Damping of elastic-viscoelastic beams

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DAMPING OF ELASTIC-VISCOELASTIC BEAMS

by

Ray A. West

A Thesis Submitted
in
Partial Fulfillment
of the
Requirements for the degree of
MASTER OF SCIENCE
in
Mechanical Engineering

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PERMISSION GRANTED

DAMPING OF ELASTIC - VISCOELASTIC BEAMS

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ABSTRACT

The subject of this investigation is the dynamic behavior of a composite beam consisting of a main elastic core and two viscoelastic layers. The viscoelastic layers provide increased damping of vibrations. When the beam vibrates, damping is caused by energy dissipation in the viscoelastic layers due to their shear deformations. Thus, the goal of this investigation is the development of a method to assess the dynamic characteristics, including the effectiveness of damping, in the layered elastic-viscoelastic beam. Analysis of the resulting sixth-order partial differential equation along with the appropriate boundary conditions is the key to the dynamic characterization of the beam. The developed method will allow the design of beams with predictable dynamic characteristics. In addition, the methodology and analysis will provide insights to the effectiveness of surface damping treatments in general.
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LIST OF SYMBOLS

a--dimensionless length parameter  

$a_1,a_2,a_3$--coefficients used in the definition of the cubic equation

A, B, C--coefficients used with Laplace transform technique

b--frequency parameter

c--parameter

$d_1...d_6$--coefficient

$G$--Shear modulus

$G^*$--Complex Shear modulus where $G^* = G(1 + i\eta)$

E--Young's modulus

$E^*$--Complex Young's modulus where $E^* = E(1 + i\eta)$

$h_1,h_2,h_3$--thickness parameters

$i^2 = -1$

$l^2,m^2,n^2$, parametric representations of the roots of the governing equation

M--bending moment

p--coefficient

q--coefficient

r--coefficient

s--coefficient

t--beam width

S--Shear force

$\alpha$--coefficient

$\delta$--infinitesimal displacement, coefficient

$\gamma$--Shear angle, coefficient

$\tau$--shear stress

$\omega$--frequency

$\psi$--coefficient

$\mu$--coefficient

$\beta$--coefficient
ζ--coefficient
η--modal loss factor, coefficient
δ--incremental operator
ρ--density, mass per unit length
θ--angular displacement
Ω--frequency parameter equal to ω/ω₀
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CHAPTER I -- INTRODUCTION

This investigation considers the damping of elastic-viscoelastic beams and was undertaken with the view of developing a method or methods to assess the dynamic characteristics, including the effectiveness of damping, in layered elastic-viscoelastic beams. Two case studies were implemented to quantitatively address this aim. One case study involved the effects of changes in the damping factor on the resultant frequency decay response. In the other case study the influence of the viscoelastic layer thickness on the frequency response was considered.

The beam configuration studied in this investigation is a five layer cantilever beam comprised of an elastic core sandwiched between viscoelastic layers of equal thickness and constraining layers on each viscoelastic layer. Stretching and contracting of the constraining layers is taken into account resulting into a sixth-order differential equation governing the beam’s motion which has the following form:

\[ y^{\text{iv}} - G(c + 1/h_2 h_3 E_3)y^{\text{iv}} + (\rho/\text{EI})(\partial^2 y/ \partial t^2)'' - Gp/(h_2 h_3 E_3 \text{EI})(\partial^2 y/ \partial t^2) = 0 \]  

(1)

where,

\[ c = \frac{bh_1 a}{\text{EI}}. \]
In general, the Laplace transform technique was effectively applied to obtain solutions to the governing equation over the spatial variable. The boundary conditions not prescribed at the fixed end of the beam were obtained through variational techniques and the characteristic frequency equation was obtained after application of both prescribed and natural boundary conditions. The solution of the system has the following form:

\[ y(x,t) = \sum_{j=1}^{\infty} \phi_j(x)e^{i\omega_j t} \]  

Due to the complex nature of the derived frequency equation and other factors contributing to the system's eigenfunctions, the damping assessment aspects of this thesis were limited to time response considerations. Consequently, a constant factor such that \( \phi_j(L) = 1 \) for all modes was assumed and damping effectiveness was then based upon the exponential decay term, \( e^{i\omega_j t} \), evaluated at the tip of the beam.

Case Study One shows that oscillations are damped out more rapidly with increases in the damping factor. Case Study Two illustrates the fact that the time response is independent of the viscoelastic layer thickness. A detailed documentary of these findings is contained in Chapter VI of this thesis.
BACKGROUND

A variety of vibration problems have employed surface damping treatments for their solutions, especially problems dealing with resonance and noise in aircraft structures. The most popular type of surface damping treatment is the sandwich method. This method consists of having either a viscoelastic core with two outside elastic layers, or an elastic core with a viscoelastic layer on one or both sides of the core. According to Mead and Markus[1], the theory of flexural vibration of damped sandwich beams and plates has been considered by a number of authors. Moreover, the case of alternating layers of elastic and viscoelastic materials has also received much attention in the literature.

For the case of a viscoelastic core and external elastic layers, the method is referred to as the shear type. The case consisting of viscoelastic treatment only is referred to as the unconstrained layer method. According to Nashif[2], for a given weight, the shear type damping treatment is more efficient than the unconstrained layer damping treatment.

Kerwin[3] performed the fundamental investigation into the problem of sandwich beams under no loads. In Kerwin's analysis, the normal forces acting on the viscoelastic layer were neglected because they were small compared to the normal forces acting on the elastic layer. There was also relaxation of constraints in assuming that damping was due only to the shearing of the viscoelastic layer, since the modulus of the elastic layer is several times higher than the modulus of the viscoelastic layer.
The theory of shock and vibration provides a means of deducing relative damping factors. Moreover, the effectiveness of damping which is required by a given vibration- or noise-reduction problem is governed generally by two considerations. The first consideration is that of the amount of additional vibration reduction that is achievable through increasing the damping capacity of the treatment at hand. In other words, when is a point of diminishing returns reached? How much damping is achievable? The second consideration is the quantity of a damping material necessary to optimally accomplish the desired damping effectiveness outlined above, in the first consideration. Clearly both material and geometric properties will influence the damping effectiveness and general dynamic characteristics of a damped sandwich beam.

In general, a viscoelastically damped system may be represented by the analog model illustrated in Figure 1.1. Moreover, the nature of a system with viscoelastic damping is better understood by close examination of the loss factor, \( \eta \), the storage modulus, \( G_1 \), and the loss modulus, \( G_2 \), of the viscoelastic material. \( G_1 \) is the real part of the complex shear modulus and \( G_2 \) is the imaginary part of the complex shear modulus. According to Harris [4], one distinguishing characteristic of the dynamic behavior of viscoelastic materials is its strong dependence on frequency and temperature. Such dependence on temperature and frequency is further illustrated in Figure 1.2. At high frequencies (low temperatures), the storage modulus, \( G_1 \), is large and the loss modulus, \( G_2 \), is small. In this region, the material behaves, for the most part, elastically. At low frequencies (high temperatures) the behavior of the material is more rubbery. Both the storage modulus and the loss modulus are
Figure 1.1  Analog Model of Viscoelastic behavior of Rubber-like Materials
low in this region. In the transition region, the loss modulus reaches a maximum. The loss factor, $\eta$, also peaks in this region but, at a lower frequency than the loss modulus. One should note that most engineering problems fall into either the transition region or the glassy region.

K. Sato and G. Shikanai [5] studied a system consisting of a viscoelastic core and thin elastic outer layers, subjected to axial forces. Their analysis, based on stress and strain equations, yielded a sixth-order ordinary differential equation of motion based on the longitudinal strain, instead of the deflection associated with free vibration. They report that the damping efficiency, viewed as a composite loss factor, increases under compression, and decreases with increasing tension. In the absence of an axial force, there is a specific frequency at which the value of the loss factor becomes the maximum for the system. The effect of axial forces on the loss factor, appear mainly in the region of frequencies lower than the above mentioned specific frequency, with increasing effects as the frequency decreases. Inversely, this factor decreases gradually as the frequency increases beyond the above mentioned specific frequency.

Based on the assumptions used by Kerwin [3] in his analysis, DiTaranto and Blasingame [6] have established a relationship between the slope and the axial strain of the elastic layer. This relationship was the basis for the analysis used in [6] to determine the composite loss factor and natural frequencies of the system. The latter was shown to be independent of the end conditions and mode shapes. Further, it was reported by Asnani and Nakra [7] that the damping effectiveness, as described by the equation developed in [5], is dependent on the number of elastic and viscoelastic layers along with the
Low frequency or High Temperature

Rubbery

Transition

Glassy

Log Scale

Increasing Frequency at Constant Temperature
or Decreasing Temperature at Constant Frequency

Figure 1.2 Dependence of the Dynamic Behavior of a Viscoelastic Material on Temperature and Frequency
thickness ratios of the viscoelastic layers to the elastic layers. Moreover, the damping effectiveness of multilayered systems, in an unsymmetrical arrangement of layers, has been studied by Nakra [8].

Ross, Kerwin, and Ungar [9] provided an analysis for a three layer system which is usually used to handle both the extensional and shear damping treatments. This analysis was based on the assumptions that there were rigid connections between layers, and that the system was simply supported. Their equations were developed and solved using sinusoidal expansions for the modes of vibration. It was reported that for other boundary conditions, approximations must be used depending on the mode shape of the structure in question.

Nashif warns of possible misleading results when dealing with the published values for the complex moduli of viscoelastic materials. He warns that in many cases these moduli have been obtained by using the analysis of Ross, Kerwin, and Ungar [9], to estimate the value of the complex shear modulus, $G^*$, instead of using it to estimate the system loss factor, $\eta$, for which it was intended.

**PROCEDURE**

This thesis investigation began with a review of work previously done in this field. This review included technical reports, journal publications and textbooks. In addition various engineering and mathematical analysis tools were investigated for utilization in this study.
Next, the beam model configuration was established. As stated previously, a five-layer cantilevered sandwich beam configuration was considered. The equation of motion governing the deflection of this beam was determined based on the deflected beam geometry, bending stresses, shear forces, and other relevant factors.

Techniques for solving the derived equation of motion were explored in order to test the efficacy of the Laplace transform as a method of solving sixth-order cantilever beam problems. Three preliminary cantilever beam problems involving differentials of lesser order than sixth were considered. The first case involved the longitudinal vibrations of a cantilever beam. For these vibratory considerations, the equation of motion is second-order. The other two cases involved the transverse vibrations of a cantilever beam which require that the equation of motion be fourth-order. Case two considered free beam vibrations and case three involved viscoelastically damped beam vibrations.

Next, the governing equation was analyzed and the boundary conditions not specified by the cantilever support were determined. Separation of variables was implemented to replace the partial differential equation involving two independent variables by two ordinary differential equation in a single variable. Such equations are more easily solved.

The specific steps involved in obtaining solutions to the governing equation were developed. The Laplace transform technique was applied to the spatial equation. Various mathematical theorems and definitions were invoked to
determine the resultant Laplace transform of sixth-order spatial equation. Partial fraction decomposition was used to rearrange the terms of the resultant transform equation. Coefficients were equated and inverse Laplace transforms taken to obtain a general solution over the spatial variable. The appropriate boundary conditions were then applied and a characteristic frequency equation obtained. The terms of this frequency equation are very involved and complicated. The special considerations resulting from this complexity are explored.

An expression for the system response was determined. Two case studies were performed to quantitatively assess the influence of the damping factor and the viscoelastic layer thickness on the time response. Both numerical and graphical data was obtained through advanced computational tools.

Finally, conclusions and recommendations were made using the results of the engineering and mathematical analysis described above.

**FORMAT**

This chapter contains background information relevant to the subject of this investigation. The overall procedures are outlined and the thesis format is presented. The format of the remaining information is discussed below. Chapter II describes the beam configuration for this investigation. Chapter III explains how the governing equation of motion was derived, taking into account stretching and contracting of the constraining layers. Chapter IV
shows the development of solution techniques leading up to a solution of the sixth-order problem. **Chapter V** shows the analysis of the equation of motion and the derivation of the natural boundary conditions using variational techniques. **Chapter VI** discusses the solution results via the Laplace transform technique and reports in detail how these results are applied in the development of a method to assess the dynamic characterization of the specified beam. **Chapter VII** reports the major findings, conclusions, and recommendations based on the investigation and analysis done for this thesis. **References** are listed immediately after the Conclusions and Recommendations chapter. In addition, three appendices are attached to provide more detailed information on solution techniques and computer programs used in connection with this work.
CHAPTER II -- THE BEAM CONFIGURATION

The beam configuration for this investigation is a five layer composite cantilever beam which is illustrated in Figure 2.1. The central layer shown in Figure 2.1 comprises the beam’s elastic core and the viscoelastic (gray) layers are found on both sides of the elastic layer. The viscoelastic layers have complex shear moduli given by \( G^* = G(1 + i\eta) \). Thus, damping is introduced into the system via the viscoelastic layers. The outside constraining layers are linearly elastic and assumed to be thin in comparison to the internal layers. In addition, the shear deformation for the viscoelastic layers is induced by the outside constraining layers. All adjacent layers are assumed to be perfectly bonded such that there is no slipping. Finally, it is assumed that the viscoelastic layers do not contribute to the flexural stiffness of the beam, that is, they are only responsible for the system’s damping.

The author uses the term damping to indicate energy-dissipative properties of materials and systems. There are, in general, three major categories of materials whose damping properties have been extensively studied by others. These categories are:

1. viscoelastic materials
2. structural metals and nonmetals
3. surface coatings.
For the viscoelastic materials studied, a linear behavior is assumed. In addition, these materials are amenable to the law of superposition and rheological considerations such as model analog analysis, as seen in Figure 1.1 in the preceding chapter. Many polymeric materials are grouped under this heading. As previously stated in this work, it is the damping occurring in the viscoelastic layers of the sandwich beam system that is of particular interest.
In the case of structural materials, more often than not, significant nonlinearities characterize their behavior. Such nonlinear behavior is especially true when high levels of stress are involved. Consequently, structural metals and nonmetals do not usually exemplify the linear behavior assumed for viscoelastic materials. Furthermore, surface coatings are often used to reduce energy flow from structural materials. These coatings take advantage of material bond interface damping through their bond with a structural metal or nonmetal.

A hysteresis loop is formed when a material is subjected to cyclic stress or strain. A measure of the damping energy involved with such a stress or strain may be assessed from this loop. In other words, the area enclosed by the loop is proportional to the energy lost per cycle. Figure 2.2 illustrates such a hysteresis loop. Furthermore, there are two damping properties which are better understood through their relationship to the energy lost per cycle. The first property is known as the specific damping capacity and is defined as the ratio of energy lost per cycle to the peak potential energy. The second property is the loss factor which is defined as the ratio of the energy loss per radian to the peak potential energy. The loss factor is a very important damping property and will be a central aspect of this investigation.

As previously stated, the shear deformation of the viscoelastic layers is induced by the constraining elastic layers. If the neutral axes for all the layers is assumed to exhibit the same deflection and one assumes no stretching or contracting in the constraining layers, it can be easily illustrated that the governing equation takes the form of a fourth-order partial differential
FIGURE 2.2 -- Hysteresis Loop Illustrating Energy Lost Per Cycle
equation in the spatial variable. However, if one relaxes the latter assumption and allows for conceivable stretching and contracting of the constraining layers, the equation governing this more realistic system is a sixth-order partial differential equation. A detailed accounting of these considerations is presented in the following chapter.
In order to derive the governing differential equation and the associated boundary conditions, it is first necessary to understand the geometry of displacements. The deflected beam geometry shown in Figure 3.1 is based on the geometry of displacements shown by Hetnarski, Messalti, Hirschbein, and Chamis [10]. Initially, we assume that the constraining layers do not stretch and that the associated deflections are small. Consequently, Figure 3.1 illustrates that the undirected displacements in the core elastic layer and the constraining layer are respectively expressed as:

\[ m = y'h_1/2 \] and \[ n = y'h_3/2 \]

The shear angle in the viscoelastic layer, \( \gamma \), is thus

\[ \gamma = y' + (m + n)/h_2 = a y' \]

where, \( a = 1 + (h_1 + h_3)/2h_2 \)

the angle, \( \gamma \) is assumed positive when measured counterclockwise from line AB. The shear force per unit length of the beam, \( S \), acting in opposite directions on the bounding elastic layers is described by the equation:

\[ S = \gamma Gb \]
FIGURE 3.1-- Geometry of Displacements
where \( b \) is the thickness of the beam perpendicular to the plane of deformation and \( G \) is the shear modulus. The total bending moment, \( M \), in an arbitrary cross-section can be expressed as

\[
M = M_0 + M_1
\]

where \( M_0 \) is a function of the beam's flexural rigidity, or in other words

\[
M_0 = -EI \frac{\partial^2 y}{\partial x^2}.
\]

\( M_1 \) is due to the shear force, \( S \), exerted by the viscoelastic layers. In addition, the flexural rigidity of the composite beam is expressed by

\[
EI = E_1 I_1 + 2E_3 I_3
\]

if it is assumed that the damping layers do not effect the bending stiffness of the beam. The equation of motion for the beam is established by considering an infinitesimal element of the central layer as shown in Figure 3.2. Moment equilibrium requires that

\[
S h_1 dx - V dx + dM_0 = 0
\]

which results in the relation for the shear force

\[
V = dM_0/dx + Sh_1.
\]

Furthermore, the motion of the element in the vertical direction is governed by
FIGURE 3.2-- Free Body Diagram of Beam Element
\[(\rho dx)\frac{\partial^2 y}{\partial t^2} = dV\]

or equivalently,

\[\rho \frac{\partial^2 y}{\partial t^2} = \frac{dV}{dx}\]

Combining the two relationships listed above results in

\[\rho \frac{\partial^2 y}{\partial t^2} = E\frac{\partial^2 M_0}{\partial x} + h, \frac{\partial^2 S}{\partial x}\]

which after substitution and the rearranging of terms becomes

\[E\frac{\partial^4 y}{\partial x^4} - \frac{\partial bh_1 G}{\partial x^2} + \rho \frac{\partial^2 y}{\partial t^2} = 0\]  

Equation (3) governs the motion of the beam under the assumption that the outer constraining layers do not stretch. A more accurate description of the free vibrations of the composite beam is attainable when the extension and contraction of the constraining layers are taken into consideration. To mathematically model this more general case, we can rewrite equation (3) in the form

\[E\frac{\partial^4 y}{\partial x^4} - bh_1 G\gamma' + \rho \frac{\partial^2 y}{\partial t^2} = 0\]  

in which the shear angle, \(\gamma\), in the absence of longitudinal deformation of the constraining layers equates to the quantity ay'.

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There will be a longitudinal deformation in the form of extension or contraction due to the shear force exerted by the viscoelastic layers on the constraining layers. This additional flexibility affects the magnitude of the shear angle in the viscoelastic layers. The absolute value of the resulting shear angle will always be less than the absolute value of \( \gamma \). From Figure 3.1 we readily observe that the resulting shear angle is quantified by the following equation

\[
\gamma + \delta = ay' + u(x,t)/h_2
\]  

(5)

where \( u(x,t) \) is the longitudinal displacement of the constraining layer on the positive side of the \( y \)-axis. In the configuration shown in Figure 3.1, observe that \( u(x,t) \) is negative. This is consistent with the assumed sign convention for the angle, \( \gamma \). Therefore, clockwise rotation through an angle, \( \delta \), indicates a negative displacement. Substituting the expression \( \gamma + \delta \) into equation (5) for \( \gamma \) results in

\[
E I \frac{\partial^4 y}{\partial x^4} - bh_1 G(\gamma + \delta)' + \rho \frac{\partial^2 y}{\partial t^2} = 0
\]

Substituting the value of the expression \( \gamma + \delta \) from equation (5) leads to the following

\[
E I \frac{\partial^4 y}{\partial x^4} - bh_1 G(a \frac{\partial^2 y}{\partial x^2} + 1/h_2 \frac{\partial u}{\partial x}) + \rho \frac{\partial^2 y}{\partial t^2} = 0
\]  

(6)

The variables \( y \) and \( u \) are the two dependent variables in equation (6). An additional relationship between these deformations can be defined. Such a relationship between the shear stress, \( \tau \), in the viscoelastic layer and the strain energy in the constraining layer at a given position \( x \) is deduced as
\[ E_3 h_3 \partial u / \partial x = \int_x^L \tau \, dx \quad (7) \]

In this relationship, the sign of \( \tau \) is the same as the sign of \( \partial u / \partial x \).

Differentiating equation (7) results in

\[ E_3 h_3 \partial^2 u / \partial x^2 = -\tau dx \quad (8) \]

In addition, we have the relationship

\[ \tau = -G(\gamma + \delta) \quad (9) \]

Combining equation (7) and equation (8) results in

\[ E_3 h_3 \partial^2 u / \partial x^2 - G(\gamma + \delta) = 0 \quad (10) \]

Differentiating equation (6) with respect to \( x \) and substituting \( \partial^2 u / \partial x^2 \) from equation (10) leads to

\[ E ly - bh_1 G[a \partial^3 y / \partial x^3 + G(\gamma + \delta)/h_2 h_3 E_3] + \rho(\partial^2 y / \partial t^2 y)' = 0 \quad (11) \]
Equation (11) above further expands to:

\[ \text{Ely}^\nu - bh_1 G[\frac{3}{h_2} y^\nu x^3 + G(ay' + u(x,t)/h_2)/h_2 h_3 E_3] + p(\frac{3}{h_2} y/ \partial t^2)'' = 0 \]  

(12)

Differentiating equation (12) with respect to \( x \) and substituting

\[ \frac{1}{h_2} \frac{\partial u}{\partial x} = \frac{E I}{b h_1 G} y^\nu - a y'' + \frac{p}{b h_1 G} (\frac{3}{h_2} y/ \partial t^2) \]

into equation (12) gives

\[ y^\nu - G(c + 1/h_2 h_3 E_3) y^iv + \frac{(\rho/E I)(\frac{3}{h_2} y/ \partial t^2)'' - Gp/(h_2 h_3 E_3 E I)(\frac{3}{h_2} y/ \partial t^2)}{E I} = 0 \]  

(13)

where,

\[ c = \frac{b h_1 a}{E I} \]

The natural motion of the composite beam is thus governed by equation (13) which is identical to equation (1) presented in Chapter I. Moreover, this formulation considers the stretching and contracting of the constraining layers, which, as previously stated, are related to the magnitude of the shear angle in the viscoelastic layers.
CHAPTER IV -- ANALYSIS OF CANTILEVER BEAMS

This chapter deals with the development of solution strategies applicable to the sixth-order sandwich beam problem investigated in this thesis. Preliminary analysis involving problems of lower order and complexity was performed to establish a precedent to follow for the sixth-order problem. Furthermore, this exercise will help develop a more intuitive understanding of solutions to cantilever beam problems. Three beam problems were reviewed. The first problem considered the longitudinal vibrations of a cantilever beam, the second, free vibrations in the lateral direction, and the third, viscoelastically damped vibrations in the lateral direction.

This work addresses solutions to the cantilever beam problem containing both real and imaginary components. Mead and Markus[1] reported that for sandwich beam end support conditions other than simply supported, the frequency response is not purely sinusoidal. Moreover, they addressed the possibility of an entirely real function solution to this type of beam problem and concluded that an entirely real solution for \( y(x,t) \) was not a general possibility.
In this simple case we are concerned with the axial stretching along the beam as shown in Figure 4.1. The corresponding differential equation is second order in the variable $u$ which represents the displacement.

The reader will note that the beam properties contributing to the extension of the beam are the modulus of elasticity, $E$, the cross-sectional area, $A$, the mass per unit volume or density, $\rho$, and the beam length, $L$. Thus, the standard equation for the displacement of the beam along its length is:

$$\frac{E A u_{x x}}{\rho} = u_{t t}$$  \hspace{1cm} (14)
Considering a complex modulus of elasticity for this beam, results in

\[
\frac{E^* A}{\rho} u_{xx} = u_{tt}
\]

in which the complex modulus is expressed as

\[
E^* = E(1 + i\eta)
\]

where \(\eta\) is the modal loss factor.

Let

\[
\frac{E^* A}{\rho} = c^2
\]

then the governing equation is given as

\[
c^2 u_{xx} = u_{tt}
\]  \(15\)

The displacement of any point on the beam, \(u(x,t)\), can be represented by

\[
y(x)e^{i\omega t},
\]

where \(\omega\) is complex and denoted as

\[
\omega = a + bi.
\]

In addition, here \(y(x)\) is a mode shape to be determined.
Making the appropriate substitutions and differentiating we obtain

\[ c^2y'' - \omega^2 y = 0. \]

The boundary conditions are \( y(0) = 0 \) and \( y'(L) = 0 \).

Applying the Laplace Transform in the spatial variable results in

\[ c^2s^2 Y(s) - sy(0) - y'(0) - \omega^2 Y(s) = 0. \]

Thus after combining terms and simplification, the transform of the solution becomes

\[ Y(s) = \frac{y'(0)}{c^2s^2 - \omega^2} \]

This leads to the modal shape function

\[ y(x) = \frac{y'(0)}{c} \sinh(\omega x/c) \]

Applying the boundary condition at \( x = L \) gives

\[ \cosh(\omega L/c) = 0. \quad (16) \]
Since both \( \omega \) and \( c \) are complex we write equation (16) in exponential form:

\[
\frac{e^{\omega L/c} + e^{-\omega L/c}}{2} = 0.
\]

Simplifying, we obtain

\[
e^{2\omega L/c} + e^{0} = 0.
\]

Taking the natural logarithm and rearranging terms of the above expression results in

\[
e^{2\omega L/c} = -1.
\]

This concept is shown graphically in Figure 4.2 which follows:

![FIGURE 4.2--Polar Displacements in the Complex Plane](image-url)
Since the complex function is multi-valued,

\[ 2\omega L/c = \pi i \pm 2n\pi i \]
\[ = (\pi \pm 2n\pi)i \]
\[ = N\pi i, \text{ for } N \text{ odd} \]

This leads to

\[ \omega^2 L^2 = \frac{-N^2\pi^2 c^2}{4} \]

Now setting

\[ \omega = a + bi, \]

and substituting

\[ c^2 = \frac{EA(1 + i\eta)}{\rho} \]

we obtain the relationship

\[ (a + bi)^2 = \frac{-N^2\pi^2 EA(1 + i\eta)}{4L^2\rho} \]

In polar form the above can be expressed as

\[ (a + bi)^2 = \frac{-N^2\pi^2 EA\sqrt{1 + n^2} e^{i\theta}}{4L^2\rho} \]
where
\[ \theta = \tan^{-1} \eta. \]

Here we can see the necessity of expressing the solution as a complex function.

Taking the square root, we have

\[
a + bi = \pm \frac{iN\pi(EA/\rho)^{1/2}(1 + \eta^2)^{1/4} e^{i\theta/2}}{2L}
\]

\[ = \pm i\omega_0 (1 + \eta^2)^{1/4} e^{i\theta/2} \]

in which the undamped natural frequency appears as

\[ \omega_0 = \frac{N\pi(EA/\rho)^{1/2}}{2L}. \]

Utilizing the identity,

\[ e^{i\theta/2} = \cos(\theta/2) + i \sin(\theta/2) \]

results in,

\[ a + bi = \pm i\omega_0 (1 + \eta^2)^{1/4} [\cos(\theta/2) + i\sin(\theta/2)]. \]

Equating the real and imaginary parts of the above equation leads to
\[ a = \pm \omega_0 (1 + \eta^2)^{1/4} \sin (\theta/2) \]

\[ b = \pm \omega_0 (1 + \eta^2)^{1/4} \cos (\theta/2) \]

The oscillations will decay only if the real component of the complex frequency is negative, so we choose \( a < 0 \).

That is,

\[ a = -\omega_0 (1 + \eta^2)^{1/4} \sin (\theta/2) \]

Recall that \( \eta = \tan \theta \), so the real part of the frequency is given by

\[
a = -\omega_0 \left( \frac{1 + \sqrt{\eta^2 + 1}}{\sqrt{2}} \right)^{1/2}
\]

Similarly, the imaginary part is,

\[
b = \omega_0 \left( \frac{1 + \sqrt{\eta^2 + 1}}{\sqrt{2}} \right)^{1/2}
\]

The real and imaginary parts of the complex frequency are graphed against the modal loss factor \( \eta \) in Figure 4.3.
Figure 4.3--Frequency vs. Modal Loss Factor
FREE LATERAL VIBRATIONS OF A CANTILEVER BEAM

Now we consider the free vibrations of a cantilever beam. Such a beam is governed by the following fourth-order differential equation:

\[ E^* l \frac{\partial^4 w}{\partial x^4} = \rho \frac{\partial^2 w}{\partial t^2} \]  \hspace{1cm} (17)

with boundary conditions

\[ w(0) = 0, \quad w''(L) = 0 \]
\[ w''(0) = 0, \quad w'''(L) = 0. \]

Again, the modulus of elasticity is assumed to be complex and given by the equation

\[ E^* = E(1 + i\eta). \]

Through separation of variables we obtain a general solution expression of the modal shape function as:

\[ w(x) = Acoshkx + Bsinh kx + Ccoskx + Dsinkx \]
Application of the boundary conditions at the fixed end of the beam gives:

\[ A = -C \]
\[ B = -D. \]

If we eliminate \( A \) and \( B \) from the expressions for the boundary conditions at the free end of the beam we obtain the following system of equations.

\[
\begin{align*}
C(coshkL + coskL) + D(sinhkL + sinkL) &= 0. \\
C(sinhkL - sinkL) + D(coshkL + coskL) &= 0.
\end{align*}
\]

In order for a nontrivial solution to the above system to exist, the determinant of the coefficient matrix must equal zero.

Thus,

\[
\begin{vmatrix}
coshkL + coskL & sinhkL + sinkL \\
sinhkL - sinkL & coshkL + coskL
\end{vmatrix}
= 0.
\]

The resulting characteristic equation is written below:

\[ coskLcoshkL = -1. \] (18)
where

\[ k = (\rho A \omega^2/E*l)^{1/4} \]

Equation (18) requires that \( kL = N\pi \) for odd values of \( N \).

We can now substitute

\[ \omega = a + bi \]

into

\[ \omega^2 = \frac{k^4E^*l}{\rho A} \]

and thus obtain the expression

\[ (a + bi)^2 = \frac{N^4n^4E^*l(1 + i\eta)}{L^4 \rho A} \]
In polar form we have

\[(a + bi)^2 = \frac{N^4 \pi^4 E \sqrt{1 + \eta^2}}{L^4 \rho A} e^{i\theta},\]

where,

\[\theta = \tan^{-1} \eta.\]

Taking the square root of both sides of the above expression gives

\[a + bi = \pm \frac{i N^2 \pi^2 (E \sqrt{1/\rho A})^{1/2}(1 + \eta^2)^{1/4} e^{i\theta/2}}{L^2}\]

\[= \pm i \omega_0 (1 + \eta^2)^{1/4} e^{i\theta/2}\]

The undamped natural frequency of an elastic beam is given by,

\[\omega_0 = \frac{N^2 \pi^2 (E \sqrt{1/\rho A})^{1/2}}{L^2}.\]

Using the identity

\[e^{i\theta/2} = \cos(\theta/2) + i \sin(\theta/2)\]

results in the complex frequency,

\[a + bi = \pm i \omega_0 (1 + \eta^2)^{1/4} [\cos(\theta/2) + i \sin(\theta/2)].\]
Again equating the real and imaginary parts of the above equation leads to

\[ a = \pm \omega_0 (1 + \eta^2)^{1/4} \sin (\theta/2) \]

\[ b = \pm \omega_0 (1 + \eta^2)^{1/4} \cos (\theta/2). \]

As in the second-order problem, we obtain the following expressions for the real and imaginary parts of the complex frequencies:

\[ a = -\omega_0 \frac{\sqrt[1/2]{1 + \eta^2 - 1}}{\sqrt{2}} \]

\[ b = \omega_0 \frac{\sqrt[1/2]{1 + \eta^2 + 1}}{\sqrt{2}} \]
VISCOELASTICALLY DAMPED LATERAL VIBRATIONS OF A CANTILEVER BEAM

The five layer cantilever beam problem studied by Messalti [11] is reviewed. The essential boundary conditions are:

\[ Y(0) = 0 \text{ and } Y'(0) = 0. \]

From the Laplace transform technique we obtain

\[ Y(s) = \frac{d_2 s + d_3}{(s^4 - cG^2 s^2 - \Omega^2)} \]

The denominator of the expression for the Laplace transform, \( s^4 - cG^2 s^2 - \Omega^2 \), may be solved for the parameter \( s^2 \). Consequently we receive the following roots:

\[ s_1^2 = \frac{cG^* + \sqrt{c^2G^*2 + 4\Omega^2}}{2} \equiv u^2 > 0 \]

\[ s_2^2 = \frac{cG^* - \sqrt{c^2G^*2 + 4\Omega^2}}{2} \equiv -v^2 < 0 \]
The reader should note that the complex parameters, $u^2$ and $v^2$ have opposite signs.

In addition,

$$Y(s) = \frac{d_2 s + d_3}{(s^2 - u^2)(s^2 + v^2)}$$

We observe that

$$u^2 - v^2 = cG^*$$

and

$$u^2v^2 = \Omega^2$$

Applying partial fractions and taking the inverse Laplace transform results in

$$Y(x) = \frac{d_2 s}{v^2 + u^2} [\cosh u x - \sinh v x] + \frac{d_3}{v^2 + u^2} [1/u \sinh u x - 1/v \sin v x]$$

Applying the boundary condition at $x = L$ we obtain the characteristic equation in terms of $u$ and $v$. The characteristic equation becomes:

$$(u^2 + v^2)(\cosh u L - \cos v L) + 2u^2v^2 + uv(u^2 - v^2)(\sinh u L - \sin v L) = 0 \quad (19)$$
Let
\[ r = (c^2G^* + 4\Omega^2)^{1/2} \]
thus,
\[ u = (cG^*/2 + r)^{1/2} \]
and
\[ v = (-cG^*/2 + r)^{1/2} \]
Substituting into (19) we deduce the characteristic equation in terms of the frequency parameter, \( \Omega^2 \).

\[
\begin{align*}
&\left(2\sqrt{c^2G^* + 4\Omega^2}\right) \left(\cosh\sqrt{cG^*/2 + \sqrt{c^2G^* + 4\Omega^2}} - \cos\sqrt{cG^*/2 + \sqrt{c^2G^* + 4\Omega^2}L}\right) \\
&+ 2\Omega^2 + \Omega cG^* \left(\sinh\sqrt{cG^*/2 + \sqrt{c^2G^* + 4\Omega^2}L} - \sin\sqrt{cG^*/2 + \sqrt{c^2G^* + 4\Omega^2}L}\right) = 0 \quad (20)
\end{align*}
\]
The relationship between the damped natural frequency in the \(j\)th mode and the undamped natural frequency is as follows:

\[
\omega_j^2 = \Omega^2 \omega_0^2. 
\]

Again, the complex moduli are given by

\[
E^* = E(1 + i\eta) \\
G^* = G(1 + i\eta) 
\]

Thus,

\[
\omega_j^2 = \Omega_j^2 \frac{E \sqrt{1 + \eta^2} e^{i\theta}}{\rho} 
\]

Substituting

\[
\omega = (a + bi) 
\]

into equation (21) gives

\[
(a + bi)^2 = \Omega^2 \frac{E \sqrt{1 + \eta^2} e^{i\theta}}{\rho} 
\]
Taking the square root of both sides of equation (22) results in

\[ (a + bi) = \pm i\Omega E(1 + \eta^2)^{1/4} e^{i\theta/2} \]  \hspace{1cm} (23)

Again substituting for \( e^{i\theta/2} \) and equating the real and imaginary parts of the above equation leads to

\[ a = \pm \Omega E(1 + \eta^2)^{1/4} \sin(\theta/2) \]  \hspace{1cm} \rho

\[ b = \pm \Omega E(1 + \eta^2)^{1/4} \cos(\theta/2). \]  \hspace{1cm} \rho

Similar to the preceding cases, we obtain the following expressions for the real and imaginary parts of the complex frequencies:

\[ a = \Omega E(\sqrt{1 + \eta^2} + 1)^{1/2} \]  \hspace{1cm} \rho \sqrt{2}

\[ b = \Omega E(\sqrt{1 + \eta^2} + 1)^{1/2} \]  \hspace{1cm} \rho \sqrt{2}
In summary, we have found that the Laplace transform technique can be effectively applied to cantilever beam problems and we are thus given a model to follow when dealing with the sixth-order governing equation for this thesis. In addition, the implications of complex frequencies have been considered.
CHAPTER V - ANALYSIS OF THE EQUATION OF MOTION

In Chapter III, the following differential equation governing the flexural motion of the composite beam was derived:

\[ \dddot{y} - \frac{G}{h_2 h_3 E_3} \dddot{y} + \frac{(p/EI)(\dddot{y}/\ddot{t}^2)''}{G \frac{h_2 h_3 E_3 E}{EI}} = 0 \quad (24) \]

In the preceding chapter solution to cantilever beam problems were investigated. In this chapter we analyze the equation of motion and deduce the natural boundary conditions associated with the support conditions. The method of separation of variables is implemented. In addition, variational techniques are used to derive boundary conditions not specified at the fixed end of the beam.

The idea behind the method of separation of variables is to convert the partial differential equation into a set of ordinary differential equations which are more readily solved. More explicitly, we set

\[ y(x,t) = Y(x)e^{\text{i}\omega t} \]

and substitute into equation (22). Background information pertaining to the selection of this solution for \( y(x,t) \) was presented in the preceding chapter. The result is two ordinary differential equations.
\[ Y^{VI} - G(c + 1/(h_2 h_3 E_3)) Y^{IV} - \rho \omega^2/(EI) Y'' + G \rho \omega^2/(h_2 h_3 E_3) Y = 0 \]  
\[ T'' + \omega^2 T = 0 \]

The reader will note a sign change for the last two terms of equation (25) compared to equation (24) due to the second time derivative of

\[ y(x,t) = Y(x)e^{i\omega t}. \]

Further, the determination of the solution of the original partial differential equation is now reduced to the determination of solutions of two ordinary differential equations. Since \( e^{i\omega t} \) is an admissible solution to equation (26), the problem is essentially simplified to the determination of appropriate solutions to one ordinary differential equation.

Recall from Chapter II, that the beam considered in this investigation is a five-layer cantilever beam. The prescribed boundary conditions for this beam are:

\[ Y(0) = 0 \text{ and } Y'(0) = 0. \]  

In total, six boundary conditions must be specified in order to solve the equation of motion.

The natural boundary conditions, that is, those not specified \textit{a priori}, are derived using variational principles. The principles of variational calculus require that one extremize a functional, \( V \), with respect to a family of admissible functions close to \( y(x) \). Such extremization also requires that the extremal function, \( y(x) \), extremize \( V \) with respect to all subsets of the family of
admissible functions. These subsets of the family of admissible functions are given by the equation:

$$\tilde{y}(x) = y(x) + \xi \eta(x)$$

for any continuously differentiable function, $\eta$, satisfying the homogeneous boundary conditions $\eta(x_1) = \eta(x_2) = 0$. Figure 5.1 shows $y(x)$ and admissible variations, $\tilde{y}(x)$.

![Figure 5.1-- Variations of $y(x)$](image-url)
In addition, the outcome of the extremization process is a differential equation and a set of kinematic or rigid boundary conditions, natural boundary conditions, and a combination of kinematic and natural boundary conditions, which help facilitate extremizing the functional. For a large group of functionals the resulting boundary value problem is both self-adjoint and positive definite. Boundary value problems having these properties are considered *properly posed*. In other words, the variational process leads to a properly posed boundary value problem for which a unique solution exists.

The weak form of equation (25) may be expressed as

\[
\int_0^L [Y'' + a_4 Y'' + a_2 Y' + a_0 Y] \delta Y \, dx = 0 \tag{28}
\]

where,

\[
a_4 = -G(c + 1/(h_2 h_3 E_3))
\]
\[
a_2 = -\rho \omega^2/(EI)
\]
\[
a_0 = G\rho \omega^2/(h_2 h_3 E_3 E I)
\]

Integrating equation (26) by parts gives:

\[
\int_0^L [-Y'' \delta Y' - a_4 Y''' \delta Y' - a_2 Y'' \delta Y' + a_0 Y] \, dx + [Y' \delta Y + a_4 Y''' \delta Y + a_2 Y'' \delta Y] = 0
\]

Integration by parts several more times results in the symmetric form:

\[
\int_0^L [-Y''' \delta Y'' + a_4 Y'' \delta Y'' - a_2 Y' \delta Y' + a_0 Y] \, dx + [Y' \delta Y + a_4 Y''' \delta Y + a_2 Y'' \delta Y] \left[ Y'' \delta Y' + a_4 Y'' \delta Y' \right] + [Y''' \delta Y'''] = 0 \tag{29}
\]
The specified boundary conditions require that

\[ \delta Y(0) = 0 \text{ and } \delta Y'(0) = 0 \]  

(30)

However, equation (28) must hold for all variations satisfying the constraint conditions (30) described above.

Since \( \delta Y(L) \), \( \delta Y'(L) \), \( \delta Y''(0) \), and \( \delta Y''(L) \) are considered arbitrary, the natural boundary conditions are deduced from equation (29) as follows:

\[
\begin{align*}
(Y^v + a_4 Y''' + a_2 Y')_x &= L = 0 \\
(Y^v + a_4 Y'')_x &= L = 0 \\
Y'''(0) &= 0 \\
Y'''(L) &= 0
\end{align*}
\]  

(31)
CHAPTER VI--DISCUSSION OF THE SOLUTION METHOD

This chapter reports the details of the solution of the derived sixth-order differential equation. The Laplace transform method greatly reduced the algebraic complexity of this endeavor. Once a general solution to equation (25) in Chapter V was obtained and the boundary conditions prescribed at the beam's fixed end were applied, the sixth-order system was reduced to a fourth-order system. Consequently, the resultant characteristic frequency equation is a fourth-degree equation.

The damping effectiveness aspects of this thesis were limited to time considerations. This was done because even after the reduction in complexity facilitated by the Laplace transform technique was obtained, the frequency equation and the associated expressions involving the natural frequency were still quite complicated. Therefore, it became necessary to reduce the focus of this investigation. A constant factor such that \( \phi_j(L) = 1 \) for all modes was assumed and damping effectiveness was based on the term \( e^{i\omega t} \). In addition, DiTaranto and Blasingame[6] reported that the natural frequencies for sandwich beams are independent of the end conditions and mode shapes.
THE LAPLACE TRANSFORM TECHNIQUE

The Laplace transform is a standard tool for the solution of initial value problems, that is, problems for which appropriate values are prescribed at a fixed instant in time. By definition the Laplace transform of a function, $f(t)$, is

$$L(f) = \int_{0}^{\infty} f(t) e^{-pt} \, dt$$

where $p = a + i \omega$ is a complex number. Moreover, the Laplace transform of a function approaches the Fourier transform when the real part of the complex variable, $p$, approaches zero. For most shock vibration problems, the Fourier and Laplace transforms provide similar results, however, the Laplace transform has certain advantages of use over the Fourier transform and there are more extensive tables of Laplace transforms available than there are tables of Fourier transforms. In addition, a significant disadvantage of the Fourier transform is that the defining integral of the Fourier transform sometimes does not converge unless an additional convergence factor is employed.
Ordinarily, Laplace transforms are used to solve initial value problems given by $n$th-order linear differential equations of the form:

$$b_n \frac{d^n y}{dx^n} + b_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + b_1 \frac{dy}{dx} + b_0 y = g(x)$$

with specified initial conditions

$$y(0) = c_0, \ y'(0) = c_1, \ \ldots \ y^{n-1}(0) = c_{n-1}$$

Nevertheless, the Laplace transform technique can be successfully applied to solve boundary value problems. In order to effectively perform such an application, the domain of the problem is assumed to be infinite and transformations are taken over the space variable, not time. Figure 6.1 is a graphical representation of the relationship between the beam boundaries and required domain of the spatial variable, $x$. The boundary conditions at one end of the beam are enforced as initial conditions, and the remaining initial conditions are carried through as unknowns to be solved for by applying boundary conditions at the beam's terminal end, $x = L$. 
From the previous chapter, we note that the spatial component of the solution satisfies the equation,

\[ Y^{vi} + a_4 Y^{iv} + a_2^2 Y'' + a_0 Y = 0 \]  

(32)

where

\[
\begin{align*}
    a_4 &= -G(c + 1/(h_2 h_3 E_3)) \\
    a_2 &= \rho \omega^2/(E I) \\
    a_0 &= G_p \omega^2/(h_2 h_3 E_3 E I)
\end{align*}
\]

with boundary conditions at the beam's end:

\[ y(0) = 0, \ y'(0) = 0 \] (essential boundary conditions)
APPLYING THE LAPLACE TRANSFORM TECHNIQUE

Let \( Y(s) \) denote \( L\{Y\} \), then

\[
L\{Y''\} = s^6Y(s) - s^5Y(0) - s^4Y'(0) - s^3Y''(0) - s^2Y'''(0) - sYiv(0) - Yv(0)
\]

\[
L\{a_4Y''\} = a_4[s^4Y(s) - s^3Y(0) - s^2Y'(0) - sY''(0) - Y'''(0)]
\]

\[
L\{a_2Y''\} = a_2[s^2Y(s) - sY(0) - Y'(0)]
\]

\[
L\{a_0Y\} = a_0Y(s)
\]

Applying the boundary conditions at the fixed end and combining the terms gives the Laplace transform of equation (26) as

\[
L\{Y'' + a_4Y''' + a_2Y'' + a_0Y\} = s^6Y(s) - s^5Y''(0) - sYiv(0) - Yv(0) + a_4s^4Y(s) - a_4sY''(0) + a_2s^2Y(s) + a_0Y(s) = 0
\]

The resultant transform, \( Y(s) \), after terms are grouped and rearranged is as follows

\[
Y(s) = L[Y(s)] = \frac{s^3Y''(0) + s(Yiv(0) + a_4Y''(0)) + Yv(0)}{s^6 + a_4s^4 + a_2s^2 + a_0}
\]

(33)
Please note, that at this point the initial conditions in equation (33) are unknowns to be determined. The denominator of the right hand side of equation (33) is cubic in $s^2$ and can be solved using standard techniques for solving cubic equations. This procedure is reported in Appendix A. In addition, the Maple[20] computer code used to symbolically solve the cubic equation is contained in Appendix C.

The general solution to the governing spatial equation is of the following form:

$$Y(x) = a_1 L^{-1}[Y_1] + a_2 L^{-1}[Y_2] + a_3 L^{-1}[Y_3]$$  \hspace{1cm} (34)

Using the method of partial fractions we can express the Laplace transform as

$$f_1(s) = \frac{A s^3 + B s + C}{(s^2 - l^2)(s^2 - m^2)(s^2 - n^2)} = \frac{d_1 s + d_2}{(s^2 - l^2)} + \frac{d_3 s + d_4}{(s^2 - m^2)} + \frac{d_5 s + d_6}{(s^2 - n^2)}$$

$$f_2(s) = \frac{A s^3 + B s + C}{(s^2 - l^2)(s^2 - m^2)(s^2 - n^2)} = \frac{d_1 s + d_2}{(s^2 - l^2)(s^2 - m^2)(s^2 - n^2)}$$

It follows that

$$\frac{A s^3 + B s + C}{(s^2 - m^2)(s^2 - n^2)} = \frac{d_1 s + d_2}{(s^2 - m^2)(s^2 - n^2)}$$

$$\frac{A s^3 + B s + C}{(s^2 - l^2)(s^2 - n^2)} = \frac{d_3 s + d_4}{(s^2 - l^2)(s^2 - n^2)}$$

$$\frac{A s^3 + B s + C}{(s^2 - l^2)(s^2 - m^2)} = \frac{d_5 s + d_6}{(s^2 - l^2)(s^2 - m^2)}$$
Equating the coefficients, one obtains the results listed below

\[ d_1 = \frac{A l^2 + B}{(l^2-m^2)(l^2-n^2)} = \frac{Y''(0)l^2 + Y''\nu(0) + a_4 Y''(0)}{(l^2-m^2)(l^2-n^2)} \]

\[ d_2 = \frac{C/l}{(l^2-m^2)(l^2-n^2)} = \frac{Y'\nu(0)/l}{(l^2-m^2)(l^2-n^2)} \]

\[ d_3 = \frac{A m^2 + B}{(m^2-l^2)(m^2-n^2)} = \frac{Y''(0)m^2 + Y''\nu(0) + a_4 Y''(0)}{(m^2-l^2)(m^2-n^2)} \]

\[ d_4 = \frac{C/m}{(m^2-l^2)(m^2-n^2)} = \frac{Y'\nu(0)/m}{(m^2-l^2)(m^2-n^2)} \]

\[ d_5 = \frac{A n^2 + B}{(n^2-l^2)(n^2-m^2)} = \frac{Y''(0)n^2 + Y''\nu(0) + a_4 Y''(0)}{(n^2-l^2)(n^2-m^2)} \]

\[ d_6 = \frac{C/n}{(n^2-l^2)(n^2-m^2)} = \frac{Y'\nu(0)/n}{(n^2-l^2)(n^2-m^2)} \]
Taking the inverse Laplace transform results in the following expression for the general solution over the spatial variable:

\[ Y(x) = d_1 \cosh lx + d_2 \sinh lx + d_3 \cosh mx + d_4 \sinh mx + d_5 \cosh nx + d_6 \sinh nx \quad (35) \]

A mathematical representation of \( l, m, \) and \( n \) is given below and on the immediately following pages.

\[ 1^2 = \sqrt{\frac{3}{54} \left( \frac{9a - 27a^2 - 2a^2}{9} \right) + \left( \frac{3a - a^2}{54} \right)^3 + \left( \frac{a^2 - 27a - 2a^2}{54} \right)^2} \]

\[ - \frac{a_4}{3} \]
\[
\begin{align*}
\mathbf{m}^2 &= -\frac{1}{2} \left( \sqrt[3]{\frac{9a_a - 27a^2 - 2a^2}{54} + \left( \frac{3a_a}{9} \right)^3 + \left( \frac{9a_a - 27a^2 - 2a^2}{54} \right)^2} \right) \\
&\quad + \frac{1}{2} \sqrt{3} \left( \sqrt[3]{\frac{9a_a - 27a^2 - 2a^2}{54} + \left( \frac{3a_a}{9} \right)^3 + \left( \frac{9a_a - 27a^2 - 2a^2}{54} \right)^2} \right) \\
&\quad - \sqrt[3]{\frac{9a_a - 27a^2 - 2a^2}{54} + \left( \frac{3a_a}{9} \right)^3 + \left( \frac{9a_a - 27a^2 - 2a^2}{54} \right)^2} \right)
\end{align*}
\]
\[ n^2 = -\frac{1}{2} \left( \sqrt[3]{\frac{9a^2 - 27a - 2a^2}{54} + \left( \frac{3a - a^2}{9} \right)^3 + \left( \frac{9a^2 - 27a - 2a^2}{54} \right)^2} + \right. \]
\[ \left. \sqrt[3]{\frac{9a^2 - 27a - 2a^2}{54} - \left( \frac{3a - a^2}{9} \right)^3 + \left( \frac{9a^2 - 27a - 2a^2}{54} \right)^2} \right) - \frac{a^4}{3} \]

\[-\frac{1}{2} \sqrt{3} I \left( \sqrt[3]{\frac{9a^2 - 27a - 2a^2}{54} + \left( \frac{3a - a^2}{9} \right)^3 + \left( \frac{9a^2 - 27a - 2a^2}{54} \right)^2} \right) \]
\[ - \sqrt[3]{\frac{9a^2 - 27a - 2a^2}{54} - \left( \frac{3a - a^2}{9} \right)^3 + \left( \frac{9a^2 - 27a - 2a^2}{54} \right)^2} \right) \]
Application of boundary conditions (27) reduces the sixth-order system to a fourth-order system. This requires the solution of a fourth-degree characteristic frequency equation as opposed to a sixth-degree equation. In addition, the characteristic equation is further determined by application of the natural boundary conditions (31) which are listed below in an expanded format. These equations represent the remaining coefficients found in equation (35) after conditions (27) are applied.

The first natural boundary condition is

$$(Y'' + a_4 Y'''' + a_2 Y')_{x=L} = 0.$$ 

More explicitly,

$$-d_3 l^5 \sinh l L - d_6 l^5 \sinh L - d_4 l^5 \cosh L - d_6 l^5 \cosh L$$
$$+ d_3 m^5 \sinh m L + d_4 m^5 \cosh m L + d_5 n^5 \sinh n L + d_6 n^5 \cosh n L +$$
$$a_4 (-d_3 l^3 \sinh l L - d_5 l^3 \sinh L - d_4 l^3 \cosh L -$$
$$d_6 l^3 \cosh h L + d_3 m^3 \sinh m L + d_4 m^3 \cosh m L + d_5 n^3 \sinh n L +$$
$$d_6 n^3 \cosh n L) + a_2 (-d_3 l \sinh l L - d_5 l \sinh L - d_4 l^3 \cosh L - d_6 l^3 \cosh L$$
$$+ d_3 m \sinh m L + d_4 m \cosh m L + d_5 n \sinh n L + d_6 n \cosh n L) = 0.$$ 

The second natural boundary condition is

$$(Y'' + a_2 Y'')_{x=L} = 0.$$
More explicitly,

\[-d_3 l^4 \cosh L - d_5 l^4 \cosh L - d_4 l^4 \sinh L - d_6 l^4 \sinh L + d_3 m^4 \cosh mL + d_4 m^4 \sinh mL + d_5 n^4 \cosh mL + d_6 n^4 \sinh mL + a_2 (-d_3 l^2 \cosh lL - d_5 l^2 \cosh lL - d_4 l^2 \sinh lL + d_6 l^2 \sinh lL + d_3 m^2 \cosh mL + d_4 m^2 \sinh mL + d_5 n^2 \cosh mL + d_6 n^2 \sinh mL) = 0.\]

The third natural boundary condition is

\[Y'''(0) = 0.\]

More explicitly,

\[-d_4 l^3 - d_6 l^3 + d_4 m^3 + d_6 n^3 = 0\]

The fourth natural boundary condition is

\[Y'''(L) = 0.\]

More explicitly,

\[-d_3 l^3 \sinh L - d_5 l^3 \sinh L - d_4 l^3 \cosh hL - d_6 l^3 \cosh hL + d_3 m^3 \sinh mL + d_4 m^3 \cosh mL + d_5 n^3 \sinh mL + d_6 n^3 \cosh mL = 0.\]
Consequently, the fourth-order system may be expressed in the following matrix form.

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44}
\end{bmatrix}
\begin{bmatrix}
d_3 \\
d_4 \\
d_5 \\
d_6
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

From the theory of matrix algebra we know that the eigenvalues of an \( n \times n \) real or complex matrix \( A \), are real or complex numbers, \( \lambda \), for which there is a nonzero \( x \) with \( Ax = \lambda x \). In addition, there is a very useful theorem of linear algebra relating eigenvalues and singularities. Namely, \( \lambda \) is an eigenvalue of \( A \) if and only if \( A - \lambda I \) is singular, which holds if and only if the determinant of \( A - \lambda I \) equals zero. In other words, the characteristic equation of \( A \) is given by the expression: \( \det(A - \lambda I) = 0 \). Information pertaining to the determination of eigensystem matrices and eigenvalues is found in appreciably greater detail in reference [12]. Consequently, the characteristic frequency equation is
obtained by equating to zero the determinant of the above coefficient matrix and is written below.

\[
\begin{align*}
& a_{13}a_{24}a_{32}a_{41} + a_{12}a_{24}a_{33}a_{41} + a_{13}a_{24}a_{31}a_{42} + a_{11}a_{24}a_{33}a_{42} + a_{12}a_{24}a_{31}a_{43} + \\
& a_{11}a_{24}a_{32}a_{43} - a_{13}a_{22}a_{31}a_{44} + a_{12}a_{22}a_{31}a_{44} + a_{13}a_{21}a_{32}a_{44} + a_{11}a_{23}a_{32}a_{44} - \\
& a_{12}a_{21}a_{33}a_{44} + a_{11}a_{22}a_{33}a_{44} = 0
\end{align*}
\]

where,

\[
\begin{align*}
& a_{11} = -l^5 \sinh L + m^5 \sinh m L - a_4 l^3 \sinh L + a_4 m^3 \sinh m L - a_2 l \sinh L + a_2 m \sinh m L \\
& a_{12} = -l^4 \cosh L + m^4 \cosh m L - a_2 l^3 \cosh L + a_2 m^3 \cosh m L \\
& a_{13} = -l^3 \sinh L + m^3 \sinh m L \\
& a_{14} = 0 \\
& a_{21} = -l^5 \cosh L + m^5 \cosh m L - a_4 l^3 \cosh L + a_4 m^3 \cosh m L - a_2 l \cosh L + a_2 m \cosh m L \\
& a_{22} = -l^4 \sinh L + m^4 \sinh m L - a_2 l \sinh L + a_2 m \sinh m L \\
& a_{23} = -l^3 \cosh L + m^3 \cosh m L \\
& a_{24} = -l^3 + m^3 \\
& a_{31} = -l^5 \sinh L + n^5 \sinh n L - a_4 l^3 \sinh L + a_4 n^3 \sinh n L - a_2 l \sinh L + a_2 n \sinh n L \\
& a_{32} = -l^4 \cosh L + n^4 \cosh n L - a_2 l \cosh L + a_2 n \cosh n L \\
& a_{33} = -l^3 \cosh L + n^3 \cosh n L \\
& a_{34} = 0 \\
& a_{41} = -l^5 \cosh L + n^5 \cosh n L - a_4 l^3 \cosh L + a_4 n^3 \cosh n L - a_2 l \cosh L + a_2 n \cosh n L \\
& a_{42} = -l^4 \sinh L + n^4 \sinh n L - a_2 l \sinh L + a_2 n \sinh n L \\
& a_{43} = -l^3 \cosh L + n^3 \cosh n L \\
& a_{44} = -l^3 + n^3
\end{align*}
\]
It is clear that equation (36) is very complicated. In addition, the algebraic manipulations required to solve such an equation are extensive. For this reason, mathematical analysis software was employed whenever possible to reduce such algebraic complexity. The analysis package Maple [13] was used for much of the initial investigations. Subsequently, Mathematica [14] and Fortran programs were used to solve the equations and expressions involving complex variables. Mathematica [14] was especially beneficial in its ability to handle complex arguments to the hyperbolic functions of sine and cosine. For more information concerning the computer algorithms and programs corresponding to the mathematical analysis performed for this thesis refer to Appendix C.

SYSTEM RESPONSE

In general, the system response for the beam configuration studied in this investigation is governed by the following equation:

\[ y(x,t) = \sum_{j=1}^{\infty} \phi_j(x)e^{iu_j t} \quad (37) \]
where,

\[ \Phi(x) = d_1 \cosh lx + d_2 \sinh lx + d_3 \cosh mx + d_4 \sinh mx + d_5 \cosh nx + d_6 \sinh nx \]  

(38)

It is obvious that there are six coefficients in equation (38). Therefore, for a specific solution, six equations are required to determine the coefficients. Four of these equations were given previously in this chapter. The remaining two coefficients are obtained from Fourier expansions of the initial displacement and initial velocity in a particular beam problem. In addition, the reader will recall that the parameters l, m, and n were previously defined.

The mathematical computations required to begin an assessment of the effectiveness of damping were reduced by condensing the focus of this investigation to time considerations only and basing assessments on the term \( e^{i\omega t} \). However, future investigations with stricter emphasis on computational techniques could encompass both spatial and time components.

**NATURAL FREQUENCY AND THE MODAL LOSS FACTOR**

In Chapter II we defined the loss factor as the ratio of the energy lost per unit radian to the total potential energy. Then, in Chapter IV we discussed that the damped natural frequency of the composite beam in the \( j^{th} \) mode of vibration is related to the undamped frequency, \( \omega_0^2 \), and a frequency parameter, \( \Omega^2 \), in the following manner.
\[ \omega_j^2 = \Omega^2 \omega_0^2 \]

Therefore,

\[ \omega_j^2 = \frac{\Omega^2 EI \sqrt{1 + \eta^2 e^{i\theta}}}{\rho} \quad (39) \]

The real and imaginary components of the complex frequency were also discussed in Chapter IV. Such information provides additional insight into the nature of the complex natural frequency study in this investigation. As a consequence a family of curves for various values of \( \Omega^2 \) was obtained using equation (39). These curves are shown in Figure 6.5
FIGURE 6.2  Family of Curves
Frequency versus Modal Loss Factor
CASE STUDY ONE: DETERMINATION OF THE EFFECTS OF $\eta$ ON $y(t)$

<table>
<thead>
<tr>
<th>$h_1$ = 1.0 in.</th>
<th>$h_2$ = 0.1 in.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_3$ = 0.01 in.</td>
<td>$\eta$ = various (see below)</td>
</tr>
<tr>
<td>$x$ = 24 in.</td>
<td>$= .06, .24, .36, .54, .96$</td>
</tr>
<tr>
<td>$c = 42.112 \times 10^{-6} \text{ lb}$</td>
<td>$G^* = 48.35(1 + i\eta)$</td>
</tr>
<tr>
<td>$E_3 = 30 \times 10^6 \text{ lb/in.}^2$</td>
<td>$EI = 2.5 \times 10^6 \text{ lb-in}$</td>
</tr>
<tr>
<td>$\rho = 7.4562 \times 10^{-4} \text{ lb-ft.sec}^2/\text{in.}^3$</td>
<td>$\omega = \sqrt{E(1 + i\eta)/\rho}$</td>
</tr>
<tr>
<td>$\Omega = 1$</td>
<td>$\omega_j = \Omega(\sqrt{E(1 + i\eta)/\rho})$</td>
</tr>
</tbody>
</table>

An investigation was conducted to further determine what effect varying the damping factor, $\eta$, has on the system response. Five values for $\eta$ were selected as shown above. The thickness of the elastic core considered for this investigation was one inch and the thicknesses of the viscoelastic and constraining layers were one-tenth inch and one one-hundreth inch respectively. The elastic core material considered was steel as indicated by the density and modulus values shown above. In addition, the viscoelastic material was rubber with complex modulus specifications defined above.

The general procedures used in this investigation may be used for subsequent studies and analysis. The computer programs used to generate both tabular and graphical data are contained in Appendix C for this purpose. These programs are straightforward and based fundamentally on equations (25), and (26) found in Chapter V and equations (32) and (37) contained in this chapter.
The parameters l, m, and n, the undamped natural frequency, $\omega_0$, and the damped natural frequency of the system, $\omega_j$, were computed for each value of the damping factor. The parametric relationship described by equation (39) was used to compute $\omega_j$. Given the system's natural circular frequency and the parameters: l, m, and n, a solution to equation (37) becomes straightforward. The real component of the exponential term expressed in equation (37) causes dissipation whereas the imaginary term in that equation causes oscillation.

Graphical output depicting the real and imaginary parts of the response function were obtained. As stated earlier, Mathematica [14] was used to generate response data and graphs. Figures 6.3 through 6.7 show these responses for specific values of the damping factor. Figure 6.3 illustrates the response when the damping factor is 0.06. Figures 6.4 through 6.7 show the response when the damping factor is 0.24, 0.36, 0.54, 0.96 respectively. A pattern is easily observed.

As the damping factor increased, the time for the response to be damped out was significantly reduced. For example, the absolute value of the amplitude of both the real and imaginary curves shown in Figure 6.3, where $\eta$ is 0.06, is greater than 0.5 unit after forty-three seconds. In Figure 6.5, where $\eta$ is 0.36, the absolute value of the amplitude of both the real and imaginary curves decays to less than 0.5 unit after only thirteen seconds. In Figure 6.7, where $\eta$ is 0.96, the absolute value of the amplitude of both the real and imaginary curves is reduced to less than 0.25 unit after only nine seconds. From these illustrations, it is obvious that the damping factor significantly affects the response curve. As suspected, the oscillations decay more rapidly with
increases in the damping factor. In other words, the time required for the oscillations to decay is reduced.
\[ \eta = 0.06 \]

- - - Real

--- Imaginary

FIGURE 6.3 -- Response vs. Time
FIGURE 6.4 -- Response vs. Time

\[ \eta = 0.24 \]

- - - Real

- - - Imaginary
$\eta = 0.36$

- - - Real

- - - Imaginary

FIGURE 6.5 -- Response vs. Time
\[ \eta = 0.54 \]

- \( \cdots \cdots \) Real
- \( \text{Imaginary} \)

**FIGURE 6.6 -- Response vs. Time**
\[ \eta = 0.96 \]

- - - Real

- - - - Imaginary

**FIGURE 6.7 -- Response vs. Time**
CASE STUDY TWO: THE EFFECTS OF $h_2$ ON $y(t)$,

\[
\begin{align*}
  h_1 &= 1.0 \text{ in.} \\
  x &= 24 \text{ in.} \\
  h_3 &= 0.01 \text{ in.} \\
  c &= 42.112 \times 10^{-6} \text{ lb} \\
  E_3 &= 30 \times 10^6 \\
  \rho &= 7.4562 \times 10^{-4} \text{ b.lbf sec}^2/\text{in.}^3 \\
  h_2 &= \text{various (see below)} = .025, .1, .3, .5, .8 \text{ in.} \\
  \eta &= .54 \\
  G^* &= 48.35 + 2.93i \\
  EI &= 2.5 \times 10^6 \text{ b.lbf-in} \\
  \omega &= \sqrt{E(1 + i\eta)I/\rho}
\end{align*}
\]

Another study was initiated to evaluate the effects of the viscoelastic layer thickness, $h_2$, on the time response. Messalti[11] found that the complex frequencies were weak functions of the constraining and viscoelastic layer thicknesses. Similar to the previous case study involving the damping factor, five critical values for the viscoelastic layer thickness were selected for review. The damping factor was held constant at 0.54 for this investigation. The various values for $h_2$ and the other contributory parameters are listed in the table above.

Graphical output was also obtained and this output is illustrated in Figures 6.8 through 6.12. This case study revealed that the time decay is independent of the thickness of the viscoelastic layers. Therefore, each of the curves is exactly the same despite the changes in the value for $h_2$ input to the analysis program. A closer review of the equations used to compute the time response shows that the viscoelastic layer thickness is indeed not a factor.
\( h_2 = 0.025 \)

- \( \cdots \cdots \) Real
- \( \cdots \cdots \) Imaginary

**FIGURE 6.8 -- Response vs. Time**
$h_2 = 0.1$

- - - - Real

- - - - Imaginary

FIGURE 6.9 -- Response vs. Time
$h_2 = 0.3$

- - - - Real

- - - - Imaginary

FIGURE 6.10 -- Response vs. Time
FIGURE 6.11 -- Response vs. Time
FIGURE 6.12 -- Response vs. Time

$h_2 = 0.8$

- - - - Real

--- - Imaginary

FIGURE 6.12 -- Response vs. Time
SUMMARY

In summary for the sandwich beam method which is the most popular type of surface damping treatment, a sixth-order differential equation is required to describe the beam's flexural motion when stretching and contraction in the outer constraining layers is taken into account. The Laplace transform technique lends itself to the solution of such a sixth-order equation for an assumed infinite domain with transformations taken over the spatial variable and not time. In total, six boundary conditions are necessary to fully define the beam constraints. Two of these conditions are specified via the cantilever support conditions prescribed at the beam's fixed end. The other four boundary conditions are deduced by using variational techniques. Such techniques provide a basis for the application of approximation solution methods. Application of the natural boundary conditions results in a derived frequency equation.

The system's normal modes are specified by the characteristic or eigenfunctions obtained from the frequency equation. A graphical relationship between the frequency parameters and the modal loss factor of the composite beam material was obtained.

Two case studies were implemented to quantitatively address a method or methods of assessing the dynamic characteristics, including the effectiveness of damping in elastic-viscoelastic beams. Case Study One showed that oscillations are damped out more rapidly with increases in the damping factor. Case Study Two showed that the time response is independent of the
viscoelastic layer thickness. However, the spatial term of equation (37) is influenced by the viscoelastic layer thickness. The extent of this influence is the topic of future investigation with regard to damping of elastic/viscoelastic beams.

Insight to the dynamic behavior of a sandwich beam system is gained through understanding the previously discussed association between the characteristic equation, the natural frequency, the mode shapes, and the damping factor. Such insight may be applied for a better understanding of the contributory influence a specified factor has on other factors and/or the total system response.

In addition, future investigative studies relative to this topic will be more effective if a procedure similar to the one outlined below is followed after a careful review of this document and pertinent Reference items.

1. Beam Geometry Defined
2. Governing Equation(s) Derived and Boundary Conditions Established
3. Calculus Techniques Used to Obtain General Expression of Solution in the Desired Variable(s) (e.g. The Laplace Transform Technique)
4. Boundary Conditions Applied to Obtain Frequency Equation(s)
5. The Real and Imaginary Components of the Complex Frequency Separated
6. Expressions For Time and Space Components of the General Solution Determined
8. Latitude of Desired Outcome of Case Studies Established
9. Case Studies Conducted
10. Assessment of Beam's Dynamic Characteristics Performed Based on Case Study Specifications
The author would like to note that in order to successfully complete many of the above-mentioned steps it is necessary to customize computer code for the specified problem and the desired results. A very important factor to consider is one's list of assumptions for such analysis. While it is not possible for the author to specify future inputs exactly, well-documented programming techniques should be followed. As far as commercial software is concerned, determination of the software's ability to provide the desired results must be done on a case-by-case basis. The computer programs contained in Appendix C provide an excellent starting point for such work.
CHAPTER VII -- CONCLUSIONS AND RECOMMENDATIONS

This chapter contains conclusions and recommendations based on this investigation. The chapter is divided into two subsections below.

CONCLUSIONS

The Laplace transform technique was effectively applied to the solution of the sixth-order beam problem.

The characteristic frequency equation for the sixth-order sandwich beam problem was obtained.

A proportional relationship was shown to exist between damping factor, $\eta$, and the damped natural frequency.

The sixth-order sandwich beam system response was found to be not strictly sinusoidal.

The damping factor affects the response such that oscillations decay more rapidly as the damping factor is increased.

The time response was found to be independent of the viscoelastic layer thickness.
RECOMMENDATIONS

Investigate the application of additional numerical analysis techniques in order to characterize the sixth-order frequency equation in terms of various $\Omega^2$.

Investigate additional boundary conditions.

Rigorously, explore applications of approximation methods which may be used further reduce and simplify the characteristic frequency equation.

Employ computer software to analyze the characteristic equation obtained in this thesis for numerical data relative to the complex parameters and graphical data plotted in the complex plane.

Continuously pursue computer programs or software packages with the capabilities of directly solving the frequency equation derived in this investigation.
REFERENCES


APPENDIX A

Solution of the Cubic Equation
Solution of the Cubic Equation

For the general cubic expression \( r^3 + a_1 r^2 + a_2 r + a_3 = 0 \) the solutions are as follows:

\[
\begin{align*}
    r_1 &= S + T - \frac{1}{3}a_1 \\
    r_2 &= -\frac{1}{3}(S + T) - \frac{1}{3}a_1 + \frac{i}{3}\sqrt{3}(S - T) \\
    r_3 &= -\frac{1}{3}(S + T) - \frac{1}{3}a_1 - \frac{i}{3}\sqrt{3}(S - T)
\end{align*}
\]

for

\[
\begin{align*}
Q &= 3a_2 - a_1^2 \\
R &= \frac{9a_4 - 27a_2 - 2a_1^2}{54}
\end{align*}
\]

\[
\begin{align*}
    S &= \sqrt[3]{R + \sqrt{Q^3 + R^2}} \\
    T &= \sqrt[3]{R - \sqrt{Q^3 + R^2}}
\end{align*}
\]

If \( a_1, a_2, a_3 \) are real and if the discriminant is \( D = Q^3 + R^2 \) then

(i) \( D > 0 \) one root is real and two complex conjugate

(ii) \( D = 0 \) all roots are real and at least two are equal

(iii) \( D < 0 \) all roots are real and unequal

The appropriate values for \( a_4, a_2, \) and \( a_0 \) can be substituted into the cubic expression giving
\[ a_1 = G(c + (1/h_2 h_3 E_3)) \]

\[ a_2 = (\rho \omega^2)/EI \]

\[ a_3 = (G\rho \omega^2)/(h_2 h_3 E_3 EI) \]

The cubic expression representing the governing equation for the sixth-order beam problem is thusly written

\[(s^2)^3 + G(c + (1/h_2 h_3 E_3))(s^2)^2 + (\rho \omega^2)/EI(s^2) + (G\rho \omega^2)/(h_2 h_3 E_3 EI) \quad (39)\]

We receive the three roots of (39) as \( l^2, m^2, \) and \( n^2 \) respectively.

\[ l^2 = r_1 \]
\[ m^2 = r_2 \]
\[ n^2 = r_3 \]
APPENDIX B

Properties of Rubber
(Source: Shock and Vibration Handbook)
DYNAMIC PROPERTIES OF RUBBER

MODULUS AND DAMPING. Rubber is not perfectly elastic—it exhibits internal damping and its stiffness tends to increase as the frequency of loading is increased. The action of rubber can be represented by an idealized mathematical model to which the measured performance can be compared. The idealized behavior most nearly approximating that of rubberlike materials is known as linear viscoelastic behavior. The mechanical model used for mathematical derivation is shown in Fig. 35.6. (Also see Chap. 29 for a discussion of electrical models.) If this model is subjected to a sinusoidal force \( F = F_0 \sin \omega t \), the response is given by

\[
x = \frac{F_0}{k_1 + k_2} \left[ \sin \omega t \frac{k_2 \sin \omega t}{k_2(1 + \omega^2 \psi^2)} + \frac{k_2 \omega \sin \omega t}{k_2(1 + \omega^2 \psi^2)} \right]
\]

(35.2)

where \( k_1 \) is the stiffness of spring 1, \( k_2 \) is the stiffness of spring 2, \( c \) is the viscous damping coefficient, and

\[
\psi = \frac{\omega}{k_2}(1 + \frac{1}{k_1 + k_2})
\]

(35.3)

The terms in brackets in Eq. (35.2) represent components of the model response. The first term represents the ordinary elastic response which is in-phase with the force and is independent of frequency; the second term is also in-phase with the force, but is frequency-dependent and represents the elastic component of viscoelastic response; the third term is frequency-dependent and is 90° out-of-phase with the force; it is responsible for the energy losses.

The term dynamic modulus is used to indicate the mean stress-strain ratio of a rubber specimen during a period of dynamic stress. A dynamic modulus measurement may be considered as one made under repeated stress cycles at a frequency of over 6 cpm. As frequency is increased from zero, the dynamic modulus of a rubber compound increases rather rapidly at first and then reaches a plateau in which it remains relatively constant or increases slightly with increased frequency. The frequency at which the plateau area begins varies from 2 to 20 cps, depending on the polymer used, compounding ingredients, and specimen temperature. Above

![Fig. 35.7. The effect of amplitude of deformation on the dynamic modulus of an SBR tread stock. (After Gehman.19)](image-url)
per at progressively rubber a dynamic effect increase in modulus.

An increase in the amplitude of deformation applied to a rubber specimen causes a decrease in dynamic modulus. This decrease may be as much as 25 per cent during the first 2.5 per cent of deformation, but thereafter is small. Figure 35.7 illustrates this phenomenon. These results were obtained by measuring the dynamic modulus of a specimen at a very low amplitude in compression, stopping the test machine, increasing the amplitude slightly, and repeating the measurement until a strain of 2.3 per cent was reached; then measurements were made at progressively smaller amplitudes of deformation, producing the results shown in the return curves. Increasing the filler loading of a compound causes an increase in the dynamic modulus.

The dynamic load-deflection curve of a typical rubber isolator is shown in Fig. 35.8.

The per cent damping $\delta$ of such an isolator is defined as

$$\delta = \frac{S_1}{S} \times 100 \quad \text{per cent} \quad (35.4)$$

where $S_1$ represents the area within the hysteresis loop and $S$ represents the area under the loading curve (top part of loop). Most rubber compounds show a definite increase in damping as frequency is increased from zero. As the frequency is further increased, the increase in damping becomes progressively smaller and damping may become essentially constant over a broad frequency range. Above $10^4$ cps, damping increases and reaches a maximum at approximately $10^6$ cps.

As temperature is increased from normal, the damping of most rubberlike materials decreases somewhat. As temperature is decreased from normal, a marked increase in damping occurs. Damping reaches a maximum at the second-order transition temperature. In general, damping in rubber compounds is more sensitive to temperature than is the dynamic modulus.

Damping decreases somewhat with increased specimen elongation, but this effect is small. Damping is decreased as the modulus of a particular compound is increased by increasing its molecular weight. This effect is illustrated in Fig. 35.9. The opposite effect is shown (damping is increased) if the modulus increase is obtained by increased filler loading. Damping is influenced more by filler loading than by dynamic modulus.

A great number of test methods and machines have been used to measure the dynamic properties of rubber. These have varied considerably in frequency and deformation applied to the sample. A method which is widely used employs a mechanical oscillograph. A specimen 0.50 in. (12.7 mm) 10 cgs, the modulus increases to a much higher level. High temperatures decrease dynamic modulus slightly, and low temperatures cause a rapid increase in stiffness (as in the case of static modulus).

![Fig. 35.8. Dynamic load-deflection curve of a rubber isolator.](image)

![Fig. 35.9. Dynamic resilience vs dynamic modulus for several gum rubber compounds. (From unpublished work of K. E. Gus.)](image)
in diameter and 0.75 in. (19.1 mm) high is located in a manner such that when an unbalanced beam is released, its fall is resisted by the specimen and a free vibration occurs. A chronograph records the free oscillations. Then, the dynamic resilience, computed from the record, is given by

$$\frac{\delta_f}{\delta_1} \times 100 \text{ per cent}$$  \hspace{1cm} (35.5)

where $\delta_f$ is the recorded deflection of the beam during its first downward cycle and $\delta_1$ is the recorded deflection during the first rebound cycle. The frequency of oscillation usually is a few cycles per second. Impact resilience may be determined by means of the Bashore Resiliometer, in which a small weight is dropped on a rubber sample and the rebound height is noted. Many similar methods are in use. Figures 35.10.A and 35.10.B illustrate the effect of temperature on the resilience of various polymers at low and high frequency, respectively, as determined from the tests described.

The ratio between two adjacent displacement peaks of a free vibration may be expressed as

$$\frac{\delta_2}{\delta_1} = \exp \left[ \frac{-2\pi (c/c_e)}{[1 - (c/c_e)^2]^{1/2}} \right]$$  \hspace{1cm} (35.6)

where $c/c_e$ is the ratio of the viscous damping coefficient to the coefficient for critical damping, or the fraction of critical damping. Illum shows computation of $c/c_e$.

* The designation $\exp [A]$ is used here for $e^A$. 

Fig. 35.10. The effect of temperature on the resilience of several rubber compounds as measured by (A) the mechanical oscillograph and (B) the Bashore Resiliometer.
the damping ratio of rubber compounds from resilience measurements. Representative values of the fraction of critical damping are given in Table 35.4. The range given for each polymer is necessary because of differences in properties brought about by hardness and by different compounding ingredients, and does not include the effect of temperature.

**Fatigue.** A rubber specimen subjected to a tensile stress approaching its tensile strength will continue to elongate with time and eventually will rupture. This process is known as static fatigue; it is the end result of the creep process. Dynamic fatigue occurs in a specimen subjected to an alternating stress centered about zero. In most vibration isolators, the fatigue process is some combination of static and dynamic fatigue. The results of an investigation of the dynamic fatigue characteristics of 50 durometer isolators strained in tension and compression are shown in Fig. 35.11. Dynamic fatigue life is plotted as a function of the per cent minimum strain, for fixed values of dynamic strain. The latter two parameters are computed in the following way. The per cent minimum strain is

\[
\frac{l_{\text{min}} - l_0}{l_0} \times 100 \% \quad (100)
\]

(35.7)

![Diagram](image_url)

* SEE EQ (35.7)

** SEE EQ (35.8)

Fig. 35.11. The effect of strain on fatigue life of rubber specimens tested in tension and compression. *(After Caldwell, Merrill, Slioman, and Yeat)*
and the per cent dynamic strain is
\[
\frac{l_{\text{max}} - l_{\text{min}}}{l_0} \times 100 \quad \text{per cent}
\]
(35.8)

where \(l_0\) is the unstrained length of the specimen, \(l_{\text{min}}\) is the minimum strained length, and \(l_{\text{max}}\) is the maximum strained length. Figure 35.11 shows that (1) for small strains there is a pronounced minimum in the fatigue life when the sample is returned to zero at the end of the stroke; (2) the point of maximum fatigue life shifts toward lower minimum strain as the dynamic strain is increased; and (3) fatigue life decreases as dynamic strain is increased. This latter effect may be caused partially by the greater heat generated. Specimens tested in shear also have a minimum life when returned to zero strain at one end of the stroke. A summary of the fatigue life of shear specimens as a function of dynamic strain is given in Table 35.5.

### Table 35.5. Fatigue Life in Cycles of Shear Specimens as a Function of Dynamic Strain for Various Lateral Strains

<table>
<thead>
<tr>
<th>Dynamic strain, per cent</th>
<th>Lateral strain</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Zero</td>
</tr>
<tr>
<td>-25 to +25</td>
<td>7 \times 10^4</td>
</tr>
<tr>
<td>0 to 50</td>
<td>1 \times 10^4</td>
</tr>
<tr>
<td>75 to 125</td>
<td>13 \times 10^4</td>
</tr>
</tbody>
</table>

Stress concentrations reduce the fatigue life of rubber isolators. Sudden changes in rubber section, sharp-edged inserts, protruding boltheads, and weld flash should be avoided. In many cases, fatigue failures will develop inside the rubber at some distance from the sharp edge of a metal insert.

**DYNAMIC AGING.** Dynamic stresses accelerate most aging effects, because dynamic strain exposes new areas of the molecular structure to attack. The cracks which form in a rubber flex test specimen are caused by a combination of dynamic fatigue and oxidation or ozone attack. One mechanism of protection against ozone is the inclusion of a wax in the rubber compound. The solubility of the wax is low and it *blooms* (i.e., migrates to the surface) and forms a protective coating. Such a compound has poor ozone resistance under dynamic conditions because the wax film is broken. A rubber compound containing wax usually is not used for bonded vibration isolators, since wax has a detrimental effect on rubber-to-metal adhesion. Flexible coatings are sometimes used when rubber isolators are exposed to abnormal ozone concentrations. Another method of surface protection is surface chlorination, in which chlorine is added at the molecular double bonds. This treatment also greatly reduces the coefficient of friction of the rubber on smooth, dry hard surfaces.
APPENDIX C

Computer Analysis Programs and Output Data
MAPLE Program for Sixth-Order Differential Equation

de:=diff(diff(diff(diff(diff(y(x),x),x),x),x),x)-\(1^2-m^2-n^2\)*diff(diff(diff(diff(y(x),x),x),x),x)+\(1^2-m^2+1^2+n^2\)*diff(diff(diff(y(x),x),x),x)-\(1^2+m^2+n^2\)*y(x) = 0;

# boundary conditions

#
d1:=(A*1^2+B)/(\(1^2-m^2\))*(\(1^2-n^2\));
d2:=(C/l)/(\(1^2-m^2\))*(\(1^2-n^2\));
d3:=(A*m^2+B)/(\(m^2-1^2\))*(\(m^2-n^2\));
d4:=(C/m)/(\(m^2-1^2\))*(\(m^2-n^2\));
d5:=(A*n^2+B)/(\(n^2-1^2\))*(\(n^2-m^2\));
d6:=(C/n)/(\(n^2-1^2\))*(\(n^2-m^2\));

# first boundary condition--
bc1:=d1*l^5*sinh(l*L)+d2*l^5*cosh(l*L)+d3*m^5*sinh(m*L)+d4*m^5*cosh(m*L)-d5*n^5*sinh(n*L)+d6*n^5*cosh(n*L)-\(1^2-m^2+n^2\)*d1*l^3*sinh(l*L)*d2*l^3*cosh(l*L)*d3*m^3*sinh(m*L)*d4*m^3*cosh(m*L)+d5*n^3*sinh(n*L)*d6*n^3*cosh(n*L)+\(1^2-l^2+m^2+n^2\)*(d1*l^1*sinh(l*L)+d2*l^1*cosh(l*L)+d3*m^1*sinh(m*L)+d4*m^1*cosh(m*L)+d5*n^1*sinh(n*L)+d6*n^1*cosh(n*L))^2 = 0;

# second boundary condition--
bc2:=d1*l^4*cosh(l*L)+d2*l^4*sinh(l*L)+d3*m^4*cosh(m*L)+d4*m^4*sinh(m*L)-d5*n^4*cosh(n*L)+d6*n^4*sinh(n*L)-\(1^2-m^2+n^2\)*d1*l^2*cosh(l*L)+d2*l^2*sinh(l*L)+d3*m^2*cosh(m*L)+d4*m^2*sinh(m*L)+d5*n^2*cosh(n*L)+d6*n^2*sinh(n*L)+\(1^2-m^2+n^2\)*d1*l^0+cosh(l*L)+d2*l^0*sinh(l*L)+d3*m^0*cosh(m*L)+d4*m^0*sinh(m*L)+d5*n^0*cosh(n*L)+d6*n^0*sinh(n*L)= 0;

# third boundary condition--
bc3:=(d1*l^3*sinh(l*L)+d2*l^3*cosh(l*L)+d3*m^3*sinh(m*L)+d4*m^3*cosh(m*L)+d5*n^3*sinh(n*L)+d6*n^3*cosh(n*L))^3 = 0;
MAPLE Program for Sixth-Order Differential Equation

# Solution of the Cubic Equation
#
hl:=1.0;
h2:=0.1;
h3:=0.01;
E3=30*10^6
EI=2.5*10^6*t*(hl^3+2*h3^3)
p=7.45*10^-4*t*(hl+2*h3)
r=1/(h2*h3*e3)
c1=4.2112*10^-5
eta=0.55
Gl=800*(w/20)^.234*(1 + eta*I)
a4=-Gl*(cl+r);
a2=-z
a0=Gl*z*r;
#
# G=modulus of rigidity-- for viscoelastic case G is complex
# z=omega squared
# r=geometric parameter l/h2h3E3
#
Q=(3*a2 - a4^2)/9;
R=(9*a4*a2 - 27*a0 - 2*a4^3)/54;
S=(R + Sqrt[(Q^3+R^2)])^(1/3);
T=(R - Sqrt[(Q^3+R^2)])^(1/3);
#
r1=S+T -a4/3;
r2=-.5*(S+T) -a4/3 +.5I*Sqrt[3]*(S-T);
r3=-.5*(S+T) -a4/3 -.5I*Sqrt[3]*(S-T);

l=(r1)^0.5;
simplify("); 
evaluate(");
#

m=(r2)^0.5;
simplify("); 
evaluate(");
#
n=(r3)^0.5;
simplify("); 
evaluate(");
MATHEMATICAL PROGRAMS FOR CASE STUDY ONE
Mathematica Input File: \( \eta = 0.06 \)

\[
\begin{align*}
h_1 &= 1.0 \\
h_2 &= 0.1 \\
h_3 &= 0.01 \\
\gamma &= 1 \\
e_3 &= 30 \times 10^6 \\
e_i &= 2.5 \times 10^6 t (h_1^3 + 2h_3^3) \\
p &= 7.45 \times 10^{-4} t (h_1 + 2h_3) \\
r &= 1/(h_2 h_3 e_3) \\
cl &= 4.2112 \times 10^{-5} \\
\eta &= 0.06 \\
\theta &= \text{ArcTan}[\eta] \\
w_0 &= \sqrt{e_i/p(1+\eta^2) - 5E^2(I*\theta)} \\
w &= \gamma w_0 \\
g_1 &= 48.35(1 + \eta I) \\
A_4 &= -g_1(c_l + r) //N; \\
a_2 &= -p w/e_i //N; \\
a_0 &= g_1 r p w/e_i //N; \\
Q &= (3a_2 - a_4^2)/9; \\
R &= (9a_4 a_2 - 27a_0 - 2a_4^3)/54; \\
S &= (R + \sqrt{[Q^3+R^2]})^{1/3}; \\
T &= (R - \sqrt{[Q^3+R^2]})^{1/3}; \\
12 &= S + T - a_4/3 //N; \\
m_2 &= -0.5(S+T) - a_4/3 + 0.5I\sqrt{3}(S-T) //N; \\
n_2 &= -0.5(S+T) - a_4/3 - 0.5I\sqrt{3}(S-T) //N; \\
l &= \sqrt{12} \\
m &= \sqrt{m_2} \\
n &= \sqrt{n_2};
\end{align*}
\]
Mathematica Input File:  \( \eta = 0.24 \)

\[
\begin{align*}
h_1 &= 1.0 \\
h_2 &= 0.1 \\
h_3 &= 0.01 \\
gamma &= 1 \\
e_3 &= 30 \times 10^6 \\
ei &= 2.5 \times 10^6 t (h_1^3 + 2h_3^3) \\
p &= 7.45 \times 10^{-4} t (h_1 + 2h_3) \\
r &= 1/(h_2 h_3 e_3) \\
c_1 &= 4.2112 \times 10^{-5} \\
\eta &= .24 \\
\theta &= \text{ArcTan}[\eta] \\
w_0 &= \sqrt{e_i (1+\eta^2)/(l+\eta)} \\
w &= \gamma w_0 \\
z &= \sqrt{e_i/p (l+\eta^2) (l+\eta)/(l+\eta^2)} \\
g_1 &= 48.35 \times (1 + \eta) \\
a_2 &= -p w/e_i \\
a_0 &= g_1 (c_1 + r) \\
Q &= (3a_2 - a_4^2)/9 \\
R &= (9a_4 a_2 - 27a_0 - 2a_4^3)/54 \\
S &= (R + \sqrt{(Q^3 + R^2)})^{(1/3)} \\
T &= (R - \sqrt{(Q^3 + R^2)})^{(1/3)} \\
l_2 &= S + T - a_4/3 \\
m_2 &= -0.5(S+T) - a_4/3 + 0.5I*\sqrt{3}(S-T) \\
n_2 &= -0.5(S+T) - a_4/3 - 0.5I*\sqrt{3}(S-T) \\
l &= \sqrt{l_2} \\
m &= \sqrt{m_2} \\
n &= \sqrt{n_2}.
\end{align*}
\]
\[ h_1 = 1.0 \]
\[ h_2 = 0.1 \]
\[ h_3 = 0.01 \]
\[ \gamma = 1 \]
\[ e_3 = 30 \times 10^6 \]
\[ e_i = 2.5 \times 10^6 t (h_1^3 + 2 h_3^3) \]
\[ p = 7.45 \times 10^{-4} t (h_1^2 + 2 h_3) \]
\[ r = 1 / (h_2 h_3 e_3) \]
\[ c_1 = 4.2112 \times 10^{-5} \]
\[ \eta = 0.36 \]
\[ \theta = \arctan(\eta) \]
\[ w_0 = \sqrt{e_i (1 + \eta I) / p} \]
\[ w = \gamma w_0 \]
\[ z = \sqrt{e_i / p (1 + \eta^2 + 0.5 \eta I \theta)} \]
\[ g_1 = 48.35 (1 + \eta I) \]
\[ a_4 = -g_1 (c_1 + r) / N \]
\[ a_2 = -p w / e_i / N \]
\[ a_0 = g_1 r p w / e_i / N \]
\[ Q = (3 a_2 - a_4^2) / 9 \]
\[ R = (9 a_4^2 a_2 - 27 a_0 - 2 a_4^3) / 54 \]
\[ S = (R + \sqrt{Q^2 + R^2})^{(1/3)} \]
\[ T = (R - \sqrt{Q^2 + R^2})^{(1/3)} \]
\[ 12 = S + T - a_4 / 3 / N \]
\[ m_2 = -0.5 (S + T) - a_4 / 3 + 0.5 i \sqrt{3} (S - T) / N \]
\[ n_2 = -0.5 (S + T) - a_4 / 3 - 0.5 i \sqrt{3} (S - T) / N \]
\[ l = \sqrt{Q} \]
\[ m = \sqrt{m_2} \]
\[ n = \sqrt{n_2} \]
Mathematica Input File: \( \eta = 0.54 \)

\[
\begin{align*}
h_1 &= 1.0 \\
h_2 &= 0.1 \\
h_3 &= 0.01 \\
gamma &= 1 \\
e_3 &= 30 \times 10^6 \\
ei &= 2.5 \times 10^6 t (h_1^3 + 2h_3^3) \\
p &= 7.45 \times 10^{-4} t (h_1 + 2h_3) \\
r &= 1/(h_2 h_3 e_3) \\
c_1 &= 4.2112 \times 10^{-5} \\
et &= 0.54 \\
theta &= \text{ArcTan}[\eta] \\
w_0 &= \sqrt{\frac{e_i (1+\eta^2)}{p}} \\
w &= \text{gamma} \ w_0 \\
z &= \sqrt{\frac{e_i}{p} (1+\eta^2)} \ \frac{\eta}{\theta} \\
\frac{g_1}{48.35} &= (1 + \eta^2) \\
a_4 &= -g_1 (c_1 + r) \ //N; \\
a_2 &= -p w / e_i \ //N; \\
a_0 &= g_1 r p w / e_i \ //N; \\
Q = (3 a_2 - a_4^2) / 9; \\
R &= (9 a_4 a_2 - 27 a_0 - 2 a_4^3) / 54; \\
S &= (R + \sqrt{[Q^3 + R^2]})^{(1/3)}; \\
T &= (R - \sqrt{[Q^3 + R^2]})^{(1/3)}; \\
12 &= S + T - a_4 / 3 \ //N; \\
m_2 &= -0.5 (S + T) - a_4 / 3 + 0.5 \sqrt{3} (S - T) \ //N; \\
n_2 &= -0.5 (S + T) - a_4 / 3 - 0.5 \sqrt{3} (S - T) \ //N; \\
l &= \sqrt{12} \\
m &= \sqrt{m_2} \\
n &= \sqrt{n_2};
\end{align*}
\]
Mathematica Input File:  \( \eta = 0.96 \)

\[
\begin{align*}
  h_1 &= 1.0 \\
  h_2 &= 0.1 \\
  h_3 &= 0.01 \\
  \gamma &= 1 \\
  e_3 &= 30 \times 10^6 \\
  e_i &= 2.5 \times 10^6 t (h_1^3 + 2h_3^3) \\
  p &= 7.45 \times 10^{-4} t (h_1 + 2h_3) \\
  r &= 1/(h_2 h_3 e_3) \\
  c_1 &= 4.2112 \times 10^{-5} \\
  \eta &= 0.96 \\
  \theta &= \arctan(\eta) \\
  w_0 &= \sqrt{e_i (1 + \eta I)/p} \\
  w &= \gamma w_0 \\
  z &= \sqrt{e_i/p (1 + \eta I)} \\
  g_1 &= 48.35 (1 + \eta I) \\
  a_4 &= -g_1 (c_1 + r) \quad //N; \\
  a_2 &= -p w / e_i \quad //N; \\
  a_0 &= g_1 r p w / e_i \quad //N; \\
  Q &= (3a_2 - a_4^2)/9; \\
  R &= (9a_4 a_2 - 27a_0 - 2a_4^3)/54; \\
  S &= (R + \sqrt{(Q^3 + R^2)})^{1/3}; \\
  T &= (R - \sqrt{(Q^3 + R^2)})^{1/3}; \\
  l &= S + T - a_4/3 \quad //N; \\
  m_2 &= 0.5 (S + T) - a_4/3 + 0.5 I \sqrt{3} (S - T) \quad //N; \\
  n_2 &= 0.5 (S + T) - a_4/3 - 0.5 I \sqrt{3} (S - T) \quad //N; \\
  \eta &= \sqrt{l} \\
  m &= \sqrt{m_2} \\
  n &= \sqrt{n_2};
\end{align*}
\]
MATHEMATICAL PROGRAMS FOR CASE STUDY TWO
Mathematica Input File:  \( h_4 = 0.025 \)

\[
\begin{align*}
\text{h1} &= 1.0 \\
\text{h2} &= 0.025 \\
\text{h3} &= 0.01 \\
\gamma &= 1 \\
\text{e3} &= 30 \times 10^6 \\
ei &= 2.5 \times 10^6 t \times (h1^3 + 2h3^3) \\
p &= 7.45 \times 10^{-4} t \times (h1 + 2h3) \\
r &= 1 / (h2 \times h3 \times e3) \\
c1 &= 4.2112 \times 10^{-5} \\
\eta &= .54 \\
\theta &= \text{ArcTan} \[ \eta \] \\
w0 &= \text{Sqrt} \[ ei \times (1 + \eta I) / p \] \\
w &= \text{gamma} \times w0 \\
z &= \text{Sqrt} \left[ ei / p \times (1 + \eta I)^2 \right] \\
g1 &= 48.35 \times (1 + \eta I) \\
a4 &= -g1 \times (c1 + r) \div N; \\
a2 &= -p \times w / ei \div N; \\
a0 &= g1 \times r \times p \times w / ei \div N; \\
Q &= (3 \times a2 - a4^2) / 9; \\
R &= (9 \times a4 \times a2 - 27 \times a0 - 2 \times a4^3) / 54; \\
S &= (R + \text{Sqrt} \left[ (Q^3 + R^2) \right])^{1/3}; \\
T &= (R - \text{Sqrt} \left[ (Q^3 + R^2) \right])^{1/3}; \\
l2 &= S + T - a4 / 3 \div N; \\
m2 &= -0.5 \times (S + T) - a4 / 3 + 0.5 \times \text{Sqrt} \left[ 3 \right] \times (S - T) \div N; \\
n2 &= -0.5 \times (S + T) - a4 / 3 - 0.5 \times \text{Sqrt} \left[ 3 \right] \times (S - T) \div N; \\
l &= \text{Sqrt} \left[ l2 \right]; \\
m &= \text{Sqrt} \left[ m2 \right]; \\
n &= \text{Sqrt} \left[ n2 \right];
\end{align*}
\]
Mathematica Input File:  \( h_2 = 0.1 \)

\[
\begin{align*}
  &h_1=1.0 \\
  &h_2=0.1 \\
  &h_3=0.01 \\
  &\text{gamma} = 1 \\
  &e_3 = 30 \times 10^6 \\
  &e_i = 2.5 \times 10^6 t \times (h_1^3 + 2h_3^3) \\
  &p = 7.45 \times 10^{-4} t \times (h_1 + 2h_3) \\
  &r = 1/(h_2^3e_3) \\
  &c_1 = 4.2112 \times 10^{-5} \\
  &\text{eta} = 0.54 \\
  &\text{theta} = \text{ArcTan}[\text{eta}] \\
  &w_0 = \sqrt{e_i \times (1 + \text{eta I}) / p} \\
  &w = \text{gamma} \times w_0 \\
  &z = \sqrt{e_i / p \times (1 + \text{eta}^2)^{0.5} \times (1 + \text{theta})} \\
  &c_1 = 48.35 \times (1 + \text{eta I}) \\
  &a_4 = -c_1 \times (c_1 + r) \quad /N; \\
  &a_2 = -p \times w / e_i \quad /N; \\
  &a_0 = c_1 \times r \times p \times w / e_i \quad /N; \\
  &Q = (3 \times a_2 - a_4^2) / 9; \\
  &R = (9 \times a_4 \times a_2 - 27 \times a_0 - 2 \times a_4^3) / 54; \\
  &S = (R + \text{Sqrt}[(Q^3 + R^2)]) \times (1/3); \\
  &T = (R - \text{Sqrt}[(Q^3 + R^2)]) \times (1/3); \\
  &l_2 = S + T - a_4 / 3 \quad /N; \\
  &m_2 = -0.5 \times (S + T) - a_4 / 3 + 0.5 \times \text{Sqrt}[3] \times (S - T) \quad /N; \\
  &n_2 = -0.5 \times (S + T) - a_4 / 3 - 0.5 \times \text{Sqrt}[3] \times (S - T) \quad /N; \\
  &l = \text{Sqrt}[l_2] \\
  &m = \text{Sqrt}[m_2] \\
  &n = \text{Sqrt}[n_2];
\end{align*}
\]
Mathematica Input File: \( h_2 = 0.3 \)

\( h_1 = 1.0 \)
\( h_2 = 0.3 \)
\( h_3 = 0.01 \)
\( \text{gamma} = 1 \)
\( e_3 = 30 \times 10^{-6} \)
\( e_i = 2.5 \times 10^{-6} t \times (h_1^3 + 2h_3^3) \)
\( p = 7.45 \times 10^{-4} t \times (h_1 + 2h_3) \)
\( r = 1 / (h_2^2 h_3^3 e_3) \)
\( c_1 = 4.2112 \times 10^{-5} \)
\( \eta = 0.54 \)
\( \theta = \text{ArcTan}[\eta] \)
\( w_0 = \sqrt{e_i (1 + \eta I) / p} \)
\( w = \text{gamma} \times w_0 \)
\( z = \sqrt{\eta I \times p / (1 + \eta I)} \times 5 \times E^{(I \times \theta)} \)
\( g_1 = 48.35 \times (1 + \eta I) \)
\( a_4 = g_1 \times (c_1 + r) / N \)
\( a_2 = -p \times w / e_i / N \)
\( a_0 = g_1 \times r \times p \times w / e_i / N \)
\( Q = (3 \times a_2 - a_4^2) / 9 \)
\( R = (9 \times a_4 \times a_2 - 27 \times a_0 - 2 \times a_4^3) / 54 \)
\( S = (R + \sqrt{(Q^3 + R^2)}) \times (1/3) \)
\( T = (R - \sqrt{(Q^3 + R^2)}) \times (1/3) \)
\( l_2 = S + T - a_4 / 3 / N \)
\( m_2 = -0.5 \times (S + T) - a_4 / 3 + 0.5 \times \sqrt{3} \times \sqrt{3} + (S - T) / N \)
\( n_2 = -0.5 \times (S + T) - a_4 / 3 - 0.5 \times \sqrt{3} \times \sqrt{3} + (S - T) / N \)
\( l = \sqrt{l_2} \)
\( m = \sqrt{m_2} \)
\( n = \sqrt{n_2} \)
Mathematica Input File:  h_4 = 0.5

h1=1.0
h2=0.5
h3=0.01
gamma=1
e3=30*10^6
ei=2.5*10^6*t*(h1^3+2*h3^3)
p=7.45*10^-4*t*(h1+2*h3)
r=1/(h2*h3*e3)
c1=4.2112*10^-5
eta=.54
theta=ArcTan[eta]
w0=Sqrt[ei*(1+eta I)/p]
w=gamma*w0
z=Sqrt[ei/p*(1+eta^2)^.5*E^(I*theta)]
g1=48.35*(1 + eta I)
a4=-g1*(c1+r) //N;
a2=-p*w/ei //N;
a0=g1*r*p*w/ei //N;
Q=(3*a2 - a4^2)/9;
R=(9*a4*a2 - 27*a0 - 2*a4^3)/54;
S=(R + Sqrt[(Q^3+R^4)])^(1/3);
T=(R - Sqrt[(Q^3+R^4)])^(1/3);
l2=S+T -a4/3 //N;
m2=-.5*(S+T) -a4/3 +.5*I*Sqrt[3]*(S-T) //N;
n2=-.5*(S+T) -a4/3 -.5*I*Sqrt[3]*(S-T) //N;
l=Sqrt[l2]
m=Sqrt[m2]
n=Sqrt[n2];
Mathematica Input File:  \( h_2 = 0.8 \)

\[
\begin{align*}
h_1 &= 1.0 \\
h_2 &= 0.8 \\
h_3 &= 0.01 \\
gamma &= 1 \\
e_3 &= 30 \times 10^6 \\
ei &= 2.5 \times 10^6 t (h_1^3 + 2h_3^3) \\
p &= 7.45 \times 10^{-6} t (h_1^2 + 2h_3^2) \\
r &= 1/(h_2h_3e_3) \\
c_1 &= 4.2112 \times 10^{-5} \\
et &= 0.54 \\
theta &= \text{ArcTan} [\eta] \\
w_0 &= \text{Sqrt} [\text{ei} \times (1+\eta I)/p] \\
w &= \text{gamma} \times w_0 \\
z &= \text{Sqrt} [\text{ei}/p \times (1 + \eta^2)^{1.5} \times (1 + \eta I)] \\
g_1 &= 48.35 \times (1 + \eta I) \\
a_4 &= -g_1 \times (c_1 + r) /N; \\
a_2 &= -p \times w_0/ei//N; \\
a_0 &= g_1 \times r \times p \times w_0/ei //N; \\
Q &= (3a_2 - a_4^2)/9; \\
R &= (9a_4^2a_2 - 27a_0 - 2a_4^3)/54; \\
S &= (R + \text{Sqrt} [(Q^3R^2)])^{1/3}; \\
T &= (R - \text{Sqrt} [(Q^3R^2)])^{1/3}; \\
l_2 &= S + T - a_4/3 /N; \\
m_2 &= -0.5 \times (S + T) - a_4/3 + 0.5 \times \text{Sqrt} [3] \times (S - T) /N; \\
n_2 &= -0.5 \times (S + T) - a_4/3 - 0.5 \times \text{Sqrt} [3] \times (S - T) /N; \\
l &= \text{Sqrt}[l_2]; \\
m &= \text{Sqrt}[m_2]; \\
n &= \text{Sqrt}[n_2];
\end{align*}
\]