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Deflections of a Ring Due to Normal Loads Using Energy Method and Stiffness Matrix Method

A.J. Rabinowitz

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DEFLECTIONS OF A RING DUE TO NORMAL LOADS USING ENERGY METHOD AND STIFFNESS MATRIX METHOD

by

A. J. Rabinowitz

A Thesis Submitted
in
Partial Fulfillment
of the
Requirements for the Degree of
MASTER OF SCIENCE
in
Mechanical Engineering

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Deflections and reactions of a ring of 1.855" mean diameter, fixed at one point, simply supported at another point and loaded normally at an arbitrary point were determined using the energy method and the stiffness matrix method. These results were verified experimentally and it was found that the values of displacements and reactions obtained by the two theoretical methods were within 4.5% and 0.5% respectively of experimental results. When effect of transverse shear was included in the computations done using the energy method it increased the values of deflections by less than 1.5% thus confirming that the effect of transverse shear is negligible on deflections of a normally loaded ring. Finally, the investigation established that either the energy method or the stiffness matrix method, can be successfully used to predict deflections and reactions of an arbitrarily normally loaded ring.
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\[ J = \text{polar moment of inertia} \]

\[ k_{ij} = \text{stiffness coefficient, force in the } i \text{ direction due to a unit displacement in the } j \text{ direction} \]

\[ k^c_{ij} = \text{stiffness coefficient associated with constraint energy} \]

\[ k^s_{ij} = \text{stiffness coefficient associated with strain energy} \]

\[ [K] = \text{stiffness matrix, } n \times n \text{ matrix of stiffness coefficients} \]

\[ l = \text{length of beam} \]

\[ M_1, M_2, M_3 = \text{moments} \]

\[ N, S, X, Y, Z, C, L = \text{locations on ring} \]

\[ P, F_1, F_2, Q_s = \text{normal loads on ring} \]

\[ q_i = \text{generalized coordinates, displacements, } i = 1, 2, \ldots n \]

\[ \{q\} = \text{displacement vector, } n \times 1 \text{ column matrix of generalized displacements} \]

\[ q^a_i = \text{displacement in the } i \text{ direction of node } a \text{ of an element} \]

\[ q^b_i = \text{displacement in the } i \text{ direction of node } b \text{ of an element} \]

\[ \{q_s\} = n \times 1 \text{ column matrix of structural displacements, displacements at one end of an element with respect to structural orthogonal axes} \]

\[ \{q_e\} = n \times 1 \text{ column matrix of elemental displacements, displacements at one end of an element with respect to an orthogonal set of axes applicable to the particular element} \]

\[ Q_i = \text{generalized loads, loads in the } i \text{ direction} \]
\( \{Q\} \) = force vector, \( n \times 1 \) column matrix of generalized forces

\( Q_i^a \) = force in the \( i \) direction of node \( a \) of an element

\( Q_i^b \) = force in the \( i \) direction of node \( b \) of an element

\( \{Q_s\} \) = \( n \times 1 \) column matrix of forces and moments applied at one end of an element with respect to the coordinate system for the structure

\( \{Q_e\} \) = \( n \times 1 \) column matrix of forces and moments applied at one end of an element with respect to the coordinate system of the element

\( R \) = mean radius of curvature of circular beam or ring

\( [R_{s-e}] \) = rotation matrix which transforms a structural system into the elemental coordinate system

\( s \) = length along curved beam

\( T_1, T_2, T_3 \) = Torques

\( U \) = energy

\( U_s \) = strain energy

\( U_c \) = constraint energy

\( U_B \) = strain energy due to bending

\( U_T \) = strain energy due to torsion

\( U_{TS} \) = strain energy due to transverse shear

\( u_i \) = displacement in the \( i \) direction

\( \{u\} \) = displacement vector, \( n \times 1 \) column matrix of displacements

\( V_1, V_2, V_3 \) = vertical shearing forces

\( W \) = work

\( W_{1-7} \) = simplifying coefficients
x, y, z = position variables in a three dimensional rectangular coordinate reference system

\( X_1 - X_{11} \) = simplifying coefficients

\( \alpha, \beta, \gamma, \omega \) = angular locations on ring

\( \gamma_0 \) = variable angular location on ring

\( \delta_S, \delta'_S \) = deflections at location S on ring

\( \delta \) = deflection

\( \lambda_i \) = Lagrangian multiplier, reaction force in the i direction

\( \Theta_S, \Theta'_S \) = angular displacements, slope at location S on ring

\( \Theta_\theta \) = angular rotation of beam due to bending moment

\( \Theta_\phi \) = angular rotation of beam due to twisting torque

\( \tau \) = shearing stress

\( \nu \) = Poisson's ratio

\( \rho \) = shearing strain energy per unit volume
INTRODUCTION

A ring is universally defined as a curved bar or beam whose cross-sectional area dimensions are small in comparison with its radius of curvature.

Virtually every industry involved with the design of mechanical components of structures is faced with the need to analyze the stresses and strains in rings subjected to various combinations of loads. In submarines, for example, the hull is fabricated from a number of rings. These rings must be capable of withstanding radial forces due to shock and water pressure. The rings also have to withstand normal forces due to thermal expansions. Other examples of ring applications are to be found in frameworks of aircraft and missiles, bearings, radar, etc.

Further applications are found in the construction industry where rings are used as supporting foundations for reinforced concrete water tanks. Rings are also useful in the design of high temperature piping such as used in the engine room and reactor room of atomic submarines.

With such wide applications it becomes imperative to know, in detail, the state of stress and strain in ring structures.

Three methods can be used to determine displacements of rings subjected to normal loads. One method employs
energy principles, starting with Castigliano's second theorem (principle of least work) to find the external and internal redundants, and then employs either Castigliano's first theorem, or principle of virtual displacements to find the displacements of the ring.

A second method employs stress, strain, and equilibrium relationships to develop a differential equation. The equation is then solved in conjunction with the appropriate boundary conditions.

A third method, combines the principles and techniques of both aforementioned methods into a matrix oriented approach which is very useful in solving complex structures that are difficult to handle by exact analytical approaches. This method, which is particularly adaptable to computer calculations, is commonly referred to as the stiffness matrix approach, or direct stiffness method, or finite element method. It involves replacement of the continuous structure by one composed of finite elements. The stiffness matrix approach is relatively new; in fact it has only become increasingly popular in the last 20 years during the advent of the "computer age". Its popularity is particularly high in the aviation and space industries where it is used to analyze stresses and strains in numerous structures such as wings and missile components.
It is the objective of this thesis to investigate the deflections in a normally loaded ring by the classical energy method and the modern stiffness matrix method and further, to verify results obtained by actual experimentation.
Several authors have dealt with the problem of normally loaded rings. Volterra, for example, used methods of harmonic analysis to determine deflections of circular beams resting on an elastic foundation and loaded normally. He obtained his results in a closed form under the assumption that the foundation reacts following the classical Winkler-Zimmerman hypothesis. Rodriguez analyzed normally loaded rings by manipulating the classical equilibrium equations. Patel, Kamel and Reddy, have analyzed normally loaded curved beams but have not used their results to analyze rings specifically.

Levy used a combination of Castigliano's theorems and classical techniques of solid body mechanics to derive displacement coefficients for circular rings loaded normally. His analysis is limited to rings which are supported at opposite ends of a diameter.

The above authors neglected deformations produced by transverse shearing and/or limited their analysis to symmetrical loading. The analysis presented herein, includes effects of transverse shear, and deals with the use of two approaches mentioned in the introduction in solving symmetrical as well as unsymmetrical loadings.

* Superscripts refer to the references listed in Bibliography.
STATEMENT OF THE PROBLEM

We seek the displacements at various points in a normally loaded ring which is constrained at several points along its circumference by fixed and/or simple supports. The points of application of external forces are completely arbitrary. We also wish to determine the effect of transverse shear on the magnitude of displacements.

The results obtained by using an energy method will be compared with those obtained by using a stiffness matrix approach and with those obtained by experimental methods.

The theoretical basis for the solution of the problem is given below.
THEORY

When a conservative structure is subjected to forces which are applied gradually so that at any instant in the loading history the structure is in static equilibrium, the kinetic energy of the structure will be zero. All the work \( W \) done by the forces acting through certain displacements will be stored in the structure in the form of strain energy \( U \). This energy depends on the final configuration of the structure and not on the load-path history, or the manner in which the final configuration is reached \(^7,^8,^9\). If the force-displacement relationship for the structure under consideration is linear, the work done by force \( F_i \) when it causes a final displacement of \( u_i \) is given by

\[
W = \int_0^{u_i} F_i \, du_i = \int_0^{u_i} k_{ii} u_i \, du_i = \frac{1}{2} k_{ii} u_i^2 = \frac{1}{2} F_i u_i \quad (1)
\]

where \( k_{ii} \) equals the stiffness coefficient of the structure, or the force at location \( i \) due to a unit displacement at location \( i \). When the structure is acted on by a number of forces \( F_i \) through a number of displacements, the total strain energy can be represented by

\[
U = \frac{1}{2} \sum_i F_i u_i \quad (2)
\]

which in matrix form is \( U = \frac{1}{2} \{ u \}^t \{ F \}, \quad (3) \)

where \( \{ u \}^t \) is the transpose of \( \{ u \} \).
Since \( \{F\} = [K]\{u\} = \sum_j k_{ij} u_j \)

\[
U = \frac{1}{2} \{u\}^T [K]\{u\} = \frac{1}{2} \sum_i \sum_j k_{ij} u_i u_j \tag{4}\]

which can be written in the following form:

\[
U = \frac{1}{2} \begin{bmatrix} u_1 & u_2 & \ldots & u_i & \ldots & u_n \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} & \ldots & k_{1i} & \ldots & k_{1n} \\ k_{21} & k_{22} & \ldots & k_{2i} & \ldots & k_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_{i1} & k_{i2} & \ldots & k_{ii} & \ldots & k_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_{n1} & k_{n2} & \ldots & k_{ni} & \ldots & k_{nn} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_i \\ \vdots \\ u_n \end{bmatrix} \tag{4a}\]

If we take a partial derivative of the strain energy \( U \) with respect to displacement \( u_i \), the only nonzero terms in the multiplication of the right side of equation (4a) will be terms containing \( u_i \). Also, since \( k_{ij} = k_{ji} \)

\[
\frac{\partial U}{\partial u_i} = \sum_j k_{ij} u_j \tag{5}
\]

or

\[
\frac{\partial U}{\partial u_i} = F_i \tag{6}
\]

Equation (6) is a statement of Castigliano's first theorem.

It can also be expressed as follows:

\[
\frac{\partial U}{\partial q_i} = Q_i \tag{7}
\]

where \( q_i \) are generalized coordinates and \( Q_i \) are generalized forces. For an element which is loaded externally and which is held in equilibrium by reaction forces, the \( U \) in the above equation consists of strain energy plus constraint energy.
Equation (4) for strain energy can be expressed in quadratic form as

\[ U_s = \frac{1}{2} \sum_i \sum_j k_{ij}^s q_i q_j \text{ where } k_{ij}^s = k_{ji}^s \]  (8)

This form is positive definite since strain energy can assume only positive values for any deflections. \( k_{ij}^s \) represents the stiffness coefficient for the element, that is, the generalized force at a point acting in the i direction due to a unit displacement acting in the j direction. \( q_i \) represents generalized coordinates, that is, three translational and three rotational displacements from equilibrium position for each node in the structure.

The constraint energy can be expressed through the Lagrangian multiplier in a quadratic form as

\[ U_c = \frac{1}{2} \sum_i \sum j C_{ij} \lambda_i q_j \]  (9)

The constraints satisfy the equation

\[ \sum_j C_{ij} q_j = 0 \]  (9a)

which linearly involves generalized coordinates \( q_i \) and represents a constraint condition. The Lagrangian multiplier \( \lambda_i \) has the physical significance that it provides the generalized forces of reaction in the i direction on a constrained node. The coefficient \( C_{ij} \) represents the deflection at node i due to a deflection at node j.
The introduction of the constraint energy renders equation (7) as an independent set. One realizes that if the strain energy alone were used in equation (7), the partial derivatives would result in a dependent set of equations because all of the \( q_i \) would not be independent quantities.

A comparison of strain and constraint energy equations show that they both have the same form if the Lagrangian multiplier \( \lambda_i \) is considered as additional unknown generalized coordinate and which, therefore, can be called \( q_i \). Thus, the total potential energy in the structure can be expressed as

\[
U = U_c + U_s = \frac{1}{2} \sum_{ij} k_{ij} q_i q_j + \frac{1}{2} \sum_{ij} k^s_{ij} q_i q_j = \frac{1}{2} \sum_{ij} k_{ij} q_i q_j
\]

(10)

where \( k_{ij} \) now represents stiffness as well as constraint coefficients, and the \( q_i \) represent displacements as well as reactions.

Differentiating equation (10) with respect to \( q_k \),

\[
\frac{\partial U}{\partial q_k} = \frac{1}{2} \sum_i \sum_j \left\{ k_{ij} \frac{\partial q_i}{\partial q_k} + k_{ji} q_i \frac{\partial q_j}{\partial q_k} \right\}, \text{ since } k_{ij} = k_{ji}
\]

Replacing \( k_{ij} \) by \( \frac{\partial F_i}{\partial q_j} \) and \( k_{ji} \) by \( \frac{\partial F_j}{\partial q_i} \), the above equation becomes
\[
\frac{\partial U}{\partial q_k} = \frac{1}{2} \sum_i \sum_j \left\{ \frac{\partial F_i}{\partial q_k} \frac{\partial q_j}{\partial q_k} q_j + \frac{\partial F_i}{\partial q_j} \frac{\partial q_i}{\partial q_k} q_i \right\}
\]

= \frac{1}{2} \sum \sum \left\{ K_{ij} q_j + K_{ik} q_i \right\}

= \Sigma_j K_{ij} q_j

Or, restated,

\[
\frac{\partial U}{\partial q_i} = \Sigma_j K_{ij} q_j = Q_i
\]

or

\[
[K] \{q\} = \{Q\}
\]

In above, \([K]\) is a matrix of stiffness coefficients for the structure, \(\{q\}\) is a vector of generalized coordinates of any point in the structure with respect to the structural coordinate system, and \(\{Q\}\) is a vector of applied generalized force components of which are three orthogonal forces and three orthogonal moments applied to each point in the structure or three mutually perpendicular rotations initially applied to a constrained point with respect to the structural coordinate system.

Equation (2) can also be written in matrix form as

\[
U = \frac{1}{2} \{F\}^t \{u\}
\]

(13)

and since \(\{u\} = [a]\{F\}\)

\[
U = \frac{1}{2} \{F\}^t [a]\{F\}
\]

(14)
where $a_{ij}$ is the flexibility coefficient or the deflection in the $i$ direction due to a unit force in the $j$ direction.

If we take a partial derivative of the strain energy as represented by equation (14), with respect to any force $F_i$ (with all $F_i$ considered to be independent and with $a_{ij} = a_{ji}$) we obtain,

$$\frac{\partial U}{\partial F_i} = \sum_j a_{ij}F_j$$  \hspace{1cm} (15)

or

$$\frac{\partial U}{\partial F_i} = u_i$$  \hspace{1cm} (16)

which is an expression for Castigliano's second theorem.

We now proceed to apply the above general theory to the case of a ring.
Consider a force $P$ acting on a ring at angle $\psi$ and in a direction normal to plane $X-Y$ (see Figure 1).

The ring is fixed at point $S$ and is simply supported at point $N$. The angle $\psi_0$ is a variable angle. Displacements in the horizontal plane are considered to be small quantities of a high order and are therefore neglected.

We wish to find the vertical displacement of the ring at angle $\psi_0$ when $\psi_0$ lies between $\psi$ and $\alpha$.

Consider a free body diagram for the section to the right side of fixed support $S$. The resultant beam is a curved cantilever. We substitute a vertical force $F_1$ for the simple support. The force and the moment exerted by
the portion of the ring removed at S are represented by $F_2$ and $H_2$. The curved cantilever and applied forces are shown in Figure 2.

![Figure 2](image)

From equation (16), the vertical deflection $\delta_s$ at point S equals

$$\delta_s = \frac{\partial U_S}{\partial F_s}$$

(17)

where $F_s$ = the vertical force acting at point S.

$U_S$ = the strain energy transferred to the beam.

$$U_S = U_B + U_T + U_{TS}$$

(18)

$U_B$ = strain energy due to bending (Figure 3).

$U_T$ = strain energy due to torsion (Figure 4).

$U_{TS}$ = strain energy due to transverse shear (Figure 5).

Strain energy due to bending is given by
Strain energy due to torsional shear is given by

\[ U_t = \frac{1}{2} T \Theta_e \]
\[ = \frac{T^2 l}{2 JG} \]

since \( \Theta_e = \frac{T l}{JG} \)

-14-
Strain energy due to transverse shear is given by

\[
\frac{dU}{2} = \frac{T dy dx}{2} = \frac{T dx dy w x}{2G} = \frac{T^2 dx dy w}{2G}
\]  
\text{(21a)}

\text{Since } \tau = G \mu

From equation (21a), the shearing strain energy per unit volume can be expressed as

\[
\rho = \frac{T^2}{2G}
\]

Since shearing stress \tau at any point \( y \) when a vertical force \( V \) is acting is given by

\[
\tau = \frac{V}{2I_{zz}} \left( \frac{\mu^2}{4} - y^2 \right)
\]  
\text{(21b)}

the energy for the elemental volume \( b \ dy \ dx \) becomes

\[
\Delta U_{ts} = \frac{V^2}{8G I_{zz}} \left( \frac{\mu^2}{4} - y^2 \right) b \ dy \ dx
\]  
\text{(21c)}

\[
\therefore U_{ts} = \int_{-\frac{\mu}{2}}^{\frac{\mu}{2}} \int_{0}^{1} \frac{V^2}{8G I_{zz}} \left( \frac{\mu^2}{4} - y^2 \right) b \ dy \ dx
\]  
\text{(21d)}

\[
= \int_{0}^{1} \frac{V^2 \mu^2}{2G I_{zz}} \ dy
\]  
\text{(21e)}
To obtain the deflection at point S due to force $F_1$ and $P$ acting alone (neglecting $F_2$ and $H_2$) it is first necessary to assume an imaginary force $Q_s$ acting at S. The deflection at S will then be equal to the partial derivative with respect to $Q_s$ of the total energy in the beam (bending, torsional and transverse shear) contributed by $F_1$, $P$ and the imaginary force $Q_s$. Thus, from equations (17), (18), (19), (20) and (21e), the deflection at S due to force $P$ and $F_1$ can be written as

$$\delta_s = \int_0^\beta \frac{M_1}{EI} \frac{\partial M_1}{\partial Q_s} \, ds + \int_0^\beta \frac{M_2}{EI} \frac{\partial M_2}{\partial Q_s} \, ds + \int_0^\beta \frac{M_3}{EI} \frac{\partial M_3}{\partial Q_s} \, ds$$

$$+ \int_0^\beta \frac{T_1}{JG} \frac{\partial T_1}{\partial Q_s} \, ds + \int_0^\beta \frac{T_2}{JG} \frac{\partial T_2}{\partial Q_s} \, ds + \int_0^\beta \frac{T_3}{JG} \frac{\partial T_3}{\partial Q_s} \, ds$$

$$+ \int_0^\beta \frac{R^2 V_1}{10GI} \frac{\partial V_1}{\partial Q_s} \, ds + \int_0^\beta \frac{R^2 V_2}{10GI} \frac{\partial V_2}{\partial Q_s} \, ds + \int_0^\beta \frac{R^2 V_3}{10GI} \frac{\partial V_3}{\partial Q_s} \, ds$$

(22)

where

$$M_1 = Q_s \, R \sin \theta_0$$

$$M_2 = Q_s \, R \sin \theta_0 - F_1 \, R \sin (\theta_0 - \beta)$$

$$M_3 = Q_s \, R \sin \theta_0 - F_1 \, R \sin (\theta_0 - \beta) + PR \sin (\theta_0 - \beta)$$

$$T_1 = Q_s \, R \, (1-\cos \theta_0)$$

$$T_2 = Q_s \, R \, (1-\cos \theta_0) - F_1 \, R \, (1-\cos (\theta_0 - \beta))$$

$$T_3 = Q_s \, R \, (1-\cos \theta_0) - F_1 \, R \, (1-\cos (\theta_0 - \beta)) + PR \, (1-\cos (\theta_0 - \beta))$$
\[ V_1 = Q_s \]
\[ V_2 = Q_s - F_1 \]
\[ V_3 = Q_s - F_1 + P \]

Substituting \( M_1, M_2, M_3, T_1, T_2, T_3, V_1, V_2, V_3 \) into equation (22), replacing \( ds \) by \( R d\theta \) and integrating, the resulting deflection after setting \( Q_s \) to zero is given by

\[
\sum_s = -F_1 (D_1 + D_2 + D_3 + D_4) + P (D_5 + D_6 + D_7) \quad (23)
\]

where

\[
D_1 = \frac{R^3}{2EI} (-\beta \cos \beta + \sin \beta + \alpha \cos \beta - \cos \beta \sin \alpha \cos \alpha + \sin \beta \sin \alpha) \quad (24)
\]
\[
D_2 = \frac{R^3}{2EI} (\alpha \cos \alpha - \cos \alpha \sin \alpha \cos \alpha - \sin \alpha \sin \alpha - \beta \cos \alpha + \sin \alpha) \quad (25)
\]
\[
D_3 = \frac{R^3 R (\alpha - \beta)}{10 GI} \quad (26)
\]
\[
D_4 = \frac{R^3 R (\alpha - \beta)}{10 GI} \quad (27)
\]
\[
D_5 = \frac{R^3}{2EI} (\alpha \cos \alpha - \cos \alpha \sin \alpha \cos \alpha - \sin \alpha \sin \alpha - \beta \cos \alpha + \sin \alpha) \quad (28)
\]
\[
D_6 = \frac{R^3}{16} (\alpha - \sin \alpha \cos \alpha + \cos \alpha \sin \alpha + \alpha \cos \alpha + \sin \alpha \cos \alpha - \frac{1}{2} \cos \alpha - \frac{1}{2} \sin \alpha \cos \alpha - \frac{1}{2} \sin \alpha \sin \alpha) \quad (29)
\]
\[
D_7 = \frac{R^3 R (\alpha - \beta)}{10 GI} \quad (30)
\]

Using the same procedure, it is found that the deflection at \( S \) due to \( F_1 \) and \( H_2 \) acting alone is given by

\[
\sum_s' = F_2 (D_8 + D_9 + D_{10}) - H_2 (D_{11} + D_{12}) \quad (31)
\]

where

-17-
\[ D_8 = \frac{R^3}{2EI} \left( \alpha - \sin \alpha \cos \alpha \right) \]  

\( (32) \)

\[ D_9 = \frac{R^3}{JG} \left( \alpha - 2 \sin \alpha + \frac{\alpha \sin \alpha \cos \alpha}{2} \right) \]  

\( (33) \)

\[ D_{10} = \frac{R^2 R \alpha}{10GI} \]  

\( (34) \)

\[ D_{11} = \frac{R^2 \sin^2 \alpha}{2EI} \]  

\( (35) \)

\[ D_{12} = \frac{R^2}{JG} \left( - \cos \alpha - \frac{\sin \alpha}{2} + 1 \right) \]  

\( (36) \)

Furthermore, the slope at \( S \) due to \( F_1 \) and \( P \) acting alone is given by

\[ \Theta_s = F_1 \left( D_{13} + D_{14} \right) + P \left( D_{15} + D_{16} \right) \]  

\( (37) \)

where

\[ D_{13} = -\frac{R^2}{2EI} \left( \beta \sin \beta + \cos \beta \sin^2 \alpha - \alpha \sin \beta - \sin \beta \sin \alpha \cos \alpha \right) \]  

\( (38) \)

\[ D_{14} = -\frac{R^2}{2JG} \left( 2 \cos \beta - 2 \cos \alpha + \beta \sin \beta - \cos \beta \sin^2 \alpha - \alpha \sin \beta \right. \]  

\[ \left. + \sin \beta \sin \alpha \cos \alpha \right) \]  

\( (39) \)

\[ D_{15} = \frac{R^2}{2EI} \left( \cos \theta \sin^2 \alpha - \alpha \sin \theta - \sin \theta \sin \alpha \cos \alpha + \theta \sin \theta \right) \]  

\( (40) \)

\[ D_{16} = \frac{R^2}{2JG} \left( 2 \cos \theta - 2 \cos \alpha - \cos \theta \sin^2 \alpha + \sin \theta (3 - \alpha) + \sin \theta \sin \alpha \cos \alpha \right) \]  

\( (40a) \)

Also, the slope at \( S \) due to \( F_2 \) and \( H_2 \) acting alone is given by

\[ \Theta_s' = F_2 \left( D_{17} + D_{18} \right) - H_2 \left( D_{19} + D_{20} \right) \]  

\( (41) \)

where
\[ D_{17} = \frac{R^2 \sin^2 \alpha}{2EI} \]  
\[ D_{18} = \frac{R^2}{JG} \left( \cos \alpha + \frac{\sin^2 \alpha}{2} - 1 \right) \]  
\[ D_{19} = \frac{R}{2EI} \left( \alpha + \sin \alpha \cos \alpha \right) \]  
\[ D_{20} = \frac{R}{2JG} \left( \alpha - \sin \alpha \cos \alpha \right) \]  

We also know that
\[ \delta_s + \delta'_s = 0 \]  
\[ \Theta_s + \Theta'_s = 0 \]  
\[ \sum F_z = 0 \]

Substituting equations (23), (31), (37), (41) into equations (46), (47), and using the equilibrium equation (48), the constraints can be found to be

\[ F_1 = \frac{W_1 - P(W_2)}{W_3 + W_1} \]  
\[ F_2 = 1 - F_1 \]  
\[ H_2 = \frac{F_1(W_4) + P(W_5)}{W_7} \]

where

-19-
\[ W_1 = D_1 + D_2 + D_3 + D_4 \]  
\[ W_2 = D_5 + D_6 + D_7 \]  
\[ W_3 = D_8 + D_9 + D_{10} \]  
\[ W_4 = D_{13} + D_{14} \]  
\[ W_5 = D_{15} + D_{16} \]  
\[ W_6 = D_{17} + D_{18} \]  
\[ W_7 = D_{19} + D_{20} \]  

Once the constraints \( F_1 \), \( F_2 \) and \( H_2 \) are known, the deflection at \( S \) is given by

\[ \text{Def} = X_1 + X_2 + X_3 - X_4 - X_5 - X_6 + X_7 + X_8 + X_9 + X_{10} - X_{11} \]  

where,

\[ X_1 = \frac{F_AR^3}{2EI} \left[ \cos w (\alpha - \sin \alpha \cos \alpha) - \sin w \sin^2 \alpha \right] \]  
\[ - \cos w (w - \sin w \cos w) - \sin^3 \alpha \]  

\[ X_2 = \frac{P_R^3}{2EI} \left[ \cos \alpha \cos w (\alpha - \sin \alpha \cos \alpha) - \sin \alpha \cos \omega \sin^2 \alpha \right. \]  
\[ - \sin \omega \cos \omega \sin^2 \alpha + \sin \omega \sin (\omega - \sin \omega \cos \omega) \]  
\[ + \cos \omega \sin \omega \sin^2 \omega - \sin \omega \sin (w + \sin \omega \cos \omega) \]  

\[ X_3 = \frac{F_AR^3}{3EI} \left[ \alpha - \cos \omega \sin \alpha + \sin \omega \cos \omega \alpha - \sin \alpha \right. \]  
\[ + \cos \omega (\alpha + \sin \alpha \cos \omega) + \sin \omega \sin^2 \alpha - \omega \]  
\[ + \cos \omega \sin \omega - \sin \omega \cos \omega + \sin \omega - \cos \omega \left( \omega + \sin \omega \cos \omega \right) \]  
\[ - \frac{\sin \omega (\sin^2 \omega)}{2} \]
\[
x_4 = \frac{H_2 R^2}{E I} \left[ \frac{\cos w \sin^2 \alpha}{2} - \frac{\sin w (\alpha + \sin \alpha \cos \alpha)}{2} \\
- \frac{\cos w}{2} \sin^2 w + \frac{\sin w}{2} (w + \sin w \cos w) \right] \tag{63}
\]

\[
x_5 = \frac{P R^3}{2 J G} \left[ 2 \alpha - 2 \cos w \sin \alpha + 2 \cos \alpha \sin w - 2 \cos \beta \sin \alpha \\
+ \cos \beta \cos w (\alpha + \sin \alpha \cos \alpha) + \sin w \cos \beta \sin \alpha \\
+ \sin \beta \sin w (\alpha - \sin \alpha \cos \alpha + \sin \beta \cos w \sin^2 \alpha \\
+ 2 \sin \beta \cos \alpha \right] \tag{64}
\]

\[
x_6 = \frac{P R^3}{2 J G} \left[ -2w + 2 \cos w \sin w - 2 \cos w \sin w + 2 \cos \beta \sin w \\
- \cos \beta \sin w (w + \sin w \cos w) - \sin w \cos \beta \sin^2 w \\
- \sin \beta \sin w (w - \sin w \cos w) - \sin \beta \cos w \sin^2 w \\
- 2 \sin \beta \cos w \right] \tag{64a}
\]

\[
x_7 = \frac{PR^3}{2 J G} \left[ 2 \alpha - 2 \cos w \sin \alpha + 2 \sin w \cos \alpha - 2 \cos \beta \sin \alpha \\
+ \cos \beta \cos w (\alpha + \sin \alpha \cos \alpha) + \sin w \cos \beta \sin \alpha \\
+ \sin \beta \sin w (\alpha - \sin \alpha \cos \alpha + \sin \beta \cos w \sin^2 \alpha \\
+ \sin \beta \cos \alpha \right] \tag{65}
\]

\[
x_8 = - \frac{PR^3}{2 J G} \left[ 2w - 2 \cos w \sin w + 2 \sin w \cos w - 2 \cos \beta \sin w \\
+ \cos \beta \cos w (w + \sin w \cos w) + \sin w \cos \beta \sin^2 w \\
+ \sin \beta \sin w (w - \sin w \cos w + \sin \beta \cos w \sin^2 w \\
+ 2 \sin \beta \cos \alpha \right] \tag{66}
\]

\[
x_9 = \frac{H_2 R^2}{2 J G} \left[ -2 \cos \alpha \cos w \sin^2 \alpha - \sin w (\alpha - \sin \alpha \cos \alpha) \\
+ 2 \cos w - \cos w \sin^2 w - \sin w (w - \sin w \cos w) \right] \tag{67}
\]
\[ x_{10} = \frac{A^2 R}{10G} \left( F_2 - F_1 + p \right) (\alpha - w) \] (68)

\[ x_{11} = \frac{F_1 R^3}{2EI} \left[ \cos \beta \cos \omega (\alpha - \sin \alpha \cos \omega) - \sin \beta \cos w \sin^2 \alpha \\
- \sin \omega \cos \beta \sin^2 \alpha + \sin \beta \sin \omega (\alpha + \sin \alpha \cos \omega) \\
- \cos \beta \cos w (\omega - \sin \omega \cos \omega) + \sin \beta \cos w \sin^2 \omega \\
+ \sin \omega \cos \beta \sin^2 \omega - \sin \beta \sin \omega (\omega + \sin \omega \cos \omega) \right] \] (69)

In the foregoing expressions \( w \) represents a specific value of \( \theta_0 \). Next, we consider the theory for stiffness matrix method.
STIFFNESS MATRIX METHOD

In order to solve for displacements using the general theory stated in section IV with values of applied forces as known quantities, it would be necessary to use a stiffness matrix for the structure. Since it is very difficult to obtain such a stiffness matrix, one prefers to obtain a stiffness matrix for each element in the structure analytically and to incorporate these stiffness matrices for the elements into one overall matrix $[K]$ which can be used in equation (12). Thus for each element, the following equation can be written $^{12,13}$

$$
\begin{bmatrix}
[K_e] \\
\end{bmatrix}
\begin{bmatrix}
q_e \\
\end{bmatrix} =
\begin{bmatrix}
Q_e \\
\end{bmatrix}
$$

(70)

where $[K_e]$ is a matrix of stiffness coefficients for an element, $\{q_e\}$ is a vector of 6 deflections at each end of an element with respect to an orthogonal set of axes applicable to the particular element, and $\{Q_e\}$ is a vector of 6 components consisting of three orthogonal forces and three orthogonal moments applied at each end of an element with respect to the coordinate system for the element.

The assumptions used in defining $[K_e]$ are:

1) Deflections are small.
2) The physical properties of the elements are uniform.
3) The elements are elastic.
Since equation (70) is in terms of the element coordinate system, its terms must be transformed into the structural coordinate system in order to be consistent with equation (12). Using a rotation matrix which transforms one coordinate system into another, the following equations can be written:

$$\{q_e\} = [R_s-e] \{q_s\} \quad (71)$$
$$\{Q_e\} = [R_s-e] \{Q_s\} \quad (72)$$

where $[R_s-e]$ is a rotational matrix which transforms a structural coordinate system into the element coordinate system.

Substituting equations (71) and (72) in equation (70) will give:

$$[K_e] [R_s-e] \{q_s\} = [R_s-e] \{Q_s\} \quad (73)$$

Multiplying both sides of equation (73) by the inverse of $[R_s-e]$ which also happens to be the transpose of $[R_s-e]$ will give:

$$[R_s-e]^t [K_e] [R_s-e] \{q_s\} = [R_s-e]^t [R_s-e] \{Q_s\}$$

or

$$\{K_s\} \{q_s\} = \{Q_s\} \quad (74)$$

where $[K_s] = [R_s-e]^t [K_e] [R_s-e]$.
\[ [K_s] \] is a matrix of either stiffness or constraint coefficients for an element and will be referred to as an element stiffness matrix.

Since equation (74) now contains terms with respect to the structural coordinate system, it can be substituted into equation (12), resulting in a set of simultaneous linear equations. The \([K]\) matrix of equation (12) contains the \([K_s]\) matrices for all the elements and is called the structural matrix.

Solution of equation (12) for the displacements \(\{q\}\) can be obtained by employing any convenient method like matrix inversion, Gauss Jordon method etc. The solution may be formally written as
\[
\{q\} = [K]^{-1}\{Q\}
\] (75)

After the unknown displacements are obtained, the end reactions with respect to the structural coordinate system of each element can be determined by substituting the appropriate displacements into equation (74).

The above theory is applicable to any type of structure, i.e., it is general in nature. For a specific structure, one first has to decide on the shape of elements into which the structure will be broken up. In case of a ring, it can be readily broken up into small curved beams or into small straight beams. Curved elements would more closely idealize
the structure than the straight elements. However, the stiffness matrix of a curved beam is considerably more complex than that of a rectangular beam. With sufficiently large number of finite elements, a ring made up of straight rectangular beams should closely approach the actual configuration. Hence for the specific problem discussed in this thesis it was decided to break up the ring into straight rectangular beams.
Let us consider the rectangular beam shown in Figure 6.

Nodes a and b are located at the cross-sectional centroids of the left and right faces of the beam respectively. Displacements of node a and node b in relation to a reference set of orthogonal axes are represented by $q_i^a$ and $q_i^b$ where $i$ varies from 1 to 6.

$q_1^a$, $q_2^a$ and $q_3^a$ refer to the deflections of node a in the x, y, and z directions respectively, while $q_4^a$, $q_5^a$ and $q_6^a$ refer to the slopes or angular rotations of node a about the x, y and z axes respectively. Similarly, $q_1^b$, $q_2^b$, $q_3^b$ represent the deflections of node b in the x, y and z directions res-
pectively; and $q^b_4$, $q^b_5$ and $q^b_6$ represent angular rotations of node b about the x, y and z axes respectively.

Loads are represented by $Q_i$ where $i$ varies from 1 to 6. Loads $Q^a_1$, $Q^a_2$, $Q^a_3$, for example, represent forces acting on node a in the x, y and z directions respectively; and loads $Q^a_4$, $Q^a_5$, $Q^a_6$ represent moments acting at node a about the x, y and z axes respectively.

To develop the stiffness matrix for the beam, let us first consider axial loads acting on nodes a and b. The force displacement relationships are:

\[ Q^b_1 = \frac{AE}{L} (q^b_1 - q^a_1) \]  \hspace{1cm} (76)

\[ Q^b_4 = \frac{kG}{L} (q^b_4 - q^a_4) \]  \hspace{1cm} (77)

\[ Q^a_1 = \frac{AE}{L} (q^a_1 - q^b_1) \]  \hspace{1cm} (78)

\[ Q^a_4 = \frac{kG}{L} (q^a_4 - q^b_4) \]  \hspace{1cm} (79)

$(q^b_1 - q^a_1)$ represents the axial displacement of node b in relation to node a, or compressive strain of the beam.

$(q^b_4 - q^a_4)$ represents the angular rotation of node b in relation to node a, or torsional twist of the beam about its longitudinal axis.

$(q^a_1 - q^b_1)$ represents the axial displacement of node a in relation to node b, or the tensile strain of the beam.

$(q^a_4 - q^b_4)$ represents the angular rotation of node a in relation to node b, or torsional twist of the beam about its
longitudinal axis.

Now application of loads at node b such that the beam is displaced in the X-Y plane (Figure 7) leads to the following force-displacement relationships:

\[ q_2^b = q_2^a + L q_6^a + \frac{Q_2 L^2}{3EI} + \frac{Q_2 FL}{AG} + \frac{Q_6 L^2}{2EI} \]  \hspace{1cm} (80)

where

\[ q_2^b \]  is the total vertical or Y deflection of node b in relation to the structural reference coordinate system.
\( q^a_2 \) is the initial vertical deflection of node b, which would be equal to the vertical deflection of node a.

\( Lq^a_6 \) is the initial vertical deflection of node b contributed by the slope at node a. It is a product of the length of the beam and the slope at node a.

\( Q^b_2L^3/3EI \) is the vertical deflection of node b due to bending of the beam. That is, the vertical deflection at node b due to a vertical force \( Q^b_2 \) applied at node b while node a is fixed.

\( Q^b_6L^2/2EI \) is the vertical deflection of node b due to a moment \( Q^b_6 \) applied at node b while node a is fixed.

\( Q^b_2FL/AG \) is the vertical deflection of the beam due to transverse shear, which will be discussed later.

Furthermore

\[
q^b_6 = q^a_6 + \frac{Q^b_2L^2}{2EI_1} + \frac{Q^b_6L}{EI_2} \tag{81}
\]

where

\( q^b_6 \) is the total slope at node b after application of loads \( Q^b_2 \) and \( Q^b_6 \).

\( q^a_6 \) is the initial slope at node b which is equal to the slope at node a.

\( Q^b_2L^2/2EI \) is the slope at node b due to application of load \( Q^b_2 \) while node a is fixed.

\( Q^b_6L/EI \) is the slope at node b due to application of load \( Q^b_6 \) while node a is fixed.
Equations (80) and (81) may be written in the following matrix forms:

\[
\begin{bmatrix}
\phi^b - \phi^a - \phi^e \\
\phi^b - \phi^e
\end{bmatrix}
= \begin{bmatrix}
\frac{l}{3EI} \left( 1 + \frac{3EI}{l^2} \right) & \frac{l^3}{2EI} \\
\frac{l^2}{2EI} & \frac{l}{EI^2}
\end{bmatrix}
\begin{bmatrix}
Q^b \\
Q^e
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
Q^b \\
Q^e
\end{bmatrix}
= \begin{bmatrix}
\frac{12S_{z2}}{l^2} & -\frac{6S_{z1}}{l} \\
-\frac{6S_{z1}}{l} & 4S_{z2}
\end{bmatrix}
\begin{bmatrix}
\phi^b - \phi^a - \phi^e \\
\phi^b - \phi^e
\end{bmatrix}
\]  
(82)

or

\[
Q^b = \frac{-12S_{z1}\phi^a}{l^2} - \frac{6S_{z1}\phi^e}{l} + \frac{12S_{z1}\phi^e}{l^2} - \frac{6S_{z1}\phi^b}{l}
\]  
(83)

\[
Q^e = \frac{6S_{z2}\phi^a}{l^2} + 2S_{z2}\phi^e - \frac{6S_{z1}\phi^b}{l} + 4S_{z2}\phi^e
\]  
(84)

In the same manner as above, by considering the same configuration but with loads on node a we find that:

\[
Q^a = \frac{12S_{z1}\phi^a}{l^2} + \frac{6S_{z1}\phi^e}{l} - \frac{12S_{z1}\phi^e}{l^2} + \frac{6S_{z1}\phi^b}{l}
\]  
(85)

\[
Q^e = \frac{6S_{z1}\phi^a}{l} + 4S_{z2}\phi^e - \frac{6S_{z1}\phi^b}{l} + 2S_{z2}\phi^e
\]  
(86)

For analyzing deflection in the x-z plane, we can substitute the following changes in the nomenclature of equations (83), (84), (85), and (86):
\[
\begin{align*}
q_3^a &= q_2^a \\
\quad - q_5^a &= q_6^a \\
Q_3^a &= Q_2^a \\
Q_5^a &= -Q_6^a \\
q_3^b &= q_2^b \\
\quad - q_5^b &= q_6^b \\
Q_3^b &= Q_2^b \\
Q_5^b &= Q_6^b
\end{align*}
\]

which essentially implies replacing \( y \) by \( z \) and \( z \) by \(-y\).

From eq. (83)

\[
Q_3^b = \frac{-12S_{y1}q_3^a}{L^2} + \frac{6S_{y1}q_6^a}{L} + \frac{12S_{y1}q_3^b}{L^2} + \frac{6S_{y1}q_6^b}{L} 
\]  
(87)

From eq. (84)

\[
-\frac{Q_5^b}{L} = \frac{6S_{y1}q_3^b}{L} - 2S_{y3}q_5^a - \frac{6S_{y1}q_3^b}{L} - 4S_{y2}q_5^b 
\]  
(88)

From eq. (85)

\[
Q_3^a = \frac{12S_{y1}q_3^a}{L^2} - \frac{6S_{y1}q_6^a}{L} - \frac{12S_{y1}q_3^b}{L^2} - \frac{6S_{y1}q_6^b}{L} 
\]  
(89)

From eq. (86)

\[
-\frac{Q_5^a}{L} = \frac{6S_{y1}q_3^a}{L} - \frac{4S_{y2}q_5^a}{L} - \frac{6S_{y1}q_3^b}{L} - 2S_{y3}q_5^b 
\]  
(90)

Equations (76), (77), (78), (79), (83), (84), (85), (86), (87), (88), (89), (90), have twelve unknown displacements: \( q_1^a, q_2^a, q_3^a, q_4^a, q_5^a, q_6^a, q_1^b, q_2^b, q_3^b, q_4^b, q_5^b, q_6^b \).

These equations can be conveniently expressed in the following matrix form where the coefficient matrix becomes the stiffness matrix that we are looking for:
In above,

\[ S_t = \frac{AE}{L}; \quad S_{ts} = \frac{Ee}{L} \]

\[ S_y(z) = \frac{EI}{L} yy(z) \quad C_1; \quad C_1 = \frac{1}{1 + 4S_y(z)}; \quad C_2 = \frac{1 + S_y(z)}{1 + 4S_y(z)}; \quad C_3 = \frac{1 - 2S_y(z)}{1 + 4S_y(z)} \]

\[ S_{sy}(z) = \frac{3EI}{AGL^2} yy(z) f_y(z) \]
The above matrix equation is in effect equation (70) and it pertains to one element in the structure being analyzed. Each element in the structure will have a similar matrix equation. All of these equations can be converted into forms represented by equation (74) which in turn can be consolidated into one giant matrix equation represented by equation (12). The resultant stiffness matrix \([K_s]\) of equation (74), for example, can be considered as one element of the general stiffness matrix \([K]\) in equation (12); likewise, the displacement matrix and applied force matrix of equation (74) for one element can be considered as one element each of the general displacement matrix and applied force matrix of equation (12) respectively.

Thus, in a structure which is broken into \(n\) number of beams, the stiffness matrix of the general equation (12) expressing relationships of forces and displacement would be an \(n \times n\) matrix with each element in this matrix being a \(12 \times 12\) matrix. The displacement matrix and force matrix of equation (12) would be an \(n \times 1\) matrix and \(n \times 1\) matrix respectively, with each element in these matrices being a \(12 \times 1\) matrix.

To obtain displacements, equation (12) can be manipulated to provide:

\[
\begin{bmatrix} q \end{bmatrix} = \begin{bmatrix} K \end{bmatrix}^{-1} \begin{bmatrix} Q \end{bmatrix}
\]

Equation (91) can be solved using any of several methods.
such as the Gauss Jordon method to give displacements at any node in the structure.
APPLICATION OF THE TWO METHODS TO A SPECIFIC RING

We now proceed to apply the two methods of obtaining displacements viz. the Energy Method and Stiffness Matrix Method to a specific numerical example involving ring structure.

Consider a ring which is fixed at point S (Figure 9), simply supported at point L, normally loaded at point C. The normal load $P$ equals 1 lb. The physical properties and dimensions of the ring are taken as follows:

$D_0$ = Outer Diameter = 1.945 in.

$D_i$ = Inner Diameter = 1.765 in.

$R$ = Mean Radius = .927 in.

$E$ = Modulus of Elasticity = $30 \times 10^6$ lbs/in$^2$

$b$ = Thickness = .090 in.

$n$ = No. of Nodes or Elements Into Which Structure is Broken = 36

$I_{xx}$ = Moment of Inertia About the Element's $x$-$x$ Axis = $5.47 \times 10^{-6}$ in.$^4$

$I_{yy}$ = Moment of Inertia About the Element's $y$-$y$ Axis = $5.47 \times 10^{-6}$ in.$^4$

$I_{zz}$ = Polar Moment of Inertia of Element = $10.94 \times 10^{-6}$ in.$^4$

$f$ = Lateral Shear Shape Factor = 1.177

$e$ = Torsional Shear Shape Factor = $22 \times 10^{-6}$
A ring such as the one described above has been used as a supporting structure for a safety and arming device in proximity fuzes used in anti-aircraft projectiles, where deflections of supporting rings are critical.

Figure 9

The lateral shear shape factor $f$ is a dimensionless quantity, dependent on the shape of the cross-section, which is introduced to account for the fact that the strain, as determined by dividing the average shear stress on a cross-section by its shear modulus, is not uniformly distributed over the cross-section.

The lateral shear shape factor can be calculated from the formula

$$f = \frac{L + 11\nu}{10 + (1+\nu)}$$

where $\nu$ = the Poisson's ratio.

This formula was derived by Cowper because a similar shape coefficient derived by Timoshenko has led to unsatisfactory
results. Cowper's results agree fairly closely with other authors who have also questioned Timoshenko's shear factor and derived their own. Cowper derived his formula by integrating the equations of three dimensional elasticity theory.

The torsional shear shape factor \( e \) is a factor in the well known equation for angle of twist \([e.\text{q.}\, (20)]\).

\[
\Theta = \frac{Tl}{eG}
\]

where for circular cross-section, \( e \) is the polar moment of inertia \( J \). For other cross-sectional areas, this factor will change. For rectangular cross-sections, for example:

\[
e = ab^3 \left[ \frac{16}{3} - 3.36 \frac{b}{a} \left( 1 - \frac{b^4}{12a^4} \right) \right]
\]

\[
e = .1406a^4, \text{ if } a = b
\]

Deflection at point \( X \) on the ring shown in Figure 9 was obtained through the energy method by substituting the above constants into equation (22) and solving for \( \Delta \). To facilitate calculation, the problem was programmed and run on an 1800 MPX operating system. The language used was Basic Fortran. Running time for the program was nineteen seconds. A copy of the program is contained in the appendix.

Deflection and forces found by this method, are presented in Table I:
Table I

Deflection and Forces
(Energy Method - transverse shear included)

<table>
<thead>
<tr>
<th>Def. (in.) at X</th>
<th>F₁ (lb.)</th>
<th>F₂ (lb.)</th>
<th>H₂ (in.-lb.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>.0048279</td>
<td>.76019</td>
<td>.23980</td>
<td>.49055</td>
</tr>
</tbody>
</table>

The results when the terms that contribute to transverse shear are removed from equation (29) are given in Table II.

Table II

Deflection and Forces
(Energy Method - transverse shear neglected)

<table>
<thead>
<tr>
<th>Def. (in.) at X</th>
<th>F₁ (lb.)</th>
<th>F₂ (lb.)</th>
<th>H₂ (in.-lb.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>.0048202</td>
<td>.76008</td>
<td>.23991</td>
<td>.49047</td>
</tr>
</tbody>
</table>

To obtain a deflection at a similar point by means of the stiffness matrix method, the mean circumference of the ring is divided into 36 equal straight beams and the nodal points are designated as shown in Figure 10.
The origin of the coordinate system for the ring is established at node 36 with the x-y plane being coplanar with the plane of the ring.

The \((x,y)\) coordinates of \(j^{th}\) node in relation to the ring structure coordinate system may be expressed as:

\[
X_j = R - R \cos \left[ \frac{\pi}{18} j \right]
\]

\[
Y_j = R \sin \left[ \frac{\pi}{18} j \right]
\]

with \(j\) varying from 1 to 36.

Figure 10
Node 36 is considered as a ground node. All displacements at this node are therefore zero. At node 13 which is simply supported only, the deflection in the Z direction is zero. A force of one pound is applied at node 22.

To solve for deflections by the stiffness matrix approach, an existing computer program called the Strap 3 program is used. This program, which was developed at Eastman Kodak Company, is based on the mathematical formulation discussed earlier. The program consists of many subroutines, or modules, linked together by a monitoring system. These modules are written in Fortran IV for an IBM 360 computer. Deflections of the ring (Figure 4) obtained using the above program are shown in the following table:

**Table III**

**Deflections at Nodes - Stiffness Matrix Method**

(Due to 1 lb. Force at Node 22)

<table>
<thead>
<tr>
<th>Node Number</th>
<th>Vertical Deflections (in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-.0000385</td>
</tr>
<tr>
<td>2</td>
<td>-.0001496</td>
</tr>
<tr>
<td>3</td>
<td>-.0003218</td>
</tr>
<tr>
<td>4</td>
<td>-.0005348</td>
</tr>
<tr>
<td>5</td>
<td>-.0007661</td>
</tr>
<tr>
<td>6</td>
<td>-.0009907</td>
</tr>
<tr>
<td>7</td>
<td>-.0011720</td>
</tr>
<tr>
<td>Node Number</td>
<td>Vertical Deflections (in.)</td>
</tr>
<tr>
<td>-------------</td>
<td>---------------------------</td>
</tr>
<tr>
<td>8</td>
<td>-.0012850</td>
</tr>
<tr>
<td>9</td>
<td>-.0012999</td>
</tr>
<tr>
<td>10</td>
<td>-.0011942</td>
</tr>
<tr>
<td>11</td>
<td>-.0009483</td>
</tr>
<tr>
<td>12</td>
<td>-.0005536</td>
</tr>
<tr>
<td>13</td>
<td>.0000000</td>
</tr>
<tr>
<td>14</td>
<td>.0006903</td>
</tr>
<tr>
<td>15</td>
<td>.0014904</td>
</tr>
<tr>
<td>16</td>
<td>.0023595</td>
</tr>
<tr>
<td>17</td>
<td>.0032430</td>
</tr>
<tr>
<td>18</td>
<td>.0040942</td>
</tr>
<tr>
<td>19</td>
<td>.0048545</td>
</tr>
<tr>
<td>20</td>
<td>.0054827</td>
</tr>
<tr>
<td>21</td>
<td>.0059401</td>
</tr>
<tr>
<td>22</td>
<td>.0061826</td>
</tr>
<tr>
<td>23</td>
<td>.0061993</td>
</tr>
<tr>
<td>24</td>
<td>.0060071</td>
</tr>
<tr>
<td>25</td>
<td>.0056154</td>
</tr>
<tr>
<td>26</td>
<td>.0050737</td>
</tr>
<tr>
<td>27</td>
<td>.0044175</td>
</tr>
<tr>
<td>28</td>
<td>.0036939</td>
</tr>
<tr>
<td>29</td>
<td>.0036939</td>
</tr>
<tr>
<td>30</td>
<td>.0022336</td>
</tr>
<tr>
<td>Node Number</td>
<td>Vertical Deflections (in.)</td>
</tr>
<tr>
<td>-------------</td>
<td>---------------------------</td>
</tr>
<tr>
<td>31</td>
<td>0.0015607</td>
</tr>
<tr>
<td>32</td>
<td>0.0009930</td>
</tr>
<tr>
<td>33</td>
<td>0.0005485</td>
</tr>
<tr>
<td>34</td>
<td>0.0002342</td>
</tr>
<tr>
<td>35</td>
<td>0.0000548</td>
</tr>
<tr>
<td>36</td>
<td>0.0000000</td>
</tr>
</tbody>
</table>

The constraints were found to be:

\[
\begin{array}{ccc}
F_1 \text{ (lb.)} & F_2 \text{ (lb.)} & H_2 \text{ (in.-lb.)} \\
0.760989 & 0.239010 & 0.492000 \\
\end{array}
\]

Running time for the above program was 11 seconds.

Excerpts of the program are contained in the appendix. The entire program, which comprises over 150 pages of subprograms, is available at Eastman Kodak Company, Rochester, New York.
EXPERIMENTAL VERIFICATION

A test was conducted to verify deflections due to normal loads on the ring analyzed by the two methods. Normal loads were applied by a Chatillon push-pull meter, model DPP-30 at 3 locations. The experimental set-up was as shown in Figure 11.

The fixed point of the ring was held rigidly by a heavy duty vise. Points X, Y and Z and the simple support point were located by a toolmaker's microscope (Gaertner). The simple support consisted of a .060" diameter pin.

Deflections were read at points X, Y and Z upon application of a force at point C, using a Starrett Number 711
Results of the experimental test were as follows:

### Table IV

**Deflections - Experimental Method**

<table>
<thead>
<tr>
<th>Experimenterd Run Number</th>
<th>Normal Deflection/lb. (in./lb.)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pt. X</td>
</tr>
<tr>
<td>1</td>
<td>0.0040</td>
</tr>
<tr>
<td>2</td>
<td>0.0042</td>
</tr>
<tr>
<td>3</td>
<td>0.0046</td>
</tr>
<tr>
<td>4</td>
<td>0.0050</td>
</tr>
<tr>
<td>5</td>
<td>0.0050</td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td>0.0046</td>
</tr>
</tbody>
</table>
DISCUSSION

Deflections and Reactions

The average deflection at Pt. X, (Figure 11) which is identical in location to node 27 in the finite matrix scheme, was found experimentally to be .0046". Pt. Y in the experimental test was similar in location to a position between node 31 and node 32 in the stiffness matrix analysis. And Pt. Z in the experimental test was similar in location to node 9 in the stiffness matrix analysis. A comparison of the results for both approaches is shown in the following table:

<table>
<thead>
<tr>
<th></th>
<th>Matrix Analysis</th>
<th>Experimental Test*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Node 9</td>
<td>-.0012999</td>
<td>-.0012</td>
</tr>
<tr>
<td>Node 27</td>
<td>.0044175</td>
<td>.0046</td>
</tr>
<tr>
<td>Node 31½</td>
<td>.0012768</td>
<td>.0013</td>
</tr>
</tbody>
</table>

* Average value of 5 trials.

The deflections obtained by the stiffness matrix approach and the experimental method are thus seen to be within 4%.

The deflection and constraints found at location X on a ring (Figure 11) by the three different methods discussed in this thesis were as follows:
Table VI
Comparison of Results Obtained by Matrix, Energy & Experimental Methods

<table>
<thead>
<tr>
<th>Method</th>
<th>Def. (in.)</th>
<th>F1 (lb.)</th>
<th>F2 (lb.)</th>
<th>H2 (in-lb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Experimental</td>
<td>.0046</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
<td>Stiffness Matrix</td>
<td>.0044</td>
<td>.760989</td>
<td>.239010</td>
<td>.492000</td>
</tr>
<tr>
<td>Energy</td>
<td>.0048</td>
<td>.76019</td>
<td>.23980</td>
<td>.49055</td>
</tr>
</tbody>
</table>

The above table indicates that the deflections obtained by the energy method and the stiffness matrix method are within 4½% of the results obtained from the experimental method while those obtained by the two theoretical methods are within 9% of each other. Such close agreement of the results justifies the idealization of the problem. It is of interest to note that the energy method tends to give higher values of deflection while the stiffness matrix method gives values lower than those obtained by the experimental method. Some variation in results is to be expected due to lack of precise values of the physical properties of the ring such as Young's modulus of elasticity, Poisson's ratio etc. It is also expected that stiffness matrix method should yield values lower than those obtained by the energy method due to the fact that the former method uses a foreshortened circumference resulting from a ring made up of straight beams instead of curved beams. The deviation in the values of deflection obtained by the two theoretical methods can be reduced by either, using
curved beams in the stiffness matrix method while keeping the number of nodes fixed, or by employing greater number of nodes, say 100, while retaining the straight beam approximation.

The values of reactions and moments acting on the ring obtained by the two analytical methods are in excellent agreement, with deviation of less than one-half percent, which can be attributed to the round-off and the truncation errors in computation.

**Effects of Transverse Shear**

The following table contains a comparison of the deflections and constraints at point X as determined by the energy method when transverse shear is included and when transverse shear is neglected.

<table>
<thead>
<tr>
<th></th>
<th>Def. (in.)</th>
<th>$F_1$ (lb.)</th>
<th>$F_2$ (lb.)</th>
<th>$H_2$ (in.-lb.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>With Shear</td>
<td>.0048279</td>
<td>.76019</td>
<td>.23980</td>
<td>.49055</td>
</tr>
<tr>
<td>Without Shear</td>
<td>.0048202</td>
<td>.76008</td>
<td>.23991</td>
<td>.49047</td>
</tr>
</tbody>
</table>

The above table clearly shows that the effects of transverse shear on deflection are less than one and a half percent and therefore indeed negligible. This agrees with the observation by Seely and Smith that except for relatively short, deep beams, the deflection caused by transverse shear may be neglected without introducing serious error.
Since the effects of transverse shear are so negligible incorporation of terms accounting for it into the stiffness matrix program becomes merely a matter of academic interest.

Comparison of the Energy Method and the Stiffness Matrix Method

Equations in chapter V for the energy method were formulated for a specific loading configuration. Any change in the loading scheme would necessitate reformulating the governing equations. Also, the equations in the thesis were formulated to predict deflection at a specific point on the ring. If one is interested in predicting deflection at any arbitrary point on the ring a formidable amount of work would indeed be required. On the other hand, equations for the stiffness matrix method are fewer and less involved and yet permit incorporation of any combination of loading scheme. It has also the advantage of predicting deflections at all the nodes almost simultaneously. It does have the disadvantage that it requires solution of $12 \times n$ number of equations where $n$ is the number of nodes chosen, thus requiring significant amount of computer time, especially if one needs high accuracy.
CONCLUSION

The results of this investigation can be summerized as follows:

(a) The energy method and the stiffness matrix method as applied to a normally loaded ring give values of deflections which are within $4\frac{1}{2}\%$ of those obtained by the experimental method.

(b) The values of reactions and moments obtained by the two theoretical methods are within 0.5%.

(c) Effects of transverse shear on deflection of a normally loaded ring is negligible, less than $1\frac{1}{2}\%$.

(d) The energy method is suitable when computer time is at premium and one is interested in a specific loading configuration and in values of deflections and reactions at a specific point on the ring.

(e) The stiffness matrix method is a versatile one capable of incorporating various kinds of loading conditions. It never requires expensive computer time.
BIBLIOGRAPHY


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14. G. R. Cowper


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17. Enrico Volterra


18. Francis B. Hildebrand


19. R. J. Melosh


20. Eric Ressoner

FOR CC673

OCS(CARD, 1443 PRINTER)

IST ALL

DEFLECTIONS OF NORMALLY LOADED RINGS

1 A=6.283
   EL=3.840
   B=2.26B
   W=4.712
   P=1.
   R=.927
   E=30. E+06
   AIX=5.47E-06
   AJ=10.94E-06
   G=11.5E+06
   H=0.900
   C1=(P*R**3)/(2.*E*AIX)
   D1=C1*(COS(EL)*COS(B)*(A-EL-(SIN(A)*COS(A))+(COS(EL)*SIN(EL))))
   1+C1*(SIN(EL)*SIN(B)*(A-EL)-SIN(B)*SIN(A)*(SIN(A)*COS(EL)-COS(A))
   2*SIN(EL))-SIN(EL)*COS(B)*SIN(A)*SIN(A)+SIN(EL)*COS(B)*SIN(EL)*
   3*SIN(EL))
   C2=(P*R**3)/(AJ*G)
   D2=C2*(A-EL+SIN(B)*COS(A)-COS(B)*SIN(A)+SIN(EL)*COS(A)-COS(EL)*
   1*SIN(A)+(A/2.)*SIN(B)*SIN(EL)+(A/2.)*COS(B)*COS(EL)+SIN(A)*SIN(A)/
   22.*0)*(SIN(B)*COS(EL)+COS(B)*SIN(EL))+(SIN(A)*COS(A))/2.*SIN(B)*
   3*SIN(EL)*SIN(EL)-SIN(B)*SIN(EL))-(EL/2.*SIN(B)*SIN(EL))-SIN(B)*COS(EL)+
   4*COS(B)*SIN(EL)-COS(EL)*SIN(EL)/2.-SIN(B)*COS(EL)*SIN(EL)/2.)
   D3=R**3/(2.*E*AIX)*(A+2.*SIN(A)*COS(A)*SIN(A)*SIN(B)*SIN(A)-SIN(A)*
   COS(B))
   C3=P*R**3/(2.*A-EL))
   D4=C2*(A-EL+SIN(B)*COS(A)-COS(B)*SIN(A)+SIN(EL)*COS(A)-COS(EL)*
   1*SIN(A)+(A/2.)*SIN(B)*SIN(EL)+(A/2.)*COS(B)*COS(EL)+SIN(A)*SIN(A)/
   22.*0)*(SIN(B)*COS(EL)+COS(B)*SIN(EL))+(SIN(A)*COS(A))/2.*SIN(B)*
   3*SIN(EL)*SIN(EL)-SIN(B)*SIN(EL))-(EL/2.*SIN(B)*SIN(EL))-SIN(B)*COS(EL)+
   4*COS(B)*SIN(EL)-COS(EL)*SIN(EL)/2.-SIN(B)*COS(EL)*SIN(EL)/2.)
   D5=R**3/(2.*E*AIX)*(A+2.*SIN(A)*COS(A)*SIN(A)*SIN(B)*SIN(A)-SIN(A)*
   COS(B))
   C4=(R**3)/(2.*A-EL))
   D6=1-SIN(EL)-COS(EL)+SIN(EL))
   D9=(R**3)/(2.*E*AIX)*(A+COS(EL)-COS(EL)*SIN(A)*COS(A)-SIN(A)*SIN(EL)
   1(A)*
   1*SIN(EL)-EL*COS(EL)+SIN(EL))
   D10=(R**3)/(AJ*G)*(SIN(B)-B-(5*COS(B))/2.-
   1*(SIN(B)*COS(B)*COS(B))/2.-SIN(B)*SIN(B)*SIN(B))/2.-SIN(B)*COS(B)
   2-COS(B)*SIN(B)+A-SIN(A)*COS(A)+COS(A)*SIN(B)-SIN(A)+(A*COS(B))/2.
   3+(SIN(A)*COS(A)*COS(B))/2.+(SIN(A)*SIN(A)*SIN(B))/2.)
   D11=(R**3)/(2.*AJ*G)*(2.*A-2.*SIN(A)*COS(EL)+COS(A)*SIN(EL)/2.-2.
   1*SIN(A)+A+COS(EL)+SIN(A)*COS(A)+COS(A)*COS(EL)+(SIN(A))/2.*SIN(EL)-2.
   2+EL/2.*SIN(EL)-EL*COS(EL)-SIN(EL)*COS(EL))/2.-SIN(EL))/2.3)
   D12=(H**2*R**(EL-B))/10.*G*AIX)
   D13=H**2*R**(A-EL))/10.*G*AIX)
   D14=(R**2)/(2.*E*AIX)*B+SIN(B)+COS(B)*SIN(EL)+SIN(A)*SIN(A)-A*SIN(B)-
   1*SIN(B)+SIN(A)*COS(A)
   D15=(R**2)/(2.*E*AIX)*(COS(EL)*SIN(A)*SIN(A)-A*SIN(EL)-SIN(EL)*
   SINA)*COS(A)+EL*SIN(EL))
16 = (R * R * 2 / (2. * A * J * G) * (2. * COS(E) - 2. * COS(A) - COS(E) * SIN(A) * SIN(A) + SIN(E) * (E - A) + SIN(E) * SIN(A) * COS(A))

17 = (-R * R * 2 / (2. * A * J * G) * (2. * COS(B) - 2. * COS(A) + B * SIN(B) - COS(B) * SIN(A))


19 = (-R * R * 2 * SIN(A) * SIN(A) / (2. * E * A * I)


21 = (-R * R * 2 / (2. * A * J * G) * (2. * (-COS(A)) - SIN(A) * SIN(A) + 2.)

22 = (H * R * 2 * R * A) / (10. * G * A * I)

23 = (-R * R * 2 * SIN(A) * SIN(A) / (2. * E * A * I)

24 = (R * (A + SIN(A) * COS(A))) / (2. * E * A * I)

25 = (R * R * 2 / (2. * A * J * G) * (2. * COS(A) + SIN(A) * SIN(A) - 2.)

26 = R / (2. * A * J * G) * (A - SIN(A) * COS(A))

Y1 = D8 + D10 + D12 + D22

Y2 = D9 + D11 + D13

Y3 = D18 + D20 + D22

Y4 = D14 + D17

Y5 = D15 + D16

Y6 = D23 + D25

Y7 = D24 + D26

Y8 = D8 + D10

Y9 = D9 + D11

Y10 = D18 + D20

F2 = ((-Y1) + P * Y2) / (Y3 + Y1) * (-1.)

F1 = 1. - F2

H2 = (F1 * Y4 + P * Y5) / Y7

F2 R = ((Y8) + P * Y9) / (Y10 + Y8) * (-1.)

F1 R = 1. - F2 R

H2 R = (F1 R * Y4 + P * Y5) / Y7


B * COS(B) * SIN(A) + 2. * COS(B) * COS(W) * (A + SIN(A) * COS(A)) + SIN(W) * 9. * COS(B) * SIN(A) * SIN(A) + SIN(B) * SIN(W) * (A - SIN(A) * COS(A)) + SIN(B) * COS(W)

1. * SIN(A) + (A + 2. * SIN(B) * COS(A))

4*\sin(W)\sin(W)-2*\sin(B)\cos(W)

X7 = (PR**3)/(2*AJ*G)*(2*A-2*\cos(W)\sin(A)+2*\sin(W)\cos(A)-
6*\cos(EL)\sin(A)\cos(EL)\cos(W)\sin(A)\cos(A)+\sin(W)\cos(EL)\sin(A)\sin(W)
7*\sin(A)\sin(A)\sin(W)\sin(A)-\sin(A)\cos(A)\sin(W)\sin(EL)\cos(W)*
8*\sin(A)\sin(A)\sin(W)\sin(A)\sin(W))

X8 = (PR**3)/(2*AJ*G)*(-2*W+2*\cos(W)\sin(W)-2*\sin(W)
9*\cos(W)+2*\cos(EL)\sin(W)-\cos(EL)\cos(W)\sin(W)\sin(W)\cos(W))
1-\sin(EL)\cos(W)\sin(W)\sin(W)\cos(W))
1-\sin(EL)\cos(W)\sin(W)\sin(W)\cos(W)-2*\sin(EL)\cos(W))

X9 = (H**2*R)/(10*G*AI*F2-F1+P)*(A-W)

DEF = X1+X2+X3-X4-X5-X6+X7+X8-X9+X10-X11

WRITE (3,5) F1
WRITE (3,5) F2
WRITE (3,5) H2
WRITE (3,5) DEF
WRITE (3,5) F1R
WRITE (3,5) F2R
WRITE (3,5) H2R
WRITE (3,5) DEFR
WRITE (3,5) C1
WRITE (3,5) D1
WRITE (3,5) C2
WRITE (3,5) D2
WRITE (3,5) C3
WRITE (3,5) C6
WRITE (3,5) D5
WRITE (3,5) C3
WRITE (3,5) C4
WRITE (3,5) D8
WRITE (3,5) D9
WRITE (3,5) D10
WRITE (3,5) D11
WRITE (3,5) D12
WRITE (3,5) D13
WRITE (3,5) D14
WRITE (3,5) D15
WRITE (3,5) D16
WRITE (3,5) D17
WRITE (3,5) D18
WRITE (3,5) D19
WRITE (3,5) D20
WRITE(3,5)D21
WRITE(3,5)D22
WRITE(3,5)D23
WRITE(3,5)D24
WRITE(3,5)D25
WRITE(3,5)D26
WRITE(3,5)Y1
WRITE(3,5)Y2
WRITE(3,5)Y3
WRITE(3,5)Y4
WRITE(3,5)Y5
WRITE(3,5)Y6
WRITE(3,5)Y7
WRITE(3,5)Y8
WRITE(3,5)Y9
WRITE(3,5)Y10
WRITE(3,5)X1
WRITE(3,5)X11
WRITE(3,5)X2
WRITE(3,5)X3
WRITE(3,5)X4
WRITE(3,5)X5
WRITE(3,5)X6
WRITE(3,5)X7
WRITE(3,5)X8
WRITE(3,5)X9
WRITE(3,5)X10
CALL EXIT
END

VARIABLE ALLOCATIONS
A(R )=0000  E(R )=000C  B(R )=0004  W(R )=0006
C(R )=000E  AIX(R )=0010  A(R )=0010  G(R )=0012
D1(R )=0018  C2(R )=001A  D2(R )=001C  D3(R )=001E
C3(R )=0024  C4(R )=0026  D8(R )=0028  D9(R )=002A
D12(R )=0030  D13(R )=0032  D14(R )=0034  D15(R )=0036
D18(R )=003C  D19(R )=003E  D20(R )=0040  D21(R )=0042
D24(R )=0048  D25(R )=004A  D26(R )=004C  Y1(R )=004E
Y4(R )=0054  Y5(R )=0056  Y6(R )=0058  Y7(R )=005A
Y10(R )=0060  F2(R )=0062  F1(R )=0064  H2(R )=0066
H2R(R )=006C  X1(R )=006E  X11(R )=0070  X2(R )=0072
X5(R )=0078  X6(R )=007A  X7(R )=007C  X8(R )=007E
DEF(R )=0084  X1R(R )=0086  X11R(R )=0088  X9R(R )=008A
X5R(R )=0090  X6R(R )=0092  DEFR(R )=0094

REFERENCED STATEMENTS
1

STATEMENT ALLOCATIONS
5 =00FE 1 =0100

FEATURES SUPPORTED
ONE WORDD INTEGERS
0.7601943 DEF
0.2398057 (DEFLECTION AT NODE 27 INCLUDING TRANSVERSE SHEAR)
0.4905528
0.0482799
0.7600817
0.2399184
0.4904707
0.0482020DEF (DEFLECTION AT NODE 27 WITHOUT TRANSVERSE SHEAR)
0.0024271 C1
0.0009953 C2
0.0063317 C3
0.0212644
0.0085502
0.0031658
0.0493982
0.0000291
0.0000479
-0.0043966
-0.0061022
0.0245426
0.0034388
0.0000187
0.0000291
0.0080589
0.041126
-0.0066963
0.0217286
0.0152503
-0.0000000
0.0596753
-0.0000000
0.0000749
-0.0000000
0.0177457
-0.0000000
0.0231480
0.021939
-0.0026342
-0.00750007
0.0297875
-0.0025837
-0.0000000
0.0408938
0.0201460
-0.0026634
0.0749257
0.0005816
-0.0010365
0.0043115
0.001076
0.0020175
0.0175294 \times 5
-0.0124427 \times 6
0.0421747 \times 7
-0.0370085 \times 8
-0.0007197 \times 9
0.0000089 \times 10
# Appendix B
## Matrix Method

**Structural Analysis Program**

### Output: Structural Description

<table>
<thead>
<tr>
<th>Type</th>
<th>Upper</th>
<th><em>CCORDINATE</em></th>
<th><em>X</em></th>
<th><em>Y</em></th>
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*      * NJOYT  * CF LOAD  * CF LOAD  * AND ELEN COORD SYSTEMS
*      * LOAD  * 22  * 25  * 1.000000  * C  *  C  *  C

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WALL TIME =  2.78 SECONDS
CPU TIME =  1.97 SECONDS

*COMPUTE STRUCTURAL MATRIX
WALL TIME =  2.78 SECONDS
CPU TIME =  1.97 SECONDS

*SOLVE STRUCTURAL EQUATIONS SIMULTANEOUSLY
WALL TIME =  2.74 SECONDS
CPU TIME =  2.02 SECONDS

*OUTPUT STATIC DISPLACEMENTS AND VECTORS
WALL TIME =  15.62 SECONDS
CPU TIME =  4.72 SECONDS
EXECUTE STRAP FOR DISPLACEMENTS AND REACTIONS

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CPU TIME = 0.20 SECONDS

INPUT STRUCTURAL DESCRIPTION

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OUTPUT STRUCTURAL DESCRIPTION

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