On Chromatic Polynomial and Ordinomial

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1 Introduction

Suppose $\Gamma$ is a graph with $|V(\Gamma)| = n$. For $\lambda$ a positive integer, let $[\lambda] = \{1, 2, \ldots, \lambda\}$ be a set of $\lambda$ distinct colors. A $\lambda$-coloring of $\Gamma$ is a mapping $f$ from $V(\Gamma)$ to $[\lambda]$. Whenever for every two adjacent vertices $u$ and $v$, $f(u) \neq f(v)$, we will call $f$ a proper coloring of $\Gamma$; otherwise, improper. When a proper $\lambda$-coloring exists, we call $\Gamma$ a $\lambda$-colorable graph. The chromatic number of $\Gamma$, denoted by $\chi(\Gamma)$, is defined as the smallest $\lambda$ such that $\Gamma$ is $\lambda$-colorable, and if that is the case, we call $\Gamma$ a $\lambda$-chromatic graph. As we are only interested in proper colorings of graphs using $[\lambda]$ as our color set, we will drop the term “proper” and the prefix “$\lambda$” from “proper $\lambda$-coloring” throughout this thesis, unless stated otherwise. Colorings $f$ and $g$ are considered distinct, if there exists $v \in V(\Gamma)$, such that $f(v) \neq g(v)$. The chromatic function of a graph, $C(\Gamma; \lambda)$, is the number of distinct colorings of $\Gamma$.

Theorem 1.1: $C(\Gamma; \lambda)$ is a degree $n$ monic polynomial of $\lambda$.

Proof: Let $r$ be a positive integer and $m_r(\Gamma)$ denote the number of distinct $r$-color-partitions; an $r$-color-partition of $V(\Gamma)$ is a partition of vertices into $r$ nonempty subsets, known as color-classes, such that no two vertices in a subset are adjacent. Clearly, for $r$ greater than $n$, $m_r(\Gamma) = 0$. For $r$ less than or equal to $n$, we can color each $r$-color-partition in $\lambda_r = \lambda(\lambda - 1)(\lambda - 2) \cdots (\lambda - r + 1)$ ways; Hence, as there are $m_r(\Gamma)$ such partitions, we have

$$C(\Gamma; \lambda) = \sum_{r=1}^{n} m_r(\Gamma) \lambda_r. \quad (1)$$

It is clear that for every $r$, $\lambda_r$ is a polynomial of $\lambda$ which implies $C(\Gamma; \lambda)$ is also a polynomial of $\lambda$. Furthermore, as there is one color-partition of $V(\Gamma)$ into $n$ color-classes, the coefficient $m_n(\Gamma)$ of $\lambda_n$ (a polynomial of degree $n$ that has the highest degree among $\lambda_r$) is equal to 1. This proves that $C(\Gamma; \lambda)$ is monic with degree $n$.\[1\]

From now on, we will refer to $C(\Gamma; \lambda)$ as the chromatic polynomial of graph $\Gamma$. Furthermore, whenever $\Gamma$ contains a loop, no color-partition is possible and we have $m_r(\Gamma) = 0$, for $1 \leq r \leq n$; Hence, by using the explicit expression 1.1, known as the factorial form of $C(\Gamma; \lambda)$, $C(\Gamma; \lambda) = 0$. In addition to that, as multiple edges have no effect on color-partitions, we will assume that graphs under consideration have no multiple edges. Due to these observations, graphs of interest are loop-less graphs with no multiple edges.

Conventionally, we will assume that $C(\Gamma; 0) = 0$ which implies $\lambda$ is a factor of chromatic polynomial of any graph. Moreover, $\Gamma$ is $\lambda$-colorable if and only if $C(\Gamma; \lambda) > 0$. From this remark, it is obvious that $\chi(\Gamma) = \min \{\lambda \in \mathbb{N} \mid C(\Gamma; \lambda) > 0\}$ and the four-color theorem can be rephrased as follows:
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If \( \Gamma \) is a planar graph, then \( C(\Gamma; 4) > 0 \).

One important remark which we have to make here is: by \( \mathbb{N} \) we mean the set of all positive integers, while by \( \mathbb{N}_0 \) the set of all natural numbers is meant. Clearly, \( \mathbb{N} = \mathbb{N}_0 - \{0\} \).

We will find chromatic polynomials of some familiar graphs and introduce methods that are useful in finding them. Our next proposition deals with chromatic polynomial of complete graphs:

Proposition 1.2: If \( K_n \) denotes the complete graphs of order \( n \), \( C(K_n; \lambda) = \lambda^n \).

Proof: Because every two vertices in \( K_n \) are adjacent, there is no color-partition of this graph into \( r \) color-classes, for \( 1 \leq r < n \), by using the Pigeon-hole Principle; in other words, \( m_r(K_n) = 0 \). On the other hand, we can uniquely color-partition \( K_n \) into \( n \) color-classes, each containing only one vertex, so \( m_n(K_n) = 1 \); Therefore, \( C(K_n; \lambda) = \sum_{r=1}^{n} m_r(K_n) \lambda^r = \lambda^n \).

Let \( \Gamma \) be a graph having two components, \( \Gamma_1 \) and \( \Gamma_2 \). As \( V(\Gamma_1) \) and \( V(\Gamma_2) \) are disconnected, we can color them independently; Hence,

\[
C(\Gamma; \lambda) = C(\Gamma_1; \lambda) C(\Gamma_2; \lambda).
\]

Similarly, we have

Theorem 1.3: Let \( \Gamma \) be a graph with \( k \) components, denoted by \( \Gamma_1, \Gamma_2, \ldots, \Gamma_k \). Then,

\[
C(\Gamma; \lambda) = \prod_{i=1}^{k} C(\Gamma_i; \lambda).
\]

This implies the following two corollaries:

Corollary 1.4: If \( N_n \) denotes the null graph of order \( n \), then \( C(N_n; \lambda) = \lambda^n \).

Corollary 1.5: Let \( \Gamma \) be a graph with \( c \) components. Then \( \lambda^c \) is factor of \( C(\Gamma; \lambda) \).

Now, we will state a very important theorem which enables us to compute the chromatic polynomial of a graph by computing chromatic polynomial of "smaller" and "simpler" graphs. In order to do so, we need to introduce some notation: suppose \( e \) is an edge in \( \Gamma \). By deleting \( e \) from \( \Gamma \), we will obtain a graph, denoted by \( \Gamma^{(e)} \), which has the same vertices as \( \Gamma \) but edge-set \( E(\Gamma) - \{e\} \). By contracting \( e \) in \( \Gamma \), we mean the graph which results from \( \Gamma^{(e)} \) by identifying the two vertices which \( e \) had them as its ends in \( \Gamma \); Let \( \Gamma^{(e)} \) denote this graph. Using the above notation, we have the so-called Deletion-contraction Theorem:
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Figure 1.1: $\Gamma, \Gamma^{(e)}, \Gamma_{(e)}$

Theorem 1.6: For every edge $e$ in any graph $\Gamma$,

$$C(\Gamma; \lambda) = C(\Gamma^{(e)}; \lambda) - C(\Gamma_{(e)}; \lambda).$$ (2)

Proof: Let $u$ and $v$ be the two ends of $e$ in $\Gamma$. Let $C_1$ and $C_2$, respectively, be the set of colorings of $\Gamma^{(e)}$ in which $u$ and $v$ are colored differently and have the same colors. Clearly, there is a 1-1 correspondence from $C_1$ to colorings of $\Gamma$; Also, from $C_2$ to colorings of $\Gamma_{(e)}$. As we have $C(\Gamma; \lambda) = |C_1| + |C_2| = C(\Gamma; \lambda) + C(\Gamma_{(e)}; \lambda)$, thus the result. 

Theorem 1.7: (Addition-contraction Theorem) Let $\Gamma$ be a graph and $u$ and $v$ two non-adjacent vertices in $\Gamma$. If $\Gamma + uv$ and $\Gamma_{(uv)}$ denote graphs obtained by, respectively, linking and contracting $u$ and $v$ in $\Gamma$, then $C(\Gamma; \lambda) = C(\Gamma + uv; \lambda) + C(\Gamma_{(uv)}; \lambda)$.

In using Deletion-contraction Theorem or Addition-contraction Theorem, whenever a multiple edge occurs, as we previously made this remark on multiple edges, we will delete all edges but one in a multiple edge. As applications of the Deletion-contraction Theorem, the following two propositions are quite classic:

Proposition 1.8: For every tree $T$ of order $n$, $C(T; \lambda) = \lambda(\lambda - 1)^{n-1}$.

Proof: Proof by induction on $n$: When $n = 2$, it is clear that $C(T; \lambda) = C(K_2; \lambda) = \lambda(2) = \lambda(\lambda - 1)$. Now, assume for every tree of order $n - 1$ it is true, and let $T$ be a tree of order $n$. Let $v$ be a leaf in $T$ and $e$ the edge incident to $v$. It is obvious that $T^{(e)}$ has two components: the isolated vertex $v$ and a tree of degree $n - 1$. By the induction hypothesis and Theorem 1.3, $C(T^{(e)}; \lambda) = \lambda \cdot \lambda(\lambda - 1)^{n-2} = \lambda^2(\lambda - 1)^{n-2}$. On the other hand, $T_{(e)}$ is also a tree of order $n - 1$ and $C(T_{(e)}; \lambda) = \lambda(\lambda - 1)^{n-2}$. Now, by using the Deletion-contraction Theorem, we have

$$C(T; \lambda) = C(T^{(e)}; \lambda) - C(T_{(e)}; \lambda) = \lambda^2(\lambda - 1)^{n-2} - \lambda(\lambda - 1)^{n-2} = \lambda(\lambda - 1)^{n-1}.$$

By using Theorem 1.3 and Proposition 1.8, we have the following:

Corollary 1.9: If $F$ is a forest of order $n$ with $c$ components, then $C(F; \lambda) = \lambda^c(\lambda - 1)^{n-c}$.
Proposition 1.10: If $C_n$ denotes the cycle of order $n$, then $C(C_n; \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1)$.

**Proof:** Proof by induction on $n$: When $n = 3$, it is clear that $C(C_3; \lambda) = C(K_3; \lambda) = \lambda(\lambda - 1)(\lambda - 2) = (\lambda - 1)^3 - (\lambda - 1)$. By assuming that this is true for $C_{n-1}$, we will show it is also true for $C_n$. Let $e$ be an edge in $C_n$. By deleting and contracting $e$, we will have $P_{n-1}$, path of length $n - 1$, and $C_{n-1}$, respectively. By the induction hypothesis, the Deletion-contraction Theorem, and the fact that every path is a tree, we have

$$C(C_n; \lambda) = C(P_{n-1}; \lambda) - C(C_{n-1}; \lambda) =$$

$$\lambda(\lambda - 1)^{n-1} - [(\lambda - 1)^{n-1} + (-1)^{n-1}(\lambda - 1)] =$$

$$\lambda(\lambda - 1)^n + (-1)^n(\lambda - 1).$$

We will introduce another useful method, but before that, some notation is required. Let $\Gamma_1$ and $\Gamma_2$ be two graphs and $r$ be a natural number less than or equal to minimum of $\omega(\Gamma_1)$ and $\omega(\Gamma_2)$ (note that $\omega(\Gamma)$ denotes the clique number of a graph $\Gamma$). It follow that both these graphs have a copy of $K_r$ as a subgraph and the graph $\Gamma$ obtained from $\Gamma_1$ and $\Gamma_2$ by identifying these two copies of $K_r$ is called a $K_r$-gluing of $\Gamma_1$ and $\Gamma_2$. Clearly, when $r$ is equal to zero, the gluing is just the disjoint union of the two graphs. By using the aforementioned notation, we have the following theorem:

Theorem 1.11: If $\Gamma$ is a $K_r$-gluing of the graphs $\Gamma_1$ and $\Gamma_2$, then

$$C(\Gamma; \lambda) = \frac{C(\Gamma_1; \lambda)C(\Gamma_2; \lambda)}{C(K_r; \lambda)}.$$ \hspace{1cm} (3)

**Proof:** When $r$ is equal to zero, the validity of this formula has been verified in Theorem 1.3, for $k = 2$. Now, assume that $r$ is a positive integer. The number of distinct ways to color $\Gamma$ with $\lambda$ colors is equal to $C(\Gamma_1; \lambda)$ times $C(\Gamma_2; \lambda)/C(K_r; \lambda)$, the number of ways we can extend distinct $\lambda$-colorings of $K_r$ to distinct $\lambda$-colorings of $\Gamma_2$. This proves the result.\]

Corollary 1.12: Let $\Gamma'$ be graph containing $K_r$, for $r$ a positive integer. For graph $\Gamma$ obtained from $\Gamma'$ by adding a new vertex $v$ and connecting $v$ to vertices of $K_r$, we have $C(\Gamma; \lambda) = C(\Gamma'; \lambda)(\lambda - r)$.

Let $\Gamma$ be a graph and for $k \geq 4$, a copy of $C_k$ in $\Gamma$, say $\Gamma'$, is called a pure cycle when there are no edges in $\Gamma$ linking two non-consecutive vertices in $\Gamma'$. If such an edge exists, it is called a chord in $\Gamma'$. A graph $\Gamma$ in which every cycle has a chord is called a chordal graph (see Figure 1.2) and it can be shown that any chordal graph of order $n$ is constructed recursively from $N_1$ by applying repeatedly the rule we have used in Corollary 1.12, $(n - 1)$ times. This implies that if $\Gamma$ is a chordal graphs and for $0 \leq i \leq k = \chi(\Gamma) - 1$, $r_i \in \mathbb{N}$ such
that $\sum_{i=0}^{k} r_i = n$, then $C(\Gamma; \lambda) = \lambda^n (\lambda - 1)^n (\lambda - 2)^r \cdots (\lambda - k)^{r_k}$. There is a well-known example that the converse of the above remark is not true (see Figure 1.3).

For $q \in \mathbb{N}$, we will define the class of $q$-trees recursively as follows:

i) $K_q$ is a $q$-tree;

ii) A $q$-tree of order $(n + 1)$ is obtained from $\Gamma$, a $q$-tree of order $n$, provided that $n \geq q$, by adding a new vertex and linking it to a $K_q$ in $\Gamma$.

Clearly, as one may see how a $q$-tree of order $n$ is constructed, every $q$-tree is a chordal graph and we have the following proposition on their chromatic polynomial using induction:

If $\Gamma$ is a $q$-tree of order $n$ ($n \geq q$), then $C(\Gamma; \lambda) = \lambda(\lambda - 1) \cdots (\lambda - q + 1)(\lambda - q)^{n-q}$.

One interesting example that we will return to later in Section 5 is $L_n = K_2 \times P_n$, for $n \in \mathbb{N}$ (see Figure 1.4). We know from Proposition 1.10 that $C(C_4; \lambda) = (\lambda - 1)^4 + (-1)^4(\lambda - 1) = \lambda(\lambda - 1)(\lambda^2 - 3\lambda + 3)$. On the other hand, it is easy to verify that $L_1$ is the cycle $C_4$. In general, as $L_n$ is an edge-gluing of $L_{n-1}$ and $C_4$, chromatic polynomial of $L_n$ is equal to $\lambda(\lambda - 1)(\lambda^2 - 3\lambda + 3)^n$, using induction and Theorem 1.11.
Another interesting example is the graph $\Delta_n = K_3 \times P_n$, for $n \in \mathbb{N}$ (see Figure 1.5). First, we will find the chromatic polynomial of $\Delta_1$ (see Figure 1.6) and then as $\Delta_n$ is a $K_3$-gluing of $\Delta_{n-1}$ and $\Delta_1$, we will use (1.3) to write a recursive formula for $C(\Delta_n; \lambda)$.

\begin{align*}
C(\Delta_1; \lambda) &= \left( \frac{C(K_2; \lambda) C(K_3; \lambda)}{C(K_1; \lambda)^2} - \frac{C(K_3; \lambda)}{C(K_1; \lambda)} + 2 \frac{C(K_3; \lambda)}{C(K_2; \lambda)} - 1 \right) C(K_3; \lambda) = \\
&= (\lambda^3 - 5\lambda^2 + 9\lambda - 5) C(K_3; \lambda) = \lambda(\lambda - 1)(\lambda - 2)(\lambda^3 - 5\lambda^2 + 9\lambda - 5).
\end{align*}

On the other hand, for $n \geq 2$,

\[ C(\Delta_n; \lambda) = \frac{C(\Delta_1; \lambda) C(\Delta_{n-1}; \lambda)}{C(K_3; \lambda)} = (\lambda^3 - 5\lambda^2 + 9\lambda - 5) C(\Delta_{n-1}; \lambda) = \]
\[(\lambda^3 - 5\lambda^2 + 9\lambda - 5)^{n-1} C(\Delta_1; \lambda) = \lambda(\lambda - 1)(\lambda - 2)(\lambda^3 - 5\lambda^2 + 9\lambda - 5)^n.\]

Hence, for \(n \in \mathbb{N}\), we have

\[C(\Delta_n; \lambda) = \lambda(\lambda - 1)(\lambda - 2)(\lambda^3 - 5\lambda^2 + 9\lambda - 5)^n.\]

**Theorem 1.13:** Let \(\Gamma'\) be a connected graph, and \(u\) and \(v\) two non-adjacent distinct vertices of \(\Gamma'\). Let \(\Gamma\) be the graph obtained from \(\Gamma'\) by adding a \(uv\)-path of length \(k\). Then,

\[C(\Gamma; \lambda) = \frac{C(C_{k+1}; \lambda) C(\Gamma'; \lambda)}{\lambda(\lambda - 1)} + (-1)^k C(\Gamma_{(uv)}; \lambda).\]

**Proof:** From Corollary 1.7, we have \(C(\Gamma; \lambda) = C(\Gamma + uv; \lambda) + C(\Gamma_{(uv)}; \lambda)\). Clearly, \(\Gamma + uv\) is an edge-gluing of \(\Gamma' + uv\) and \(C_{k+1}\). On the other hand, \(\Gamma_{(uv)}\) is a vertex-gluing of \(\Gamma_{(uv)}'\) and \(C_k\). As so, by using Theorem 1.11, we have the following:

\[C(\Gamma; \lambda) = \frac{C(C_{k+1}; \lambda) C(\Gamma' + uv; \lambda)}{\lambda(\lambda - 1)} + \frac{C(C_k; \lambda) C(\Gamma_{(uv)}'; \lambda)}{\lambda} = \]

\[\frac{C(C_{k+1}; \lambda) [C(\Gamma'; \lambda) - C(\Gamma_{(uv)}'; \lambda)]}{\lambda(\lambda - 1)} + \frac{C(C_k; \lambda) C(\Gamma_{(uv)}'; \lambda)}{\lambda} = \]

\[\frac{C(C_{k+1}; \lambda) C(\Gamma'; \lambda)}{\lambda(\lambda - 1)} + C(\Gamma_{(uv)}'; \lambda) \left( \frac{C(C_k; \lambda)}{\lambda} - \frac{C(C_{k+1}; \lambda)}{\lambda(\lambda - 1)} \right) = \]

\[\frac{C(C_{k+1}; \lambda) C(\Gamma'; \lambda)}{\lambda(\lambda - 1)} + \frac{C(C_{k+1}; \lambda) C(\Gamma'; \lambda)}{\lambda(\lambda - 1)} + (-1)^k C(\Gamma_{(uv)}'; \lambda).\]

For \(r, s, t \in \mathbb{N}\), a **theta** graph \(\theta(r, s, t)\) is constructed from \(N_2\), by connecting the two vertices by three disjoint paths of length \(r, s,\) and \(t\) (see Figure 1.7). By applying Theorem 1.13, we have

\[C(\theta(r, s, t); \lambda) = \frac{C(C_{r+1}; \lambda) C(C_{s+t}; \lambda)}{\lambda(\lambda - 1)} + (-1)^r \frac{C(C_r; \lambda) C(C_t; \lambda)}{\lambda}.\]
More generally, for \( k \geq 3 \) and \( s_1, s_2, \ldots, s_k \in \mathbb{N} \), a theta graph \( \theta(s_1, s_2, \ldots, s_k) \) is similarly constructed from \( N_2 \), but this time, by using \( k \) number of disjoint paths of length \( s_1, s_2, \ldots, s_k \). By using induction and Theorem 1.13, we can easily prove the following:

\[
C(\theta(s_1, s_2, \ldots, s_k); \lambda) = \prod_{i=1}^{k} \frac{C(C_{s_i+1}; \lambda)}{\lambda^{k-1}(\lambda - 1)^{k-1}} + \prod_{i=1}^{k} \frac{C(C_{s_i}; \lambda)}{\lambda^{k-1}}.
\]

Let \( \Gamma_1 \) and \( \Gamma_2 \) be two disjoint graphs. Suppose \( |V(\Gamma_i)| = n_i \), for \( i = 1, 2 \). By the join graph of these two graphs, denoted by \( \Gamma_1 \cup \Gamma_2 \), we mean the graph obtained by connecting every vertex in one graph to all vertices in the other. Now, assume that \( C(\Gamma_1; \lambda) \) and \( C(\Gamma_2; \lambda) \) are expressed in factorial form, \( \sum_{r=1}^{n_1} m_r(\Gamma_1) \lambda(r) \) and \( \sum_{s=1}^{n_2} m_s(\Gamma_2) \lambda(s) \), respectively. The umbral product of these two forms, denoted by \( C(\Gamma_1; \lambda) \odot C(\Gamma_2; \lambda) \), is also a factorial form, obtained by applying the standard polynomial product, as if treating the factorials \( \lambda(r) \) as powers \( \lambda^r \); more precisely, \( \lambda(r) \odot \lambda(s) = \lambda(r+s) \).

Theorem 1.14: Let \( \Gamma_1 \) and \( \Gamma_2 \) be graphs with chromatic polynomials expressed in factorial form. The chromatic polynomial of their join, \( \Gamma_1 \cup \Gamma_2 \), is the umbral product of their chromatic polynomials

\[
C(\Gamma_1 \cup \Gamma_2; \lambda) = C(\Gamma_1; \lambda) \odot C(\Gamma_2; \lambda).
\]

Proof: To prove this, first we will prove that \( m_t(\Gamma) \) (1 \( \leq t \leq n_1 + n_2 \)), the color-partitions of \( \Gamma \), is equal to \( \sum_{r+s=t} m_r(\Gamma_1) m_s(\Gamma_2) \), for 1 \( \leq r \leq n_1 \) and 1 \( \leq s \leq n_2 \). A color-class of \( \Gamma \) is either a color-class of \( \Gamma_1 \) or a color-class of \( \Gamma_2 \), since every vertex in \( \Gamma_1 \) is adjacent to every vertex in \( \Gamma_2 \), and vice versa. Due to this fact, as stated before, the number of color-partitions of \( \Gamma \) into \( t \) color-classes is equal to the sum of color-partitions of \( \Gamma_1 \) and \( \Gamma_2 \) into \( r \) and \( s \) color-classes, respectively, provided that \( t = r + s \). Now, we can write

\[
C(\Gamma_1; \lambda) \odot C(\Gamma_2; \lambda) = \left( \sum_{r=1}^{n_1} m_r(\Gamma_1) \lambda(r) \right) \odot \left( \sum_{s=1}^{n_2} m_s(\Gamma_2) \lambda(s) \right) =
\]
\[ \sum_{r=1}^{n_1} \sum_{s=1}^{n_2} m_r(\Gamma_1) m_s(\Gamma_2) \left( \lambda_r \odot \lambda_s \right) = \sum_{r=1}^{n_1} \sum_{s=1}^{n_2} m_r(\Gamma_1) m_s(\Gamma_2) \lambda_{r+s} = \]
\[ \sum_{t=1}^{n_1+n_2} m_t(\Gamma) \lambda_t = C(\Gamma_1 + \Gamma_2; \lambda). \]

**Theorem 1.15:** Let \( \Gamma \) be a graph. For \( N_1 + \Gamma \) and \( N_2 + \Gamma \) which are called the *cone* and *suspension* of \( \Gamma \), respectively, we have \( C(N_1 + \Gamma; \lambda) = \lambda C(\Gamma; \lambda - 1) \) and \( C(N_2 + \Gamma; \lambda) = \lambda C(\Gamma; \lambda - 1) + \lambda(\lambda - 1) C(\Gamma; \lambda - 2) \).

**Proof:** For the former, we can write,
\[ C(N_1 + \Gamma; \lambda) = C(N_1; \lambda) \odot C(\Gamma; \lambda) = \]
\[ \lambda(1) \odot \sum_{r=1}^{n} m_r(\Gamma) \lambda_r = \sum_{r=1}^{n} m_r(\Gamma) \left( \lambda(1) \odot \lambda_r \right) = \]
\[ \sum_{r=1}^{n} m_r(\Gamma) \lambda_{r+1} = \sum_{r=1}^{n} m_r(\Gamma) \left( \lambda(\lambda - 1)_r \right) = \]
\[ \lambda \left( \sum_{r=1}^{n} m_r(\Gamma)(\lambda - 1)_r \right) = \lambda C(\Gamma; \lambda - 1). \]

By using the identity \( \lambda^2 = \lambda(1) + \lambda(2) \), the proof of the latter is as follows:
\[ C(N_2 + \Gamma; \lambda) = C(N_2; \lambda) \odot C(\Gamma; \lambda) = \]
\[ \left( \lambda(1) + \lambda(2) \right) \odot \sum_{r=1}^{n} m_r(\Gamma) \lambda_r = \]
\[ \sum_{r=1}^{n} m_r(\Gamma) \left( \lambda(1) \odot \lambda_r \right) + \sum_{r=1}^{n} m_r(\Gamma) \left( \lambda(2) \odot \lambda_r \right) = \]
\[ \sum_{r=1}^{n} m_r(\Gamma) \lambda_{r+1} + \sum_{r=1}^{n} m_r(\Gamma) \lambda_{r+2} = \]
\[ \sum_{r=1}^{n} m_r(\Gamma) \left( \lambda(\lambda - 1)_r \right) + \sum_{r=1}^{n} m_r(\Gamma) \left( \lambda(\lambda - 1)(\lambda - 2)_r \right) = \]
\[ \lambda \left( \sum_{r=1}^{n} m_r(\Gamma)(\lambda - 1)_r \right) + \lambda(\lambda - 1) \left( \sum_{r=1}^{n} m_r(\Gamma)(\lambda - 2)_r \right) = \]
\[ \lambda C(\Gamma; \lambda - 1) + \lambda(\lambda - 1) C(\Gamma; \lambda - 2). \]

In a same fashion, we can prove that for \( \Gamma \) a graph and \( k \in \mathbb{N} \),

\[ C(N_k + \Gamma; \lambda) = \sum_{i=1}^{k} \lambda^i C(\Gamma; \lambda - i). \]

As an application of Theorem 1.15, we will find \( C(\bigcup_{i=1}^{n} K_{k_i} + N_2; \lambda) \):

\[
C(\bigcup_{i=1}^{n} K_{k_i} + N_2; \lambda) = \\
\lambda C(\bigcup_{i=1}^{n} K_{k_i}; \lambda - 1) + \lambda(\lambda - 1) C(\bigcup_{i=1}^{n} K_{k_i}; \lambda - 2) = \\
\lambda \prod_{i=1}^{n} C(K_{k_i}; \lambda - 1) + \lambda(\lambda - 1) \prod_{i=1}^{n} C(K_{k_i}; \lambda - 2) = \\
\lambda \prod_{i=1}^{n} (\lambda - 1)(k_i) + \lambda(\lambda - 1) \prod_{i=1}^{n} (\lambda - 2)(k_i) = \\
\frac{1}{\lambda^{n-1}} \left( \lambda^n \prod_{i=1}^{n} (\lambda - 1)(k_i) \right) + \frac{1}{\lambda^{n-1}(\lambda - 1)^{n-1}} \left( \lambda^n (\lambda - 1)^n \prod_{i=1}^{n} (\lambda - 2)(k_i) \right) = \\
\frac{1}{\lambda^{n-1}} \left( \prod_{i=1}^{n} \lambda(\lambda - 1)(k_i) \right) + \frac{1}{\lambda^{n-1}(\lambda - 1)^{n-1}} \left( \prod_{i=1}^{n} \lambda(\lambda - 1)(\lambda - 2)(k_i) \right) = \\
\frac{1}{\lambda^{n-1}} \left( \prod_{i=1}^{n} \lambda(k_i+1) \right) + \frac{1}{\lambda^{n-1}(\lambda - 1)^{n-1}} \left( \prod_{i=1}^{n} \lambda(k_i+2) \right) = \\
\frac{1}{\lambda^{n-1}(\lambda - 1)^{n-1}} \left( (\lambda - 1)^{n-1} \prod_{i=1}^{n} \lambda(k_i+1) + \prod_{i=1}^{n} \lambda(k_i+2) \right) = \\
\frac{1}{\lambda^{n-1}(\lambda - 1)^{n-1}} \left( (\lambda - 1)^{n-1} \prod_{i=1}^{n} \lambda(k_i+1) + \prod_{i=1}^{n} (\lambda - k_i - 1) \lambda(k_i+1) \right) = \\
\prod_{i=1}^{n} \frac{\lambda(k_i+1)}{\lambda^{n-1}(\lambda - 1)^{n-1}} \left( (\lambda - 1)^{n-1} + \prod_{i=1}^{n} (\lambda - k_i - 1) \right).
By wheel (pyramid) and double pyramid of order \( n \), respectively, we mean the cone of \( C_{n-1} \) and suspension of \( C_{n-2} \), denoted by \( W_n \) and \( \Pi_n \). By using Theorem 1.15 and Proposition 1.10, it is clear that

\[
C(W_n; \lambda) = \lambda(\lambda - 2)^{n-1} + (-1)^{n-1}\lambda(\lambda - 2),
\]

\[
C(\Pi_n; \lambda) = \lambda(\lambda - 1)(\lambda - 3)^{n-2} + \lambda(\lambda - 2)^{n-2} + (-1)^n\lambda(\lambda^2 - 3\lambda + 1).
\]

The bi-wheel of order \( n \), denoted by \( B_n \), is the graph \( K_2 + C_{n-2} \) and by using Theorem 1.6, we have

\[
C(B_n; \lambda) = C(n; \lambda) - C(W_{B_i}; \lambda) = \lambda(\lambda - 1)(\lambda - 3)^{n-2} + (-1)^n\lambda(\lambda^2 - 3\lambda + 1).
\]

Proposition 1.16: Let \( n, k, m_1, m_2, \ldots, m_k \in \mathbb{N} \) with \( m_1 + m_2 + \ldots + m_k = n - 1 \). By a broken wheel of \( W_n \), denoted by \( W_n(m_1, m_2, \ldots, m_k) \), we mean the graph obtained from \( W_n \) by deleting \( n - 1 - k \) spokes (and so there are \( k \) spokes left) such that the number of rim edges between existing successive spokes are \( m_1, m_2, \ldots, m_k \). By letting \( Q_r(\lambda) \) being equal to \( \frac{1}{\lambda(\lambda - 1)} C(C_{r+2}; \lambda) = \frac{1}{\lambda}[(\lambda - 1)^{r+1} + (-1)^r] \), we have

\[
C(W_n(m_1, m_2, \ldots, m_k); \lambda) = \lambda \prod_{i=1}^{k} Q_{m_i}(\lambda) + (-1)^{n-1}\lambda(\lambda - 2).
\]

**Proof:** We will prove this proposition using induction on \( k \). For \( k = 1 \), we have \( m_1 = n - 1 \) and \( W_n(n - 1) \) is a vertex-gluing of \( C_{n-1} \) and \( K_2 \). Hence, by Theorem 1.11, \( C(W_n(n - 1); \lambda) \) is equal to \( \frac{1}{\lambda} C(C_{n-1}; \lambda) C(K_2; \lambda) \). It is an easy task to verify that \( C(W_n(n - 1); \lambda) \) is equal to \( \lambda Q_{n-1} + (-1)^{n-1}\lambda(\lambda - 2) \).

It can be verified that when the wheel is not broken, namely when for every \( 1 \leq i \leq k, m_i = 1 \), (1.4) is valid. If that is the case, \( k = n - 1, W_n(m_1, m_2, \ldots, m_k) \) is basically \( W_n \), and \( C(W_n(m_1, m_2, \ldots, m_k); \lambda) \) is equal to \( \lambda(\lambda - 2)^{n-1} + (-1)^{n-1}\lambda(\lambda - 2) \). On the other hand, \( \lambda \prod_{i=1}^{k} Q_{m_i}(\lambda) + (-1)^{n-1}\lambda(\lambda - 2) = \lambda \prod_{i=1}^{n-1} Q_1(\lambda) + (-1)^{n-1}\lambda(\lambda - 2) \). By substituting \( Q_1(\lambda) = (\lambda - 2) \) and simplifying, we will have \( \lambda(\lambda - 2)^{n-1} + (-1)^{n-1}\lambda(\lambda - 2) \), as expected.
Now, assume that for $k$, the proposition is true and we will prove it is also true for $k + 1$. Let $W_n(m_1, m_2, \ldots, m_{k+1})$ be a broken wheel with $m_1 + m_2 + \ldots + m_{k+1} = n - 1$. Without lacking any generality, we can assume that $m_{k+1}$ is greater than 1, unless for every $1 \leq i \leq k + 1$, $m_i = 1$ which we talked about in the previous paragraph. Now, let $u$ and $v$ be the non-adjacent vertices at the consecutive existing spokes that are connected by $P$, this path of length $m_{k+1}$. It is clear that $W_n(m_1, m_2, \ldots, m_{k+1}) = W_{n-m_k}(m_1, m_2, \ldots, m_k)$ and our induction hypothesis implies that (1.4) is valid for this graph. On the other hand, if $W'$ denotes the graph from which $W_n(m_1, m_2, \ldots, m_{k+1})$ can be constructed by connecting $u$ and $v$ using $P$, then we have the following, using Theorem 1.13:

$$C(W_n(m_1, m_2, \ldots, m_{k+1}); \lambda) =$$

$$\frac{C(C_{m_{k+1}}; \lambda) C(W'; \lambda)}{\lambda(\lambda - 1)} + (-1)^{m_{k+1}} C(W_n(m_1, m_2, \ldots, m_{k+1})(uv); \lambda).$$

It is not hard to see that $C(W'; \lambda) = \prod_{i=1}^{k} C(C_{m_i+2}; \lambda)/[\lambda(\lambda - 1)]^{k-1}$. This fact enables us to write

$$C(W_n(m_1, m_2, \ldots, m_{k+1}); \lambda) =$$

$$\frac{C(C_{m_{k+1}}; \lambda) \prod_{i=1}^{k} C(C_{m_i+2}; \lambda)}{\lambda^k(\lambda - 1)^k} + (-1)^{m_{k+1}} \prod_{i=1}^{k} Q_{m_i}(\lambda) + (-1)^{n-m_{k+1}+1} \lambda(\lambda - 2) =$$

$$C(C_{m_{k+1}}; \lambda) \prod_{i=1}^{k} \frac{C(C_{m_i+2}; \lambda)}{\lambda(\lambda - 1)} + (-1)^{m_{k+1}} \lambda \prod_{i=1}^{k} Q_{m_i}(\lambda) + (-1)^{n-1} \lambda(\lambda - 2) =$$

$$C(C_{m_{k+1}}; \lambda) \prod_{i=1}^{k} Q_{m_i} + (-1)^{m_{k+1}} \lambda \prod_{i=1}^{k} Q_{m_i}(\lambda) + (-1)^{n-1} \lambda(\lambda - 2) =$$

$$(C(C_{m_{k+1}}; \lambda) + (-1)^{m_{k+1}} \lambda) \prod_{i=1}^{k} Q_{m_i}(\lambda) + (-1)^{n-1} \lambda(\lambda - 2) =$$

$$\lambda Q_{m_{k+1}} \prod_{i=1}^{k} Q_{m_i}(\lambda) + (-1)^{n-1} \lambda(\lambda - 2).$$

We will finish this section by indicating this remark that, although in this section we assumed $\lambda$ to be an positive integer representing the number of colors available to properly color the vertices of a graph $\Gamma$ with, but from now on, we will assume $\lambda$ to be any complex number, and as a result, chromatic polynomial of any graph will be an element of $\mathbb{C}[\lambda]$. 
2 Ordinal Numbers and Their Arithmetic

In order to introduce the algebraic machinery which will be used in the fourth section, we first need to give a brief review of ordinal numbers.

We will start with some definitions which are required to define ordinal numbers. We will assume that \((X, <)\) denotes an ordered set \(X\) with order \(<\) (an order is a relation on \(X\) that is irreflexive, antisymmetric and transitive). An ordered set has a total order when the condition of trichotomy is satisfies: for all \(x, y \in X\), one and only one of \(x < y\), \(x = y\), and \(y < x\) holds. An element \(x \in X\) is called a least element when for all \(y \in X\), \(x < y\). When an element \(x \in X\) exists such that for all \(y \in X\), \(y < x\) implies \(y = x\), we call \(x\) a minimal element. Regarding least and minimal elements we have the following theorem:

**Theorem 2.1:** Let \((X, <)\) be an ordered set.
(i) If a least element exists in \(X\), then it is the unique minimal element.
(ii) If \(X\) is totally ordered, then any minimal element of \(X\) is a least element.

We call a set \((X, <)\) with a total order a well-ordered set when every non-empty subset of \(X\) has a least element with respect to \(<\). Let \((X, <)\) be a totally ordered set and an element of \(X\), say \(y\), is given. A section \(X_y\) is defined as the set of all element in \(X\) less than \(y\) with respect to \(<\): \(X_y = \{x \in X \mid x < y\}\). An ordinal is a well-ordered set \((X, <)\) such that for all \(y \in X\), \(X_y = y\). The following is a very important theorem, proving that if ordinals exist, they are unique up to isomorphism - in set-theoretic terminology, an isomorphism between two ordered sets is a bijection which preserves the order both ways.

**Theorem 2.2:** Any well-ordered set is isomorphic to a unique ordinal.

Now, we will show that ordinals exist. The starting ordinals are defined as follows which are identified with natural numbers:

\[
\begin{align*}
0 &= \emptyset \\
1 &= \{\emptyset\} = \{0\} \\
2 &= \{\emptyset, \{\emptyset\}\} = \{0, 1\} \\
3 &= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{0, 1, 2\} \\
&\vdots \\
n + 1 &= \{0, 1, 2, \ldots, n\} \\
&\vdots
\end{align*}
\]

To expand our list of ordinal numbers we need the following theorem:
Theorem 2.3:
(i) If $\alpha$ is an ordinal number, so is $\alpha \cup \{\alpha\}$.
(ii) The union of a set of ordinals is an ordinal.

Based on this theorem we define three type of ordinals. The first type consists only of zero. The second type are called successor ordinals which are constructed by using (i) in the above theorem. If $\alpha$ is an ordinal, then the successor of $\alpha$, denoted by $s(\alpha)$, is the smallest ordinal bigger than $\alpha$. So, natural number $n + 1$ is the successor of $n$ and the smallest ordinal bigger than $n$. Finally, the third type of ordinals are those constructed by using (ii). A non-zero ordinal which is the union of all its predecessors is called a limit ordinal. If $\gamma$ is a limit ordinal, then $\gamma = \bigcup_{\alpha < \gamma} \alpha$. The smallest ordinal number after the natural numbers is the union of all natural numbers denoted by $\omega$ and the next ordinal after $\omega$ is $\omega \cup \{\omega\} = s(\omega)$. With regard to this classification, we have the following theorem:

Theorem 2.4: Any non-zero ordinal is either a successor ordinal or a limit ordinal.

A worth mentioning remark that might clarify confusions regarding ordinals is that for ordinals $\alpha$ and $\beta$, $\alpha < \beta$, $\alpha \subset \beta$, and $\alpha \in \beta$ are equivalent. In addition to that, for any ordinals $\alpha$ and $\beta$ exactly one of $\alpha < \beta$, $\alpha = \beta$, and $\beta < \alpha$ holds.

In order to make expanding our list of ordinals much easier, we will define addition, multiplication, and exponentiation for ordinal numbers. Let $\alpha$ and $\beta$ be ordinal numbers.

- Addition:
  - $\alpha + 0 = \alpha$,
  - $\alpha + s(\beta) = s(\alpha + \beta)$,
  - If $\gamma$ is a limit ordinal, $\alpha + \gamma = \bigcup_{\beta < \gamma} \alpha + \beta$.

- Multiplication:
  - $\alpha \cdot 0 = 0$,
  - $\alpha \cdot s(\beta) = \alpha \cdot \beta + \alpha$,
  - If $\gamma$ is a limit ordinal, $\alpha \cdot \gamma = \bigcup_{\beta < \gamma} \alpha \cdot \beta$.

- Exponentiation:
  - $\alpha^0 = 1$,
  - $\alpha^{s(\beta)} = \alpha^\beta \cdot \alpha$,
  - If $\gamma$ is a limit ordinal, $\alpha^\gamma = \bigcup_{\beta < \gamma} \alpha^\beta$.

Using transfinite induction which we will not discuss in this thesis, the first three properties in the following theorem can be proved. The fourth and fifth properties are proved by using apropos set-theoretic isomorphisms.
Theorem 2.5: For ordinals $\alpha$, $\beta$, and $\gamma$,

- $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$;
- $(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma$;
- $\alpha^{\beta + \gamma} = \alpha^{\beta} \cdot \alpha^{\gamma}$;
- If $\gamma + \alpha = \gamma + \beta$, then $\alpha = \beta$;
- If $\gamma \cdot \alpha = \gamma \cdot \beta$ and $\gamma \neq 0$, then $\alpha = \beta$.

It is noteworthy that addition and multiplication of ordinals are not commutative operations. For example, it is not difficult to verify that $1 + \omega = \omega \neq \omega + 1$ and $2 \cdot \omega = \omega \neq \omega \cdot 2 = \omega + \omega$. In addition to that, right cancelation laws are not necessarily valid for addition and multiplication. As an example, from $1 + \omega = 2 + \omega = \omega$ and $1 \cdot \omega = 2 \cdot \omega = \omega$, we can not cancel $\omega$ from the right and conclude that $1 = 2$, which clearly is false.

Now, with quite an ease, we have

\[
\begin{align*}
0, 1, 2, 3, \ldots, n, \ldots \\
\omega, \omega + 1, \omega + 2, \omega + 3, \ldots, \omega + n, \ldots \\
\omega + \omega = \omega \cdot 2, \omega \cdot 2 + 1, \omega \cdot 2 + 2, \omega \cdot 2 + 3, \ldots, \omega \cdot 2 + n, \ldots \\
\omega \cdot 2 + \omega = \omega + \omega + \omega = \omega \cdot 3 \\
\vdots \\
\omega \cdot n \\
\vdots \\
\omega^2 = \omega \cdot \omega = \bigcup_{n \in \mathbb{N}} \omega \cdot n, \omega^2 + 1, \omega^2 + 2, \ldots, \omega^2 + n, \ldots \\
\omega^2 + \omega = \bigcup_{n \in \mathbb{N}} \omega^2 + n, \omega^2 + \omega + 1, \omega^2 + \omega + 2, \ldots, \omega^2 + \omega + n, \ldots \\
\omega^2 + \omega \cdot 2 = \omega^2 + \omega + \omega \\
\vdots \\
\omega^n \\
\vdots \\
\omega^\omega = \bigcup_{n \in \mathbb{N}} \omega^n, \omega^\omega + 1, \omega^\omega + 2, \ldots, \omega^\omega + n, \ldots \\
\omega^\omega + \omega = \bigcup_{n \in \mathbb{N}} \omega^\omega + n \\
\vdots
\end{align*}
\]
\[
\begin{align*}
\omega^\omega + \omega \cdot 2 \\
\vdots \\
\omega^\omega + \omega \cdot n \\
\vdots \\
\omega^\omega + \omega^2, \omega^\omega + \omega^2 + 1, \omega^\omega + \omega^2 + 2, \ldots, \omega^\omega + \omega^2 + n, \ldots \\
\omega^\omega + \omega^2 + \omega \\
\vdots \\
\omega^\omega + \omega^\omega = \omega^\omega \cdot 2 \\
\vdots \\
\omega^\omega \cdot n \\
\vdots \\
\omega^{\omega+1} = \omega^\omega \cdot \omega = \bigcup_{n \in \mathbb{N}} \omega^\omega \cdot n \\
\vdots \\
\omega^{\omega \cdot \omega} = \omega^{\omega^2} \\
\vdots \\
\omega^{\omega^n} \\
\vdots \\
\omega^{\omega^\omega} = \bigcup_{n \in \mathbb{N}} \omega^{\omega^n} \\
\vdots \\
\end{align*}
\]

We can extend this list even further by defining towers of \( \omega \) as follows (Note: \( \gamma \) is a limit ordinal):

\[
\begin{align*}
\omega_0 &= \omega; \\
\omega_{\sigma(\alpha)} &= \omega^{\omega_{\alpha}}; \\
\omega_{\gamma} &= \bigcup_{\alpha < \gamma} \omega_{\alpha}.
\end{align*}
\]
As one may see, the infinite tower $\omega^\omega = \omega^\omega^\cdots$, known also as $\epsilon_0$, is the smallest solution to the equation $\omega^\varepsilon = \varepsilon$. Other solutions of the aforementioned equation can be found by using the limit operation and the so-called *epsilon function* which is not in the scope of this thesis. Moreover, the unimaginably monstrous ordinal $\omega_\omega$ is a solution to the equation $\omega_\varepsilon = \varepsilon$.

The following theorem and lemma can be proved by applying transfinite induction:

**Theorem 2.6:**

i) Any infinite ordinal can be written in the form $\gamma + n$, where $\gamma$ is a limit ordinal and $n$ a natural number.

ii) Any limit ordinal can be written in the form $\omega \cdot \alpha$ for some ordinal $\alpha$.

**Corollary 2.7:** Any ordinal number can be written in the form $\omega \cdot \alpha + n$, where $\alpha$ is an ordinal and $n$ a natural number.

**Lemma 2.8:** For any ordinal $\alpha > 0$, $1 + \omega \cdot \alpha = \omega \cdot \alpha$.

**Theorem 2.9:** For every natural number $k$ and every infinite ordinal $\gamma$, we have $k + \gamma = \gamma$.

*Proof:* Using the above result, there exist $\alpha > 0$ and $n \in \mathbb{N}_0$ such that $\gamma = \omega \cdot \alpha + n$. Our proof is by induction on $k$. When $k = 0$, the result is obvious. Now, assume that the result is true for $k$, we will show that it is also true for $k + 1$:

$$(k + 1) + \gamma = (k + 1) + (\omega \cdot \alpha + n) = ((k + 1) + \omega \cdot \alpha) + n = (k + (1 + \omega \cdot \alpha)) + n = (k + \omega \cdot \alpha) + n = \omega \cdot \alpha + n = \gamma.$$  

More generally, by using Corollary 2.7, we have the following theorem:

**Theorem 2.10:** For every ordinal numbers $\alpha$ and $\beta$, if $\beta \leq \alpha$, then there exist ordinal number $\gamma$ such that $\beta + \gamma = \alpha$.

We will finish this section by stating an interesting theorem which is an assertion to avoid a variant of Russell's Paradox, known as Burali-Forti Paradox:

**Theorem 2.11:** The ordinal numbers do not form a set (but a class, denoted by $\mathcal{O}$).
3 The Ring of Ordinomials

In this section, we will step-by-step develop an algebraic structure which we will utilize in section 5 in order to define partial and universal chromatic ordinomials of infinite graphs. Please be informed that by $O^*$ we mean $O - \{0\}$.

First, we will define a free multiplication on the set $B = \{(\lambda - \zeta)^n | \zeta \in C, \alpha \in O^* \} \cup \{1\}$ of formal objects by putting these building blocks next to each other, satisfying the following axioms:

$A_1)(\forall x_i \in B) (x_1 x_2 = x_2 x_1)$;

$A_2)(\forall x_i \in B) ((x_1 x_2) x_3 = x_1 (x_2 x_3))$;

$A_3)(\forall x \in B) (x 1 = x)$;

$A_4)(\forall \zeta \in C)(\forall \alpha_i \in O^*)
\left(\alpha_1 \leq \alpha_2 \right) \rightarrow \left((\lambda - \zeta)^{\alpha_1}(\lambda - \zeta)^{\alpha_2} = (\lambda - \zeta)^{\alpha_1+\alpha_2}\right)$. 

To clarify how $A_4$ works, we will have a few examples:

i) For $k, l \in N$ such that $k \leq l$, $(\lambda - \zeta)^k(\lambda - \zeta)^l = (\lambda - \zeta)^l(\lambda - \zeta)^k = (\lambda - \zeta)^{l+k}$;

ii) For $k \in N$, $(\lambda - \zeta)^{\omega}(\lambda - \zeta)^k = (\lambda - \zeta)^k(\lambda - \zeta)^{\omega} = (\lambda - \zeta)^{k+\omega} = (\lambda - \zeta)^{\omega}$;

iii) $(\lambda - \zeta)^{\omega}(\lambda - \zeta)^{\omega^2} = (\lambda - \zeta)^{\omega^2}(\lambda - \zeta)^{\omega} = (\lambda - \zeta)^{\omega^2+\omega} = (\lambda - \zeta)^{\omega^3}$;

iv) $(\lambda - \zeta)^{\omega}(\lambda - \zeta)^{\omega^2} = (\lambda - \zeta)^{\omega^2}(\lambda - \zeta)^{\omega} = (\lambda - \zeta)^{\omega^2+\omega} = (\lambda - \zeta)^{\omega^3}$;

v) By Theorem 2.9, for $k \in N$ and $\gamma$ an infinite ordinal, $(\lambda - \zeta)^{\gamma} = (\lambda - \zeta)^{k+\gamma} = (\lambda - \zeta)^k(\lambda - \zeta)^\gamma$.

The set $B$ is totally ordered in the sense that, for all $\zeta$ and $\alpha$, $1 < (\lambda - \zeta)^\alpha$, and if $\alpha_1 < \alpha_2$, then $(\lambda - \zeta)^{\alpha_1} < (\lambda - \zeta)^{\alpha_2}$. In addition to that, if $\zeta_1 < \zeta_2$, then for every $\alpha_1, \alpha_2 \in O^*$, $(\lambda - \zeta_1)^{\alpha_1} < (\lambda - \zeta_2)^{\alpha_2}$; we say $\zeta_1 = a + i b < \zeta_2 = c + i d$, when either $a < c$ or when $a = c$, then $b < d$.

Our first theorem in this section is

Theorem 3.1: If $B[\lambda] = \{ \prod_{i=1}^n x_i | n \in N, x_i \in B \}$, then $B[\lambda]$ is a free commutative monoid.

Proof: Let $n$ and $m$ be arbitrary positive integer and for $1 \leq i \leq n$ and $1 \leq j \leq m$, $x_i, y_j \in B[\lambda]$. We have

$$\left( \prod_{i=1}^n x_i \right) \left( \prod_{j=1}^m y_j \right) = \prod_{k=1}^{n+m} z_k,$$

in which, for $1 \leq k \leq n$, $z_k = x_k$ and for $n + 1 \leq k \leq n + m$, $z_k = y_{k-n}$. It is clear from axioms $A_1$-$A_3$ that $B[\lambda]$ is associative and commutative and has 1 as its identity.
An element \( x \) of \( \mathbb{B}[\lambda] \) is called a \textit{reduced element}, when \( x \) is either 1 or \( \prod_{i=1}^{n} (\lambda - \zeta_i)^{\alpha_i} \), for distinct \( \zeta_i \)'s. By \textit{reduced form} of \( y \in \mathbb{B}[\lambda] \), we mean a reduced element \( x \), such that \( y \) is equal to, by applying axioms \( A_1 \), \( A_3 \), and \( A_4 \), finitely many times. Consequently, reduced elements \( \prod_{i=1}^{n} (\lambda - \zeta_i)^{\alpha_i} \) and \( \prod_{j=1}^{m} (\lambda - \xi_j)^{\beta_j} \) are equal, when \( n = m \) and there exist \( \sigma \in S_n \), such that \( \zeta_i = \xi_{\sigma(i)} \) and \( \alpha_i = \beta_{\sigma(i)} \), for \( i \in \{1, 2, \ldots, n\} \). By convention, we will assume that 1 is not equal to any other reduced element.

**Theorem 3.2:** Every element of \( \mathbb{B}[\lambda] \) has a reduced form which is unique.

**Proof:** As we made a clear remark in previous paragraph, uniqueness follows from existence. Let \( x \) be an element of \( \mathbb{B}[\lambda] \) equal to \( \prod_{i=1}^{n} x_i \), with \( x_i \in \mathbb{B} \). Our proof is based on induction on \( n \). When \( n = 1 \), either \( x = 1 \) or \( x = (\lambda - \zeta)^{\alpha} \), which in either case by definition \( x \) is a reduced element. Now, assume that every element \( \prod_{i=1}^{k} y_i \) has a reduced form. We will show that, every element \( \prod_{i=1}^{k+1} x_i \) also has a reduced form. For \( 1 \leq i \leq k+1 \), we can assume that \( x_i = (\lambda - \zeta_i)^{\alpha_i} \), because if there exist \( i \in \{1, 2, \ldots, k+1\} \) such that \( x_i = 1 \), without loss of generality assume \( x_{k+1} = 1 \), then \( x = \prod_{i=1}^{k+1} x_i = \prod_{i=1}^{k} x_i \) which by induction hypothesis has a reduced form. Thus, two cases may occur:

i) For \( i \in \{1, 2, \ldots, k+1\} \), \( \zeta_i \)'s are distinct complex numbers which by definition \( x \) is a reduced element.

ii) For \( i \) and \( j \) in \( \{1, 2, \ldots, k+1\} \), \( \zeta_i = \zeta_j \). By using \( A_1 \) and \( A_4 \), finitely many times, we will decrease the number of \( x_i \)'s occurring in \( x \) and then by induction hypothesis \( x \) has a reduced form.

From now on, we will assume every element \( x \in \mathbb{B}[\lambda] \) is a reduced element. For every \( \lambda \in \mathbb{B}[\lambda] \), \( 1 \mid x \) and if \( x = \prod_{i=1}^{n} (\lambda - \zeta_i)^{\alpha_i} \), then for \( \beta \in \mathbb{N}^* \) and \( \zeta \in \mathbb{C} \), \( (\lambda - \zeta)^{\beta} \mid x \), when there exist \( i \in \{1, 2, \ldots, n\} \), such that \( \zeta = \zeta_i \) and \( \beta \leq \alpha_i \). Let \( y = \prod_{j=1}^{m} y_j \), then \( y \mid x \), if \( m \leq n \) and \( y_j \mid x \), for every \( j \). By convention, if \( x \mid 1 \), then \( x = 1 \). If \( x, y \in \mathbb{B}[\lambda] \), then we will define the greatest common divisor of \( x \) and \( y \), denoted by \( (x, y) \), as the element \( z \) such that \( z \mid x \) and \( z \mid y \), and if \( w \mid x \) and \( w \mid y \), then \( w \mid z \).

Assume that for \( y = \prod_{j=1}^{m} (\lambda - \zeta_j)^{\beta_j} \) and \( x = \prod_{i=1}^{n} (\lambda - \zeta_i)^{\alpha_i} \), \( y \mid x \). So, by definition \( m \leq n \) and \( (\lambda - \zeta_j)^{\beta_j} \mid \prod_{i=1}^{n} (\lambda - \zeta_i)^{\alpha_i} \), for every \( j \). Without loss of generality, assume that \( \xi_j = \zeta_j \) and as a result, \( \beta_j \leq \alpha_j \). Then, by Theorem 2.10, there exist \( \gamma_j \) such that \( \beta_j + \gamma_j = \alpha_j \). By using these fact, we have \( x = y(\prod_{i=1}^{m} (\lambda - \zeta_i)^{\gamma_i} \prod_{i=m+1}^{n} (\lambda - \zeta_i)^{\alpha_i}) \). So, when \( y \mid x \), there exist \( z \) such that \( x = yz \).

Our next goal is to define a degree function, denoted by \( \deg(x) \), for every element \( x \) of \( \mathbb{B}[\lambda] \). Conventionally, \( \deg(1) = 0 \). For \( x = \prod_{i=1}^{n} (\lambda - \zeta_i)^{\alpha_i} \), let \( \beta_1, \beta_2, \ldots, \beta_n \) be the rearrangement of \( \alpha_1, \alpha_2, \ldots, \alpha_n \), such that \( \beta_1 \leq \beta_2 \leq \ldots \leq \beta_n \). Then, \( \deg(x) \) is equal to the
sum $\beta_1 + \beta_2 + \beta_3 + \ldots + \beta_n$. We will call $x$ purely infinite, when $\beta_1$ is an infinite ordinal, which clearly implies $\deg(x)$ is infinite. As $\deg(\ )$ is now a well-defined mapping from $\mathbb{B}[\lambda]$ to $\mathbb{O}$, it is clear that for $x, y \in \mathbb{B}[\lambda]$, if $\deg(x) \neq \deg(y)$, then $x \neq y$.

Now assume, for $x \in \mathbb{B}[\lambda]$, $\deg(x)$ is infinite. It is easy to see that $x$ can be uniquely decomposed into the form $yz$ such that $y$ is purely infinite, $\deg(z)$ is infinite. As $\deg(\ )$ is now a well-defined mapping from $\mathbb{B}[\lambda]$ to $\mathbb{O}$, it is clear that for $x, y \in \mathbb{B}[\lambda]$, if $\deg(x) \leq \deg(y)$, then $x \leq y$.

Now it is time to equip $\mathbb{B}[\lambda]$ with a complex scalar product, defined as $r \cdot x = (r, x) \in C \times \mathbb{B}[\lambda]$ and having the following properties:

\[ P_1(\forall \tau \in C)(\forall x \in \mathbb{B}[\lambda]) \left( (\deg(x) \in \mathbb{N}_0) \rightarrow (\tau \cdot x = \tau x \in \mathbb{C}[\lambda]) \right); \]

Particularly, $\tau \cdot 1 = \tau$ and to be more specific, $0 = 0 \cdot 1$ and $1 = 1 \cdot 1$.

\[ P_2(\forall x \in \mathbb{B}[\lambda]) \left( (0 \cdot x = 0) \land (1 \cdot x = x) \right); \]

\[ P_3(\forall \tau \in C)(\forall x \in \mathbb{B}[\lambda]) \left( (\tau \cdot x = 0) \rightarrow (\tau = 0) \right); \]

\[ P_4(\forall \tau_i \in \mathbb{C}^*)(\forall x_i \in \mathbb{B}[\lambda]) \left( (\tau_1 \cdot x_1 = \tau_2 \cdot x_2) \rightarrow ((\tau_1 = \tau_2) \land (x_1 = x_2)) \right). \]

We should assert that, from $P_1$ and $P_3$, it is clear that $1 = 1 \cdot 1 \neq 0 \cdot 1 = 0$.

We will define a free addition and a multiplication on elements of $C \times \mathbb{B}[\lambda]$, having the following properties in order to introduce the desired free ring:

Addition:

\[ P_1^+(\forall \tau_i \in C)(\forall x_i \in \mathbb{B}[\lambda]) \left( \tau_1 \cdot x_1 + \tau_2 \cdot x_2 = \tau_2 \cdot x_2 + \tau_1 \cdot x_1 \right); \]

\[ P_2^+(\forall \tau_i \in C)(\forall x_i \in \mathbb{B}[\lambda]) \left( \tau_1 \cdot x_1 + (\tau_2 \cdot x_2 + \tau_3 \cdot x_3) = (\tau_1 \cdot x_1 + \tau_2 \cdot x_2) + \tau_3 \cdot x_3 \right); \]

\[ P_3^+(\forall \tau_i \in C)(\forall x \in \mathbb{B}[\lambda]) \left( \tau_1 \cdot x + \tau_2 \cdot x = (\tau_1 + \tau_2) \cdot x \right); \]

\[ P_4^+(\forall \tau \in C)(\forall x \in \mathbb{B}[\lambda]) \left( \tau \cdot x + 0 = \tau \cdot x \right); \]

\[ P_5^+(\forall \tau \in C)(\forall x \in \mathbb{B}[\lambda]) \left( \tau \cdot x + (-\tau) \cdot x = 0 \right). \]

Multiplication:

\[ P_1^*(\forall \tau_i \in C)(\forall x_i \in \mathbb{B}[\lambda]) \left( (\tau_1 \cdot x_1) \times (\tau_2 \cdot x_2) = (\tau_1 \tau_2) \cdot (x_1 x_2) \right); \]
The free ring we are looking for is defined as
\[ \mathcal{O}^*[\lambda] = \left\{ \sum_{i=1}^{n} \tau_i \cdot x_i \middle| n \in \mathbb{N}, \tau_i \in \mathbb{C}, x_i \in \mathbb{B}[\lambda] \right\}. \]

Theorem 3.3: \( \mathcal{O}^*[\lambda] \) is a free commutative ring having a multiplicative identity.

**Proof:** Let \( \sum_{i=1}^{n} \tau_i \cdot x_i \) and \( \sum_{j=1}^{m} \phi_j \cdot y_j \) be elements of \( \mathcal{O}^*[\lambda] \). We will first prove that both addition is a closed operation. We have
\[ \left( \sum_{i=1}^{n} \tau_i \cdot x_i \right) + \left( \sum_{j=1}^{m} \phi_j \cdot y_j \right) = \sum_{k=1}^{n+m} \psi_k \cdot z_k, \]
in which \( \psi_k = \tau_k \) and \( z_k = x_k \), for \( 1 \leq k \leq n \), and \( \psi_k = \phi_{k-n} \) and \( z_k = y_{k-n} \), for \( n+1 \leq k \leq n+m \).

Being an associative and commutative operation and having 0 as the additive identity element are inherited from \( P_1^+, P_2^+ \), and \( P_4^+ \). Finally, for every element \( \sum_{i=1}^{n} \tau_i \cdot x_i \), there exist an additive inverse, namely \( \sum_{i=1}^{n} (-\tau_i) \cdot x_i \).

Now we will show that multiplication is a closed operation:
\[ \left( \sum_{i=1}^{n} \tau_i \cdot x_i \right) \times \left( \sum_{j=1}^{m} \phi_j \cdot y_j \right) = \sum_{k=1}^{nm} \psi_k \cdot z_k, \]
such that, for \( 1 \leq l \leq n \) and \( 1 \leq k \leq nm \), when \( (l-1)m+1 \leq k \leq lm \), then \( \psi_k = \tau_l \phi_{k-(l-1)m} \) and \( z_k = x_l y_{k-(l-1)m} \).
Being a distributive operation with respect to addition and having associative and commutative properties are inherited from properties of $P_2^x$, $P_4^x$, and $P_5^x$. Finally, $1 = 1 \cdot 1$ is the identity of multiplication.

From now on, an element of $O^*\{\lambda\}$ will be called an ordinal and the ring itself will be the ring of ordinals as it is a free ring constructed in a similar fashion as the familiar polynomial rings were and it also utilizes ordinal arithmetic.

**Theorem 3.4:** $C[\lambda]$ is a subring of $O^*\{\lambda\}$.

**Proof:** The fundamental theorem of algebra states that every monic polynomial of degree $n$ with complex roots has $n$ complex roots (not necessarily all distinct). So, we can write every element $p(\lambda)$ of $C[\lambda]$ in the form $\tau(\lambda - \xi_1)(\lambda - \xi_2)\cdots(\lambda - \xi_n)$ with $\xi_i, \tau \in C$. After rewriting it in the form $\tau(\lambda - \zeta_1)^{n_1}(\lambda - \zeta_2)^{n_2}\cdots(\lambda - \zeta_m)^{n_m}$ (where $m, n_1, n_2, \ldots, n_m \in N$ and $\zeta_i$'s are all distinct complex numbers), it is clear from $P_1$ that $p(\lambda) \in O^*\{\lambda\}$, and thus $C[\lambda] \subseteq O^*\{\lambda\}$.

Now, define $g : C[\lambda] \rightarrow O^*\{\lambda\}$ that maps every element $p(\lambda) = \tau_n \lambda^n + \cdots + \tau_2 \lambda^2 + \tau_1 \lambda + \tau_0$ ($n \in N, \tau_i \in C$ and $\tau_n \neq 0$) of $C[\lambda]$ to $\tau_n \cdot \lambda^n + \cdots + \tau_2 \cdot \lambda^2 + \tau_1 \cdot \lambda + \tau_0 \cdot 1$ of $O^*\{\lambda\}$. It is easy to check that $g$ is a well-defined mapping and for every $p(\lambda)$ and $q(\lambda)$, $g(p(\lambda) + q(\lambda)) = g(p(\lambda)) + g(q(\lambda))$ and $g(p(\lambda)q(\lambda)) = g(p(\lambda)) \times g(q(\lambda))$.

Let's assume that for $x = \sum_{i=1}^{n} \tau_i \cdot x_i$, $F_x \subseteq \{1, 2, \ldots, n\}$ is the set of all $i$'s such that $\text{deg}(x_i)$ is finite. If $F_x$ is non-empty, then for $i \in F_x$, $\tau_i \cdot x_i$ is in $C[\lambda]$ and so is $\sum_{i \in F_x} \tau_i \cdot x_i$. Without loss of generality, assume that, for $1 \leq k \leq n$, $F_x = \{k, k + 1, \ldots, n\}$. As $\sum_{i \in F_x} \tau_i \cdot x_i$ is in $C[\lambda]$, three cases may occur: it is equal to zero, a non-zero complex number, or a polynomial.

If it is equal to zero, then we will omit it from $x$ and rewrite $x$ as $\sum_{i=1}^{k-1} \tau_i \cdot x_i$. Clearly, when $k = 1$, $x = 0$.

If it is a non-zero complex number, say $\tau$, we will rewrite $x$ as $\sum_{i=1}^{k-1} \tau_i \cdot x_i + \tau \cdot 1$. Obviously, when $k = 1$, $x = \tau \cdot 1 = \tau$.

Finally, if it is a polynomial $\tau(\lambda - \zeta_1)^{n_1}(\lambda - \zeta_2)^{n_2}\cdots(\lambda - \zeta_m)^{n_m}$ in which $m, n_1, n_2, \ldots, n_m \in N$ and $\zeta_i$'s are distinct complex numbers, then clearly $y = (\lambda - \zeta_1)^{n_1}(\lambda - \zeta_2)^{n_2}\cdots(\lambda - \zeta_m)^{n_m}$ is a reduced element of $B[\lambda]$ and we can rewrite $x$ as $\sum_{i=1}^{n} \tau \cdot x_i = \sum_{i=1}^{k-1} \tau \cdot x_i + \tau \cdot y$. Surely, when $k = 1$, $x = \tau \cdot y$.

With regard to the above observation, we call an ordinal $x$ polynomially reduced, when the set $F_x$ defined above is either empty or a singleton. From definition of $0 = 0 \cdot 1$, $F_0 = \{1\}$. By polynomially reduced form of $y$, we mean a polynomially reduced ordinal $x$ which $y$ is equal to using the above process.

The above lines proves the following theorem:
Theorem 3.5: Every element of $O^*[\lambda]$ is equal to a polynomially reduced ordinomial. $lacksquare$

Now, let $x = \sum_{i=1}^{n} \tau_i \cdot x_i$ be a polynomially reduced ordinomial such that either $F_x$ is empty, or if not, $n > 1$. So, either $\text{deg}(x_i)$ is infinite for all $i$ or for all except one index which we assume it is equal to $n$. If the former happens, we know that every $x_i$ can be decomposed into $y_i z_i$ such that $\text{deg}(y_i)$ is infinite, $\text{deg}(z_i) \in N_0$, and $(y_i, z_i) = 1$. If the latter happens, we will do the same except for $x_n$. If all $y_i$'s are distinct, we will call $x$ a reduced ordinomial. Now, assume they are not all distinct. We will partition the set of indices into $S_1, S_2, \ldots, S_m$ while $S_1$ contains all the indices $i$ such that $y_i$'s are equal; in latter case, we will assume that $S_m = \{n\}$. Then, we have $x = \sum_{i \in S_1} \tau_i \cdot x_i + \sum_{i \in S_2} \tau_i \cdot x_i + \cdots + \sum_{i \in S_m} \tau_i \cdot x_i$. We assume for $1 \leq j \leq m$, $y_j' = y_i$ such that $i \in S_j$; in latter case, we will assume that $y_m' = 1$. So, we can rewrite $x$ as $y_1' \times (\sum_{i \in S_1} \tau_i \cdot z_i) + y_2' \times (\sum_{i \in S_2} \tau_i \cdot z_i) + \cdots + y_m' \times (\sum_{i \in S_m} \tau_i \cdot z_i)$. For $1 \leq j \leq m$, $\sum_{i \in S_j} \tau_i \cdot z_i$ is a member of $C[\lambda]$: if it is zero, then $y_j' \times (\sum_{i \in S_j} \tau_i \cdot z_i) = 0$; if it is a non-zero complex $\phi_j$, then $y_j' \times (\sum_{i \in S_j} \tau_i \cdot z_i) = y_j' \times \phi_j = \phi_j \cdot y_j'$. Finally, it is equal to a polynomial $\phi_j \cdot z_j'$, then $y_j' \times (\sum_{i \in S_j} \tau_i \cdot z_i) = y_j' \times \phi_j \cdot z_j' = \phi_j \cdot y_j' z_j'^\prime$ in which $z_j'^\prime$ is an element of $B[\lambda]$ such that $y_j' z_j'^\prime$ is the reduced form of $y_j' z_j'$ and $(y_j', z_j'^\prime) = 1$. By rewriting $x$ as described, $x$ will either be equal to zero or a reduced ordinomial. When $n = 1$ and $F_x$ is not empty, then $x$ is a polynomially reduced element of $C[\lambda]$ which would be a reduced ordinomial by definition; Hence, 0 is a reduced ordinomial.

By the reduced form of an ordinomial $y$, we mean a reduced ordinomial $x$ such that $y$ is equal to $x$ by going through the processes described above. If $y$ becomes zero, we will call $y$ a degenerate ordinomial; otherwise, it is called a non-degenerate ordinomial.

The above lines proves the following theorem:

Theorem 3.6: Every element of $O^*[\lambda]$ is equal to a reduced ordinomial.$

\text{Let } x = \sum_{i=1}^{n} \tau_i \cdot x_i \text{ be a non-degenerate ordinomial such that } F_x \text{ is empty and for every } i, x_i \text{ is a purely infinite element of } B[\lambda]; \text{ then, we will call } x \text{ a purely infinite ordinomial. When, only } F_x \text{ is empty, } x \text{ is just called an infinite ordinomial.}$

If $\sum_{i=1}^{n} \tau_i \cdot x_i$ is a non-degenerate ordinomial, then so is $\sum_{i=1}^{n} (-\tau_i) \cdot x_i$. Moreover, It is clear that when one is purely infinite (or just infinite), so would be the other.

Theorem 3.7: Every element of $O^*[\lambda]$ is either equal to 0 or a unique non-degenerate ordinomial up to permutation of indices.

\textbf{Proof:} From the previous theorem, we know that every ordinomial is equal to a reduced element and from definition, its is either degenerate or not. So, we only have to prove that
if non-degenerate ordinals \( x = \sum_{i=1}^{n} \tau_i \cdot x_i \) and \( y = \sum_{j=1}^{m} \phi_j \cdot y_j \) are equal, then \( n = m \) and there exist \( \sigma \in S_n \) such that \( \tau_i = \phi_{\sigma(i)} \) and \( x_i = y_{\sigma(i)} \) for all \( i \).

Not only these two ordinals are reduced, but also they are polynomially reduced. Three cases may occur: 1) \( F_x \) and \( F_y \) are both empty which is the case when they are both infinite ordinals; 2) \( F_x \) and \( F_y \) are non-empty sets and we assume \( F_x = \{n\} \) and \( F_y = \{m\} \); 3) Just one of \( F_x \) or \( F_y \) is empty. We start with the first case:

Let \( \sum_{i=1}^{n} \tau_i \cdot x_i \) and \( \sum_{j=1}^{m} \phi_j \cdot y_j \) be infinite non-degenerate ordinals such that \( x_i = x'_ix''_i \) and \( y_j = y'_jy''_j \) are the unique decomposition of \( x_i \) and \( y_j \); clearly, \( (x'_i, x''_i) = 1 \) and \( (y'_j, y''_j) = 1 \). Now assume that \( x'_i \)'s and \( y'_j \)'s are all distinct elements of \( \mathbb{B}[\lambda] \). It is not hard to see that the sum \( \sum_{i=1}^{n} \tau_i \cdot x_i + \sum_{j=1}^{m} \phi_j \cdot y_j = \sum_{i=1}^{n} \tau_i \cdot (x'_ix''_i) + \sum_{j=1}^{m} \phi_j \cdot (y'_jy''_j) \) is also an infinite non-degenerate ordinal.

Now, assume that infinite non-degenerate ordinals \( \sum_{i=1}^{n} \tau_i \cdot x_i \) and \( \sum_{j=1}^{m} \phi_j \cdot y_j \) are equal to each other. Let \( x'_i, x''_i, y'_j \) and \( y''_j \) be defined as was defined in the previous paragraph. We can write \( \sum_{i=1}^{n} \tau_i \cdot x_i + \sum_{j=1}^{m} (-\phi_j) \cdot y_j = \sum_{i=1}^{n} \tau_i \cdot (x'_ix''_i) + \sum_{j=1}^{m} (-\phi_j) \cdot (y'_jy''_j) = 0 \), and clearly there exist \( i_1 \) and \( j_1 \), such that \( x'_{i_1} = y'_{j_1} \). Because if \( x'_{i_1} \)'s and \( y'_{j_1} \)'s were all distinct, then one side of the equation was degenerated, while the other side is non-degenerate which clearly is not possible. As \( x'_{i_1} \)'s and \( y'_{j_1} \)'s are all distinct, \( x'_{i_1} \) is only equal to \( y'_{j_1} \), and vice versa. Without loss of generality, assume \( i_1 = n \) and \( j_1 = m \) and \( z = x'_{n} = y'_{m} \). So, \( \sum_{i=1}^{n-1} \tau_i \cdot (x'_ix''_i) + \sum_{j=1}^{m-1} (-\phi_j) \cdot (y'_jy''_j) + z \times (\tau_n \cdot x''_n + (\phi_m) \cdot y''_m) = 0 \). If \( \tau_n \cdot x''_n + (\phi_m) \cdot y''_m \) is not equal to zero, because \( z \) is distinct from the rest of \( x'_{i_1} \)'s and \( y'_{j_1} \)'s, the sum cannot be degenerated to 0. So, \( \tau_n \cdot x''_n = \phi_m \cdot y''_m \) and as a result \( \tau_n = \phi_m \) and \( x''_n = y''_m \); Hence, \( x_n = y_m \). \( \sum_{i=1}^{n-1} \tau_i \cdot (x'_ix''_i) + \sum_{j=1}^{m-1} (-\phi_j) \cdot (y'_jy''_j) = 0 \). By repeating the argument given above we can prove that \( n = m \) and there exist \( \sigma \in S_n \) such that \( y_i = x_{\sigma(i)} \) and \( \phi_i = \tau_{\sigma(i)} \).

Now, assume for \( x = \sum_{i=1}^{n} \tau_i \cdot x_i \) and \( y = \sum_{j=1}^{m} \phi_j \cdot y_j \), we have \( x = y, F_x = \{n\} \), and \( F_y = \{m\} \). So, we can write \( \sum_{i=1}^{n-1} \tau_i \cdot x_i + \sum_{j=1}^{m-1} (-\phi_j) \cdot y_j + \tau_n \cdot x_n + (\phi_m) \cdot y_m = 0 \). Assume, \( \tau_n \cdot x_n \) and \( (\phi_m) \cdot y_m \) are not equal. Then, on one side of the above equation we have a non-degenerate ordinals, while the other side is zero which is impossible. It follows that \( \tau_n \cdot x_n = \phi_m \cdot y_m \) and as a result \( \tau_n = \phi_m \) and \( x_n = y_m \). So, we can rewrite the above equation as \( \sum_{i=1}^{n-1} \tau_i \cdot x_i + \sum_{j=1}^{m-1} (-\phi_j) \cdot y_j = 0 \). If \( n \) and \( m \) are both greater than 1, then we can argue as we did for the first case; If one was 1 while the other one was greater than 1, we would have a non-degenerate ordinal equal to 0 which surely is not possible; Finally, if they are both equal to one we are not left with anything to argue about.

At the end, for non-degenerate ordinals \( x \) and \( y \), assume that \( x = y \) and \( F_x = \{n\} \) while \( F_y \) is empty. We will show this case is not possible. We have \( \sum_{i=1}^{n-1} \tau_i \cdot x_i + \sum_{j=1}^{m-1} (-\phi_j) \cdot y_j + (\tau_n \cdot x_n) = 0 \). We know \( \tau_n \neq 0 \), but then we have a non-degenerate ordinals equal to
0 which is not possible. ■

From now on, by an ordinomial we mean either 0 or a non-degenerate ordinomial. The degree of an ordinomial \( x = \sum_{i=1}^{n} \tau_i \cdot x_i \), denoted by \( \text{deg}(x) \), is defined as the largest ordinal occurring in the set \( \{ \text{deg}(x_i) \mid i = 1, 2, \ldots, n \} \) and as \( 0 = 0 \cdot 1 \), have \( \text{deg}(0) = 0 \). Theorem 3.7 asserts that \( \text{deg}(x) \) is a well-defined mapping from \( \mathcal{O}^\ast[\lambda] \) to \( \mathcal{O} \). So, for \( x \) and \( y \) ordinals, if \( \text{deg}(x) \neq \text{deg}(y) \), then \( x \neq y \). An ordinomial \( x \) is called a monic ordinomial, when for all \( i, \tau_i = 1 \) and \( \text{deg}(x) > 0 \).

The subset \( \{1 \cdot x \mid x \in \mathcal{B}[\lambda]\} \) of \( \mathcal{O}^\ast[\lambda] \) denoted by \( \mathcal{B}[\lambda] \) and armed with \( \cdot \), is a monoid isomorphic to \( \mathcal{B}[\lambda] \). From \( P_3 \) it is clear that if \( 1 \cdot x \) and \( 1 \cdot y \) are elements of \( \mathcal{B}[\lambda] \), then \( 1 \cdot x \cdot 1 \cdot y = 1 \cdot xy \) is not equal to zero. It is clear from definition that every element of \( \mathcal{B}[\lambda] \) is a monic ordinomial. On the other hand, if \( \mathcal{M}[\lambda] \) is the set all monic polynomials in \( \mathcal{C}[\lambda] \), we know that with polynomial multiplication it is a submonoid of \( \mathcal{C}[\lambda] \) homomorphic to \( \mathcal{B}[\lambda] \). This observation will be used in section 5 when we talk about chromatic ordinals.

In general, as one may see in the following example \( \mathcal{O}^\ast[\lambda] \) is not an integral domain:

\[
(\lambda - \zeta)^{\omega^3} \times ((\lambda - \zeta)^{\omega^1} + (-1) \cdot (\lambda - \zeta)^{\omega^2}) = \\
(\lambda - \zeta)^{\omega^1+1}(\lambda - \zeta)^{\omega^3} + (-1) \cdot (\lambda - \zeta)^{\omega^2}(\lambda - \zeta)^{\omega^3} = \\
(\lambda - \zeta)^{\omega^2+\omega^3} + (-1) \cdot (\lambda - \zeta)^{\omega^2+\omega^3} = \\
(\lambda - \zeta)^{\omega^3} + (-1) \cdot (\lambda - \zeta)^{\omega^3} = 0.
\]

For \( x \in \mathcal{O}^\ast[\lambda], 1 \mid x \), and if \( x = \sum_{i=1}^{n} \tau_i \cdot x_i \), then \( y \in \mathcal{B}[\lambda] \) divides \( x \), if \( y \mid x_i \), for every \( i \). By convention, for every \( x \in \mathcal{B}[\lambda], x \mid 0 \). Assume that for a non-degenerate ordinomial \( x \) and \( y \in \mathcal{B}[\lambda], y \mid x = \sum_{i=1}^{n} \tau \cdot x_i \). By definition for every \( i \), \( y \mid x_i \), and as a result there exist \( z_i \) such that \( x = y z_i \). Therefore, \( x = y \times (\sum_{i=1}^{n} \tau \cdot z_i) \). Finally, if \( F_x \) is not empty and by assumption equal to \( \{ n \} \), then by definition \( y \mid x_n \) and as \( x_n \) is a polynomial, \( \text{deg}(y) \in \mathcal{N}_0 \).

We will now develop a limit operation which would be the foundation of how chromatic ordinals will be defined for infinite graphs in section 5. Let \( f : \mathcal{N} \rightarrow \mathcal{N} (f : \omega \rightarrow \omega) \) be an order preserving mapping, meaning that if \( k_1 < k_2 \), then \( f(k_1) \leq f(k_2) \). We will define \( \lim_{k \rightarrow \omega}(\lambda - \zeta)^{f(k)} \) as follows:

\[
\lim_{k \rightarrow \omega}(\lambda - \zeta)^{f(k)} = \lim_{k \leq \omega}(\lambda - \zeta)^{f(k)} = (\lambda - \zeta)^{\left( \bigcup_{f(k)} f(k) \right)}.
\]

One may clearly see that

i) \( \lim_{k \rightarrow \omega}(\lambda - \zeta)^{f(k)} = (\lambda - \zeta)^{k_0} \), when there exist \( M \in \mathcal{N} \) such that for \( k > M \) and \( k_0 \in \mathcal{N} \), \( f(k) = k_0 \);
ii) $\lim_{k \to \omega} (\lambda - \zeta)^{f(k)} = (\lambda - \zeta)^\omega$, otherwise.

This limit operation has the properties

$$\lim_{k \to \omega} \left( \tau \cdot \prod_{i=1}^n (\lambda - \zeta_i)^{f_i(k)} \right) = \tau \cdot \prod_{i=1}^n \left( \lim_{k \to \omega} (\lambda - \zeta_i)^{f_i(k)} \right)$$

for distinct complex numbers $\zeta_i$ and order preserving mappings $f_i : \mathbb{N} \to \mathbb{N}$, and

$$\lim_{k \to \omega} \left( \sum_{i=1}^n \tau_i \cdot \prod_{j=1}^{m_i} (\lambda - \zeta_{ij})^{f_{ij}(k)} \right) = \sum_{i=1}^n \left( \lim_{k \to \omega} (\tau_i \cdot \prod_{j=1}^{m_j} (\lambda - \zeta_{ij})^{f_{ij}(k)}) \right),$$

such that for every $i$, $\zeta_{ij}$ are distinct complex numbers and $f_{ij} : \mathbb{N} \to \mathbb{N}$ are order preserving mappings.

If $f(k)$ can be decomposed into the form $f_1(k) + f_2(k)$ in which $f_1, f_2 : \mathbb{N} \to \mathbb{N}$ are two other order preserving mappings, then following is not necessarily valid

$$\lim_{k \to \omega} (\lambda - \zeta)^{f(k)} = \left( \lim_{k \to \omega} (\lambda - \zeta)^{f_1(k)} \right) \left( \lim_{k \to \omega} (\lambda - \zeta)^{f_2(k)} \right).$$

For example, we know $\lim_{k \to \omega} (\lambda - \zeta)^{2k} = (\lambda - \zeta)^\omega$. On the other hand, we can decompose $2n$ into the form $k + k$. So, we have $\lim_{k \to \omega} (\lambda - \zeta)^k (\lim_{n \to \omega} (\lambda - \zeta)^k) = (\lambda - \zeta)^\omega (\lambda - \zeta)^\omega = (\lambda - \zeta)^{\omega^2}$ which is not equal to $(\lambda - \zeta)^\omega$.

We would finish this section with the following question: What would happen, if we replace $A_4$ with the axiom below?

$A_4' \forall \zeta \in \mathbb{C} \forall \alpha_i \in \mathbb{O}^*$

$$\left( \alpha_2 \leq \alpha_1 \right) \rightarrow \left( (\lambda - \zeta)^{\alpha_1} (\lambda - \zeta)^{\alpha_2} = (\lambda - \zeta)^{\alpha_2} (\lambda - \zeta)^{\alpha_1} = (\lambda - \zeta)^{\alpha_1 + \alpha_2} \right).$$
4 Chromatic Polynomial of Some Chains of Graphs

In the first section we saw the chromatic polynomial of some families of graphs such as null graphs, trees, cycles, complete graphs, chordal graphs, $q$-trees, $n$-ladders, $\Delta_n$ graphs, theta graphs, wheels, double/pyramids, bi-wheels, and broken wheels. In this section we will first define what a sequence of graphs is and afterwards try to find chromatic polynomial of some interesting sequences of graphs, either in closed or recursive form.

By a sequence of graphs, we mean the family $\{\Gamma_n\}_{n \in \mathbb{N}}$ in which $\Gamma_n$ is a finite graph. We call a sequence, a chain of graphs provided that $\Gamma_1 \subset \Gamma_2 \subset \cdots \subset \Gamma_n \subset \cdots$ and for all $n \in \mathbb{N}$, $|V(\Gamma_n)| = f(n)$ in which $f(n)$ is a strictly increasing function from $\mathbb{N}$ to $\mathbb{N}$. When for all $n \in \mathbb{N}$, $\Gamma_n$ is connected, the sequence is called a sequence of connected graphs. On the other hand, if there exists $M \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $\Delta(\Gamma_n) < M$, then we call $\{\Gamma_n\}_{n \in \mathbb{N}}$ a finite-degree sequence of graphs. It is obvious from our definition that $\{N_n\}_{n \in \mathbb{N}}$ is a finite-degree chain of (disconnected) graphs while $\{K_n\}_{n \in \mathbb{N}}$ is a (non-finite-degree) chain of connected graphs. Finally, a chain $\{\Gamma_n\}_{n \in \mathbb{N}}$ is called non-oscillating provided that for all $n \in \mathbb{N}$ and for all $v \in V(\Gamma_n) \setminus V(\Gamma_{n-1})$, for every $m > n$, $d_{\Gamma_m}(v) = d_{\Gamma_{m+1}}(v)$.

A chain is called chromatically conformal, when for $m \in \mathbb{N}$, there exist distinct $\zeta_1, \zeta_2, \ldots, \zeta_m$ in $\mathbb{C}$ such that for all $n \in \mathbb{N}$, $C(\Gamma_n; \lambda) = (\lambda - \zeta_1)^{f_1(n)} (\lambda - \zeta_2)^{f_2(n)} \cdots (\lambda - \zeta_m)^{f_m(n)}$, in which for $1 \leq i \leq m, f_i(n): \mathbb{N} \to \mathbb{N}$ is order preserving and $f(n) = f_1(n) + f_2(n) + \cdots + f_m(n)$. Furthermore, for $r \in \mathbb{N}$ a chain is called chromatically recursive of degree $r$, when for all $n \in \mathbb{N}$, $C(\Gamma_{n+r}; \lambda) = p_1(\lambda) C(\Gamma_{n+r-1}; \lambda) + p_2(\lambda) C(\Gamma_{n+r-2}; \lambda) + \cdots + p_r(\lambda) C(\Gamma_n; \lambda)$, in which for $1 \leq i \leq r$, $p_i(n)$ is an element of $\mathbb{C}[\lambda]$ which are not dependent on $n$, and $r$ the smallest natural number such a recursion holds. Moreover, we will assume that $p_1(\lambda) + p_2(\lambda) + \cdots + p_r(\lambda)$ is a non-constant element of $\mathbb{C}[\lambda]$. As one may see, $\{L_n\}_{n \in \mathbb{N}}$ is a finite-degree chain of connected graphs which is chromatically conformal and chromatically recursive of degree $1$:

$$C(L_n; \lambda) = \lambda(\lambda - 1)(\lambda - \left(3 + i\sqrt{3}\right))^{n}(\lambda - \left(3 - i\sqrt{3}\right))^{n},$$

$$C(L_{n+1}; \lambda) = (\lambda^2 - 3\lambda + 3)C(L_n; \lambda).$$

The first example we would discuss in this section is defined as follows: Assume $\Psi_1$ is the graph in Figure 4.1(a), and $\Psi_n$ is recursively built from $\Psi_{n-1}$ by identifying the vertices $a_{n-1}, b_{n-1}, c_{n-1}, d_{n-1}$ in $\Psi_{n-1}$ with vertices $a_{n-1}, b_{n-1}, c_{n-1}, d_{n-1}$ of $\Psi_n$, respectively. One may also see in Figure 4.1(b) how $\Psi_n$ looks like. We know that

$$C(\Psi_1; \lambda) = \frac{C(C_8; \lambda)C(C_4; \lambda)^2}{\lambda^2(\lambda - 1)^2} = \lambda(\lambda - 1)(\lambda^2 - 3\lambda + 3)^2(\lambda^6 - 7\lambda^5 + 21\lambda^4 - 35\lambda^3 + 35\lambda^2 - 21\lambda + 7).$$
In order to find the chromatic polynomial of $\Psi_n$ for $n \geq 2$, we need to find the chromatic polynomial of $\Psi'_n$ and $\Psi''_n$ which are built from $\Psi_{n-1}$ by identifying vertices $a_{n-1}, b_{n-1}, c_{n-1}, d_{n-1}$ with vertices $a_1, b_1, c_1, d_1$ of $\Psi'_n$ and $\Psi''_n$ (see Figure 4.1(a)), respectively. In Figure 4.2(a) and 4.2(b), we have drawn $\Psi'_n$ and $\Psi''_n$ for $n \geq 2$. $\Psi'_1$ and $\Psi''_1$ can be found in Figure 4.2(c).

For $n \geq 2$, we have the following relations between the chromatic polynomial of $\Psi'_n$ and $\Psi''_n$:

$$C(\Psi'_n; \lambda) = \frac{C(\Omega_1; \lambda)}{\lambda (\lambda - 1)} C(\Psi_{n-1}; \lambda) + \frac{C(\Omega_2; \lambda)}{\lambda} C(\Psi''_{n-1}; \lambda),$$

$$C(\Psi''_n; \lambda) = \frac{C(\Omega_3; \lambda)}{\lambda (\lambda - 1)} C(\Psi_{n-1}; \lambda) + \frac{C(\Omega_4; \lambda)}{\lambda} C(\Psi''_{n-1}; \lambda),$$

in which $\Omega_i$, for $1 \leq i \leq 6$, are graphs in Figure 4.3(a) and 4.3(b). The recursive relations above can be written in matrix form

$$\begin{bmatrix} C(\Psi'_n; \lambda) \\ C(\Psi''_n; \lambda) \end{bmatrix} = \begin{bmatrix} \frac{C(\Omega_1; \lambda)}{\lambda (\lambda - 1)} & \frac{C(\Omega_2; \lambda)}{\lambda} \\ \frac{C(\Omega_3; \lambda)}{\lambda (\lambda - 1)} & \frac{C(\Omega_4; \lambda)}{\lambda} \end{bmatrix} \begin{bmatrix} C(\Psi'_{n-1}; \lambda) \\ C(\Psi''_{n-1}; \lambda) \end{bmatrix}.$$
and as a result,

\[
\begin{bmatrix}
C(\Psi_n'; \lambda) \\
C(\Psi_n''; \lambda)
\end{bmatrix} = \left[ \frac{C(\Omega_1; \lambda)}{\lambda(\lambda-1)} \frac{C(\Omega_2; \lambda)}{\lambda} \right]^{n-1} \begin{bmatrix}
C(\Psi_1'; \lambda) \\
C(\Psi_1''; \lambda)
\end{bmatrix}.
\]

On the other hand, it easily can be seen that for \( n \geq 2 \),

\[
C(\Psi_n; \lambda) = \frac{C(\Psi_{n-1}'; \lambda)C(\Omega_1; \lambda)}{\lambda(\lambda - 1)} + \frac{C(\Psi_{n-1}''; \lambda)C(\Omega_2; \lambda)}{\lambda},
\]

which can be rewritten as

\[
C(\Psi_n; \lambda) = \left[ \frac{C(\Omega_1; \lambda)}{\lambda(\lambda - 1)} \frac{C(\Omega_2; \lambda)}{\lambda} \right] \begin{bmatrix}
C(\Psi_{n-1}'; \lambda) \\
C(\Psi_{n-1}''; \lambda)
\end{bmatrix}.
\]
These observations enable us to write the chromatic polynomial of \( \Psi_n \) in the matrix from

\[
C(\Psi_n; \lambda) = \left[ \begin{array}{cc}
C(\Omega_1; \lambda) & C(\Omega_2; \lambda) \\
\lambda^{-1} & \lambda
\end{array} \right] \left[ \begin{array}{cc}
\frac{C(\Omega_3; \lambda)}{\lambda^{-1}} & \frac{C(\Omega_4; \lambda)}{\lambda} \\
\frac{C(\Omega_5; \lambda)}{\lambda^{-1}} & \frac{C(\Omega_6; \lambda)}{\lambda}
\end{array} \right]^{n-2} \left[ \begin{array}{c}
C(\Psi_1'; \lambda) \\
C(\Psi_2'; \lambda)
\end{array} \right].
\] (5)

In order to compute \( C(\Omega_i; \lambda) \), for \( 1 \leq i \leq 6 \), we will apply Theorem 1.13. So, we have

\[
C(\Omega_1; \lambda) = \frac{C(C_6; \lambda) C(C_7; \lambda)}{\lambda(\lambda - 1)} + (-1)^5 C(\Omega_9; \lambda),
\]

\[
C(\Omega_2; \lambda) = \frac{C(C_6; \lambda) C(C_8; \lambda)}{\lambda(\lambda - 1)} + (-1)^5 C(\Omega_{10}; \lambda),
\]

\[
C(\Omega_3; \lambda) = \frac{C(C_4; \lambda) C(C_7; \lambda)}{\lambda(\lambda - 1)} + (-1)^3 C(\Omega_9; \lambda),
\]

\[
C(\Omega_4; \lambda) = \frac{C(C_4; \lambda) C(C_8; \lambda)}{\lambda(\lambda - 1)} + (-1)^3 C(\Omega_{10}; \lambda),
\]

\[
C(\Omega_5; \lambda) = \frac{C(C_3; \lambda) C(C_7; \lambda)}{\lambda(\lambda - 1)} + (-1)^2 C(\Omega_9; \lambda),
\]

\[
C(\Omega_6; \lambda) = \frac{C(C_3; \lambda) C(C_8; \lambda)}{\lambda(\lambda - 1)} + (-1)^2 C(\Omega_{10}; \lambda),
\]

in which \( \Omega_9 \) and \( \Omega_{10} \) are graphs in Figure 4.3(c). To compute \( C(\Omega_9; \lambda) \) and \( C(\Omega_{10}; \lambda) \) one may notice that \( C(\Omega_9; \lambda) = C(\Omega_8; \lambda) - C(\Omega_{10}; \lambda) \) and \( C(\Omega_{10}; \lambda) = C(\Omega_{11}; \lambda) + C(\Omega_{12}; \lambda) \) (See Figure 4.4). By using Theorem 1.11, we have

\[
C(\Omega_{11}; \lambda) = \frac{C(K_3; \lambda)^2 C(K_4; \lambda)}{\lambda^2(\lambda - 1)^2} = \lambda(\lambda - 1)(\lambda - 2)^3(\lambda - 3),
\]

\[
C(\Omega_{12}; \lambda) = \frac{C(K_2; \lambda)^2 C(K_3; \lambda)}{\lambda^2} = \lambda(\lambda - 1)^3(\lambda - 2),
\]

\[
C(\Omega_7; \lambda) = \frac{C(C_4; \lambda)^3}{\lambda^2(\lambda - 1)^2} = \lambda(\lambda - 1)(\lambda^2 - 3\lambda + 3)^3,
\]

\[
C(\Omega_8; \lambda) = \frac{C(C_4; \lambda)^2 C(C_3; \lambda)}{\lambda^2(\lambda - 1)^2} = \lambda(\lambda - 1)(\lambda - 2)(\lambda^2 - 3\lambda + 5)^2.
\]

Consequently,

\[
C(\Omega_{10}; \lambda) = \lambda(\lambda - 1)(\lambda - 2)(\lambda^3 - 6\lambda^2 + 14\lambda - 11),
\]

\[
C(\Omega_9; \lambda) = \lambda(\lambda - 1)(\lambda - 2)^3(\lambda^2 - 3\lambda + 5).
\]
Finally, in order to be able to compute $C(\Psi_n; \lambda)$, we need to plug the following polynomials into (4.5):

$$\frac{C(\Omega_6; \lambda)}{\lambda} = (\lambda - 1)(\lambda - 2)(\lambda^5 - 8\lambda^4 + 28\lambda^3 - 54\lambda^2 + 59\lambda - 29),$$

$$\frac{C(\Omega_5; \lambda)}{\lambda(\lambda - 1)} = (\lambda - 2)(\lambda^6 - 9\lambda^5 + 37\lambda^4 - 88\lambda^3 + 129\lambda^2 - 133\lambda + 47),$$

$$\frac{C(\Omega_4; \lambda)}{\lambda} = (\lambda - 1)(\lambda - 2)^2(\lambda^5 - 7\lambda^4 + 22\lambda^3 - 38\lambda^2 + 38\lambda - 19),$$

$$\frac{C(\Omega_3; \lambda)}{\lambda(\lambda - 1)} = \lambda^8 - 12\lambda^7 + 66\lambda^6 - 217\lambda^5 + 468\lambda^4 - 683\lambda^3 + 668\lambda^2 - 408\lambda + 121.$$
\[
\frac{C(\Omega_2; \lambda)}{\lambda} = (\lambda - 1)(\lambda - 2)^2(\lambda^7 - 9\lambda^6 + 37\lambda^5 - 89\lambda^4 + 136\lambda^3 - 134\lambda^2 + 83\lambda - 28),
\]
\[
\frac{C(\Omega_4; \lambda)}{\lambda(\lambda - 1)} = \lambda^{10} - 14\lambda^9 + 91\lambda^8 - 361\lambda^7 + 968\lambda^6 - 1837\lambda^5 + 2511\lambda^4 - 2465\lambda^3 + 1694\lambda^2 - 759\lambda + 175,
\]
\[
C(\Psi'_1) = \frac{C(C_5; \lambda) C(C_4; \lambda^2)}{\lambda^2(\lambda - 1)^2} = \lambda(\lambda - 1)(\lambda^2 - 3\lambda + 3)(\lambda^4 - 5\lambda^3 + 10\lambda^2 - 10\lambda + 5),
\]
\[
C(\Psi''_1; \lambda) = \frac{C(C_5; \lambda) C(C_4; \lambda^2)}{\lambda^2(\lambda - 1)^2} = \lambda(\lambda - 1)(\lambda - 2)(\lambda^2 - 3\lambda + 3)(\lambda^2 - 3\lambda + 2).
\]

Now, we would look at the chromatic polynomial of the chain \(\Theta_n = C_4 \times P_n\) (see Figure 4.4(a)). By applying Theorem 1.6 on \(\Theta_n\) – more precisely, deleting and contracting the edges connecting \(\Theta_{n-1}\) to last copy of \(C_4\) – one can write chromatic polynomial of this graph in terms of \(C(\Theta_{n-1}; \lambda)\) and \(C(\Theta'_{n-1}; \lambda)\); \(\Theta_n\) is obtained from \(\Theta_n\) by connecting two non-adjacent vertices in the last copy of \(C_4\) (see Figure 4.4(b)). Similarly, \(C(\Theta_n; \lambda)\) can be written in terms of chromatic polynomial of \(\Theta_{n-1}\) and \(\Theta'_{n-1}\). By doing so, for \(n \geq 2\) we will have
\[
C(\Theta_n; \lambda) = (\lambda^4 - 8\lambda^3 + 28\lambda^2 - 47\lambda + 31) C(\Theta_{n-1}; \lambda) - 2(2\lambda - 5) C(\Theta'_{n-1}; \lambda),
\]
\[
C(\Theta'_n; \lambda) = (\lambda - 2)(\lambda^3 - 7\lambda^2 + 19\lambda - 19) C(\Theta_{n-1}; \lambda) + (\lambda - 5)(\lambda - 3) C(\Theta'_{n-1}; \lambda).
\]
By letting \(\theta_{11} = (\lambda^4 - 8\lambda^3 + 28\lambda^2 - 47\lambda + 31), \theta_{12} = -2(2\lambda - 5), \theta_{21} = (\lambda - 2)(\lambda^3 - 7\lambda^2 + 19\lambda - 19), \theta_{22} = (\lambda - 5)(\lambda - 3)\), the recursive relations above can be written in matrix form as one may find on the following page.

(a) \(\Theta_n\)

(b) \(\Theta'_n\)

Figure 4.4
4 Chromatic Polynomial of Some Chains of Graphs

\[
\begin{bmatrix}
C(\Theta_n; \lambda) \\
C(\Theta'_n; \lambda)
\end{bmatrix} = 
\begin{bmatrix}
\theta_{11} & \theta_{12} \\
\theta_{21} & \theta_{22}
\end{bmatrix} 
\begin{bmatrix}
C(\Theta_{n-1}; \lambda) \\
C(\Theta'_{n-1}; \lambda)
\end{bmatrix},
\]

and as a result,

\[
\begin{bmatrix}
C(\Theta_n; \lambda) \\
C(\Theta'_n; \lambda)
\end{bmatrix} = 
\begin{bmatrix}
\theta_{11} & \theta_{12} \\
\theta_{21} & \theta_{22}
\end{bmatrix}^{n-1} 
\begin{bmatrix}
C(\Theta_1; \lambda) \\
C(\Theta'_1; \lambda)
\end{bmatrix}.
\]

Finally, by substituting the chromatic polynomial of \(\Theta_1\) and \(\Theta'_1\) (see Figure 4.5) in (4.6) we will have \(C(\Theta_n; \lambda)\) and \(C(\Theta'_n; \lambda)\), for all \(n\), in matrix form.

\[
C(\Theta_1; \lambda) = \lambda(\lambda - 1)(\lambda^5 - 11\lambda^5 + 55\lambda^4 - 159\lambda^3 + 282\lambda^2 - 290\lambda + 133),
\]

\[
C(\Theta'_1; \lambda) = \lambda(\lambda - 1)(\lambda - 2)(\lambda^5 - 10\lambda^4 + 44\lambda^3 - 107\lambda^2 + 145\lambda - 87).
\]

![Figure 4.5](image)

Our last example in this section is the chain \(\Phi_n = C_5 \times P_n\), for \(n \in \mathbb{N}\) (see Figure 4.6(a)). Similar to the previous example, chromatic polynomial of \(\Phi_n\) can be computed by deleting and contracting the edges connecting \(\Phi_{n-1}\) to last copy of \(C_4\). With some effort, one may check that chromatic polynomial of this graph can be written in terms of \(C(\Phi_{n-1}; \lambda)\), \(C(\Phi'_{n-1}; \lambda)\), and \(C(\Phi''_{n-1}; \lambda)\); \(\Phi'_n\) and \(\Phi''_n\) are graphs obtained from \(\Phi_n\) by linking two non-adjacent vertices and one vertex to two vertices non-adjacent to it, respectively, in the last copy of \(C_4\) (see Figure 4.6(b) and 4.6(c)). Similarly, \(C(\Phi'_n; \lambda)\) and \(C(\Phi''_n; \lambda)\) can be written in terms of chromatic polynomial of \(\Phi_{n-1}, \Phi'_{n-1}, \) and \(\Phi''_{n-1}\). By doing so, for \(n \geq 2\) we will have

\[
C(\Phi_n; \lambda) = (\lambda - 2)^3(\lambda^2 - 4\lambda + 9) C(\Phi_{n-1}; \lambda)
\]

\[
-(5\lambda^2 - 28\lambda - 38) C(\Phi'_{n-1}; \lambda) + (\lambda - 6) C(\Phi''_{n-1}; \lambda),
\]

\[
C(\Phi'_n; \lambda) = (\lambda^5 - 11\lambda^4 + 53\lambda^3 - 135\lambda^2 + 177\lambda - 95) C(\Phi_{n-1}; \lambda) +
\]

\[
(\lambda - 2)(\lambda - 3)(\lambda - 9) C(\Phi'_{n-1}; \lambda) + 2(2\lambda - 7) C(\Phi''_{n-1}; \lambda),
\]

\[
C(\Phi''_n; \lambda) = (\lambda^5 - 11\lambda^4 + 53\lambda^3 - 135\lambda^2 + 177\lambda - 95) C(\Phi_{n-1}; \lambda) +
\]

\[
(\lambda - 2)(\lambda - 3)(\lambda - 9) C(\Phi'_{n-1}; \lambda) + 2(2\lambda - 7) C(\Phi''_{n-1}; \lambda),
\]
Finally, by substituting the chromatic polynomial of $\Phi_1$, $\Phi'_1$, and $\Phi''_1$ (see Figure 4.7) in (4.7) we will have $C(\Phi_n; \lambda)$, $C(\Phi'_n; \lambda)$, and $C(\Phi''_n; \lambda)$ for all $n$ in matrix form.

\[ C(\Phi_1; \lambda) = \lambda(\lambda - 1)(\lambda - 2)(\lambda^7 - 12\lambda^6 + 67\lambda^5 - 225\lambda^4 + 494\lambda^3 - 719\lambda^2 + 650\lambda - 282), \]
\[ C(\Phi'_1; \lambda) = \lambda(\lambda - 1)(\lambda - 2)(\lambda^7 - 13\lambda^6 + 78\lambda^5 - 280\lambda^4 + 653\lambda^3 - 998\lambda^2 + 931\lambda - 408), \]
\[ C(\Phi''_1; \lambda) = \lambda(\lambda - 1)(\lambda - 2)(\lambda^7 - 14\lambda^6 + 89\lambda^5 - 335\lambda^4 + 812\lambda^3 - 1277\lambda^2 + 1212\lambda - 534). \]

(d) $\Phi_1$, $\Phi'_1$, $\Phi''_1$

Figure 4.7

We will finish this section by the following questions: Using the same method, what is chromatic polynomial of $C_6 \times P_n$? In general, what is the chromatic polynomial of $C_m \times P_n$ for $m, n \in \mathbb{N}$?
5 Chromatic Ordinomial of Some Infinite Graphs

We call a graph *infinite* when the number of vertices is infinite. All the infinite graphs under investigation are countable graphs meaning that both their vertex set and edge set are countable sets. An infinite graph $\Gamma$ is called *locally finite* when for every vertex $v$, $d(v)$ is finite. Moreover, if there exist a positive integer $M$ such that for every vertex $v$ in $\Gamma$, $d(v) < M$, then $\Gamma$ is a graph of *finite degree*. We assume all infinite graphs we are interested in are of a finite-degree. If $\Gamma$ and $\Gamma'$ are two graphs, $\Gamma \subseteq \Gamma'$ means that there exists an isomorphism between $\Gamma$ and a subgraph of $\Gamma'$. An infinite graph $\Gamma$ is connected when for $u, v \in V(\Gamma)$, there exist a path from $u$ to $v$ of a finite length.

In general, a sequence is not necessarily convergent to a graph, meaning that there exist a graph $\Gamma$ such that for every $n \in \mathbb{N}$, $\Gamma_n \subseteq \Gamma$ and if there exists a graph $\Gamma'$ such that $\Gamma_n \subseteq \Gamma'$ for every $n \in \mathbb{N}$, then $\Gamma \subseteq \Gamma'$.

**Theorem 5.1:** When a sequence $\{\Gamma_n\}_{n \in \mathbb{N}}$ is a chain, it is convergent to $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n$ which is an infinite graph. Moreover, when there exist $m \in \mathbb{N}$ such that for every $n \geq m$, $\Gamma_n$ is connected, then $\Gamma$ is connected.

**Proof:** It is clear from definition that $\Gamma$ is the limit of the chain $\{\Gamma_n\}_{n \in \mathbb{N}}$. We only have to prove that the number of its vertices is infinite and to do so we will use Axiom of Choice. Let $v_1$ be a vertex in $\Gamma_1$. As $\Gamma_1 \subseteq \Gamma_2$, there exists $v_2$ in $\Gamma_2 \setminus \Gamma_1$. In a similar way, for every $n \in \mathbb{N}$, because $\Gamma_n \subseteq \Gamma_{n+1}$, there exist $v_{n+1}$ in $\Gamma_{n+1} \setminus \Gamma_n$. It is obvious that the set $\{v_1, v_2, \ldots, v_n, \ldots\}$ of infinitely many distinct vertices is a subset of $V(\Gamma)$; Hence, $\Gamma$ is an infinite graph.

Now, assume that there exist $m \in \mathbb{N}$ such that for every $n \geq m$, $\Gamma_n$ is connected. Let $u$ and $v$ be vertices in $\Gamma$. We know that there exist $n_1$ and $n_2$ in $\mathbb{N}$ such that $u \in \Gamma_{n_1}$ and $v \in \Gamma_{n_2}$. Now let $M$ be the maximum of $n_1$, $n_2$, and $m$. Clearly, $\Gamma_M$ is a connected finite graph which contains $\Gamma_{n_1}$ and $\Gamma_{n_2}$, and as a result, there exist a path of finite length in $\Gamma_M$ connecting $u$ and $v$. The same path in $\Gamma$ connects the two vertices, and thus $\Gamma$ is connected.\]

From now on, whenever a sequence $\{\Gamma_n\}_{n \in \mathbb{N}}$ converges to a limit $\Gamma$, we will use the notation $\Gamma = \lim_{n \to \omega} \Gamma_n$ (interchangeable with $\lim_{n \in \mathbb{N}} \Gamma_n$, $\lim_{n \in \omega} \Gamma_n$, or $\lim_{n < \omega} \Gamma_n$). Moreover, as Theorem 5.1 guarantees that there exists a unique limit for a chain, our attention would be more focused on such families of graphs.

**Corollary 5.2:** If $\{\Gamma_n\}_{n \in \mathbb{N}}$ is a chain of connected graphs, then $\Gamma = \lim_{n \to \omega} \Gamma_n$ is connected.

**Theorem 5.3:** Suppose $\{\Gamma_n\}_{n \in \mathbb{N}}$ is a finite degree chain, then $\Gamma = \lim_{n \to \omega} \Gamma_n$ is a graph of finite degree.
Proof: For every \( v \) in \( \Gamma \), there exists \( n \) such that \( v \in V(\Gamma_n) \). Because the chain is of finite degree, there exists \( M \in \mathbb{N} \) such that for all \( n \in \mathbb{N} \), for every \( v \in V(\Gamma_n) \), \( d(v) < M \). Putting these two facts together, we can conclude that there exists \( M \in \mathbb{N} \) such that for every \( v \in V(\Gamma) \), \( d(v) < M \).

After this brief introduction, partial chromatic ordinals of infinite graphs will be defined first. Let \( \Gamma \) be an infinite graph and \( \{\Gamma_n\}_{n \in \mathbb{N}} \) be a finite-degree chain of connected graphs such that \( \Gamma = \lim_{n \to \omega} \Gamma_n \). Throughout this section, by a chain of graphs, a non-oscillating finite-degree chain of connected graphs is meant, unless stated otherwise. If the limit \( \lim_{n \to \omega} C(\Gamma_n; \lambda) \) exists, then chromatic ordninal of \( \Gamma \) is partially defined and we have, \( C_p(\Gamma; \lambda) = \lim_{n \to \omega} C(\Gamma_n; \lambda) \). Clearly, when \( \{\Gamma_n\}_{n \in \mathbb{N}} \) is a chromatically conformal, \( C_p(\Gamma; \lambda) \) is defined. As chromatic polynomial of \( \Gamma_n \) is monic, partial chromatic ordninal of \( \Gamma \) is also monic and consequently in \( \mathbb{B}[\lambda] \), due to properties of the limit operation we introduced in the end of Section 3.

Let \( \{P_n\}_{n \in \mathbb{N}} \) be the chain of paths of length \( n \) and \( P = \lim_{n \to \omega} P_n \) is the one-way infinite path. We know that \( \{P_n\}_{n \in \mathbb{N}} \) is chromatically conformal, \( C(P_n; \lambda) = \lambda(\lambda - 1)^n \), and chromatic ordninal of \( P \) is partially defined: \( C_p(P; \lambda) = \lim_{n \to \omega} C(P_n; \lambda) = \lim_{n \to \omega} \lambda(\lambda - 1)^n = \lambda(\lambda - 1)^\omega \). As another example, let \( L \) be \( \lim_{n \to \omega} L_n \). This example is also chromatically conformal and we have \( C_p(L; \lambda) = \lim_{n \to \omega} C(L_n; \lambda) = \lim_{n \to \omega} \lambda(\lambda - 1)(\lambda^2 - 3\lambda + 3)^n = \lambda(\lambda - 1)(\lambda^2 - 3\lambda + 3)^\omega \).

If for every chain of graphs \( \{\Gamma_n\}_{n \in \mathbb{N}} \) such that \( \Gamma = \lim_{n \to \omega} \Gamma_n \), partial chromatic ordninal of \( \Gamma \), \( C_p(\Gamma; \lambda) \), exist and for every two chain having the aforementioned properties this ordninal is equal, then chromatic ordninal of \( \Gamma \) is universally defined and we have \( C(\Gamma; \lambda) = \lim_{n \to \omega} C(\Gamma_n; \lambda) \). It is obvious, universal chromatic ordninal is monic. As our definition for universal chromatic ordninal is quite naive and refinement is needed, we will focus our attention on partial chromatic ordninal.

By convention, chromatic ordninal of finite graphs are partially and universally defined and it is equal to their chromatic polynomial. Clearly, chromatic ordninal of finite graphs are monic and elements of \( \mathbb{B}[\lambda] \).

Proposition 5.4: Chromatic Ordninal of \( P \) is universally defined.

Proof: The proof uses the fact that every finite connected subgraph of \( P \) is a path of a finite length; Hence, for every chain \( \{\Gamma_n\}_{n \in \mathbb{N}} \) such that \( P = \lim_{n \to \omega} \Gamma_n \), \( \Gamma_n \) is a path of finite length with \( |V(\Gamma_n)| = f(n) \), \( f \) being a strictly increasing function. It follows that \( C(\Gamma_n; \lambda) = \lambda(\lambda - 1)^{f(n)-1} \) and as a result, \( C(P; \lambda) = \lambda(\lambda - 1)^\omega \).

Now if \( \Gamma \) is an infinite graph with \( k \) components \( \Gamma^1, \Gamma^2, \ldots, \Gamma^k \) such that for \( 1 \leq l \leq k \), \( \Gamma^1, \Gamma^2, \ldots, \Gamma^l \) are infinite graphs and the rest of the components are finite graphs. If chromatic
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ordinomial of $\Gamma^1, \Gamma^2, \ldots, \Gamma^t$ are partially defined, the chromatic ordinomial of $\Gamma$ is partially defined and we have,

$$C_p(\Gamma; \lambda) = C_p(\Gamma^1; \lambda) C_p(\Gamma^2; \lambda) \cdots C_p(\Gamma^t; \lambda) C(\Gamma^{t+1}; \lambda) \cdots C(\Gamma^k; \lambda).$$

Whenever chromatic polynomial of $\Gamma^1, \Gamma^2, \ldots, \Gamma^t$ is universally defined, then $C(\Gamma; \lambda)$ is also universally defined and we have

$$C(\Gamma; \lambda) = C(\Gamma^1; \lambda) C(\Gamma^2; \lambda) \cdots C(\Gamma^k; \lambda).$$

From the remark we made in Section 3, as all the factors on the right-hand side of the expressions we have for $C_p(\Gamma; \lambda)$ and $C(\Gamma; \lambda)$ are elements of $B[\lambda]$, these ordinomials are also elements of $B[\lambda]$ and as a result not equal to zero.

Because of the properties of the limiting operation we introduce in Section 3, an infinite version of Theorem 1.11 is not valid, either for partial or universal chromatic ordinomial.

Now let's assume $\{\Gamma_n\}_{n \in \mathbb{N}}$ is a chromatically recursive chain of graphs of degree $r$. By definition, for all $n \in \mathbb{N}$, we have

$$C(\Gamma_{n+r}; \lambda) = p_1(\lambda) C(\Gamma_{n+r-1}; \lambda) + p_2(\lambda) C(\Gamma_{n+r-2}; \lambda) + \cdots + p_r(\lambda) C(\Gamma_n; \lambda),$$

in which for $1 \leq i \leq r$, $p_i(n)$ is an element of $\mathbb{C}[\lambda]$ which are not dependent on $n$, and $r$ the smallest natural number such a recursion holds. Provided that $C_p(\Gamma; \lambda)$ exists, we have

$$\lim_{n \to \omega} C(\Gamma_{n+r}; \lambda) = \lim_{n \to \omega} \left( p_1(\lambda) C(\Gamma_{n+r-1}; \lambda) + p_2(\lambda) C(\Gamma_{n+r-2}; \lambda) + \cdots + p_r(\lambda) C(\Gamma_n; \lambda) \right) =$$

$$\lim_{n \to \omega} \left( p_1(\lambda) C(\Gamma_{n+r-1}; \lambda) \right) + \lim_{n \to \omega} \left( p_2(\lambda) C(\Gamma_{n+r-2}; \lambda) \right) + \cdots + \lim_{n \to \omega} \left( p_r(\lambda) C(\Gamma_n; \lambda) \right) =$$

$$p_1(\lambda) \lim_{n \to \omega} C(\Gamma_{n+r-1}; \lambda) + p_2(\lambda) \lim_{n \to \omega} C(\Gamma_{n+r-2}; \lambda) + \cdots + p_r(\lambda) \lim_{n \to \omega} C(\Gamma_n; \lambda).$$

Thus,

$$C_p(\Gamma; \lambda) = p_1(\lambda) C_p(\Gamma; \lambda) + p_2(\lambda) C_p(\Gamma; \lambda) + \cdots + p_r(\lambda) C_p(\Gamma; \lambda) =$$

$$\left( p_1(\lambda) + p_2(\lambda) + \cdots + p_r(\lambda) \right) C_p(\Gamma; \lambda) = p(\lambda) C_p(\Gamma; \lambda).$$

We know $p(\lambda)$ is a non-constant element of $\mathbb{C}[\lambda]$ and as a result a finite degree ordinomial.

Let's assume that $p(\lambda) = \tau(\lambda - \zeta_1)^{n_1}(\lambda - \zeta_2)^{n_2} \cdots (\lambda - \zeta_m)^{n_m}$ in which $m, n_1, n_2, \ldots, n_m \in \mathbb{N}$ and $\zeta_i$'s are distinct complex numbers. As $C_p(\Gamma; \lambda)$ is monic, $\tau = 1$. Furthermore, from uniqueness of elements in $B[\lambda]$ one can proof that $(\lambda - \zeta_1 \omega)(\lambda - \zeta_2 \omega) \cdots (\lambda - \zeta_m \omega) | C_p(\Gamma; \lambda)$. So, if $C_p(\Gamma; \lambda) = x_1x_2$ is the unique decomposition of $C_p(\Gamma; \lambda)$ such that $x_1$ is purely infinite, $deg(x_2) \in \mathbb{N}$, and $x_1, x_2 = 1$, then $(\lambda - \zeta_1 \omega)(\lambda - \zeta_2 \omega) \cdots (\lambda - \zeta_m \omega) | x_1.$
Conjecture 5.5: \[ x_1 = (\lambda - \zeta_1)^\omega (\lambda - \zeta_2)^\omega \cdots (\lambda - \zeta_m)^\omega . \]

Consequently, if chromatic ordinomial of \( \Gamma \) is universally defined, we have \( C(\Gamma; \lambda) = p(\lambda) C(\Gamma; \lambda) \), and whatever we had for \( C_p(\Gamma; \lambda) \) is also true for \( C(\Gamma; \lambda) \), provided that it exists.

This thesis will be finished by asking the following question: How can an infinite version of the Deletion-contraction Theorem be formulated?
6 Acknowledgement

In writing the first section of this thesis which is an introduction to chromatic polynomial, the author has a great owe to [1], [2] and [3]. For the second section which is a brief review on ordinal numbers, my main reference was [4], although I consulted [5] occasionally. In section 4, although the author came across to $\Psi_n$ in [6], his approach to the problem of finding the chromatic polynomial of this chain was independent and different from those of [6]. Finally, [7] was consulted occasionally for the section on infinite graphs.

References


