4.1. This call requires a setup structure that contains all of the boundary and initial condition constraints. In addition, the continuous and endpoint functions must be declared, as in line 3 and 4.

\begin{verbatim}
Listing 4.1: GPOPS-II Function Call
1  setup.name = strcat(fileName,stamp,'.');
2  % Function names dynamics, cost, etc.
3  setup.functions.continuous = @rimlessWheelContinuous;
4  setup.functions.endpoint = @rimlessWheelEndpoint;
5  % Data, bounds and guess
6  setup.auxdata = auxdata;
7  setup.bounds = bounds;
8  setup.guess = guess;
\end{verbatim}
The rimlessWheelContinuous.m function contains the dynamic equations of the system and the cost function. The choice of cost function has a significant impact on the locomotion trajectories of the system and will be detailed for each individual situation. The dynamic equations used in this function were derived in sections 3.1 through 3.3.

The rimlessWheelEndpoint.m has the algebraic conditions that linked the multi-phase problems together. In the event of a colliding plastic or elastic step, the equations derived in section 3.4 are used to update the states between phases.

4.2 System Parameters

The walking wheel system has a number of physical parameters. Table 4.1 summarized the nominal values for each of the physical parameters of the system. \( m_1, I_1, \) and \( L_1 \) are the mass, moment of inertia, and leg length.
of the rimless wheel. \( m_2, I_2, \) and \( L_2 \) are the mass, moment of inertia, and half length of the pendulum. (e.g., distance from pivot to center of mass.)

**Remark 3** *The moments of inertia depend on the geometry, material and axis of rotation of each element in motion, so \( I_1 \) and \( m_1 \) are not independent and neither are \( I_2 \) and \( m_2 \). In general it is not possible to give explicit closed form expressions for the inertias, however, in the case of the pendulum actuator it is easy to show that*

\[
I_2 = \frac{1}{3} m_2 L_2^2
\]

(4.2)

**Table 4.1: Nominal Parameter Values**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_1 [\text{kg}] )</td>
<td>1</td>
<td>( m_2 [\text{kg}] )</td>
<td>1</td>
</tr>
<tr>
<td>( I_1 [\text{kg}^2] )</td>
<td>1</td>
<td>( I_2 [\text{kgm}^2] )</td>
<td>1</td>
</tr>
<tr>
<td>( L_1 [\text{m}] )</td>
<td>1</td>
<td>( L_2 [\text{m}] )</td>
<td>0.5</td>
</tr>
</tbody>
</table>

The internal pendulum is driven by a geared DC motor attached to the rimless wheel. The dynamic equation of the motor driven are derived in section 3.3. The parameters of the motor are given in Table 4.2 and were selected from manufacturer specifications data sheets.

**Table 4.2: Motor Parameters**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K_m [\text{Nm/A}] ), ( \frac{\text{V}}{\text{rad/sec}} )</td>
<td>0.0048</td>
</tr>
<tr>
<td>( N )</td>
<td>505.9</td>
</tr>
<tr>
<td>( R_m [\Omega] )</td>
<td>5</td>
</tr>
</tbody>
</table>
Unless explicitly stated, the parameters outlined above will be used for each simulation.

4.3 Program Setup

In this thesis, a number of numerical experiments were run. The numerical solution of the optimal control problem requires us to set bounds on all quantities optimized for each phase. These bounds are critical for the convergence of the optimization problem to a solution. Listing 4.2 summarizes the bounds used in the numerical optimization problems. In this context a phase is a segment of the evolution of the system until a predefined event occurs. In our case a phase is the period between starting and finishing a step.

Listing 4.2: GPOPS-II Bounds Declaration

```plaintext
% GPOPSII bound structure

%% Phase 1 bounds
iphase = 1;

% initial time bounds
bounds.phase(iphase).initialtime.lower = t0Min;
bounds.phase(iphase).initialtime.upper = t0Max;

% final time bounds
bounds.phase(iphase).finaltime.lower = tfMin;
bounds.phase(iphase).finaltime.upper = tfMax/numSteps;

% initial state bounds
bounds.phase(iphase).initialstate.lower = [x10, x20min, x30min, x40min];
bounds.phase(iphase).initialstate.upper = [x10, x20max, x30max, x40max];

% Final State bounds
bounds.phase(iphase).finalstate.lower = [x1f, x2min, x3min, x4min];
```
bounds.phase(iphase).finalstate.upper = [x1f, x2max, x3max, x4max];

% states bounds during phase
bounds.phase(iphase).state.lower = [x1min, x2min, x3min, x4min];
bounds.phase(iphase).state.upper = [x1max, x2max, x3max, x4max];

% controls bounds during phase
bounds.phase(iphase).control.lower = uMin;
bounds.phase(iphase).control.upper = uMax;

% cost bounds during phase
bounds.phase(iphase).integral.lower = 0;
bounds.phase(iphase).integral.upper = 1000;

% parameter bounds
bounds.parameter.lower = [m2Min];
bounds.parameter.upper = [m2Max];

4.4 Experiments

To gain a better understanding of the motion of a rimless wheel, the following questions needed to be answered.

1. Can we generate a step with no limiting constraints on initial or final conditions?

2. Can we generate a collisionless step with no limiting constraints on initial conditions?

3. Can we generate two colliding steps without strict collision modeling?
4. Can we generate three colliding steps without strict collision modeling?

5. Can we generate two steps while integrating accurate collision modeling?

6. Is there an alternative to multiphase numerical optimization that can be explored?

These six goals layout the road map to the exploration of the motion characteristics of the rimless wheel and serve to define the different experiments performed. In the next sections the details of these experiments will be presented. The rimless wheel studied in this work has \( n = 5 \) legs and the angle between legs is \( \alpha = \frac{2}{n} \pi \). Also it was assumed for the simulations that the motor was attached to a fixed frame. This implies that \( \dot{\psi} = \dot{\theta} \).

### 4.4.1 Single Colliding Step

This was the first experiment. The motivation was to determine a set of initial and final conditions that will provide us with a successful dynamic step. Table 1 summarizes the state bounds used in the numerical solution of this experiment.

The first two rows show the constraints on the initial conditions \( \mathbf{x}(t_0) \). As can be seen in the table, the lower an upper bound for \( \phi(t_0) \) are equal.
Table 4.3: Experiment 1 State Constraints

<table>
<thead>
<tr>
<th>State</th>
<th>φ(t)</th>
<th>˙φ(t)</th>
<th>θ(t)</th>
<th>˙θ(t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimum Initial</td>
<td>(\frac{\pi - \alpha}{2})</td>
<td>0</td>
<td>-2\pi</td>
<td>-3\pi</td>
</tr>
<tr>
<td>Maximum Initial</td>
<td>(\frac{\pi - \alpha}{2})</td>
<td>(\pi)</td>
<td>2\pi</td>
<td>3\pi</td>
</tr>
<tr>
<td>Minimum Bound</td>
<td>(\frac{\pi - \alpha}{2})</td>
<td>-2\pi</td>
<td>-2\pi</td>
<td>-3\pi</td>
</tr>
<tr>
<td>Maximum Bound</td>
<td>(\frac{\pi + \alpha}{2})</td>
<td>4\pi</td>
<td>2\pi</td>
<td>3\pi</td>
</tr>
<tr>
<td>Minimum Final State</td>
<td>(\frac{\pi + \alpha}{2})</td>
<td>-2\pi</td>
<td>-2\pi</td>
<td>-3\pi</td>
</tr>
<tr>
<td>Maximum Final State</td>
<td>(\frac{\pi + \alpha}{2})</td>
<td>4\pi</td>
<td>2\pi</td>
<td>3\pi</td>
</tr>
</tbody>
</table>

This indicates to the numerical solver that \(\phi(t_0)\) is fixed. Another important constraint is \(\dot{\phi}(t_0)\). Since we are only interested in forward steps \(\dot{\phi}(t_0)\) is constrained to be non-negative. The other constraints on the remaining initial condition are given to prevent the system from deviating from physically safe values.

The third and fourth rows of Table 1 show state trajectory bounds. Any state trajectory produced by the solver must remain between these values. The bounds for \(\phi(t)\) were selected to allow free motion in a forward step. Each of the other state bounds were selected to keep the system in a physically meaningful state.

Constraints on the final state \(x(t_o)\) are imposed only on the angle \(\phi(t_f)\) which defines when a step is completed.

In addition to the above state bounds it is also necessary to specify bounds on the control input and the final time. Recall that if the final time is fixed its upper and lower bounds must be equal. In this experiment the final
time $t_f$ was free.

<table>
<thead>
<tr>
<th>Variable</th>
<th>$V$</th>
<th>$t_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimum</td>
<td>-20</td>
<td>0.1</td>
</tr>
<tr>
<td>Maximum</td>
<td>+20</td>
<td>100</td>
</tr>
</tbody>
</table>

One of the factors that affects more dramatically the trajectories of the rimless wheel is the choice of cost function. The cost function initially used to generate an optimal trajectory was the integral of the square of the input voltage (4.3).

$$J(t_f) = \int_{t_o}^{t_f} V(t)^2 \, dt \quad (4.3)$$

Once the cost function was selected, the optimal control problem is completely defined. The dynamic equations, system bounds, initialization information, and cost function were encoded in the files required by GPOPS-II and after setting additional information about the nonlinear solvers the optimization problem was executed.

As shown in Figure 4.1, we can see that a colliding step was achieved, but the step required significant amount of time. Furthermore the control action attempted to stabilize the wheel at its unstable equilibrium before completing the step. This is definitely not an efficient locomotion gait. The lesson learned from this simple experiment is that the choice of cost function plays a very important role in the generation of practical stepping patterns.
To achieve a better “coordination” a cost penalizing all the states was used. A revised cost function was created as seen in equation (4.4). This cost function penalized the \( \phi, \dot{\phi} \) and \( \theta \) 10000 times less than the input voltage, and \( \dot{\theta} \) 1000 times less than the input voltage.

\[
J = \int_{t_0}^{t_f} \left( \frac{1}{100} \phi(t)^2 + \frac{1}{100} \theta(t)^2 + \frac{1}{100} \dot{\phi}(t)^2 + \frac{1}{10} \dot{\theta}(t)^2 + 100V(t)^2 \right) dt \quad (4.4)
\]

The resulting trajectory is shown in Figure 4.2. Note that now the input voltage is essentially zero (on the order of \( 10^{-6} \)) and the step is swift.

The above cost function penalizes excessively the control action and as a result the best action is no control and the motion is completely passive.
The optimal solution provides the initial conditions that give the wheel just enough velocity for the wheel to lift–off and keeps the internal pendulum close to its stable equilibrium since that does not require much input voltage.

An energy analysis of this input trajectory can be seen in Figure 4.3. In this graph the power output of the motor on the system is plotted. To determine whether the control signal was efficient, the integral of the power output is calculated. This is done by bypassing the motor and determining the power of the torque on the system, as produced by the motor. This system required 0.0016 J from the motor.
4.4.2 Single Collisionless Step

The second experiment performed focused on creating a collisionless step. This is a step where the wheel impacts the ground at zero velocity, e.g., $\phi(t_f) = 0$. The goal of this experiment was to determine the type of motion, initial conditions, and terminal conditions that would be required to create this collisionless scenario.

The bound for this experiment are given in Table 4.4. As can be seen in the table, $\phi(t_0)$ is fixed and $\dot{\phi}(t_0)$ is not negative. Theses constraints are common and define a forward step. A relaxation of the angular velocity
values from the previous experiment was required for improved numerical convergence.

The third and fourth rows of Table 4.4 show the state bounds for this experiment. Again, the range for $\dot{\theta}(t)$ had to be increased as well. The main difference from experiment one is that the constraint on the velocity $\dot{\phi}(t_f)$ must be 0 for a collisionless step.

In addition to the above state bounds it is also necessary to specify bounds on the control input and the final time.

<table>
<thead>
<tr>
<th>Variable</th>
<th>$V$</th>
<th>$t_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimum</td>
<td>-20</td>
<td>0.1</td>
</tr>
<tr>
<td>Maximum</td>
<td>+20</td>
<td>100</td>
</tr>
</tbody>
</table>

Seeking a single collisionless step the cost function in (4.5) was examined. This cost is very similar to the cost used for the first experiment but weighted all four states as 10000 times less significant than the input
signal. This cost generates a collisionless step.

\[ J = \int_{t_o}^{t_f} \left( \frac{1}{100} \dot{\phi}^2 + \frac{1}{100} \dot{\theta}^2 + \frac{1}{100} \theta^2 + \frac{1}{100} \phi^2 + 100V^2 \right) dt \]  \hspace{1cm} (4.5)

Once this cost function was selected trajectories were produced. The system bounds, initialization information, and cost function were provided to GPOPS-II and the optimization was executed.

The trajectories that are produced by this cost function and configuration can be seen in Figure 4.4. As intended, each trajectory is continuous, low cost, and implementable. Some concern can be pointed at the \(-7.14\pi\) [radian/s] inner pendulum velocity. This large negative value could cause issues for a physical attempt of this trajectory.

![Figure 4.4: Single Collisionless Step](image-url)
When this control trajectory was interpolated into a control signal with a 1 kHz sampling rate, a near collisionless step resulted. As seen in Figure 4.5 the final rimless wheel step appears to be collisionless. Though very near collisionless, the wheel actually met the ground with 0.002756 [radians/s] of rotational velocity.

An energy analysis of this input trajectory is given in Figure 4.6. In this graph the power output of the motor on the system is plotted. To determine whether the control signal was efficient, the integral of the output power was calculated. As before, this was done by bypassing the motor and determining the energy of the torque on the system, as produced by the motor. This system required 139.8551 J of energy from the motor.
The somewhat larger amount of energy input can be attributed to several things. The speed of the arm in this experiment is quite high. The power was calculated by multiplying the absolute values of the $\dot{\phi}$ and $T$ values for each time index. This provided a curve which was then integrated to find energy.

This larger energy value is a result of the motor expending more power to apply the same amount of torque to a fast moving system. Since the $\dot{\phi}$ terms are quite large in this example, the integral grew as well.

Despite the somewhat larger energy requirement, a near collisionless step was achieved. This type of step is a better candidate for a practical collisionless low energy step, but the energy required to produce this step far
exceeds the amount of energy saved by avoiding the collision. Though encouraging, these results oversimplify the effort required to generate a collisionless step. This experiment was the most computationally intensive experiment to run. When a different set of values was used for the parameters of the system, the computation time increased significantly to the point of becoming impractical. This suggests that the numerical optimal control problem is also very sensitive to the parameters of the system.

4.4.3 Two Perfectly Elastic Colliding Steps

The third experiment attempted was to determine what type of movement would create multiple steps. This experiment produced a diverse set of trajectories based on cost function and was used to explore the affects of cost alterations on the control trajectory.

In this experiment, the collision was assumed to be completely elastic. The reason behind this decision was to experiment with multi-phase dynamic optimization without introducing multiple points of failure. To facilitate this, upon the step event the pivot point would move. Being a pin joint, the wheel could not leave the ground.

The step was produced by creating a two phase system in GPOPS-II. To make it perfectly elastic, the linkage constraints in the matlab function rimlessWheelEndpoint.m were defined to allow every state except $\phi$ to
Table 4.5: Experiment 3 State Constraints

<table>
<thead>
<tr>
<th>State</th>
<th>$\phi(t)$</th>
<th>$\dot{\phi}(t)$</th>
<th>$\theta(t)$</th>
<th>$\dot{\theta}(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimum Initial</td>
<td>$\frac{\pi - \alpha}{2}$</td>
<td>0.5</td>
<td>$-2\pi$</td>
<td>$-8$</td>
</tr>
<tr>
<td>Maximum Initial</td>
<td>$\frac{\pi - \alpha}{2}$</td>
<td>2.5</td>
<td>$2\pi$</td>
<td>8</td>
</tr>
<tr>
<td>Minimum Boundary</td>
<td>$\frac{\pi - \alpha}{2}$</td>
<td>$-5$</td>
<td>$-2\pi$</td>
<td>$-8$</td>
</tr>
<tr>
<td>Maximum Boundary</td>
<td>$\frac{\pi + \alpha}{2}$</td>
<td>10</td>
<td>$2\pi$</td>
<td>8</td>
</tr>
<tr>
<td>Minimum Final State</td>
<td>$\frac{\pi + \alpha}{2}$</td>
<td>$-5$</td>
<td>$-2\pi$</td>
<td>$-8$</td>
</tr>
<tr>
<td>Maximum Final State</td>
<td>$\frac{\pi + \alpha}{2}$</td>
<td>10</td>
<td>$2\pi$</td>
<td>8</td>
</tr>
</tbody>
</table>

maintain their values. $\phi$ was linked to force the step discontinuity in the stance leg angle from an instantaneous and elastic collision.

The collision occurs when $\phi = \frac{\pi + \alpha}{2}$. To create the positional effect of the step, the value of $\phi$ is modified to equal $\frac{\pi - \alpha}{2}$. This, in affect, creates an elastic step.

The bounds for this experiment are given in Table 4.5.

The first two rows of Table 4.5 show the constraints on the initial conditions of the problem. As can be seen in the table, the $\phi(t_0)$ state does not have any allowance for movement. Additionally, the initial condition of the $\dot{\phi}(t_0)$ state is not permitted to be less than 0.5 rad/ sec or exceed 2.5 radians/second. This is to increase the speed of convergence to a significant, nontrivial, trajectory. The other constraints are only included to prevent the system from deviating from physically safe values.
The third and fourth rows of Table 4.5 show the state bounds for the system. Any trajectory produced by the system must not exceed these values. State $\phi(t)$ values were selected as range of free motion in each moving step. Each of the other bounds were selected to keep the system in a physically safe state.

The final two rows in Table 4.5 show the terminal states of the system. There are two significant constraints here. The first constraint is on the first state, $\phi(t_f)$. This is the final position of the rimless wheel. Additionally, the wheel must complete two steps, so the final $\dot{\phi}(t_f)$ value must be greater than zero to avoid a double stance phase.

The bounds on the control input and the final time are

<table>
<thead>
<tr>
<th>Variable</th>
<th>$V$</th>
<th>$t_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimum</td>
<td>$-20$</td>
<td>$0.1$</td>
</tr>
<tr>
<td>Maximum</td>
<td>$+20$</td>
<td>$100$</td>
</tr>
</tbody>
</table>

In this experiment the mass of the internal pendulum, $m_2$ was included as an optimization parameter. This allowed GPOPS-II to alter the value of this parameter in order to further satisfy the cost function. This value could not be altered after the start of a simulation so it was only altered after the completion of an iteration.

In addition to the constraint setup, the cost function needed to be explored. This was done iteratively, as local minimum can affect the convergence. As such, the cost functions are presented with the results they
produced.

The problem setup sought two colliding steps. A number of cost functions attempted did not lead to any feasible solution within the given error tolerances. This is likely due to large numerical errors in the vicinity of the discontinuity between the two steps. This error appears in each of the multi-phase trajectories with the step discontinuity. Despite this error, control trajectories were still produced but are not optimal.

Five cost functions successfully produced control trajectories that could be used to achieve the goals of this experiment with varying degrees of success. None of the produced multi-phase cost functions could be verified within the $1 \times 10^{-8}$ second error tolerance. The cost function used to generate the first set of trajectories is (4.6).

$$J = \int_{t_o}^{t_f} \left( \frac{1}{10} \phi^2 + \dot{\phi}^2 + \theta^2 + \dot{\theta}^2 + V^2 \right) dt \quad (4.6)$$

The trajectories that are produced by this cost function can be seen in Figure 4.7. The optimal control input is bang–bang, e.g., the control signal stays at the control limits. Additionally, just before the completion of the second step, there is a sudden negative jump in the trajectory of $\dot{\phi}$.

When this bang-gang control trajectory used was re-sampled with a time step of 1 ms and used as input to the simulator of the rimless wheel. The results are shown in Figure 4.8. The first step was almost exactly as one given by GPOPS-II. The second step was about 0.5 seconds longer than
Figure 4.7: Two Colliding Steps with Low Wheel Position Cost

Figure 4.8: Two Colliding Steps with Low Wheel Position Cost Verification
the projected trajectory. This contributed to a larger impact velocity than expected. The system did successfully take two steps with this control input.

An energy analysis of this input trajectory can be seen in Figure 4.9. In this graph the power output of the motor on the system is plotted. To determine whether the control signal was efficient, the integral of the power output was calculated to determine energy. This control trajectory consumed 61.6155 J of energy.

This was an example of a controller that successfully completed two steps.
However, the second step was not executed as planned. This result motivated the search for a more suitable cost function.

The second cost function found to generate a control trajectory that could take two steps is listed in (4.7). This cost function was found by making small, consistent variations and checking the results. A number of cost functions that do not produce a result were found. Only cost functions that provide two step trajectories that could be verified were presented.

\[
J = \int_{t_0}^{t_f} \left( \frac{1}{10} \phi^2 + \frac{1}{10} \dot{\phi}^2 + \theta^2 + \dot{\theta}^2 + V^2 \right) dt \tag{4.7}
\]

The trajectories that are produced by this cost function can be seen in Figure 4.10. Again, this controller was similar to the bang–bang controller found previously. However, it appears to have a singularity at the conclusion of the second step. The “anomaly” in the \( \dot{\phi} \) trajectory is probably an artifact since the algorithm did not converge to the prescribed error tolerances.

When this control trajectory was used to drive the simulation the results were similar to the previous verification simulation as shown in Figure 4.11. The first step was almost exactly the same as the GPOPS-II and the second was about 0.2 seconds longer than the projected trajectory. This contributed to a larger impact velocity than expected. The system did successfully take two steps with this input.
Figure 4.10: Two Colliding Steps with Low Wheel Costs

Figure 4.11: Two Colliding Steps with Low Wheel Costs Verification
An energy analysis of this input trajectory can be seen in Figure 4.12. In this graph the power output of the motor on the system was plotted. To determine whether the control signal was efficient, the integral of the output power was calculated to determine energy. This system required 43.8870 J from the motor.

This optimal control trajectory was more verifiable than the previous set of trajectories. This cost function produced a more implementable signal, but a more complicated signal would require less input energy.

The the third cost function found to generate a two step control trajectory.
can be seen in (4.8).

\[
J = \int_{t_0}^{t_f} \left( \frac{1}{10} \dot{\phi}^2 + \phi^2 + \frac{1}{10} \dot{\theta}^2 + \theta^2 + V^2 \right) \, dt
\]  

(4.8)

The trajectories that are produced by this cost function can be seen in Figure 4.13. The control input seemed to attempt to stabilize the double pendulum system at its unstable equilibrium point. Again, there was an anomaly in the \( \dot{\phi} \) trajectory and as before, the algorithm again reported an inability to satisfy error tolerances.

When this control trajectory was used to drive the simulation the system was able to take two steps, but not as planned. As seen in Figure 4.14 the first step was about .3 seconds faster than the GPOPS-II prediction. The
second step was approximately 1.75 seconds faster than the projected trajectory. This yields a completely different final state at the end of the second step.

An energy analysis of this input trajectory is shown in Figure 4.15. In this graph the power output of the motor on the system was plotted. To determine whether the control signal was efficient, the integral of the output power was calculated. For this system 11.0508 J was output by the motor.
The fourth successful cost function found to generate a two step trajectory can be seen in (4.9).

$$J = \int_{t_0}^{t_f} \left( \frac{1}{10} \dot{\phi}^2 + \dot{\phi}^2 + \frac{1}{10} \dot{\theta}^2 + \frac{1}{10} \ddot{\theta}^2 + V^2 \right) dt$$ (4.9)

The trajectories that are produced by this cost function can be seen in Figure 4.16. Again, this was a control input with discontinuous derivatives. The control trajectory seemed to attempt to stabilize the double pendulum system at an unstable equilibrium point. As before, the anomaly in the $\dot{\phi}$ trajectory is due to the fact that algorithm did not converge to the given error tolerances.

Figure 4.17 shows the simulation results with the above control input. The
first step was about 2.6 seconds faster than the GPOPS-II prediction. The second step was approximately 4.8 seconds faster than the projected trajectory. This wildly different trajectory yields a completely different end state of the system.

This indicates that, if the solution to the optimal control problem does not converge to the given tolerances the results should not be used since there is no guarantee that they will result in trajectories “close” to the optimal.

An energy analysis of this input trajectory can be seen in Figure 4.18. In this graph the power output of the motor on the system was plotted. To determine whether the control signal was efficient, the integral of the
power output was calculated to determine energy. For this system 17.1577 J was output by the motor.

This control trajectory demonstrates the open loop nature of this controller. Without a feedback method, this control trajectory will perform poorly when stability over an unstable equilibrium point is required. Additionally, the length of this control trajectory allows the minute errors in the interpolation to accumulate. Though this cost function was capable of generating a two step trajectory, these results demonstrate one of the limits of numerical optimization.
The fifth and final cost function found to generate a two step elastic trajectory can be seen in (4.10).

\[ J = \int_{t_0}^{t_f} \left( \frac{1}{10} \dot{\phi}^2 + \phi^2 + \frac{1}{10} \dot{\theta}^2 + \frac{1}{10} \dot{\theta}^2 + V^2 \right) \, dt \]  

(4.10)

As before the solver could not find a solution within the prescribed tolerances. Figure 4.19 shows the trajectories at the termination of the algorithm. As before these solutions are not meaningful.

When this control trajectory was interpolated with a 1 kHz sampling rate, the system was able to take two steps. As seen in Figure 4.20 the first step was very similar to the GPOPS-II prediction. The second step took
approximately .3 seconds longer than the projected trajectory. This trajectory is about as accurate as the first two control trajectories.

An energy analysis of this input trajectory can be seen in Figure 4.21. In this graph the output power of the motor on the system was plotted. To determine whether the control signal was efficient, the integral of the output power was calculated. For this system 48.5480 J was output by the motor.

This energy integral was far better than any of the other attempted trajectories for several reasons. First, this trajectory produced a relatively short gait. The speed of the step yielded an inherent efficiency. This short
Figure 4.20: Two Colliding Steps with High Wheel Velocity Cost Verification

Figure 4.21: Two Colliding Steps with High Wheel Velocity Cost Energy
gait also assisted in reducing the effect of rounding errors in the interpolation. Additionally, this was not entirely a bang bang controller. The first step rarely approached the control limits. This made up for the large power input in the second step.

4.4.4 Three Perfectly Elastic Colliding Steps

This experiment was performed to inductively suggest that this process can be applied for many stepped problems. A successful three step gait hints by induction that trajectories can be found to create stable periodic gaits. Unless a periodic result can be found, this can not be stated as fact, but it would provide grounds for future work seeking one.

The state constraints for this experiment are identical to those used for experiment 3 and can be found in Table 4.5. The only change to the problem set up for this case is setting the number of steps to 3. All other values are the same as in 4.4.3.

A number of cost functions were attempted but few were found to produce usable control trajectories. The only verifiable three step trajectory produced was a result of the cost function outlined in equation (4.11). This is the same cost function as in equation (4.6) for the two step case.

\[
J = \int_{t_0}^{t_f} \left( \frac{1}{10} \phi^2 + \dot{\phi}^2 + \theta^2 + \dot{\theta}^2 + V^2 \right) dt
\]  

(4.11)

The trajectories that were produced by this cost function can be seen in
Figure 4.22. As in the two stage case a numerical solution accurate to the prescribed tolerance could not be obtained.

Figure 4.23 shown the results of using the above control trajectory to drive the simulation of the rimless wheel. Note that each step taken is faster than it should be. Additionally, each step taken is significantly faster than the previous step.

An energy analysis of this input trajectory can be seen in Figure 4.24. In this graph the output power of the motor on the system was plotted. Integrating the instantaneous power output leads to a total of 19.69 J output by the motor.
Figure 4.23: Two Colliding Steps with High Wheel Velocity Cost Verification

Figure 4.24: Two Colliding Steps with High Wheel Velocity Cost Energy
The next step is to use the collision dynamics of the system to attempt
to generate a multi-phase GPOPS-II trajectory that can take energy losses
into account.

4.4.5 A Plastic Collision

A plastic collision is a collision where the foot hits the ground and im-
mediately sticks and momentum, not energy, is conserved. This is how
collisions affect a system in an physical experiment. The effects of a col-
lision on a system can be mathematically determined as demonstrated
in section 3.4. Creating a trajectory that can overcome the loss of energy
due to a collision is of interest due to characterizing the true energy loss
of each step.

Finding a trajectory that can overcome the collision loss of energy will al-
low us to study the realistic dynamics of the system. It was empirically
observed that the solution to this problem was heavily dependent on the
values of some parameters of the system, e.g. the inertias. These param-
ters were adjusted to the values given in Table 4.6. The collision equations
that were determined in section 3.4 have been added to the collision.m
simulation of the system. They will be used to update the system state
between collision restarts.

We started with values corresponding to the experimental prototype of
Table 4.6: Physical Parameter Values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_1[kg]$</td>
<td>1.4</td>
<td>$m_2[kg]$</td>
<td>2.22</td>
</tr>
<tr>
<td>$I_1[kg−m^2]$</td>
<td>0.092</td>
<td>$I_2[kg−m^2]$</td>
<td>$\frac{I_2^2m_2}{3}$</td>
</tr>
<tr>
<td>$L_1[m]$</td>
<td>0.375</td>
<td>$L_2[m]$</td>
<td>0.34</td>
</tr>
</tbody>
</table>

the rimless wheel. The values of $m_2$ and $L_2$ were altered based on experimental findings. It was discovered that it was not possible to create a walking system with these parameters. These values are both four times the actual measurements.

The bounds for this experiment are defined in Table 4.7.

Table 4.7: Experiment 5 State Constraints

<table>
<thead>
<tr>
<th>State</th>
<th>$\phi(t)$</th>
<th>$\dot{\phi}(t)$</th>
<th>$\theta(t)$</th>
<th>$\dot{\theta}(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimum Initial</td>
<td>$\pi−\alpha/2$</td>
<td>0.5</td>
<td>$−5\pi$</td>
<td>$−15$</td>
</tr>
<tr>
<td>Maximum Initial</td>
<td>$\pi−\alpha/2$</td>
<td>10</td>
<td>$5\pi$</td>
<td>15</td>
</tr>
<tr>
<td>Minimum Boundary</td>
<td>$\pi−\alpha/2$</td>
<td>$−5$</td>
<td>$−5\pi$</td>
<td>$−15$</td>
</tr>
<tr>
<td>Maximum Boundary</td>
<td>$\pi+\alpha/2$</td>
<td>10</td>
<td>$5\pi$</td>
<td>15</td>
</tr>
<tr>
<td>Minimum Final State</td>
<td>$\pi+\alpha/2$</td>
<td>$−5$</td>
<td>$−5\pi$</td>
<td>$−15$</td>
</tr>
<tr>
<td>Maximum Final State</td>
<td>$\pi+\alpha/2$</td>
<td>10</td>
<td>$5\pi$</td>
<td>15</td>
</tr>
</tbody>
</table>

The first two rows of Table 4.7 show the constraints on the initial conditions of the problem. As can be seen in the table, the $\phi(t_0)$ state does not have any allowance for movement. Additionally, the initial condition of the $\dot{\phi}(t_0)$ state is not permitted to be less than 0.5 radians/second or
exceed 2.5 radians/second. This is to increase the speed of convergence to a significant, nontrivial, trajectory. The other constraints are only included to prevent the system from deviating from physically safe values. It is worth noting that this experiment needed a larger range in the $\theta$ and $\dot{\theta}$ states.

The third and fourth rows of Table 4.7 show the state bounds for the system. Any trajectory produced by the system must not exceed these values. State $\phi(t)$ values were selected as range of free motion in each moving step. Each of the other bounds were selected to keep the system in a physically safe state. It is again worth noting that this experiment needed a larger range in the $\theta$ and $\dot{\theta}$ states.

The final two rows in Table 4.5 show the terminal states of the system. There are two significant constraints here. The first constraint is on the first state, $\phi(t_f)$. This is the final position of the rimless wheel. Additionally, the wheel must complete two steps, so the final $\dot{\phi}(t_f)$ value must be greater than zero to avoid a double stance phase. Finally, experimentation demonstrated a need for a larger range in the $\theta$ and $\dot{\theta}$ states final conditions as well.

In addition to all of these values, the control signal was allowed to vary anywhere between $-65$ V and $65$ V. And the time was required to be between 0.1 seconds and 12 seconds.

These values are not consistent with the other experiments. The input
<table>
<thead>
<tr>
<th>Variable</th>
<th>V</th>
<th>$t_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimum</td>
<td>−65</td>
<td>0.1</td>
</tr>
<tr>
<td>Maximum</td>
<td>+65</td>
<td>12</td>
</tr>
</tbody>
</table>

voltage range needed to be increased to allow for convergence. These values are large enough that they would likely not work on most small commercial motors, but they are still somewhat feasible. Additionally, the time range was reduced. This was to discourage the solver from sitting on the unstable equilibrium points.

A number of cost functions were attempted but only one was found to produce a verifiable trajectory. This cost function is outlined in equation (4.12).

$$J = \int_{t_o}^{t_f} \left( \phi^2 + \frac{1}{10} \dot{\phi}^2 + \frac{1}{10} \theta^2 + \frac{1}{10} \dot{\theta}^2 + V^2 \right) d t$$  (4.12)

The trajectories that were produced by this cost function can be seen in Figure 4.25. Take note of the loss of velocity in the $\dot{\phi}$ and $\dot{\theta}$ states when the step is taken. This loss is a direct result of the impact.

The control trajectory seemed to attempt to approach the impact with speed. As before, the algorithm could not converge to the desired error tolerances.

Simulations with the input trajectory found above are shown in Figure 4.26. Note that the second step taken is faster than it should be. The first step
occurs only 0.0033 seconds earlier than it should, but the second step occurs approximately 0.2089 seconds earlier than it should and takes 0.2056 seconds less than it was expected to. But since the solution did not converge to tolerance this is no longer a surprise.

It is important to note that the trajectory produced for the $\phi$ state of the second step actually approaches its terminal state before receding. In the verification test, the simulation actually completes the step at this point. Though GPOPS-II did not converge to this state, a very nearly continuous trajectory was produced and could be verified.

An analysis of this input trajectory can be seen in Figure 4.27. In this
Figure 4.26: Two Steps with Collision Dynamics Verification

Graph the output power of the motor on the system was plotted. To determine whether the control signal was efficient, the integral of the power output was calculated. For this system 74.2061 J was output by the motor.

Upon inspection of the energy trajectory, an interesting characteristic can be seen. Just as the passive system’s power was decreasing due to the collision as expected, the power output by the motor jumped by a factor of about 2.6. This is due to the sudden increase in arm velocity and the compensation of the voltage trajectory. This large swing is spurred to overcome the collision losses and complete a second step.
Though the control input limits had to be significantly increased to create this trajectory, it shows that, if the actuator is large enough to inject back the energy lost in the collisions further trajectories can be created to enable level ground transport.

### 4.4.6 Single Step Extensions

In an attempt to mitigate the continuity issues with multi-phase optimization, two single step optimizations can be strung together. Though this does not allow for the ideal optimal control, it does produce results. Figures 4.28 shows a second trajectory generated by GPOPS-II for a single phase case using the parameters used in experiment 1. This trajectory
forced the initial conditions to be the same as the final state of the execution of experiment 1.

As can be seen in Figure 4.28, the results are very nearly the same. By concatenating these results together, a two step control trajectory was formed as seen in figure as can be seen in Figure 4.29. This control trajectory contained a discontinuity as expected. Additionally, and perhaps more significantly, the result was within error tolerances. Both collisions happened within $1 \times 10^{-8}$ sec of when they were designed to happen.

Finally, the energy analysis was obtained. Figure 4.34 shows that very little power is required to create this motion. This collision was elastic so no
Figure 4.29: Single Step Extension to Multi-Step Result, Experiment 1

energy was lost 0.0031 J of energy was required from the motor to generate this trajectory.

The equations seen in (3.85) - (3.87) can be used to determine the second step’s initial state. In this example, a colliding step will have the following initial conditions. The negative wheel velocity in state \( \dot{\phi} \) indicates that this trajectory will not be able to generate a working two step system. The wheel will not have enough energy to leave the ground. This was a common problem with GPOPS-II as well. This is a result of the decoupled moments of inertia. Due to the unrealistic inertia values, the system loses too much energy to the impact. A realistic system requires inertias that
are coupled to the mass. For this reason, the experimentally obtained parameters were examined for use used to attempt the experiment again.

Moving from the idealized parameters to the experimentally obtained parameters provided better results. Figure 4.31 shows a single step run of experiment 5. For this run, the final values that produced the working step that was discovered in experiment 6 were targeted as the end state and time. This was due to the near working nature of this previous run.

Similarly, a second trajectory was generated by GPOPS-II for a single phase case using the parameters used in experiment 5 and can be seen in Figure 4.32. This trajectory forced the initial conditions to be the same as the
The first step’s trajectory was very similar to the trajectory produced by experiment 5 and seen in Figure ex6Traj. The second step was allowed to vary in time. This created a more efficient second step.

By concatenating these two control trajectories together, a two step control trajectory was formed. As can be seen in Figure 4.33 the concatenated control trajectory performed almost exactly as specified. This control trajectory contained a discontinuity at the impact point as expected. Additionally, and perhaps more significantly, the result was within error tolerances. Both collisions happened within $1 \times 10^{-8}$ sec of when they were designed to happen.
Figure 4.32: Second Single Step Extension to Multi-Step Result, Experiment 5

Figure 4.33: Single Step Extension to Multi-Step Verification, Experiment 1
Finally, the energy analysis was obtained. Figure 4.34 shows that some power was required to create this motion. This collision was plastic so some amount of energy loss was expected. 57.6196 J of energy was required from the motor to generate this trajectory.

This trajectory is within error tolerances and creates a trajectory that is capable of level ground transport as designed.

### 4.5 Additional Analysis of Results

Experiments One and Two generated extremely promising results. Not only was it possible to create a trajectory that was capable of minimal
voltage input, but it was also possible to create a trajectory that was capable of near collisionless motion in simulation. However, this collisionless trajectory was extremely dependent on initial conditions and not control input. Additionally, experiments Three and Four demonstrated that multi-phase trajectories could be approximated. Finally, experiments five and six demonstrated the application of these concepts to systems that used realistic parameters and characteristics. However, it is important to note some of the caveats to these experiments.

In each of these experiments, the cost functions are slightly unconventional. In each experiment, we penalize the system states for having a value. As demonstrated in experiment one, this was necessary. By penalizing the state deviation from zero, the cost pushed the system to remain at reasonable velocities while moving quickly to its target. Additionally, these state deviation penalties pushed the system to avoid stalling at the unstable equilibrium points. This unintentional effect of our choice of reference angle for both the wheel and the pendulum led directly to this result.

Another interesting aspect of our cost functions was our choice of input cost. Input energy does not rely solely on input voltage. The energy consumed by the motor is a function of the current and the resistance of the motor. The reason this value is used over current is due to implementation. Retrieving the current for use in the cost proved to be resource
intensive. From the optimization point of view, it could be more conve-
nient to use torque directly, but then there would be no insight into motor characteristics.

An additional oddity in our results stems from our energy measurements. Our measure of energy consumption was taken at the output of the mo-
tor. The losses in the motor are small and consistent, however in the con-
text of this research the energy applied to the system was more significant
than the energy losses in the motor. Based on the torque and the arm
speed, the power applied to the system could be very small. Additionally,
the losses due to the collision were more apparent when measured after
the motor.

The unstable trend of the multi-state solution verification simulations
was concerning. GPOPS-II reported that a solution could not be found
within error tolerances for each of those experiments. In some rare cases,
it did not have a large effect on the system, but that unlikely accuracy
was most likely due to happenstance. The two accurate simulations were
mostly bang-bang controllers. The fact that there was no movement on
the control signal itself likely led to the unlikely accuracy.

Finally, analysis shows that there was no one ideal cost function. There
were trends that could be observed, but there was significant variation
between each system. Each time a system was changed, the cost func-
tion had to be reevaluated. The most significant pattern observed was the
heavy cost of input. By weighing the input more than the states, the input was the critical component. However, the weight of each state would completely change the system.

Single step solutions were simple to produce among many of the idealized system's initial conditions. This would indicate that creating a colliding single step solution could be used to determine the initial conditions for a second single step execution of the colliding step. This could allow for the creation of accurate multi-step models.
Chapter 5

Conclusions

5.1 Forward Work

Upon the completion of the above outlined experiments, a number of possible extensions of this work became apparent. This sections lists a number of these potential extensions. These suggestions are designed to provide a logical next step for continuation of this work.

5.1.1 Dynamic Model Revision

Simulations of a rimless wheel with plastic steps indicate that the wheel often does not retain enough energy to leave the ground after a collision. In this case, the rimless wheel will stay in a double stance state. Also, if collisionless step is achieved, the system will always enter a double stance state. Since the focus of this work was locomotion patterns, the model of the system only includes single stance dynamics.

A more realistic model requires the use of hybrid models where the system transitions between double stance and single stance states, each with
possibly different dynamic equations. This extension may reveal more sophisticated control strategies that include getting out and in to double stance states.

The models considered in this work assume the pivot foot is actually a pin joint. Though accurate for slow moving systems, at higher angular velocities the system would leave the ground. Determining at what point this would occur would lead to further accuracy in simulation and further application into experimental systems.

The model considered in this work neglects losses due viscous damping and drag. For practical implementation these losses are important and should be considered in the formulation of the optimal control problem.

Finally, the current model only considers motion in two dimensions. A three dimensional motion model would bring us to other factors relevant to the locomotion problem.

5.1.2 Physical Construction

This thesis was partially motivated by the ongoing construction of a physical rimless wheel with an internal motor driven rigid pendulum. As research progressed, it became clear that this method of numerical optimization would not be able to create effective trajectories for a wheel with the existing physical parameters. It was discovered that minor modifications to the design of that wheel would allow for the creation of working
optimal input trajectories.

To complete this facet of the research, a new rimless wheel could be created in tandem with the design of an optimal trajectory to allow experimental verification of these results. Due to the required modifications in the physical model in the simulation, our results could not be verified experimentally.

In addition to this work to develop a rimless wheel that could be used to generate optimal trajectories, experimental motor values could also be obtained and designed for. The current motor model is technically accurate, but the parameters used are not experimentally found.

5.1.3 Feedback Control

Optimal control and numerical optimization alone are considered open loop control systems. Though technically valid, any state error can lead to instability in the system, especially for highly nonlinear systems. For this reason, feedback control is desired.

The system described in this thesis is highly nonlinear. Control of nonlinear systems is inherently nontrivial. For this reason, the most simple approach to controlling this system could be a form of linearization by trajectory [35]. Applying this method would alloy a linear model of the system that is valid while in the neighborhood of the provided trajectory. Once obtained, PID control could be applied.
Other methods of nonlinear feedback control that could be applied include input-state and input-output feedback linearization. Additionally, sliding mode, or Lyapunov based, control could be applied.

### 5.2 Concluding Remarks

This thesis provided a great deal of insight concerning dynamic numerical optimization of nonlinear systems. Of particular importance was the insight into how system parameters affect the control of the system. The geometry of the rimless wheel affects significantly its ability to be controlled by an actuated internal pendulum.

In addition to the strong coupling of the design and the control, great insight into cost function design was developed. The effects of changing small details in the optimization tool's cost function were studied and found to be significant. To successfully create an optimal trajectory for a rimless wheel with an actuated internal rigid pendulum requires insight into initial conditions, system parameters and cost functions. Though independently simple, these highly coupled variables must be carefully considered before an optimal input trajectory can be found.

We were ultimately able to develop several trajectories of practical interest including collisionless and colliding plastic and elastic trajectories. These trajectories provide insight into the motion and control input that
would allow a rimless wheel to achieve energy efficient level ground transport. The results of this research suggest that set of parameters, initial conditions, and cost function exist that would produce a near collisionless system that can be experimentally verified.
Bibliography


