Balancing of the Shaking Force, Shaking Moment, Input Torque and Bearing Forces in Planar Four Bar Linkages

Timothy C. Hewitt

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BALANCING OF THE SHAKING FORCE, SHAKING MOMENT, INPUT TORQUE
AND BEARING FORCES IN PLANAR FOUR BAR LINKAGES

USING
AUTOMATED OPTIMAL DESIGN

by
Timothy C. Hewitt

A Thesis Submitted
in
Partial Fulfillment
of the
Requirements for the Degree of
MASTER OF SCIENCE
in
Mechanical Engineering

Approved by :

Prof. ____________________________ (Thesis Advisor)
Prof. ____________________________
Prof. ____________________________
Prof. ____________________________ (Department Head)

DEPARTMENT OF MECHANICAL ENGINEERING
ROCHESTER INSTITUTE OF TECHNOLOGY
ROCHESTER, NEW YORK

January, 1985
To : Whom it may Concern


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Signed,

Timothy C. Hewitt
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ABSTRACT

The theory, development and application of a computer program to balance the combined effects of the shaking force, shaking moment, input torque and individual bearing forces in four bar linkages is presented. The theory assumes the linkage to consist of rigid bodies, and is limited to balancing planar four bar linkages other than sliders. Balancing is accomplished using circular counterweights which are tangentially attached to the bearing joints. Counterweight sizes and locations are determined using nonlinear programming techniques where an objective function, dependent upon all the kinetic parameters, is minimized.

The balancing program is capable of performing diverse functions. The number of added counterweights, type and degree of numerical quadrature and regional constraints on all important balancing parameters can be varied. In addition, the program is capable of balancing linkages with offline mass distributions, and to some extent, emphasis can be placed on individual terms such as the input torque. A major limitation of the theory is the assumption of rigid links. This may not always be valid and makes the program insensitive to natural frequencies, where the amount of vibration would be excessive.
Example problems are presented to show the capabilities and application of the balancing program. The first example shows the effect of varying the degree and type of numerical integration used. The Gaussian quadrature method is shown to be most efficient, with the optimum number of sampling points determined to be 10. In example two, an inline four bar linkage operating at a constant input speed of 5000 rpm is balanced so that all kinetic quantities are reduced from 75% to 92% over the unbalanced case. Similar results are shown in example three for balancing a four bar linkage with an offline coupler mass distribution. The effect of adding from one to three counterweights is also investigated, with the results indicating that additional counterweights do not always improve the balancing situation. With just one counterweight added, the important kinetic terms are reduced an average of 87%, while the addition of three counterweights only reduces these same quantities an additional 5%.

Due to the apparent success of the developed program, the author recommends that it be extended to sliders, six bars and other practical linkages. In addition, the validity of the rigid body assumption should be experimentally and theoretically investigated.
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### Scalars

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<tr>
<td>$A_0$</td>
<td>Angular momentum about point O</td>
<td>(lb-in-s)</td>
</tr>
<tr>
<td>$C_{OR}$</td>
<td>Resultant couple about point O</td>
<td>(lb-in)</td>
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<tr>
<td>$C_p$</td>
<td>Penalty function coefficient</td>
<td></td>
</tr>
<tr>
<td>$E_T$</td>
<td>Kinetic energy</td>
<td>(lb-in)</td>
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<td>$F_{ij}$</td>
<td>Bearing joint force between joints $i$ and $j$</td>
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<td>$F_{SH}$</td>
<td>Shaking force</td>
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<td>Rms $F_{SH}$</td>
<td>(lb)</td>
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<td>Maximum allowable $F_{SHRMS}$</td>
<td>(lb)</td>
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<tr>
<td>$F_x, F_y$</td>
<td>X and Y components of the external force applied to the follower link</td>
<td>(lb)</td>
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<td>$F_R$</td>
<td>Resultant force</td>
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<td>$F_B$</td>
<td>Maximum bearing force</td>
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<td>Rms value of $F_B$</td>
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<td>Maximum allowable $F_{BRMS}$</td>
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<td>$I_{ij}$</td>
<td>Mass moment of inertia of the $i$th link about point $j$</td>
<td>(lb-in-s²)</td>
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<td>$L_i$</td>
<td>Length of link $i$</td>
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<td>Length from joint $O_3'$ to the point of external load application, point &quot;a&quot;</td>
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<td>$l_i$</td>
<td>Length to the centroid of the $i$th real link from joint $O_i$</td>
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\( l_{iC} \) - Length to the centroid of the ith counterweight from joint \( O_i \) (in)

\( l_{iT} \) - Length to the total, or combined centroid for the ith link from joint \( O_i \) (in)

\( l_{i1}, l_{i2} \) - Length to the masses of the two point mass model

\( l_{i1T}, l_{i2T} \) - Body coordinates \( r, Q, z' \), for the combined centroid location (in)

\( M_{SH/O_i} \) - Shaking moment about joint \( O_i \) (lb - in)

\( M_{SH/O_i\text{RMS}} \) - Rms value of \( M_{SH/O_i} \) (lb - in)

\( (M_{SH/O_i\text{RMS}})_{\text{max}} \) - Maximum allowable \( M_{SH/O_i\text{RMS}} \) (lb - in)

\( m_i \) - Mass of the ith real link (lb - \( s^2 \)/in)

\( m_{iC} \) - Mass of the ith counterweight (lb - \( s^2 \)/in)

\( m_{iT} \) - Mass of the ith combined centroid (lb - \( s^2 \)/in)

\( m_{i1}, m_{i2} \) - Masses of the two point mass model (lb - \( s^2 \)/in)

\( N_e \) - Number of equality constraints for optimization

\( N_r \) - Number of regional constraints for optimization

\( N_v \) - Number of design variables for optimization

\( P \) - Penalty function

\( Q \) - Constrained objective function

\( Qu \) - Unconstrained penalty function = \( Q + P \)

\( r, W \) - Gaussian quadrature abscissas and weighting factors

\( T_{a1} \) - Generalized external force (lb - in)
\( T_D \) - Input, or driving torque (lb - in)
\( T_{\text{DRMS}} \) - Rms value of \( T_D \) (lb - in)
\( (T_{\text{DRMS}})^{\text{max}} \) - Maximum allowable \( T_{\text{DRMS}} \) (lb - in)
\( T_L \) - Loading torque (lb - in)
\( x_{iT}, y_{iT} \) - Distance from the fixed coordinate system origin to the combined centroid for the \( i \)th link (in)
\( x_{iT}, y_{iT} \) - Distance from the translating coordinate system origin to the combined centroid for the \( i \)th link (in)
\( W_1, W_2 \) - Weighting factors for the objective function
\( \alpha_i \) - Angle between the \( r \) body coordinate and the \( i \)th real link centroid (rad)
\( \Theta_i \) - Angle between the \( x \) translating coordinate and the \( r \) body coordinate for the \( i \)th link (rad)
\( \Theta_{iC} \) - Angle between the \( r \) body coordinate and the \( i \)th counterweight centroid (rad)
\( \Theta_{iT} \) - Angle between the \( x \) translating coordinate and the combined centroid location for the \( i \)th link (rad)
\( \delta W \) - virtual work as a result of \( \delta \Theta_i \) (lb - in)
\( \delta \Theta_i \) - virtual displacement of the crank (rad)
Normalized Quantities

\[ \hat{A}_0 = \frac{\hat{A}_0}{L_1^2 \dot{\theta}_1} (m_{11} + m_{12}) \]

\[ C_{oR} = \frac{C_{oR}}{L_1^2 \dot{\theta}_1} (m_{11} + m_{12}) \]

\[ D_1 = \frac{L_1}{L_1} \]

\[ d_i = \frac{l_i}{L_1} \]

\[ d_a = \frac{l_a}{L_1} \]

\[ d_{i1}, d_{i2} = \frac{l_{i1}}{L_1}, \text{ and } \frac{l_{i2}}{L_1} \text{ respectively} \]

\[ E_T = \frac{E_T}{L_1^2 \dot{\theta}_1} (m_{11} + m_{12}) \]

\[ F_{ij} = \frac{F_{ij}}{L_1 \dot{\theta}_1} (m_{11} + m_{12}) \]

\[ F_{SH} = \frac{F_{SH}}{L_1 \dot{\theta}_1} (m_{11} + m_{12}) \]

\[ I_i = \frac{I_i}{L_1^2 (m_{11} + m_{12})} \]

\[ \dot{\theta}_2 / \dot{\theta}_1 \text{ and } \dot{\theta}_3 / \dot{\theta}_1 \text{ respectively} \]

\[ M_{SH/0_i} = \frac{M_{SH/0_i}}{L_1^2 \dot{\theta}_1} (m_{11} + m_{12}) \]

\[ M_{i1' M_{i2}} = \frac{m_{i1}}{(m_{11} + m_{12})} \text{ and } \frac{m_{i2}}{(m_{11} + m_{12})} \]

\[ P_{i1', P_{i2}} = M_{i1'} d_{i1} \text{ and } M_{i2'} d_{i2} \text{ respectively} \]

\[ T_D = \frac{T_D}{L_1^2 \dot{\theta}_1} (m_{11} + m_{12}) \]

\[ T_{\dot{\theta}_i} = \frac{T_{\dot{\theta}_i}}{L_1^2 \dot{\theta}_1} (m_{11} + m_{12}) \]

\[ \alpha_i = \frac{\dot{\theta}_i}{\dot{\theta}_1^2} \]

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Vectors and Matrices

$\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ - unit vectors for the body coordinates

$[E]$ - Equality constraints matrix

$[F]$ - Matrix of final items to be printed out

$i', j', k'$ - unit vectors for the fixed coordinates

$i, j, k$ - unit vectors for the translating coordinates

$[R]$ - Regional constraint matrix

$[V]$ - Scaled design variables for optimization
1.0 INTRODUCTION

Unbalanced linkages that operate at high speed, contain massive links or experience large external loads may be subjected to excessive vibration, noise and wear. Prediction of the amount of imbalance is based upon the magnitude of the shaking force, shaking moment, input torque and individual bearing forces. Reduction of some or all of these kinetic terms is desirable in balancing the linkage. There are many different methods of balancing linkages, but most have only been applied to the slider crank. Effective methods for balancing linkages other than sliders, have only been developed over the last fifteen years.

In 1969, Berkoff and Lowen [4] published a simple technique for completely eliminating the shaking force. But the method ignored the other kinetic effects, and often increased the shaking moment and input torque. Berkoff, in 1973 [5], also developed a method that completely eliminated both the shaking force and shaking moment in four bar linkages with inline mass distributions. The technique, however, required the addition of large geared inertia counterweights which ended up doubling the mass of the linkage, and most of the kinetic terms too. Reduction of the input torque has also been accomplished by a number of methods. Most notably, the use of a flywheel, which absorbs and transmits energy to smooth out input torque fluctuations. But again, most of the
methods gave little consideration to the remaining kinetic quantities. Balancing of the combined effects is the next logical step, but the equations involved are of sufficient complexity that all practical considerations are limited to computer applications. In 1975, Sadler [33] and others used nonlinear programming to balance these combined effects. Their results were in general better, and far more practical than existing closed form techniques since regional constraints could be used to limit many kinetic terms. None the less, the formulation required separate optimization trials for each quantity to be minimized, and a manual development of trade-off curves to select the best overall counterweight configuration. Additional schemes have been presented since then, but most have not handled the problem as efficiently and effectively as Lee and Cheng [21]. They chose to minimize a single objective function expressed in terms of the input torque and the ground bearing forces. Reduction of the shaking force and shaking moment is also accomplished since they are dependent upon the bearing forces. Lee and Cheng also developed an efficient, closed form solution for the kinetic analysis, and showed their method to be superior to several existing techniques.

The objective of this thesis is to balance the combined kinetic effects, using the theory presented by Lee and Cheng, and to develop a general computer program which incorporates their ideas. The analysis will be limited to balancing planar, rigid body, four bar linkages through the addition of
circular counterweights. The computer program is developed from an existing optimization program written by R.C. Johnson [18], and is modified to handle the general balancing theory.

Capabilities of the balancing program will be demonstrated using three numerical examples. First, key parameters of the program will be determined for an accurate, but efficient analysis. The optimum number of sampling points for numerical quadrature, and the degree to which emphasis may be placed on individual terms, such as the input torque, will be investigated. Results of the first example will then be used in the second example. This example presents a practical method of balancing an inline four bar linkage. The effect of placing additional counterweights on the follower and coupler links is also investigated. In the third example, a four bar linkage with an offline coupler mass is balanced. The most practical balancing scheme of the third example will then be compared to the the unbalanced linkage, the method of complete force balancing and the method of complete force and moment balancing.
2.0 LITERATURE SURVEY

The balancing of four bar linkages has been a problem for both the designer and the researcher for many years. This may be partially attributed to the complexity of the problem since many different quantities must be taken into account (ie. shaking force, shaking moment, input torque, bearing forces). Various experimental and theoretical methods exist to balance some of these parameters, but most have only been applied to the slider crank family. Techniques for balancing linkages other than sliders were not thoroughly summarized until 1968. In that year Lowen and Berkoff [22] published a survey of investigations concerning the balancing of four bar linkages with special emphasis on mechanisms other than sliders. The complete work categorized major techniques for balancing the shaking force and shaking moment, listed 119 references, and translated 11 important papers from Russian and German into English. In 1977, Berkoff supplemented the initial survey by publishing a summary of methods for determining and balancing the input torque. In that paper, 71 references were cited and major balancing techniques were described. With such comprehensive literature surveys already published, the efforts of this thesis will not try to duplicate either of the above papers. Rather, the major techniques discussed in Lowen and Berkoff's papers will be presented. In addition, important theoretical methods for balancing planar, rigid body linkages, published after the above mentioned summaries, will
be discussed.

The method of balancing the shaking force, or force balancing, has been a popular method for many years. Lowen and Berkoff [22] summarized many investigations where the shaking force was completely or partially eliminated. Complete force balancing attempts to make the center of gravity of the linkage stationary, and can be categorized into static balancing methods, method of principal vectors, cam methods and duplicate mechanism methods. On the other hand, partial balancing methods only reduce the shaking force, and may be accomplished using harmonic balancing methods or by adding springs to the linkage. Harmonic balancing methods use a Fourier series and Gaussian least squares approach to eliminate lower harmonics of the shaking force, while adding springs to the linkage alters the path through which the shaking force travels to ground. In 1969, Berkoff and Lowen [4] published a new method for completely force balancing a linkage termed the method of linear independent vectors. This method was very easy to apply, and only required the addition of two rotating counterweights fixed to the input and output links. Lowen, Trepper, and Berkoff [24] in 1973 showed the quantitative effects of complete force balancing a general inline four bar linkage on other dynamic reactions such as the input torque, shaking moment and bearing forces. The paper also provided a complete tutorial on kinetic analysis summarizing final equations and popular methods of analysis. Oldham and Walker [31] in 1978, refined the method of linearly

Although the method of force balancing significantly reduces the net force on the frame, it may actually increase the shaking moment. Lowen and Berkoff, in their initial survey, described techniques for completely and partially eliminating the shaking moment. Methods of complete shaking moment balancing include cam actuated oscillating counterweights, and the addition of a duplicate mechanism with mirror symmetry. Also, harmonic methods were popular for partially eliminating the shaking moment. Elimination of the first, and sometimes the second harmonic of the shaking moment was accomplished with two rotary masses which were synchronized 180 degrees out of phase. In 1971, Lowen and Berkoff [23] published a graphical method whereby the shaking moment was minimized while maintaining a full force balance. Complete force and moment balancing of inline four bars was given by Berkoff [4] in 1973. This method, however, doubled the total mass of the mechanism, and significantly increased all other dynamic reactions. Wiederich and Roth [43] in 1976 determined link inertial properties of a general force balanced four bar linkage so that the amount of angular momentum fluctuations, and hence the shaking moment, were
minimized. Bagci [1] in 1980 extended complete force balancing to offline four bars and some six bars using balancing idler loops. These idler loops transferred the effects of the coupler link motion to a ground bearing joint, where Lancaster type balancers eliminated the shaking moment.

In contrast to balancing the shaking force and moment, some balancing methods have concentrated on reducing the input torque. Although it can never be eliminated totally, the magnitude of torque fluctuations can be significantly reduced. In 1977, Berkoff [7] summarized available methods of torque balancing. As described by Berkoff, flywheels, spring equivalents to flywheels, and cam subsystems are all capable of smoothing out the torque curve. Each method stores and discharges energy to reduce torque fluctuations. In particular, cam subsystems can be designed to flatten out the torque to perfection. Other methods summarized by Berkoff include using torque regional constraints during kinematic synthesis (see [9] for example), and rearrangement of link geometry by internal mass redistribution.

All the previous methods discussed have focused on the elimination of one, or at most two balancing quantities. However, as shown in various published example problems, balancing of individual parameters may have an adverse effect on other dynamic quantities, such as bearing forces. During the mid 1970's researchers began attempting to balance the combined dynamic effects of the shaking force, shaking moment, input torque and bearings forces. Balancing of these combined
effects is very difficult to achieve, but was facilitated by the use of the digital computer. In 1975 Oldham [30], and Smith [36], developed similar interactive computer programs which provided full force balancing, and permitted the user to manipulate link and counterweight properties to reduce other dynamic effects. Conte, George, Mayne, and Sadler in 1975 [9] used computer aided optimal design to generate mechanisms through kinematic synthesis while minimizing individual balancing quantities. Sadler [33] in 1975 also used nonlinear programming to determine counterweight sizes and angular locations, while minimizing individual balancing quantities through a series of optimizations. Once the effects of minimizing individual parameters were known, trade-off curves were used to determine which counterweight scheme resulted in the best overall balance. Aside from digital computer applications, Trepper and Lowen [38] in 1975 developed equations for balancing the shaking force with flexible constraints on ground bearing forces. These equations, however, were of the 16th order and required extensive computer work to be solved. Elliot and Tesar [11] in 1977 developed procedures for applying existing techniques in a series of trials to determine which method provided the best overall results. These trials were time consuming and semiiterative in nature. Thus the problem of balancing the combined effects seems best suited for the computer. However, even with the aid of the computer, all of the previously mentioned procedures required some manual iteration, and at best were limited to minimizing only one quantity at a time.
This problem was partially solved by Tricamo and Lowen [40] in 1983, who used automated optimal design to minimize the maximum bearing force. The maximum bearing force was not always limited to any specific bearing location, but varied with counterweight arrangement. So, as the search process of determining a counterweight configuration was carried out, the actual location of the maximum bearing force was changing. In addition, Tricamo and Lowen's method placed regional and equality constraints on other important balancing quantities, such as the shaking force. The problem was further refined by Lee and Cheng [21] who defined a general objective function in terms of the ground bearing forces and the input torque. The main idea behind reducing the ground bearing forces is to balance the shaking force, shaking moment and two of the bearing forces in one term. The input torque term was added to incorporate sensitivity to that quantity. Parameters of the counterweights were then determined using Heurisitic computer optimization to minimize the objective function. Example problems presented by Lee and Cheng showed their method to be superior to force balancing, or force and moment balancing. Comparisons to Tricamo and Lowen's nonlinear programming method showed Lee and Cheng's method to be marginally better. The apparent success of Lee and Cheng has led to the author's interest in developing a similar program, and is presented in this thesis.
3.0 OVERVIEW OF THE THEORY

The theory of balancing the combined effects of a planar four bar linkage is divided into two parts:

1. The derivation of equations for the kinematic and kinetic analysis. In particular, general equations for the input torque, shaking force, shaking moment and bearing forces must be found.

2. Development of the optimization problem, and application of the existing automated optimization program to the specific problem of balancing four bar linkages.

Equations describing the motion of the four bar mechanism are obtained using vector analysis. This method is analogous to the complex number method as presented by Shigley and Uicker [34]. Equations for the forces and couples are derived by combining the Newtonian and Lagrangian approaches. This method allows direct evaluation of all kinetic quantities without the use of matrix solutions.

Once the equations for the kinetic analysis are known, then the general optimization problem can be formulated. A general objective function will be proposed, and a formal statement of the optimization problem will be given. With the problem defined, the techniques of automated optimal design can be applied to determine the best location of the design
variables (ie. radius and angular location of each counterweight) so that the objective function is minimized.

The key assumptions of the theory are:

1. All links are treated as rigid bodies. Therefore, the effects of stiffness, damping or natural frequencies have been ignored.

2. The effects of gravity on the linkage have been ignored.

3. The linkage and counterweights are assumed to lie in the same plane. Couples due to misalignment are not considered.

4. All links of the mechanism are assumed to have a finite length. In otherwords, the analysis does not take sliders into account.

5. Only circular counterweights of constant thickness and density are used. Further, the counterweights can only be attached tangentially to some of the bearing joints of the linkage. Effects of the counterweight attachment brackets have also been ignored.

6. All joints of the mechanism are lower pair, pin joints. The term "linkage" as defined by Shigley and Uicker [34], mandates this assumption of lower pair contact.

7. The system is conservative. Thus the effects of friction have been ignored.
The validity of each of the above assumptions depends upon the particular application and linkage to be balanced. For example, the assumption of all the links being rigid may not always be true for extremely high speed applications, or where the stiffness of each link is relatively low in comparison to the link mass. However, for many applications, these assumptions provide a close approximation to the real linkage, and facilitate a solution.
4.0 DEVELOPMENT OF THE KINEMATIC AND KINETIC EQUATIONS

4.1 Two Point Mass Model

A general model of the four bar linkage to be balanced is shown in Figure 1. It consists of three moving links and one fixed link. Link 1 is the input or crank, link 2 is considered the coupler, link 3 is the follower and link 4 is the frame. In general, counterweights may be applied to any one of the moving links at joints \(O_1', O_2',\) or \(O_3'.\) Circular counterweights of constant thickness are shown as dashed lines in Figure 1. For a general analysis, the link length, mass, thickness and counterweight thickness, radius and angular location must all be variable.

From an analysis point of view, a simpler yet dynamically equivalent model of the real counterweighted linkage is the two point mass model. First applied to mechanism balancing by Wiederich and Roth [43], the model reduces the number of variables for a general analysis. As shown in Figure 2, the model consists of two point masses rigidly attached to the same massless link. On each link, these masses are free to vary in magnitude and distance, but are restricted to lie upon the given body coordinate system axes. As shown in Figure 2, the first mass must lie on the body "r" coordinate, and the second mass must lie on the body "\(\Theta\)" coordinate. The rules of vector analysis still apply to the model, and the general configuration
Figure 1: General Four Bar Linkage with Circular Counterweights

Figure 2: Two Point Mass Model
of Figure 2 defines the positive locations of the masses.

As stated in [43], the inertial properties of any rigid body can be represented by an equivalent system of point masses. However, the model must obey all the basic laws of dynamics, and be equivalent to the real linkage. The conditions for dynamic equivalence between the real mechanism, and any equivalent model require that:

1. The total mass of the ith link must be the same for both models.
2. The center of mass of the ith link must remain in the same location for both models.
3. The total mass moment of inertia of the ith link must remain the same about the ith links centroidal location, or about a bearing joint on that link.

In the above set of remarks, statements (1.) and (2.) ensure Euler's first law of motion will be satisfied. The third statement ensures that Euler's second law of motion will be met. The additional statement in (3.) requiring the mass moment of inertia to be the same about a bearing joint, not only includes the frame joints, 0'1 and 0'3, but also applies for joints 02 and 04. This peculiarity is shown to be valid in appendix A.

The conversion from a real link with a counterweight to a two point mass model link will be shown in a two step process. First, with the real link and counterweight parameters known, an equivalent combined center of gravity location will be determined. This process is illustrated in Figure 3 for a
general link rotating about a moving joint. This general case describes link 2. However, if \( r_{00} \)' is a null vector, then the analysis can be applied to links 1 and 3. With the properties of the combined center of gravity known, the values of the two point mass model may be found as shown in Figure 4. Each of these two steps must obey the rules for dynamic conversions as stated above.

**Figure 3:** Combined C.G. Location of a General Link

with a Circular Counterweight

In both Figures 3 and 4, there are several coordinate systems: the spatial XYZ coordinates, translating xyz coordinates, and the body coordinates defined as \( r\theta z'' \). The spatial coordinates are considered to be stationary with the fixed set of stars, hence they do not have any angular velocity. The xyz coordinates also don't have any angular velocity, but
translate with the general point $O_i$. Finally, the $r\theta z''$ coordinates are fixed to the link so they translate and rotate with the body. Unit vectors for each of the coordinate systems are defined in Figures 3 and 4.

**Figure 4**: Two Point Mass Model of a Combined C.G. Location

Refering back to Figure 3, the total mass of the $i$th link is just the sum of the real link and counterweight masses, or:

$$m_{iT} = m_i + m_{ic} \quad (1)$$

In addition, the $x$ and $y$ coordinates of the combined center of gravity may be given as:

$$x_{iT} = l_{iT} \cos \theta_{iT} \quad (2)$$

$$y_{iT} = l_{iT} \sin \theta_{iT} \quad (3)$$
Or, stated in terms of the real link and counterweight parameters,

\[
x_{iT} = \frac{m_i l_i \cos(\theta_i + \Theta_{ic}) + m_i l_i \cos(\theta_i + \alpha_i)}{m_i + m_i}
\]

\[
y_{iT} = \frac{m_i l_i \sin(\theta_i + \Theta_{ic}) + m_i l_i \sin(\theta_i + \alpha_i)}{m_i + m_i}
\]

Squaring equations 2 and 3 separately, and adding the result produces:

\[
l_{iT}^2 = x_{iT}^2 + y_{iT}^2
\]

Substituting equations 4 and 5 into equation 6 for \(x_{iT}\) and \(y_{iT}\) respectively, and solving for \(l_{iT}\):

\[
l_{iT} = \frac{[m_i^2 l_i^2 + m_i l_i^2 + 2m_i l_i^2 m_i l_i \cos(\Theta_{ic} - \alpha_i)]}{m_i l_i^2}
\]

To obtain the value of \(\Theta_{iT}\), equation 3 is divided by equation 2, and equations 4 and 5 are again substituted for \(x_{iT}\) and \(y_{iT}\) respectively. The end result being:

\[
\tan(\Theta_{iT}) = \frac{m_i l_i \sin(\theta_i + \Theta_{ic}) + m_i l_i \sin(\theta_i + \alpha_i)}{m_i l_i \cos(\theta_i + \Theta_{ic}) + m_i l_i \cos(\theta_i + \alpha_i)}
\]

Last, the total mass moment of inertia of the real link and counterweight must be determined about the combined centroidal location, point \(T\). Using the parallel axis theorem to transfer the given mass moments of inertia,

\[
I_{iT}^T = I_{ic} + m_i [((x_{ic} - x_{iT})^2 + (y_{ic} - y_{iT})^2] + 2
\]
\[ I_i^R + m_i [(x_i - x_{iT})^2 + (y_i - y_{iT})^2] \] \hspace{1cm} (9)

where,

\[ x_{ic} - x_{iT} = l_{ic}\cos(\theta_i + \theta_{ic}) - l_{iT}\cos(\theta_{iT}) \] \hspace{1cm} (10)

\[ y_{ic} - y_{iT} = l_{ic}\sin(\theta_i + \theta_{ic}) - l_{iT}\sin(\theta_{iT}) \] \hspace{1cm} (11)

\[ x_i - x_{iT} = l_i\cos(\theta_i + \alpha_i) - l_{iT}\cos(\theta_{iT}) \] \hspace{1cm} (12)

\[ y_i - y_{iT} = l_i\sin(\theta_i + \alpha_i) - l_{iT}\sin(\theta_{iT}) \] \hspace{1cm} (13)

Substituting equations 10 through 13 into equation 9, and simplifying the result yields:

\[ I_{iT}^T = \{ I_{ic}^c + m_{ic} \left[ l_{ic}^2 + l_{iT}^2 - 2l_{ic}l_{iT}\cos(\theta_i + \theta_{ic} - \theta_{iT}) \right] + I_i^R + m_i \left[ l_i^2 + l_{iT}^2 - 2l_i l_{iT}\cos(\theta_i + \alpha_i - \theta_{iT}) \right] \} \] \hspace{1cm} (14)

The parameters given in equations 1, 7, 8, and 14 define the total, or combined center of gravity which is dynamically equivalent to a real link with a circular counterweight tangentially attached. The remaining portion of this section will be devoted to deriving the parameters of a general two point mass model using the combined centroidal parameters. Referring back to Figure 4, the total mass of the two point mass model is given by:

\[ m_{i1} + m_{i2} = m_{iT} \] \hspace{1cm} (15)

The centers of gravity for each of the two point masses, as shown in Figure 4, may be given along the body coordinate axis (i.e. "r" and "\theta") as:

\[ m_{i1}l_{i1} = m_{iT}l_{iT}\cos(\theta_{iT} - \theta_i) \] \hspace{1cm} (16)
m_{i2}l_{i2} = m_{iT}l_{iT}\sin(\Theta_{iT} - \Theta_{i}) \tag{17}

Finally, the mass moment of inertia about the joint $O_i$ may be given as:

\[ m_{i1}l_{i1}^2 + m_{i2}l_{i2}^2 = I_{iT} + m_{iT}l_{iT}^2 \tag{18} \]

Equations 15 through 18 have four unknowns, $m_{i1}, l_{i1}, m_{i2}, l_{i2}$. Since these four equations are independent of one another, a solution is possible and is shown in appendix B. The final equations defining the general two point mass model link are summarized here as:

\[ l_{i2} = -\frac{B \pm \sqrt{B^2 - 4AC}}{2A} \tag{19} \]

\[ m_{i2} = \frac{m_{iT}l_{iT}\sin(\Theta_{iT} - \Theta_{i})}{l_{i2}} \tag{20} \]

\[ m_{i1} = m_{iT} - m_{i2} \tag{21} \]

\[ l_{i1} = \frac{m_{iT}l_{iT}\cos(\Theta_{iT} - \Theta_{i})}{m_{i1}} \tag{22} \]

where,

\[ A = l_{i2}^2 \tag{23} \]

\[ B = -\left(2l_{i2}^2 + I_{iT}/m_{iT}\right) \tag{24} \]

\[ C = I_{iT}l_{i2}^2/m_{iT} + l_{i1}l_{i2}^2 + l_{i2}^3 \tag{25} \]
The parameters of the two point mass model should obey the equations summarized above for dynamic equivalence with the real linkage. However, solution of these equations is not always possible. There exists three special cases that are worthy of consideration. First, if \( A \) in equation 19 is zero, then \( l_{12} \) becomes infinite. This case is easily solved by rederviving the equations of the two point mass model with \( l_{12} = 0 \). Another problem arises if \( B^2 < 4AC \). In that case the model for all practical purposes is invalid, and will not provide an accurate analysis. This problem is avoided in the OPTBAL program using a regional constraint which requires \( B^2 > 4AC \). Lastly, a limiting situation arises when \( l_{12} = 0 \). As discussed by Wiederich and Roth [43], the initial conditions which dynamically define the model, will not be obeyed in a practical sense. Fortunately, it is very difficult to calculate a number of 0.0 exactly using a computer, and the problem has never occurred in actual operation the OPTBAL program. If such a situation would occur, the program writes an error message to the user.

Now that the parameters of the two point mass model have been defined, the next process in solving the overall problem is to determine the kinematic equations for the four bar linkage. This topic is covered in the next section using vector analysis.
4.2 Kinematic Analysis

The kinematic analysis of a four bar linkage will yield equations for the angular position, velocity and acceleration of links 2 and 3. For the analysis, the angular position, velocity and acceleration of link 1 are assumed to be known. The procedure will use vector analysis to derive loop equations for the position analysis. Equations for the velocity and acceleration will then be found by taking successive time derivatives of the position equations.

Referring to Figure 5, a vector loop may be taken from joint 0' to 0₂ , 0₂ to 0₄ , 0₄ to 0'₃ , and 0'₃ back to 0' . Or,

\[ \{ L₁(\cos θ₁ \hat{i}' + \sin θ₁ \hat{j}') + L₂(\cos θ₂ \hat{i}' + \sin θ₂ \hat{j}') \\
- L₃(\cos θ₃ \hat{i}' + \sin θ₃ \hat{j}') - L₄ \hat{i}' \} = 0 \]  \hspace{1cm} (26)

Dotting equation 26 with spatial unit vectors \( \hat{i}' \), and \( \hat{j}' \) respectively,

\[ L₁\cos θ₁ + L₂\cos θ₂ - L₄ = L₃\cos θ₃ \]  \hspace{1cm} (27)

\[ L₁\sin θ₁ + L₂\sin θ₂ = L₃\sin θ₃ \]  \hspace{1cm} (28)

Squaring equations 27 and 28 separately, and adding the results gives:
\[
\{ L_1^2 + L_2^2 - L_3^2 + L_4^2 + 2L_1L_2\cos(\theta_1 - \theta_2) \\
- 2L_1L_4\cos \theta_1 - 2L_2L_4\cos \theta_2 \} = 0
\] (29)

Figure 5: Stick Figure of a Four Bar Linkage

Letting,

\[
A = L_1^2 + L_2^2 - L_3^2 + L_4^2 - 2L_1L_4\cos \theta_1
\] (30)

Equation 29 now becomes,

\[
A + \cos \theta_2[2L_1L_2\cos \theta_1 - 2L_2L_4] + \sin \theta_2[2L_1L_2\sin \theta_1] = 0
\] (31)

where,

\[
\cos \theta_2 = \frac{1 - \tan^2(\frac{\theta_2}{2})}{1 + \tan^2(\frac{\theta_2}{2})}
\] (32)

\[
\sin \theta_2 = \frac{2\tan(\frac{\theta_2}{2})}{1 + \tan^2(\frac{\theta_2}{2})}
\] (33)
The analysis may be simplified even further by defining:

\[ B = 2L_1L_2\cos \Theta_1 - 2L_2L_4 \]  
\[ C = 2L_1L_2\sin \Theta_1 \]

Substituting equations 34, 35, and the 1/2 angle formulas into equation 31 produces:

\[
\frac{A + B[1 - \tan(\Theta_2/2)]}{1 + \tan^2(\Theta_2/2)} + \frac{C[2\tan(\Theta_2/2)]}{1 + \tan^2(\Theta_2/2)} = 0
\]  

(36)

Simplifying,

\[
\tan^2(\Theta_2/2)[A - B] + \tan(\Theta_2/2)[2C] + [A + B] = 0
\]  

(37)

Equation 37 is of the form where \(\tan(\Theta_2/2)\) may be solved for using the quadratic formula. Or,

\[
\tan(\Theta_2/2) = \frac{-C \pm \sqrt{C^2 - A^2 + B^2}}{A - B}
\]  

(38)

In the above equation A, B, and C are all known quantities so \(\Theta_2\) can be determined using equation 38. Once \(\Theta_2\) is known, \(\Theta_3\) can be easily found by dividing equation 28 into equation 27. Or,
Tan \( (\theta_3) \) = \[ \frac{L_1 \sin \theta_1 + L_2 \sin \theta_2}{L_1 \cos \theta_1 + L_2 \cos \theta_2 - L_4} \] (39)

Thus, equations 38 and 39 can be applied to determine angular positions of links 2 and 3 respectively if, the physical linkage parameters and the input angle are known. There are however some points worthy of further discussion. For example, note the +/- sign in equation 38 allows two mathematical solutions to exist. The two mathematical solutions actually correspond to the solution of a real linkage since any one four bar linkage may occupy two different positions depending upon the initial assemblage. An easy way of determining which sign is correct, is to verify the angular positions using a simple graphical solution.

The velocity analysis merely consists of taking a time derivative of equation 29, and solving for the angular velocity of link 2. The time derivative is given as:

\[ -L_1 L_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) + L_4 L_2 \dot{\theta}_1 \sin \theta_1 + L_4 L_2 \dot{\theta}_2 \sin \theta_2 = 0 \] (40)

Solving for \( \dot{\theta}_2 \),

\[ \dot{\theta}_2 = \dot{\theta}_1 \left[ \frac{L_4 L_2 \sin(\theta_1 - \theta_2) - L_4 \sin \theta_1}{L_2 L_1 \sin(\theta_1 - \theta_2) + L_4 \sin \theta_2} \right] \] (41)

The angular velocity of link 3 is found in a similar fashion. First, equations 27 and 28 must be arranged so that the \( L \) terms are isolated. Squaring each and adding the results will
then produce the following:

{ \frac{2}{L_1} + \frac{2}{L_3} - \frac{2}{L_2} + \frac{2}{L_4} - 2L_1L_4\cos \theta_1 \\
+ 2L_3L_4\cos \theta_3 - 2L_1L_3\cos (\theta_1 - \theta_3) } = 0 

(42)

Taking a time derivative of equation 42 will facilitate obtaining \( \dot{\theta}_3 \). Thus,

\[ L_1L_4\dot{\theta}_1\sin \theta_1 - L_3L_4\dot{\theta}_3\sin \theta_3 + L_1L_3(\ddot{\theta}_1 - \ddot{\theta}_3)\sin(\theta_1 - \theta_3) = 0 \]

(43)

Solving for \( \dot{\theta}_3 \),

\[ \dot{\theta}_3 = \dot{\theta}_1 \frac{L_3\sin(\theta_1 - \theta_3) + L_4\sin \theta_1}{L_3\sin(\theta_1 - \theta_3) + L_4\sin \theta_3} \]

(44)

The derivation of angular acceleration for links 2 and 3 is accomplished by taking time derivatives of equations 41 and 44 respectively. The derivation will be made easier with the following definitions. Let,

\[ D(\theta_1, \theta_2) = \begin{bmatrix} L_1 \sin(\theta_1 - \theta_2) - L_4 \sin \theta_1 \\ L_2 \sin(\theta_1 - \theta_2) + L_4 \sin \theta_2 \end{bmatrix} \]

(45)

\[ E(\theta_1, \theta_3) = \begin{bmatrix} L_1 \sin(\theta_1 - \theta_3) + L_4 \sin \theta_1 \\ L_3 \sin(\theta_1 - \theta_3) + L_4 \sin \theta_3 \end{bmatrix} \]

(46)

Equations 41 and 44 may now be rewritten in terms of D and E.

\[ \dot{\theta}_2 = D(\theta_1, \theta_2) \dot{\theta}_1 \]

(47)

\[ \dot{\theta}_3 = E(\theta_1, \theta_3) \dot{\theta}_1 \]

(48)

Angular accelerations can now be found by taking time derivatives of the above two equations, or:
\[ \ddot{\theta}_2 = \dot{D}(\theta_1, \theta_2) \dot{\theta}_1 + D(\theta_1, \theta_2) \ddot{\theta}_1 \quad (49) \]
\[ \ddot{\theta}_3 = \dot{E}(\theta_1, \theta_3) \dot{\theta}_1 + E(\theta_1, \theta_3) \ddot{\theta}_1 \quad (50) \]

where,

\[
\begin{align*}
D(\theta_1, \theta_2) &= \left\{ \begin{array}{l}
\frac{L_2(\dot{\theta}_1 - \dot{\theta}_2) \cos(\theta_1 - \theta_2) - L_4 \dot{\theta}_1 \cos \theta_1}{L_1 \sin(\theta_1 - \theta_2) + L_4 \sin \theta_2} + \\
\frac{L_2 \sin(\theta_1 - \theta_2) - L_4 \sin \theta_1}{L_1 \sin (\theta_1 - \theta_2) + L_4 \sin \theta_2} \frac{[L_2(\dot{\theta}_1 - \dot{\theta}_2) \cos(\theta_1 - \theta_2) + L_4 \dot{\theta}_2 \cos \theta_2]}{[L_1 \sin (\theta_1 - \theta_2) + L_4 \sin \theta_2]^2}
\end{array} \right. \\
E(\theta_1, \theta_3) &= \left\{ \begin{array}{l}
\frac{L_3(\dot{\theta}_1 - \dot{\theta}_3) \cos(\theta_1 - \theta_3) + L_4 \dot{\theta}_3 \cos \theta_1}{L_1 \sin(\theta_1 - \theta_3) + L_4 \sin \theta_3} + \\
\frac{L_3 \sin(\theta_1 - \theta_3) + L_4 \sin \theta_1}{L_1 \sin (\theta_1 - \theta_3) + L_4 \sin \theta_3} \frac{[L_3(\dot{\theta}_1 - \dot{\theta}_3) \cos(\theta_1 - \theta_3) + L_4 \dot{\theta}_3 \cos \theta_1]}{[L_1 \sin (\theta_1 - \theta_3) + L_4 \sin \theta_3]^2}
\end{array} \right. 
\end{align*}
\]

Equations for the angular position, velocity and acceleration of links 2 and 3 are given by equations 38, 39, 47, 48, 49 and 50. Once the specifications of the input and link lengths are known, then all of these kinematic parameters are capable of being determined. In the next two sections, the Newtonian and Lagrangian approaches will assume that all the kinematic parameters have already been determined.
4.3 Newtonian Approach

The Newtonian approach uses Euler's first and second laws of motion to derive equations for bearing forces and couples. The laws will be applied to each of the three moving links, which have two masses each. As shown in Figure 6, the general configuration has external forces and torques applied to it. Forces $F_x$ and $F_y$ are applied to point "a" on the follower link, while a loading torque, $T_L$, and a driving torque, $T_D$, are applied to joints $O_3'$ and $O_1'$ respectively. There are four unknown bearing forces, each with spatial $X$ and $Y$ components. Therefore, eight unknown components of the bearing forces need to be determined. In addition the input torque is assumed to be unknown. Once each of the links has been analyzed separately, the four bar linkage will be analyzed as a whole using the combined approach.

Euler's first law of motion as applied to a system of $n$ point masses may be written as:

$$ \overline{F}_R = \sum_{k=1}^{n} m_k \overline{r}_k \quad \text{(53)} $$

where,

- $\overline{F}_R$ is the resultant force vector of externally applied loads.
- $m_k$ is the mass of the $k$th point mass of the system.
- $\overline{r}_k$ is the absolute acceleration of the $k$th point mass.
Figure 6: Free Body Diagram of a General Two Point Mass Model
Considering anyone of the 3 moving links as the system, equation 53 can be rewritten as:

$$ F_{Ri} = \sum_{k=1}^{2} m_{ik} \ddot{r}_{ik} $$  \hspace{1cm} (54) \\

where \( i = 1, 2, 3 \). Euler's second law of motion for a system of \( n \) point masses may be written in general as:

$$ \dot{C}_{OR} = \sum_{k=1}^{n} \overrightarrow{r}_{0'k} \times m_k \ddot{r}_k $$  \hspace{1cm} (55) \\

where,

- \( \dot{C}_{OR} \) is the resultant couple vector, or moment about the fixed joint \( O' \).
- \( \overrightarrow{r}_{0'k} \) is the absolute position vector from point \( O' \) to the \( k \)th particle of the system.

For a system of 2 particles, or point masses, rigidly fixed together and rotating about a frame joint, Euler's second law reduces to:

$$ \dot{C}_{OR} = \sum_{k=1}^{2} \overrightarrow{r}_{ik/O'} \times m_{ik} \ddot{r}_{ik} $$  \hspace{1cm} (56) \\

where \( i = 1, 3 \). Equation 56 is adequate for the application of Euler's second law to either of links 1 or 3, since they rotate about a frame joint. However, link 2 does not rotate about a fixed joint and must be analyzed using a different equation. Appendix A shows the derivation of Euler's second law of motion for this case with the end result being:
\[ C_{o_2R} = \sum_{k=1}^{2} \left[ \frac{r}{2k/o_2} \times m_{2k} \frac{\ddot{r}}{2k/o_2} + \frac{r}{2k/o_2} \times m_{2k} \frac{\dddot{r}}{o_2} \right] \quad (57) \]

where,

- \( C_{o_2R} \) is the resultant couple vector about the point \( O_2 \).
- Point \( O_2 \) is not fixed, but may have a velocity and acceleration vector.
- \( r_{-2k/o_2} \) is the position vector of the \( k \)th point mass on link 2 with respect to the point \( O_2 \).
- \( \dddot{r}_{-2k/o_2} \) is the relative angular acceleration of the \( k \)th point mass on link 2 with respect to point \( O_2 \).
- \( \dddot{r}_{-0_2} \) is the absolute acceleration vector of point \( O_2 \).

Applying Euler's first law, as stated in equation 54, to link 1:

\[ -F_R1 = \sum_{k=1}^{2} m_{1k} \dddot{r}_{1k} \quad (58) \]

where,

\[ F_R1 = (F_{41x} + F_{21x}) \ i' + (F_{41y} + F_{21y}) \ j' \quad (59) \]

\[ \sum_{k=1}^{2} m_{1k} \dddot{r}_{1k} = m_{11} \dddot{r}_{11} + m_{12} \dddot{r}_{12} \quad (60) \]

The absolute acceleration vectors, \( \dddot{r}_{11} \), and, \( \dddot{r}_{12} \) can be found by taking time derivatives of the absolute position vectors, \( \dot{r}_{11/0_1} \), and \( \dot{r}_{12/0_1} \). Where,

\[ \dot{r}_{11/0_1} = \frac{1}{11} (\cos \theta_1 \ i' + \sin \theta_1 \ j') \quad (61) \]
\[ \frac{r_{12}/0_2}{i} = l_{12}(-\sin \Theta_1 i' + \cos \Theta_1 j') \quad (62) \]

Taking two time derivatives of equations 61 and 62 yields:

\[ \ddot{r}_{11} = l_{11} [(-\dot{\Theta}_1 \cos \Theta_1 - \dot{\Theta}_1 \sin \Theta_1)i' + (-\dot{\Theta}_1 \sin \Theta_1 + \dot{\Theta}_1 \cos \Theta_1)j'] \quad (63) \]
\[ \ddot{r}_{12} = l_{12} [(-\dot{\Theta}_1 \sin \Theta_1 + \dot{\Theta}_1 \cos \Theta_1)i' + (-\dot{\Theta}_1 \cos \Theta_1 - \dot{\Theta}_1 \sin \Theta_1)j'] \quad (64) \]

Substituting equations 59 and 60 into equation 58 gives:

\[ F_{41} + F_{21} = m_{11} \ddot{r}_{11} + m_{12} \ddot{r}_{12} \quad (65) \]

Substituting equations 63 and 64 into equation 65, after dotting with unit vectors \( \mathbf{i}' \) and \( \mathbf{j}' \) respectively produces:

\[ F_{41x} + F_{21x} = -m_{11} \ddot{r}_{11} (\dot{\Theta}_1 \cos \Theta_1 + \dot{\Theta}_1 \sin \Theta_1) + m_{12} \ddot{r}_{12} (\dot{\Theta}_1 \cos \Theta_1 - \dot{\Theta}_1 \sin \Theta_1) \quad (66) \]
\[ F_{41y} + F_{21y} = m_{11} \ddot{r}_{11} (-\dot{\Theta}_1 \sin \Theta_1 + \dot{\Theta}_1 \cos \Theta_1) - m_{12} \ddot{r}_{12} (\dot{\Theta}_1 \cos \Theta_1 + \dot{\Theta}_1 \sin \Theta_1) \quad (67) \]

To apply Euler's second law of motion to link 1, equation 56 is used, where:

\[ \sum_{k=1}^{2} \sum_{k=1}^{1} r_{1k/0_1} \cdot m_{k-1k} \ddot{r}_{k} = \sum_{k=1}^{1} r_{1k/0_1} \cdot m_{1k-1k} \ddot{r}_{1k-1k} + r_{12/0_1} \cdot m_{12} \ddot{r}_{12} \quad (69) \]

Equations for the mass-acceleration products and position vectors have previously been specified for link 1. So, equation 69 can be determined using equations.
61, 62, 63 and 64.

\[
\begin{bmatrix}
\mathbf{r}_{11/0} & \mathbf{m}_{11} & \mathbf{r}_{11} = \mathbf{m}_{11/11}
\end{bmatrix}
\begin{bmatrix}
\mathbf{i}' & \mathbf{j}' & \mathbf{k}'
\end{bmatrix}
\begin{bmatrix}
\cos\theta_1 & 0 & \sin\theta_1 \\
0 & \cdot & 0 \\
(-\dot{\theta}_1\cos\theta_1 - \dot{\theta}_1\sin\theta_1) & (-\dot{\theta}_1\sin\theta_1 + \dot{\theta}_1\cos\theta_1)
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]  

(70)

or,

\[
\mathbf{r}_{11/0} \times \mathbf{m}_{11} \mathbf{r}_{11} = \mathbf{m}_{11/11} \theta_1 \mathbf{k}'
\]  

(71)

and,

\[
\begin{bmatrix}
\mathbf{r}_{12/0} & \mathbf{m}_{12} & \mathbf{r}_{12} = \mathbf{m}_{12/12}
\end{bmatrix}
\begin{bmatrix}
\mathbf{i}' & \mathbf{j}' & \mathbf{k}'
\end{bmatrix}
\begin{bmatrix}
\sin\theta_1 & -\cos\theta_1 & 0 \\
0 & \cdot & 0 \\
(-\dot{\theta}_1\sin\theta_1 + \dot{\theta}_1\cos\theta_1) & (-\dot{\theta}_1\cos\theta_1 - \dot{\theta}_1\sin\theta_1)
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]  

(72)

or,

\[
\mathbf{r}_{12/0} \times \mathbf{m}_{12} \mathbf{r}_{12} = \mathbf{m}_{12/12} \theta_1 \mathbf{k}'
\]  

(73)

Substituting equations 68, 71 and 73 back into equation 56, and dotting with unit vector \( \mathbf{k}' \) yields:

\[
L\left[ F_{1x} \cos\theta_1 - F_{21y} \sin\theta_1 \right] + T_D = \mathbf{I}_{1}^{0}\theta_1
\]

(74)

where,

\[
\mathbf{I}_{1}^{0} = \mathbf{m}_{11/11} \mathbf{I}_{11}^2 + \mathbf{m}_{12/12} \mathbf{I}_{12}^2
\]

(75)
Applying Euler's first law of motion to link 2,

\[ F_{R2} = \sum_{k=1}^{2} m_{2k} \ddot{r}_{2k} \]  

(76)

where,

\[ F_{R2} = -(F_{21x} + F_{32x}) i' - (F_{21y} + F_{32y}) j' \]  

(77)

\[ \sum_{k=1}^{2} m_{2k} \ddot{r}_{2k} = m_{21} \ddot{r}_{21} + m_{22} \ddot{r}_{22} \]  

(78)

As with link 1, the absolute acceleration vectors of the link 2 masses are obtained by taking time derivatives of their absolute position vectors. The position vectors for the masses in link 2 are given by:

\[ \ddot{r}_{21/01}^{02} = \ddot{r}_{21}^{02} + l_{21} \left[ \cos \theta_{2} i' + \sin \theta_{2} j' \right] \]  

(79)

\[ \ddot{r}_{22/01}^{02} = \ddot{r}_{22}^{02} + l_{22} [-\sin \theta_{2} i' + \cos \theta_{2} j'] \]  

(80)

Taking two time derivatives of each of the above equations,

\[ \dddot{r}_{21} = \dddot{r}_{21}^{02} + l_{21} \left[ (-\Theta \cos \Theta - \Theta \sin \Theta) i' + (-\Theta \sin \Theta + \Theta \cos \Theta) j' \right] \]  

(81)

\[ \dddot{r}_{22} = \dddot{r}_{22}^{02} + l_{22} \left[ (\Theta \sin \Theta - \Theta \cos \Theta) i' - (\Theta \cos \Theta + \Theta \sin \Theta) j' \right] \]  

(82)

where,

\[ \dddot{r}_{02} = l_{1} \left[ (-\dot{\Theta} \cos \Theta - \dot{\Theta} \sin \Theta) i' + (-\dot{\Theta} \sin \Theta + \dot{\Theta} \cos \Theta) j' \right] \]  

(83)
Substituting equations 77 and 78 into equation 76,

\[(F_{21x} + F_{32x}) \dot{i}' + (F_{21y} + F_{32y}) \dot{j}' = -m_{21} \ddot{r}_{21} - m_{22} \ddot{r}_{22} \]  

(84)

Substituting equations 81, 82 and 83 into the above equation after dotting with unit vectors \( \hat{i}' \) and \( \hat{j}' \) respectively produces:

\[
F_{21x} + F_{32x} = \{(m + m_{21}) L \left( \ddot{\theta}_{2} \cos \theta + \ddot{\theta}_{2} \sin \theta \right) \\
+ m_{21} \dot{\theta}_{2} \cos \theta_{2} + \dot{\theta}_{2} \sin \theta_{2} \} + m \left( -\ddot{\theta}_{2} \sin \theta + \ddot{\theta}_{2} \cos \theta \right) \]  

(85)

\[
F_{21y} + F_{32y} = \{(m + m_{21}) L \left( \ddot{\theta}_{2} \sin \theta - \ddot{\theta}_{2} \cos \theta \right) \\
+ m_{21} \dot{\theta}_{2} \sin \theta_{2} - \dot{\theta}_{2} \cos \theta_{2} \} + m \left( \ddot{\theta}_{2} \cos \theta + \ddot{\theta}_{2} \sin \theta \right) \]  

(86)

Euler's second law of motion, as stated in equation 57, can be applied to link 2. Referring to the free body diagram, Figure 6, the sum of couples on link 2 at joint \( O_z \) is given by:

\[
C_{O_z} = \{ F_{32x} L \sin \theta - F_{32y} L \cos \theta \} k' \]  

(87)

The first term of equation 57 represents the time derivative of the relative angular momentum of the two point masses in link 2 about point \( O_z \), and is given by:

\[
\sum_{k=1}^{2} \left\{ \frac{x m \ddot{r}_{2k}}{2k/0_2} \right\} = \left\{ \frac{x m \ddot{r}_{21}}{21/0_2} + \frac{x m \ddot{r}_{22}}{22/0_2} \right\} \]  

(88)
The last term in equation 57 can be stated as:

\[
\sum_{k=1}^{2} \mathbf{r}_{2k/0_2} \times \mathbf{m}_{2k} \mathbf{r}_{0_2} = \mathbf{r}_{21/0_2} \times \mathbf{m}_{21} \mathbf{r}_{0_2} + \mathbf{r}_{22/0_2} \times \mathbf{m}_{22} \mathbf{r}_{0_2}
\] (89)

where,

\[
\mathbf{r}_{21/0_2} = l_{21} (\cos \theta_2 i' + \sin \theta_2 j')
\] (90)

\[
\mathbf{r}_{22/0_2} = l_{22} (-\sin \theta_2 i' + \cos \theta_2 j')
\] (91)

Taking two time derivatives of each of the above two equations produces,

\[
\ddot{\mathbf{r}}_{21/0_2} = l_{21} \left[ (\dot{\theta}_2 \cos \theta_2 - \dot{\theta}_2 \sin \theta_2) \mathbf{i} + (\dot{\theta}_2 \sin \theta_2 + \dot{\theta}_2 \cos \theta_2) \mathbf{j} \right]
\] (92)

\[
\ddot{\mathbf{r}}_{22/0_2} = l_{22} \left[ (\dot{\theta}_2 \sin \theta_2 - \dot{\theta}_2 \cos \theta_2) \mathbf{i} + (\dot{\theta}_2 \cos \theta_2 - \dot{\theta}_2 \sin \theta_2) \mathbf{j} \right]
\] (93)

The equation for the time derivative of the relative angular momentum for link 2 about point 0_2 can be found by substituting equations 90, 91, 92 and 93 into equation 88.

\[
\sum_{k=1}^{2} \mathbf{r}_{2k/0_2} \times \mathbf{m}_{2k} \mathbf{r}_{0_2} = \begin{bmatrix} m_{21} l_{21} & \mathbf{i}' & \mathbf{j}' & k' \\ l_{21} \cos \theta_2 & \sin \theta_2 & 0 \\ \dot{\theta}_2 \cos \theta_2 - \dot{\theta}_2 \sin \theta_2 & \dot{\theta}_2 \sin \theta_2 + \dot{\theta}_2 \cos \theta_2 & 0 \\ \dot{\theta}_2 \sin \theta_2 - \dot{\theta}_2 \cos \theta_2 & \dot{\theta}_2 \cos \theta_2 - \dot{\theta}_2 \sin \theta_2 & 0 \end{bmatrix} + \begin{bmatrix} m_{22} l_{22} & \mathbf{i}' & \mathbf{j}' & k' \\ -\sin \theta_2 & \cos \theta_2 & 0 \\ \dot{\theta}_2 \sin \theta_2 - \dot{\theta}_2 \cos \theta_2 & \dot{\theta}_2 \cos \theta_2 - \dot{\theta}_2 \sin \theta_2 & 0 \end{bmatrix}
\] (94)
or,

\[ \sum_{k=1}^{2} I_{2k/02} \cdot m_{2k/02} = I_{2}^{'02} \cdot k' \]  \hspace{1cm} (95)

where,

\[ I_{2}^{'02} = m_{21} l_{21}^{'2} + m_{22} l_{22}^{'2} \]  \hspace{1cm} (96)

The last term of equation 57 can be determined if equations 83, 90, and 91 are substituted into equation 89. Thus,

\[ \sum_{k=1}^{2} I_{2k/02} \cdot m_{2k/02} = m_{21} l_{21}^{'2} \cdot L_{1} \cdot \left( \begin{array}{ccc} \cos \theta_{2} & \sin \theta_{2} & 0 \\ (-\theta_{1} \cos \theta_{1} - \theta_{2} \sin \theta_{1}) & (-\theta_{1} \sin \theta_{1} + \theta_{1} \cos \theta_{1}) & 0 \\ i' & j' & k' \end{array} \right) \]  \hspace{1cm} (97)

or,

\[ \sum_{k=1}^{2} I_{2k/02} \cdot m_{2k/02} = \left\{ L_{1} \cdot m_{21} l_{21}^{'2} \cdot \left[ \theta_{1} \cos(\theta_{1} - \theta_{2}) - \theta_{2} \sin(\theta_{1} - \theta_{2}) \right] + m_{22} l_{22}^{'2} \cdot \left[ \theta_{1} \sin(\theta_{1} - \theta_{2}) - \theta_{2} \cos(\theta_{1} - \theta_{2}) \right] \right\} k' \]  \hspace{1cm} (98)

Now, equation 57 can be evaluated by back substitution of equations 87, 95 and 98.

\[ L_{2}[F_{32x} \sin \theta_{2} - F_{32y} \cos \theta_{2}] = \{ m_{21} l_{21}^{'2} \cdot L_{1} \left[ \theta_{1} \cos(\theta_{1} - \theta_{2}) - \theta_{2} \sin(\theta_{1} - \theta_{2}) \right] + m_{22} l_{22}^{'2} \cdot L_{1} \left[ \theta_{1} \sin(\theta_{1} - \theta_{2}) + \theta_{2} \cos(\theta_{1} - \theta_{2}) \right] \} k' \]  \hspace{1cm} (99)
Referring back to Figure 6, the application of Euler's first and second laws to link 3 will be very similar to the analysis already presented for link 1. Both rotate about a frame joint, and have 2 bearing forces and a torque applied to that joint. The only difference, is that link 3 has an external force applied to it. This results in an additional term. Thus, except for the external force, application of Euler's laws of motion to link 3 will provide equations of the same form as for link 1. Substituting the parameters of link 3 into equations 66, 67, 74, and taking into account the external force applied to link 3 produces the following equations.

\[
F_x + F_{43x} + F_{32x} = -m_{31}l_{31}(\theta_3 \cos \theta_3 + \theta_3 \sin \theta_3) + m_{32}l_{32}(\theta_3 \sin \theta_3 - \theta_3 \cos \theta_3)
\] (100)

\[
F_y + F_{43y} + F_{32y} = m_{31}l_{31}(\theta_3 \sin \theta_3 + \theta_3 \cos \theta_3) - m_{32}l_{32}(\theta_3 \cos \theta_3 + \theta_3 \sin \theta_3)
\] (101)

\[
l_a (-F_x \sin \theta + F_y \cos \theta) + L_1 (F_{32y} \cos \theta_3 - F_{32x} \sin \theta_3) + T_L = I_3 \ddot{\theta}_3
\] (102)

where,

\[
I_3 = m_{31}l_{31} + m_{32}l_{32}
\] (103)

The end result of the Newtonian approach is nine equations (ie. equations 66, 67, 74, 85, 86, 99, 100, 101, 102) which specify the relations between externally applied forces, and the point mass accelerations. If the physical
geometry and kinematics were known, nine variables would still remain; two components of each of the four bearing forces and the input torque. At this point a numerical elimination process, such as the Gauss-Seidel method, could be employed to find a solution to the nine unknowns. However, the solution would only be good for one geometry. The goal of this thesis is to find the optimum design, which usually requires analyzing many different geometries (often on the order of 50 to 300). Using the Gauss-Seidel method, or most other common numerical techniques, would not be very efficient for solving the optimization problem.

Lee and Cheng simplified the solution of these nine equations using a combined approach. This method, called the "direct" approach, uses the Lagrangian to first determine an expression for the input torque. The resulting equation is only a function of externally applied forces, and inertial terms generated by the linkages motion. Thus, the Lagrangian approach is independent of the Newtonian approach and provides a quick, closed form solution for the input torque. Once the input torque is known, then the bearing forces can be found using the remaining nine equations of the Newtonian approach, and a back substitution method.

To simplify the derivation of the input torque by the Lagrangian approach, normalized parameters are used. Since expressions from the Newtonian approach will later have to be combined with the Lagrangian equation, they too will have to be normalized. Definitions of these normalizing parameters
are given in the nomenclature, and are summarized below.

The normalized force, $F^*$, and the normalized torque, $T^*$, are defined as:

$$F^* = \frac{F}{(m_{11} + m_{12}) L_1 \theta_1}$$  \hspace{1cm} (104)

$$T^* = \frac{T}{(m_{11} + m_{12})^2 L_1 \dot{\theta}_1}$$  \hspace{1cm} (105)

The normalized mass, $M_{ij}^*$, is defined by:

$$M_{ij}^* = \frac{m_{ij}}{(m_{11} + m_{12})}$$  \hspace{1cm} (106)

where, $i = 1,2,3$, and, $j = 1,2$.

Normalized distance to the mass, $d_{ij}^*$, is given by:

$$d_{ij}^* = \frac{d_{ij}}{L_1}$$  \hspace{1cm} (107)

where, $i = 1,2,3$, and, $j = 1,2$.

Also, the normalized length of each link is defined as:

$$D_i^* = \frac{L_i}{L_1}$$  \hspace{1cm} (108)

where $i = 1,2,3,4$.

The equations can be further simplified by defining a normalized mass, distance product, $P_{ij}$ given by:
\[ P_{ij}^* = M_{ij}^* d_{ij} \]  

(109)

The normalized mass moment of inertia, \( I_i^* \), can be defined as:

\[ I_i^* = \frac{I_i^0}{I_i} \]  

(110)

or, in terms of the normalized two point mass model parameters,

\[ I_i^* = P_i d_{i1} + P_i d_{i2} \]  

(111)

The kinematic parameters of angular velocity and acceleration are normalized using the input angular velocity, \( \dot{\theta}_j^* \).

\[ i_j^* = \frac{\dot{\theta}_j}{\dot{\theta}_j} \]  

(112)

\[ \alpha_j^* = \frac{\ddot{\theta}_j}{\dot{\theta}_j^2} \]  

(113)

where \( i = 1, 2, 3 \).

Now that the normalized parameters have been defined, the nine equations of the Newtonian approach can be normalized. A summary of these equations is given below.

\[ F_{11}^* + F_{12}^* = -P_{11}^* (\alpha_{11} \sin \theta + \cos \theta) + P_{12}^* (-\alpha_{12} \cos \theta + \sin \theta) \]  

(114)

\[ F_{11}^* + F_{12}^* = P_{11}^* (\alpha_{11} \cos \theta - \sin \theta) - P_{12}^* (\alpha_{12} \sin \theta + \cos \theta) \]  

(115)

\[ F_{11}^* \cos \theta - F_{12}^* \sin \theta + T_{11}^* = I_{11}^* \alpha_{11}^* \]  

(116)
\[ F_{21x} + F_{32x} = \left\{ (M_{21} + M_{22}) (\alpha \sin \theta_1 + \cos \theta_1) + P_{21} (\alpha \sin \theta_2 + i \cos \theta_2^*) + P_{22} (\alpha \cos \theta_2 - i \sin \theta_2^*) \right\} \quad (117) \]

\[ F_{21y} + F_{32y} = \left\{ (M_{21} + M_{22}) (-\alpha \cos \theta_1 + \sin \theta_1) + P_{21} (-\alpha \cos \theta_2 + i \sin \theta_2^*) + P_{22} (\alpha \sin \theta_2 + i \cos \theta_2^*) \right\} \quad (118) \]

\[ F^* D^* \sin \theta - F^* D^* \cos \theta = \left\{ I_{22}^* + P_{21}^* [\alpha^* \cos (\theta_1 - \theta_2) - \sin (\theta_1 - \theta_2)] + P_{22}^* [\alpha^* \sin (\theta_1 - \theta_2) + \cos (\theta_1 - \theta_2)] \right\} \quad (119) \]

\[ F_{43x} + F_{32x} = -F_x - P_{31} (\alpha_3 \sin \theta_3 + i \cos \theta_3) + P_{32} (-\alpha_3 \cos \theta_3 + i \sin \theta_3) \quad (120) \]

\[ F_{43y} + F_{32y} = -F_y + P_{31} (\alpha_3 \cos \theta_3 - i \sin \theta_3) - P_{32} (\alpha_3 \cos \theta_3 + i \sin \theta_3) \quad (121) \]

\[ D_{3} (F_{32} \cos \theta_3 - F_{32} \sin \theta_3) = I_{3} \alpha_3^* - T_L^* + d_a (F_x \sin \beta - F_y \cos \beta) \quad (122) \]
4.4 Lagrangian Approach

The Lagrangian analysis is used to formulate an expression for the input torque which is only dependent upon the linkages inertial motion, and externally applied forces. The resulting expression is also independent of the equations generated in the Newtonian approach. For a conservative system with one degree of freedom, the general form of the Lagrange equation may be stated as:

\[
\frac{d}{dt} \left[ \frac{\partial E_T}{\partial \dot{\theta}_1} \right] - \frac{\partial E_T}{\partial \dot{\theta}_1} = T \dot{\theta}_1
\]  

(123)

where,

- \( E_T \) is the total kinetic energy of the linkage system. Potential energy is not included since the effects of gravity have been ignored.
- \( T \dot{\theta}_1 \) is the generalized external force.

The total kinetic energy is a result of the motion of links 1, 2 and 3, or,

\[
E_T = \sum_{i=1}^{3} E_i
\]  

(124)

where,

\[
E_i = \sum_{j=1}^{2} \frac{1}{2} m_{ij} (\dot{r}_{ij} \cdot \dot{r}_{ij})
\]  

(125)
The absolute velocity vector, \( \dot{r}_{ij} \), in the above equation can be obtained by taking a time derivative of the absolute position vectors derived in the Newtonian approach. The resultant expressions for the absolute velocity of the point masses may be stated as:

\[
\begin{align*}
\dot{r}_{11} &= \dot{\theta}_1 (-\sin \theta_1 \dot{i} + \cos \theta_1 \dot{j}) \\
\dot{r}_{12} &= \dot{\theta}_2 (-\cos \theta_1 \dot{i} - \sin \theta_1 \dot{j}) \\
\dot{r}_{21} &= L \dot{\theta}_1 (-\sin \theta_1 \dot{i} + \cos \theta_1 \dot{j}) + 1 \dot{\theta}_2 (-\sin \theta_1 \dot{i} + \cos \theta_1 \dot{j}) \\
\dot{r}_{22} &= L \dot{\theta}_1 (-\sin \theta_1 \dot{i} + \cos \theta_1 \dot{j}) + 1 \dot{\theta}_2 (-\cos \theta_2 \dot{i} - \sin \theta_2 \dot{j}) \\
\dot{r}_{31} &= \dot{\theta}_3 (-\sin \theta_3 \dot{i} + \cos \theta_3 \dot{j}) \\
\dot{r}_{32} &= 1 \dot{\theta}_3 (-\cos \theta_3 \dot{i} - \sin \theta_3 \dot{j})
\end{align*}
\]

Substituting the above equations into equation 125 produces:

\[
\begin{align*}
E_1 &= \frac{1}{2} \dot{\theta}_1^2 (m_{11} \dot{\theta}_1^2 + m_{12} \dot{\theta}_2^2) \\
E_2 &= \frac{1}{2} \left[ L \dot{\theta}_1 (m_{21} + m_{22}) + \dot{\theta}_2 (m_{21} \dot{\theta}_1^2 + m_{22} \dot{\theta}_2^2) + 2 L \dot{\theta}_1 \dot{\theta}_2 [m_{21} \dot{\theta}_1 \cos (\theta_1 - \theta_2) + m_{22} \dot{\theta}_2 \sin (\theta_1 - \theta_2)] \right] \\
E_3 &= \frac{1}{2} \dot{\theta}_3^2 (m_{31} \dot{\theta}_1^2 + m_{32} \dot{\theta}_2^2)
\end{align*}
\]

Realizing that the mass moment of inertia of any link about one of its bearing joints may be written as:
\[ I_i = m_i l_i^2 + m_{i2} l_{i2} \]  

(135)

Then equations 132, 133, 134 may be simplified, and combined to form an expression for the total kinetic energy of the linkage.

\[
E_T = \frac{1}{2} \left( I_1 \dot{\theta}_1^2 + I_2 \dot{\theta}_2^2 + I_3 \dot{\theta}_3^2 + L_1 \dot{\theta}_1 (m_{21} + m_{22}) + 2L_1 \dot{\theta}_1 \dot{\theta}_2 [m_{21} l_{21} \cos(\theta_1 - \theta_2) + m_{22} l_{22} \sin(\theta_1 - \theta_2)] \right)
\]

(136)

To simplify the remainder of the Lagrangian analysis, normalized equations will be used. All the normalized parameters, except for the normalized kinetic energy, were defined in equations 104 through 113 of the Newtonian approach section. The normalized kinetic energy is simply given by:

\[
E_T^* = \frac{E_T}{(m_{11} + m_{12}) L_1^2 \dot{\theta}_1^2}
\]

(137)

The normalized form of equation 136 may be written as:

\[
E_T^* = \frac{1}{2} [I_T^*]
\]

(138)

where,

\[
I_T^* = \{ I_1^* + I_2^* l_{12}^2 + I_3^* l_{33}^2 + M_{21}^* + M_{22}^* + 2l_{12}^* [p_{21}^* \cos(\theta_1 - \theta_2) + p_{22}^* \sin(\theta_1 - \theta_2)] \}
\]

(139)

Substitution of equations 137 and 138 into equation 123 will give the normalized version of the Lagrange equation.
\[
\frac{d}{dt} \left[ \frac{\partial}{\partial \dot{\Theta}_1} \left[ \frac{1}{2} (m_{11} + m_{12}) L_1 \dot{\Theta}_1 I_T^* \right] \right] - \frac{\partial}{\partial \Theta_1} \left[ \frac{1}{2} (m_{11} + m_{12}) L_1 \dot{\Theta}_1 I_T^* \right] = T_{\Theta_1}
\]

(140)

As evident from equation 139, \( I_T^* \) is not a function of \( \dot{\Theta}_1 \). Therefore, the normalized Lagrange equation may be rewritten as:

\[
(m_{11} + m_{12}) L_1 \frac{d}{dt} \left[ \dot{\Theta}_1 I_T^* \right] - \frac{1}{2} (m_{11} + m_{12}) L_1 \dot{\Theta}_1 \frac{\partial}{\partial \Theta_1} [I_T^*] = T_{\Theta_1}
\]

(141)

Performing the necessary partial and full derivatives,

\[
(m_{11} + m_{12}) L_1 \frac{d}{dt} \left[ \dot{\Theta}_1 I_T^* + \ddot{\Theta}_1 \frac{dt}{I_T^*} \right] - \frac{1}{2} (m_{11} + m_{12}) L_1 \dot{\Theta}_1 \frac{\partial}{\partial \Theta_1} \left[ I_T^* \right] = T_{\Theta_1}
\]

(142)

where the partial derivative in equation 141 becomes a full derivative since there is only one independent variable, \( \Theta_1 \).

By the chain rule,

\[
\frac{di}{dt}^* = \frac{dt}{d\Theta_1} \frac{di}{d\Theta_1} \quad \frac{d\Theta_1}{dt} = \dot{\Theta}_1 \frac{dt}{d\Theta_1}
\]

(143)

Substituting the above results into equation 142, and normalizing (ie. dividing by \( \Theta_1 L_1 (m_{11} + m_{12}) \)), produces the following expression.

\[
\alpha \dot{I}_T^* + \frac{1}{2} \frac{di}{d\Theta_1}^* \frac{dt}{d\Theta_1} = T_{\Theta_1}\]

(144)

where,

\[
T_{\Theta_1}^* = \frac{T_{\Theta_1}}{L_1 \dot{\Theta}_1 (m_{11} + m_{12})}
\]

(145)

Using equation 139, the full derivative of \( I_T^* \) with respect to \( \Theta_1 \)
may given as follows:

\[
\frac{dI^*_2}{d\theta_1} = \left\{ I_2^* \left[ 2i_2^* \frac{di_2}{d\theta_1} \right] + I_3^* \left[ 2i_3^* \frac{di_3}{d\theta_1} \right] + 2 \frac{di_2}{d\theta_1} \left[ P_{21} \cos(\theta_1 - \theta_2) + P_{22} \sin(\theta_1 - \theta_2) \right] + 2i_2(1 - i_2)[-P_{21} \sin(\theta_1 - \theta_2) + P_{22} \cos(\theta_1 - \theta_2)] \right\} \quad (146)
\]

where,

\[
\frac{di_2}{d\theta_1} = \frac{d(\dot{\theta}_2 / \dot{\theta}_1)}{d\theta_1} \quad (147)
\]

However, using the chain rule,

\[
\frac{d(\dot{\theta}_2 / \dot{\theta}_1)}{dt} = \frac{d(\dot{\theta}_2 / \dot{\theta}_1)}{d\theta_1} \cdot \frac{d\theta_1}{dt} \quad (148)
\]

Rearranging the above expression,

\[
\frac{d(\dot{\theta}_2 / \dot{\theta}_1)}{d\theta_1} = \frac{1}{\dot{\theta}_1} \frac{d(\dot{\theta}_2 / \dot{\theta}_1)}{dt} \quad (149)
\]

where,

\[
\frac{d(\dot{\theta}_2 / \dot{\theta}_1)}{dt} = \ddot{\theta}_2 \dot{\theta}_1 - \dot{\theta}_2 \ddot{\theta}_1 \quad (150)
\]

Substituting equations 149 and 150, into equation 147 yields:
\[
\frac{d\phi}{d\theta_1} = \alpha_2 - i_2\alpha_1 \quad (151)
\]

similarly for link 3,
\[
\frac{d\phi}{d\theta_1} = \alpha_3 - i_3\alpha_1 \quad (152)
\]

The full derivative of \( I^* \) with respect to the crank angle \( \theta_1 \) can now be obtained by substituting equations 151 and 152 into equation 146. Or,
\[
\frac{dI^*}{d\theta_1} = 2\{ I_2^* i_2 (\alpha_2 - i_2\alpha_1) + I_3^* i_3 (\alpha_3 - i_3\alpha_1) + \\
(\alpha_2 - i_2\alpha_1) [ P_{21} \cos(\theta_1 - \theta_2) + P_{22} \sin(\theta_1 - \theta_2) ] \\
+ i (1 - i_2) [-P^* \sin(\theta - \theta) + P^* \cos(\theta - \theta) ] \} \\
+ \frac{i^*}{2} \left[ (\alpha_2 - i_2\alpha_1) [ P_{21} \cos(\theta_1 - \theta_2) + P_{22} \sin(\theta_1 - \theta_2) ] \\
+ i (1 - i_2) [-P^* \sin(\theta - \theta) + P^* \cos(\theta - \theta) ] \right] \\
+ \frac{i^*}{2} \left[ (\alpha_2 - i_2\alpha_1) [ P_{21} \cos(\theta_1 - \theta_2) + P_{22} \sin(\theta_1 - \theta_2) ] \\
+ i (1 - i_2) [-P^* \sin(\theta - \theta) + P^* \cos(\theta - \theta) ] \right] \\
\}
\quad (153)
\]

Using the above result, the Lagrange equation can now be obtained.

\[
T^*_\theta_1 = \left\{ \alpha_1^* [ I_1^* + I_2^* i_2 + I_3^* i_3 + M_{21}^* + M_{22}^* \\
+ 2i_2 [ P_{21} \cos(\theta_1 - \theta_2) + P_{22} \sin(\theta_1 - \theta_2) ] ] \right\} + \\
+ \frac{i^*}{2} \left[ (\alpha_2^* - i_2\alpha_1^*) [ P_{21} \cos(\theta_1 - \theta_2) + P_{22} \sin(\theta_1 - \theta_2) ] + \\
+ i (1 - i_2) [-P^* \sin(\theta - \theta) + P^* \cos(\theta - \theta) ] \right] \\
+ \frac{i^*}{2} \left[ (\alpha_2^* - i_2\alpha_1^*) [ P_{21} \cos(\theta_1 - \theta_2) + P_{22} \sin(\theta_1 - \theta_2) ] + \\
+ i (1 - i_2) [-P^* \sin(\theta - \theta) + P^* \cos(\theta - \theta) ] \right] \\
\}
\quad (154)
\]

Simplifying,
\[
T^*_\theta_1 = \left\{ \alpha_1^* [ I_1^* + M_{21}^* + M_{22}^* + i_2 [ P_{21} \cos(\theta_1 - \theta_2) + P_{22} \sin(\theta_1 - \theta_2) ] \right\} + \\
+ \alpha_2^* [ I_2^* i_2 + P_{21} \cos(\theta_1 - \theta_2) + P_{22} \sin(\theta_1 - \theta_2) ] + \\
+ \alpha_3^* i_3^* i_3 + i_2 (1 - i_2) [-P^* \sin(\theta - \theta) + P^* \cos(\theta - \theta) ] \\
\}
\quad (155)
\]
The above expression represents the generalized external force in terms of the linkage motion. This generalized external force may also be expressed in terms of actual external forces through the method of virtual work.

\[ T_{\theta_1} = \frac{\delta W}{\delta \theta_1} \]  \hspace{1cm} (156)

where,

- \( \delta W_1 \) is the external work done on the four bar mechanism during a virtual displacement, \( \delta \theta_1 \).
- \( \delta \theta_1 \) is a small imaginary, or virtual rotation of the crank.

For the situation shown in Figure 7 of the combined approach section,

\[ \delta W_1 = T_L \delta \theta_3 + T_D \delta \theta_1 + l_a (-F_x \sin \beta + F_y \cos \beta) ] \delta \theta_3 \]  \hspace{1cm} (157)

The generalized external force in equation 156 may be rewritten as:

\[ T_{\theta_1} = T_D + \frac{\delta \theta_3}{\delta \theta_1} [T_L + l_a (-F_x \sin \beta + F_y \cos \beta) ] \]  \hspace{1cm} (158)

The above expression is incapable of being evaluated since the virtual displacements are imaginary. Thus, \( \delta \theta_3/\delta \theta_1 \) must be put into a more useful form. The ratio can be defined as the virtual change in \( \theta_3 \) per virtual change in \( \theta_1 \), and can be determined using kinematic equations. Equation 42 of the
kinematic analysis relates the output and input angular displacements by:

\[
\left\{ L_1 - L_2 + L_3 + L_4 - 2L_1L_4\cos\theta_1 + \\
+ 2L_3L_4\cos\theta_3 - 2L_1L_3\cos(\theta_1 - \theta_3) \right\} = 0 \tag{159}
\]

The above equation then can be differentiated to obtain a ratio of the virtual angular displacements,

\[
L_1L_4\delta\theta_1\sin\theta_1 - L_3L_4\delta\theta_3\sin\theta_3 + L_1L_3(\delta\theta_1 - \delta\theta_3)\sin(\theta_1 - \theta_3) = 0 \tag{160}
\]

Or, solving for \( \frac{\delta\theta_3}{\delta\theta_1} \),

\[
\frac{\delta\theta_3}{\delta\theta_1} = \frac{L_1}{L_3}\left[ \frac{L_3\sin(\theta_1 - \theta_3) + L_4\sin\theta_1}{L_1\sin(\theta_1 - \theta_3) + L_4\sin\theta_3} \right] \tag{161}
\]

The right hand side of the above expression is identical to the ratio of the output to the input angular velocities, which is given by equation 44 in the kinematic analysis. Therefore, the ratio of the output to input virtual angular displacements may be given as:

\[
\frac{\delta\theta_3}{\delta\theta_1} = \frac{\dot{\theta}_3}{\dot{\theta}_1} \tag{162}
\]

The final form of the generalized external force may then be stated in terms of the angular velocity ratio given in the above equation.
\[ T_{\theta_1} = T_D + \frac{d}{d\theta_1} [T_L + l_a (-F_x \sin \beta + F_y \cos \beta)] \quad (163) \]

Normalizing,

\[ T_{\theta_1}^* = T_D^* + \frac{i}{3} [t_{\theta_1}^* + d_a^* (-F_x^* \sin \beta + F_y^* \cos \beta)] \quad (164) \]

The end result of the Lagrangian analysis is an expression for the input torque. This can be found by solving the above equation for the input torque, \( T_D^* \), or:

\[ T_D^* = T_{\theta_1}^* - \frac{i}{3} [T_L^* + d_a^* (-F_x^* \sin \beta + F_y^* \cos \beta)] \quad (165) \]

where \( T_{\theta_1}^* \) can be found using equation 155.

If the external loads and the kinematics of the linkage are defined, then equation 165 provides an evaluation of the normalized input torque for any particular geometry. Equations for the bearing forces could then be determined using the nine equations of the Newtonian approach and a back substitution procedure. The combined approach, presented in the next section, shows how the principles of angular momentum can be applied to facilitate this process.
4.5 Combined Approach

The nine equations derived in the Newtonian approach, and the input torque expression obtained using the Lagrangian approach do not provide a direct evaluation of the bearing forces. In fact, these bearing forces are often solved for using a numerical procedure, such as the Gauss-Seidel method. In contrast, the "direct" method proposed by Lee and Cheng [21] provides a closed form solution of all the bearing forces using a combined formulation of both the Newtonian and Lagrangian methods.

The fundamental idea of the combined approach is to analyze the four bar linkage as a whole using Euler's second law of motion. In such a way, the Y components of each ground bearing force are determined. Lagrange's method is used to provide an expression for the input torque, \( T_D \). Expressions for the remaining bearing forces are then solved for using expressions from the Newtonian analysis coupled with a back substitution procedure.

To analyze the linkage as a whole requires a new free body diagram. Figure 7 shows a two point mass model of the linkage separated from its surroundings. Applying Euler's second law of motion to the system of three moving links about the frame joint \( 0_1' \),

\[
C_{0_1'R} = \dot{A}_{0_1'}
\]
Figure 7: Free Body Diagram of a Two Point Mass Model

Separated from its Frame

where,

\[ C_{01' R} = \{ F_{43y} L_4 + T_L + T_D - F_x l_a \sin \beta + F_y (l_a \cos \beta + L_4) \} k' \]  \hspace{1cm} (167)

\[ \dot{\mathbf{A}}_{01'} = \sum_{i=1}^{3} \dot{\mathbf{A}}_{i/01'} \]  \hspace{1cm} (168)

The above equations can be evaluated if the angular momentum of each link about joint 0' can be determined. This angular
momentum can be found using the following relations:

\[ \mathbf{A}_{i/0_1} = \mathbf{r}_{i1/0_1} \times \mathbf{m}_{i1} \hat{\mathbf{r}}_{i1} + \mathbf{r}_{i2/0_1} \times \mathbf{m}_{i2} \hat{\mathbf{r}}_{i2} \]  \hspace{1cm} (169)

where, \( i = 1, 2, 3 \). The angular momentum for link 1 can be found by letting \( i = 1 \) in the above expression, thus:

\[ \mathbf{A}_{1/0_1} = \mathbf{r}_{11/0_1} \times \mathbf{m}_{11} \hat{\mathbf{r}}_{11} + \mathbf{r}_{12/0_1} \times \mathbf{m}_{12} \hat{\mathbf{r}}_{12} \]  \hspace{1cm} (170)

or,

\[ \mathbf{A}_{1/0_1} = \mathbf{m}_{11} \begin{bmatrix} i' & j' & k' \\ \cos \theta_1 & \sin \theta_1 & 0 \\ -\dot{\theta}_1 \sin \theta_1 & \dot{\theta}_1 \cos \theta_1 & 0 \end{bmatrix} + \mathbf{m}_{12} \begin{bmatrix} i' & j' & k' \\ -\sin \theta_1 & \cos \theta_1 & 0 \\ -\dot{\theta}_1 \cos \theta_1 & -\dot{\theta}_1 \sin \theta_1 & 0 \end{bmatrix} \]  \hspace{1cm} (171)

Simplifying,

\[ \mathbf{A}_{1/0_1} = \{ (\mathbf{m}_{11} \mathbf{l}_{11} + \mathbf{m}_{12} \mathbf{l}_{12}) \dot{\theta}_1 \} k' \]  \hspace{1cm} (172)

Taking a time derivative,

\[ \dot{\mathbf{A}}_{1/0_1} = \{ I_1 \dot{\theta}_1 \} k' \]  \hspace{1cm} (173)

For the coupler, link 2, equation 169 can be written as,
\[ \mathbf{A}_{2/0} = \mathbf{r}_{21/0} \times m_{21} \mathbf{r}_{21} + \mathbf{r}_{22/0} \times m_{22} \mathbf{r}_{22} \]  

(174)

where, \( \mathbf{r}_{21/0} \) and \( \mathbf{r}_{22/0} \) have been previously defined in the Newtonian approach by equations 79, and 80. Taking a time derivative of each of these two equations yields \( \mathbf{r}'_{21} \) and \( \mathbf{r}'_{22} \):

\[ \mathbf{r}'_{21} = \left\{ \left( -L_1 \dot{\theta}_1 \sin \theta_1 - L_2 \dot{\theta}_2 \sin \theta_2 \right) \mathbf{i}' + \left( L_1 \dot{\theta}_1 \cos \theta_1 + L_2 \dot{\theta}_2 \cos \theta_2 \right) \mathbf{j}' \right\} \]  

(175)

\[ \mathbf{r}'_{22} = \left\{ \left( -L_1 \dot{\theta}_1 \sin \theta_1 - L_2 \dot{\theta}_2 \cos \theta_2 \right) \mathbf{i}' + \left( L_1 \dot{\theta}_1 \cos \theta_1 - L_2 \dot{\theta}_2 \sin \theta_2 \right) \mathbf{j}' \right\} \]  

(176)

Substituting equations 79, 80, 175 and 176 into equation 174 produces an equation for \( \mathbf{A}_{2/0}' \):

\[ \mathbf{A}_{2/0}' = m_{21} \begin{vmatrix} (L_1 \cos \theta_1 + L_2 \cos \theta_2) & (L_1 \sin \theta_1 + L_2 \sin \theta_2) & 0 \\ (-L_1 \dot{\theta}_1 \sin \theta_1 - L_2 \dot{\theta}_2 \sin \theta_2) & (L_1 \dot{\theta}_1 \cos \theta_1 + L_2 \dot{\theta}_2 \cos \theta_2) & 0 \\ \\ i' & j' & k' \end{vmatrix} + m_{22} \begin{vmatrix} (L_1 \cos \theta_1 - L_2 \sin \theta_2) & (L_1 \sin \theta_1 + L_2 \cos \theta_2) & 0 \\ (-L_1 \dot{\theta}_1 \sin \theta_1 - L_2 \dot{\theta}_2 \cos \theta_2) & (L_1 \dot{\theta}_1 \cos \theta_1 - L_2 \dot{\theta}_2 \sin \theta_2) & 0 \\ i' & j' & k' \end{vmatrix} \]  

(177)

or,

\[ \mathbf{A}_{2/0}' = \left\{ (m_{21} + m_{22}) \mathbf{L}_1 \dot{\mathbf{L}}_1 + \left( m_{21} L_1 + m_{22} L_2 \right) \dot{\mathbf{L}}_2 \right\} \]  

(178)

or,

\[ \mathbf{A}_{2/0}' = \left\{ L_1 m_{21} L_1 (\dot{\mathbf{L}}_1 - \dot{\mathbf{L}}_2) \cos (\theta_1 - \theta_2) + L_2 m_{22} L_2 (\dot{\mathbf{L}}_1 + \dot{\mathbf{L}}_2) \sin (\theta_1 - \theta_2) \right\} \]  

The second term of equation 168 can thus be obtained by taking a derivative of the above equation with respect to time. So,
\[ \dot{A}_{2/0} = \{ \ddot{\theta}_1 [(m_{21}+m_{22})L_1^2 + L_1m_{21}l_{21}\cos(\theta_1-\theta_2) + L_1m_{22}l_{22}\sin(\theta_1-\theta_2)] \]

\[ + \dot{\theta}_2 [I_2 + L_1m_{21}l_{21}\cos(\theta_1-\theta_2) + L_1m_{22}l_{22}\sin(\theta_1-\theta_2)] + \]

\[ + L_1(\dot{\theta}_1-\dot{\theta}_2)[-m_{21}l_{21}\sin(\theta_1-\theta_2) + m_{22}l_{22}\cos(\theta_1-\theta_2)] \} \dot{k}' \]  

(179)

For the follower, link 3, equation 169 can be expressed as:

\[ \dot{A}_{3/0} = r_{31/0} \times m_{31} \dot{r}_{31} + r_{32/0} \times m_{32} \dot{r}_{32} \]  

(180)

Or, in terms of magnitudes, and unit vectors:

\[ \dot{A}_{3/0} = m_{31} \begin{bmatrix} (L_4 + l_{31}\cos\theta_3) & (l_3\sin\theta_3) & 0 \\ (-l_{31}\dot{\theta}_3\sin\theta_3) & (l_3\dot{\theta}_3\cos\theta_3) & 0 \end{bmatrix} + \]

\[ + m_{32} \begin{bmatrix} (L_4 - l_{32}\sin\theta_3) & (l_3\cos\theta_3) & 0 \\ (-l_{32}\dot{\theta}_3\cos\theta_3) & (-l_3\dot{\theta}_3\sin\theta_3) & 0 \end{bmatrix} \]  

(181)

Simplifying,

\[ \dot{A}_{3/0} = \{ I_3 \ddot{\theta}_3 + L_4 \dot{\theta}_5 (m_{31}l_{31}\cos\theta_3 - m_{32}l_{32}\sin\theta_3) \} \dot{k}' \]  

(182)

The final term of equation 168, \( \dot{A}_{3/0}' \), can be obtained by taking a time derivative of the above equation.
\[\dot{\vec{A}}_{3/0_1} = \dot{\vec{s}}_3 \left[ I_3^o + L \left( m_{31} \cos \theta_3 + m_{32} \sin \theta_3 \right) \right] - L_4 \dot{\theta}_3 \left[ m_{31} l_{31} \sin \theta_3 + m_{32} l_{32} \cos \theta_3 \right] \]  

The total angular momentum of the linkage about joint 0' can now be found by substituting equations 173, 179 and 183 into equation 168.

\[\dot{\vec{A}}_{0_1} = \{ \dot{\vec{s}}_1^o [I_1 + L_1 (m_{21} + m_{22}) + L_1 (m_{21} l_{21} \cos (\theta_1 - \theta_2) + m_{22} l_{22} \sin (\theta_1 - \theta_2))] \]  

\[+ \dot{\vec{s}}_2^o [I_2 + L_1 (m_{21} l_{21} \cos (\theta_1 - \theta_2) + m_{22} l_{22} \sin (\theta_1 - \theta_2))] + \]  

\[+ \dot{\vec{s}}_3^o [I_3 + L_4 (m_{31} l_{31} \cos \theta_3 - m_{32} l_{32} \sin \theta_3)] + \]  

\[+ L_1 (\dot{\theta}_1^2 - \dot{\theta}_2^2) [-m_{21} l_{21} \sin (\theta_1 - \theta_2) + m_{22} l_{22} \cos (\theta_1 - \theta_2))] + \]  

\[- L_4 \dot{\theta}_3^2 \left[ m_{31} l_{31} \sin \theta_3 + m_{32} l_{32} \cos \theta_3 \right] \} \]  

Now, equation 166 can be used to determine the magnitude of the \(y\) bearing force in joint 0', \(F_{43y}\). Substituting equations 167, and 168 into equation 166, dotting with unit vector \(k'\), and solving for \(F_{43y}\):
\[ F_{43y} = \frac{\dot{A}_{0_1} - T_L - T_D + F_x \sin \beta - F_y (l \cos \beta + L_4)}{L_4} \]  \tag{185} 

Normalizing the above equation by the format already discussed,
\[ F_{43y} = \frac{\dot{A}_{0_1} - T_L - T_D + d_a (F_x \sin \beta - F_y \cos \beta) - D_4 F_y}{D^*_4} \]  \tag{186} 

where \( \dot{A}_{0_1} \) is the normalized magnitude of equation 184 given by,
\[ \dot{A}_{0_1} = \{ \alpha_1 [I_1 + M_21 + M_{22} + P_{21} \cos (\Theta_1 - \Theta_2) + P_{22} \sin (\Theta_1 - \Theta_2)] \] 
\[ + \alpha_2 [I_2 + P_{21} \cos (\Theta_1 - \Theta_2) + P_{22} \sin (\Theta_1 - \Theta_2)] + \] 
\[ + \alpha_3 [I_3 + D_4 (P_{31} \cos \Theta_3 - P_{32} \sin \Theta_3)] + \] 
\[ + (1 - i^2) [-P_{21} \sin (\Theta_1 - \Theta_2) + P_{22} \cos (\Theta_1 - \Theta_2)] + \] 
\[ - D^*_4 i^2 [P_{31} \sin \Theta_3 + P_{32} \cos \Theta_3] \} \]  \tag{187} 

Equation 186 represents the Y component of the normalized force \( F_{43} \) strictly in terms of known quantities. Therefore, \( F_{43} \) has a closed form solution. If Euler's second law were applied to the other ground bearing joint, a similar expression for the Y component of the bearing force in joint \( 0_1' \), \( F_{41y'} \), could be found. So, applying Euler's second law to joint \( 0_3' \),
\[ C_{03R} = \dot{A}_{03} \]  

(188)

where,

\[ \begin{align*}
C_{03R} &= \{ -F_L + T + T + 1 ( -F_x \sin \beta + F_y \cos \beta ) \} k' \quad (189) \\
\dot{A}_{03} &= \sum_{i=1}^{3} \dot{A}_{i/03} \quad (190)
\end{align*} \]

Substituting equation 189 into 188, dotting with unit vector \( k' \), and solving for \( F_{41y} \),

\[ F_{41y} = \left\{ \frac{-\dot{A}_{03} + T_D + T_L + 1_a (-F_x \sin \beta + F_y \cos \beta )}{L_4} \right\} \quad (191) \]

where \( \dot{A}_{03} \) can be obtained by a very similar analysis as that already presented for \( \dot{A}_{01} \). Normalizing equation 191,

\[ F_{41y}^* = \left\{ \frac{-A_{03}^* + T_D^* + T_L^* + d_a^* (-F_x^* \sin \beta + F_y^* \cos \beta )}{D_4^*} \right\} \quad (192) \]

where,

\[ A_{03}^* = \left\{ \alpha_1^*[I_1^* + M_{21}^* + M_{22}^* + P_{21}^* \cos(\theta_1 - \theta_2) + P_{22}^* \sin(\theta_1 - \theta_2) + \\
+ D_4^*[(-M_{21}^* - M_{22}^* - P_{11}^*) \cos \theta_1 + P_{12}^* \sin \theta_1 ] \right\} \]

\[ + \alpha_2 \left[ I_2 + P_{21} \cos(\theta_1 - \theta_2) + P_{22} \sin(\theta_1 - \theta_2) + \right. \\
- D_4 \left( P_{21} \cos \theta_2 - P_{22} \sin \theta_2 \right) \left. + \alpha_3 \right] + \\
+ (1 - i_2) \left[ -P_{21} \sin(\theta_1 - \theta_2) + P_{22} \cos(\theta_1 - \theta_2) \right] + \\
+ \left. D \left( (M_{21} + M_{22} + P_{11}^* \cos \theta + P_{12}^* \sin \theta) \right) + \\
+ D_4 i_2 \left[ P_{21} \sin \theta_2 + P_{22} \cos \theta_2 \right] \right] \] (193)

In summary, equations 165, 186 and 192 provide closed form solutions for the normalized input torque, \( T_0^* \), and normalized \( Y \) components of each of the ground bearing joints, \( F_{43y}^* \) and \( F_{41y}^* \), respectively. With these quantities known, the remaining unknown bearing forces can be easily solved for using some of the nine equations of the Newtonian approach, and a back substitution procedure. The basic procedure is described below.

Knowing \( F_{43y}^* \) and the externally applied force on the follower, equation 121 can be rearranged to yield an expression for \( F_{32y}^* \).

\[
F_{32y}^* = \left\{ -F_y^* - F_{43y}^* + P_{31} \left( \alpha_3^* \cos \theta_3 - i_3^2 \sin \theta_3 \right) \right. \\
- \left. P_{32} \left( \alpha_3^* \sin \theta_3 + i_3^2 \cos \theta_3 \right) \right\} \] (194)

Once \( F_{32y}^* \) is known using the above equation, then \( F_{32x}^* \) can be determined using equation 122.
\[ F_{32x} = \left\{ -I_3 \alpha_3 + T_L - F_x d_a \sin \beta + F_y d_a \cos \beta + F_{32y} D_3 \cos \theta_3 \right\} \] (195)

\[ D_3 \sin \theta_3 \]

Rearrangement of equation 120 provides an expression for the \( X \) component of the bearing force in joint \( O_3 \):

\[ F_{43x} = \left\{ -F_x - F_{32x} - P_{31}(\alpha_3 \sin \theta_3 + i_3 \cos \theta_3) + \right\} \]

\[ + P_{32}(-\alpha_3 \cos \theta_3 + i_3 \sin \theta_3) \} \] (196)

Also, since \( F_{41y} \) is known, \( F_{21y} \) can be obtained using equation 115.

\[ F_{21y} = \{-F_{41y} + P_{11}(\alpha_1 \cos \theta_1 - \sin \theta_1) - P_{12}(\alpha_1 \sin \theta_1 + \cos \theta_1)\} \] (197)

The unknown component of the ground bearing force, \( F_{21x} \), can be expressed in terms of \( F_{32x} \) using equation 117. Or,

\[ F_{21x} = \{-F_{32x} + (M_{21} + M_{22})(\alpha_1 \sin \theta_1 + \cos \theta_1) + \}

\[ + P_{21}(\alpha_2 \sin \theta_2 + i_2 \cos \theta_2) + P_{22}(\alpha_2 \cos \theta - i_2 \sin \theta_2)\} \] (198)

Finally, the normalized force \( F_{41x} \) can be determined by equation 114,

\[ F_{41x} = \{-F_{21x} - P_{11}(\alpha_1 \sin \theta_1 + \cos \theta_1) + P_{12}(\alpha_1 \cos \theta_1 + \sin \theta_1)\} \] (199)
Since all the bearing joint forces are known, expressions for the shaking force and shaking moments can be determined from their basic definitions. The shaking force is defined as the net force felt by the frame, while the shaking moment is equal to the time derivative of the angular momentum vector.

\[
\mathbf{F}_{\text{SH}}^* = \{- (F_{41x}^* + F_{43x}^* + F_x^*) i' - (F_{41y}^* + F_{43y}^* + F_y^*) j'\} \quad (200)
\]

\[
\mathbf{M}_{\text{SH}1}^* = - (\mathbf{A}_{01}^\prime) k' \quad (201)
\]

\[
\mathbf{M}_{\text{SH}3}^* = - (\mathbf{A}_{03}^\prime) k' \quad (202)
\]

Thus, all of the unknowns of the kinetic analysis can be solved by sequentially applying the equations derived above.

The analyses of sections 4.1 through 4.6 have provided a method of determining the input torque, shaking force, shaking moments and the bearing forces. The overall objective of balancing is to reduce or eliminate some or all of these quantities. The next section will propose an objective function which provides a qualitative evaluation of these quantities in one equation. The goal of the optimization package will then be to minimize that objective function, and thereby partially balance all these important kinetic quantities.
5.0 DEVELOPMENT OF THE OPTIMIZATION PROBLEM
AND GENERAL BALANCING PROGRAM

5.1 Objective Function Proposal

The use of automated optimal design for determining counterweight specifications requires an objective function. This function will be minimized by the optimization program, and provides part of the decision making logic. In the case of balancing four bar linkages, we desire to reduce all of the kinetic quantities as previously mentioned. Thus, a function which is equivalent to the magnitudes of all the important forces and torques is needed.

Ideally, the chosen objective function should represent a qualitative measure of the amount of imbalance that exists. For example, if the amount of imbalance increases, the objective function should increase and vice-versa. In addition the function should be flexible enough to allow for different types of balancing since, in some situations it may be desirable to balance only one quantity, such as the input torque. The objective function proposed by Lee and Cheng [21] was chosen for this thesis, as it attempts to fulfill both of these requirements. The function may be expressed as:

\[ Q = \frac{1}{2\pi} \int_0^{2\pi} [ W_1 \sqrt{F_{41}^*} + F_{43}^* ] + W_2 | T_D^* | \, d\theta \] (203)
As can be clearly seen, the function is a linear combination of an equivalent ground bearings force and the input torque. \( W_1 \) and \( W_2 \) are weighting factors which in theory, can be varied to allow for emphasis on different parameters. The integration of the function over the range from 0 to \( 2\pi \) represents an average of the objective function based on one cycle, or revolution of the crank.

At first glance, the objective function would appear only to be dependent upon the input torque and ground bearing forces. But a closer inspection of the equations that define the shaking force and shaking moment, would show that they are dependent upon both terms in the objective function. Minimizing the proposed objective function will, in effect, most likely decrease all the important kinetic balancing terms. While the proposed function does not give an exact representation of all the kinetic parameters, it does provide a numerical value which approximates the amount of imbalance in the linkage. The validity of this statement can be seen in the successful results of the example problems.

With a mathematical definition of the objective function to qualitatively measure the amount of imbalance in the mechanism, the remainder of the theory will formulate, and provide a method to solve the optimization problem.
5.2 Statement of the Optimization Problem

With the objective function defined by section 5.1, the methods of optimal design can be used to find the best size and location of the counterweights. In general, there are many different methods to find the minimum value of a function. Because of the number of variables and the complexity of the equations involved, the author has chosen the method of automated optimal design to solve the optimization problem. However, before the details of the particular optimization method are given, a formal statement of the problem is needed.

A formal statement of the optimization problem may be preliminarily stated as follows:

\[
\begin{align*}
\text{Minimize:} & \quad Q = \frac{1}{2 \pi} \int_{0}^{2\pi} \left( W_1 \sqrt{F_{41}^* + F_{43}^*} + W_2 |T_D^*| \right) \, d\theta_1 \\
\text{Subject to:} & \\
& \begin{align*}
1. & \quad 0 < l_{ic} < (l_{ic})_{\text{max}} \quad ; \quad i = 1, 2, 3 \\
2. & \quad 0 < \Theta_{ic} < (\Theta_{ic})_{\text{max}} \quad ; \quad i = 1, 2, 3 \\
3. & \quad F_{\text{SHRMS}}^* < (F_{\text{SHRMS}}^*)_{\text{max}} \\
4. & \quad M_{\text{SHRMS}}^* < (M_{\text{SHRMS}}^*)_{\text{max}} \\
5. & \quad F_{\text{BRMS}}^* < (F_{\text{BRMS}}^*)_{\text{max}} \\
6. & \quad T_{\text{DRMS}}^* < (T_{\text{DRMS}}^*)_{\text{max}} \\
\end{align*}
\end{align*}
\]

Specify:

\[
\begin{align*}
1. & \quad l_{ic} \quad \text{for} \quad i = 1, 2, 3 \\
2. & \quad \Theta_{ic} \quad \text{for} \quad i = 1, 2, 3 \\
\end{align*}
\]
where the "max " quantities are the maximum allowable limits imposed on the problem.

The preliminary statement of the problem is not suitable for efficient operation of the optimization program. For example, regional constraints are not in the standard format as shown by Johnson [18], and the variables have not been scaled. This standard format is a requirement of the preexisting optimization program used in this thesis, while scaling ensures smooth operation of the optimization search. The general form of the standard format for regional constraints may be stated as:

\[
R(I) = \text{regional constraint } \geq 0
\]

where, \( I = 1,2,3,...,\text{NR} \). Scaling of each variable is achieved by dividing by its associated maximum. For example,

\[
V(I) = \frac{\Theta T}{(\Theta T)_{\text{max}}}
\]

where, \( I = 1,2,3 \).

With the above refinements, the optimization problem can be stated in its final form as:

\[
\text{Minimize : } Q = \frac{1}{2\pi} \int_{0}^{\pi} \left[ W_1 \sqrt{F_{41}^{*2} + F_{43}^{*2}} + W_2 |T_D^{*}| \right] d\theta_1
\]

(213)

Subject to:

1. \( R(I) = V(I) \) ; \( I = 1,2,3,4,5,6 \) \hspace{1cm} (214)
2. \( R(I) = 1 - V(I) \) ; \( I = 6,7,8,9,10,11,12 \) \hspace{1cm} (215)
3. \( R(13) = (F_{\text{SHRMS}}^*)_{\text{max}} - F_{\text{SHRMS}}^* \) (216)

4. \( R(14) = (M_{\text{SHRMS}}^*)_{\text{max}} - M_{\text{SHRMS}}^* \) (217)

5. \( R(15) = (T_{\text{DRMS}}^*)_{\text{max}} - T_{\text{DRMS}}^* \) (218)

6. \( R(16) = (F_{\text{BRMS}}^*)_{\text{max}} - F_{\text{BRMS}}^* \) (219)

Specify:

1. \( V(I) = \frac{\Theta_{Ic}}{\Theta_{Ic}} \max \); \( I = 1,2,3 \) (220)

2. \( V(I+3) = \frac{1}{l_{Ic}} \max \); \( I = 1,2,3 \) (221)

In the above statements, \( Q \) is the objective function or primary design equation presented in section 5.1. Subsidiary design equations, which relate the objective function to the design variables, are not shown above. However, a complete listing of those equations has already been given in section 4.5. Also included in the final statement of the optimization problem are five types of regional constraints. The first two place lower and upper limits on the design variables, \( \Theta_{Ic} \) and \( l_{Ic} \). Thus, all the variables of the counterweights are not free, but constrained to lie within a user specified region. The last four regional constraints provide flexibility in the type of balancing, and let the user specify upper limits on important kinetic parameters. In general, three counterweights may be added to the linkage, but the developed program also permits optimization using one or two counterweights.
Many automated optimization techniques are available to solve the above defined problem. Because of its flexibility and the author's familiarity, the MODSER program developed by Johnson [18] was used. The author has chosen to rewrite the existing program in the FORTRAN language, and it is herein referred as the AOD program. A brief, but informative summary of the AOD program is given in the next section.
5.3 Summary of the AOD Program

The logic which the AOD program uses to find a solution to an optimization problem is briefly presented below. The structure of the program is the same as the MODSER program developed by Dr R. C. Johnson [18]. The version in this thesis has been rewritten in the FORTRAN programming language for use on a VAX/VMS digital computer. When the AOD program is used to solve the specific problem of balancing four bar linkages, it is referred to as the OPTBAL program. A complete summary of the OPTBAL program including input and output is given in appendix D.

Algorithms of automated optimal design basically control an iterative, computer based search using the techniques of nonlinear programming. The purpose of the search is to find an approximate minimum of an objective function, \( Q \), while maintaining user specified regional and equality constraints. These constraints are taken into account using a penalty function, \( P \), which is directly added to the objective function. The search process in the MODSER program mainly consists of two parts; determination of a search direction and a univariate line search. Both parts of the search are repeatedly applied until an approximate minimum is reached. The specifics of the penalty function, search process and termination criteria will be presented next.
Many strategies of automated optimal design are based on minimizing an unconstrained function. Since the balancing problem has regional constraints, it must be converted into an equivalent unconstrained optimization problem before it can be solved. To achieve this, the MODSER program incorporates a penalty function, $P$, which is added to the objective function when constraints are violated. Thus, the search for a minimum value of the objective function will probably avoid violating regional constraints since, if it does, the objective function is increased. The form of the penalty function used in the MODSER program may be expressed as:

$$P = C_p \eta_p \left[ \sum_{j=1}^{Ne} [E_j]^2 + \frac{1}{4} \sum_{i=1}^{Nr} [R_i - |R_i|]^2 \right] \quad (222)$$

Where $C_p$ is the penalty function coefficient which is automatically tuned by the AOD program to balance it with respect to the optimization quantity. $\eta_p$ is a factor which is increased as the solution is approached to emphasize the effect of regional constraint violations in the search process. $E_j$ is the $j$th equality constraint expressed in the form: $E_j(x) = 0$, where $x$ is the vector of the independent search variables. $R_i$ is the $i$th regional constraint expressed in the form: $R_i(x) > 0$. To incorporate the penalty function into the objective function, the two are simply added to form a new optimization quantity.

$$Q_u = Q + P \quad (223)$$

Therefore, the regional constraints have been taken into account by redefining the objective function. Although the problem is to
minimize an equivalent unconstrained function, the correct final solution should show that all of these constraints have been obeyed. In some situations, the user may specify regional constraints which are impossible to meet. Such a situation may still yield a final solution, but the results should not be considered since constraints have been violated. To alleviate this problem, regional constraints may be relaxed (ie. open up the design space), and the optimization rerun.

The basic structure of the MODSER program is shown in Figure 8. Once the problem has been set up in the standard format, the program may be run. Data is entered into the program through an external data file as shown in appendix D. After the data has been read, then a starting point for the search is either specified or automatically calculated using the random generation method. Due to the complexity of the balancing problem of this thesis, starting points were always obtained using the random number generation method. After a suitable starting point has been found, the program performs an initialization. The initial search direction is then found using the method of steepest descent (MSD) where the search direction vector, $s$, is just the negative of the gradient vector, $g$. A technique which is more efficient than MSD, is the generalized conjugate gradient method (GCG) and may be given as:

$$s_k = -g_k + w_k s_{k-1}$$

(224)
Figure 8: Flowchart of the MODSER Program

START

DATA

RANDOM GENERATION OF A GOOD STARTING POINT

MAIN PROGRAM INITIALIZATION

\( K = 0 ; ND = 1 \)

SEARCH DIRECTION

I. APPROXIMATES THE GRADIENT VECTOR USING FINITE DIFFERENCES
II. DETERMINES THE SEARCH DIRECTION VECTOR BY:
   - MSD TECHNIQUE IF ND = 1
   - GCG TECHNIQUE IF ND = 2

ND = 2

PASSED

DESCE NT DIRECTION TEST

FAILED

ND = 1

UNIVARIATE LINE SEARCH

I. STEPS ALONG SEARCH DIRECTION
II. BRACKETS NEXT BASEPOINT, \( x_{k+1} \)
III. LOCATES NEXT BASEPOINT USING INTERPOLATION, \( x_{k+1} \)

INDEXING

\( K = K+1 \)
\( x_k = x_{k+1} \)

PASSED

FUNCTION DECREASE TEST

FAILED

SEARCH TERMINATION TESTS

PASSED

FINAL ANALYSIS AND SOLUTION PRINTOUT

FAILED

REDUCE STEP SIZE

END
where,

\[
W_k = \frac{(g_k - g_{k+1}) \cdot (g_k)}{(g_k - g_{k+1}) \cdot (s_{k-1})}
\]  

(225)

The subscript \( k \) is used to designate the number of iterations. The \textit{GCG} technique requires two consecutive basepoints for its evaluation and therefore, cannot be used in the initial search direction determination. Once the search direction has been found by either of the \textit{MSD} or \textit{GCG} techniques, then a descent direction test is applied, which is given by:

\[
g_{-k} \cdot s_{-k} < 0
\]  

(226)

The above test is imposed to ensure that the search direction chosen is one in which the function decreases. If the descent direction test is passed, then a univariate line search is carried out along the calculated search direction. The end result of the line search is an approximate function minimum along that search direction. This is accomplished by finding three points within which the minimum exists, and applying an interpolation technique. After the line search is completed, a check is made on the new proposed basepoint to see if the objective function has been lowered. If the test is failed, the incremental step size is reduced and the univariate line search is repeated. If the function decrease test is passed, then a search termination test is applied. The termination test consists of three steps: the distance between successive
basepoints, the value of $\eta_p$ in comparison to $\eta_{p_{\text{max}}}$, and the rate of convergence of the solution. If anyone of the above three tests for convergence are failed, then the new basepoint, $x_{k+1}$ is chosen from the results of the line search, the counter, $k$ is indexed and a new search direction is made. If however, the termination tests are passed, then the search is halted and the results are printed out.
5.4 Summary of Analysis and Final Subroutines

The AOD computer program presented in the last section requires a subroutine to evaluate the objective function, regional constraints and equality constraints. In addition the program needs a subroutine to print out the final results. These two tasks are performed by the Analysis and Final subroutines, which are listed in appendix D for the problem of balancing four bar linkages (i.e., the OPTBAL program). A block diagram showing how information is passed from the OPTBAL main program to the Analysis and Final subroutines is shown in Figure 9. The "Pgauss" subroutine, which is also shown in that figure, provides data for numerical integration using Gaussian quadrature. The four different components were compiled separately, and then linked into one executable program. General comments about each will be presented next, while the specific options of the OPTBAL program are given in appendix D.

The analysis subroutine, as mentioned earlier, evaluates the important equations of a particular problem such as: regional constraints, equality constraints, objective function. As was shown in section 5.1, the objective function used in the OPTBAL program must be integrated over the full range of the input. Due to the complexity of this objective function, integration was done numerically using either Gaussian quadrature or a simple summation approach. Appendix C provides indepth details about each type of numerical integration. Both
methods require the evaluation of the objective function at various positions of the crank (i.e. \( \theta_i \)). However, the crank angle is not enough to specify the problem completely; values of the input speed, \( \dot{\theta}_i \), and magnitude of the crank acceleration, \( \ddot{\theta}_i \), are also needed. For simplicity, the example problems presented in this thesis assumed a constant input angular speed, hence the magnitude of the acceleration is zero. Both the input angular speed and acceleration were entered through an external data file as shown in appendix D.

Figure 9: Relationship Between the AOD Program and Analysis, Final and Pgauss Subroutines

To evaluate the kinetic balancing quantities, such as the rms value of the shaking force, different sections of the theory must be used in a particular sequence. First, the parameters of the two point mass model are determined using the theory presented in section 4.1. Next, the kinematic solution is obtained using a given crank angle and the equations of section 4.2. With the kinematic solution known, expressions for the
bearing forces, input torque, shaking force, shaking moments and the integrand of the objective function are evaluated sequentially, as summarized in section 4.5. The above two steps are repeated for various crank angles as determined by the numerical quadrature. Once this is complete, the objective function and rms quantities are determined. Finally, the regional constraints are evaluated, and the objective function may be modified by a penalty function as shown in section 5.3.

The two options for numerical integration are Gaussian quadrature (NRMS = 1), and the simple summation technique (NRMS = 2). The degree of integration for each type of quadrature can be changed using an externally specified variable, NGQ. For Gaussian quadrature, NGQ represents the number of sampling points, and can range from one to twenty-four. For integration by simple function summation, the NGQ variable is an indirect indication of the number of sampling points, and is given by the following formula.

\[
\text{number of sampling points} = \frac{360}{\text{NGQ}}
\]

The number 360 in the above expression is representative of the maximum number of sampling points using the simple summation technique (eg. NGQ = 1). At this limit, the quadrature would require evaluation of the integrand for every degree of the crank, or 360 times per cycle. As will be shown in example one, proper choice of NGQ for each type of quadrature will produce analysis results which converge. The Gaussian quadrature is
also available since it provides a relatively efficient form of integration. However, this type of quadrature is incapable of accurately predicting maximum quantities. This is acceptable for a large part of the optimization since the regional constraints and objective function are based upon rms quantities.

However, for the final design it is desirable to know the maximum values of some quantities. For example, the maximum input torque should be known in order to select a motor for the linkage. For this reason, the OPTBAL program uses simple summation integration to analyze the final results.

In addition to the Analysis subroutine, the user of the AOD program must also supply a Final subroutine to print out the results of the optimization. For the OPTBAL program, the output includes a summary of the actual dimensions and angular locations of the counterweights. A listing of the regional constraints and values of the bearing forces, shaking force, shaking moment and input torque are also provided. The optimization search is also summarized through yet another data file.

As a whole the OPTBAL program offers several important features. For instance, the user can select the number of counterweights to be added. By adjusting the number of counterweights, NC, the effects of adding from one to three counterweights can be evaluated through a series of optimization runs. In addition, the choice of no counterweights (NC=0), allows the user to analyze the unbalanced linkage. As
shown in appendix D, the only limit of the NC option is the location of the counterweights. The effects of changing the number of added counterweights is investigated in examples 2 and 3. Another feature of the program is that it allows the user to select the maximum regional constraint parameters on the rms values of important kinetic quantities, such as the shaking force or input torque. Regional constraint limits can be easily changed through the input data file and permits diversity in balancing. Investigation of the regional constraint variations are shown in example 1.
6.0 PRESENTATION OF EXAMPLE PROBLEMS

The capabilities, and effectiveness of the developed balancing program, OPTBAL, will now be demonstrated through some example problems. These example problems were taken from other published sources and provide comparisons to the unbalanced linkage, as well as other methods of balancing. A brief description of each will be presented in the next several pages, while the details and results are shown in sections 6.1 through 6.3.

The three example problems presented are:

Example 1: Capabilities of the OPTBAL Program

Example one provides some insight into the capabilities of the OPTBAL program. In the first portion, a comparison is made between the two types of numerical integration available in the program. For simplicity, the comparison is made using an unbalanced inline four bar linkage taken from Berkoff [5]. From that comparison, the optimum number of sampling points for a convergent, but efficient solution is determined. The second part of the problem focuses on the effect of varying the weighting factors, $W_1$ and $W_2$, upon the optimization results. The effect of varying the torque regional constraint upon the optimization results is also shown.
Example 2: **Balancing of an Inline Four Bar Linkage**

The results of example one are applied to balance the four bar linkage of that example in a typical design procedure. The effects of varying the number of added counterweights is shown. The results of the optimization are then compared to the unbalanced linkage, the completely force balanced linkage, and the complete force and moment balanced linkage.

Example 3: **Balancing of a Four Bar Linkage with an Offline Coupler Mass**

In this example, a graphical comparison between the method of complete force balancing, and the method of balancing linkages using the OPTBAL program is made. The linkage has been taken from Tricamo and Lowen [40], and contains a coupler link with an offline center of mass. The procedure for balancing is the same as that in example 2 since, the number of added counterweights are varied. The most practical counterweight design will then be compared graphically to the unbalanced, and completely force balanced linkage.

The full details of each of the three examples will be presented next in sections 6.1 through 6.3.
6.1 Capabilities of the OPTBAL Computer Program

In this example, the capabilities of the OPTBAL program are investigated. The information gained will provide insight as to the selection of several OPTBAL parameters for an accurate and efficient analysis. In addition, limitations of the regional constraints will be demonstrated.

A typical inline four bar linkage, taken from Berkoff [5], will be used to show these capabilities. The linkage is shown in Figure 10 and its specifications are given in Table 1. The linkage and counterweights are assumed to be made out of steel with a density of 0.283 (lb/in$^3$). The counterweights are also assumed to be circular and have a constant thickness of 0.50 inches. Counterweights may be tangentially attached to joints $O_1'$, $O_2'$ or $O_3'$, as shown in Figure 1 of section 4.1. The input speed of the linkage is assumed to be constant at 5000 rpm (523 rad/s).

The first part of this example compares the two different types of numerical integration available in the OPTBAL program: Gaussian quadrature, simple summation quadrature. Integration is used by the program to evaluate the objective function and other rms quantities. Therefore, the degree of numerical integration used is important in providing an accurate analysis. So, from an accuracy point of view, we
Figure 10: Inline Four Bar Linkage

Table 1: Parameters of the Unbalanced Inline Four Bar Linkage

<table>
<thead>
<tr>
<th>Link No., i</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length, Li (in)</td>
<td>1.000</td>
<td>4.000</td>
<td>3.000</td>
<td>3.000</td>
</tr>
<tr>
<td>Mass, mi (lb)</td>
<td>0.101</td>
<td>0.174</td>
<td>0.146</td>
<td>-</td>
</tr>
<tr>
<td>Length to C.G. Location, li (in)</td>
<td>0.500</td>
<td>2.000</td>
<td>1.500</td>
<td>-</td>
</tr>
<tr>
<td>Thickness (in)</td>
<td>0.200</td>
<td>0.200</td>
<td>0.200</td>
<td>-</td>
</tr>
<tr>
<td>Mass Moment of Inertia about the C.G. Location ( I_{11} ) (lb - in(^2))</td>
<td>0.036</td>
<td>0.433</td>
<td>0.232</td>
<td>-</td>
</tr>
</tbody>
</table>
might consider using a relatively high order of numerical quadrature (ie. a large number of sampling points). On the other hand, it is desirable to minimize the number of function evaluations required in each analysis subroutine. If the number of function evaluations are too large, then an excessive amount of computer time will be used. This part of example one will determine which degree of numerical integration is best suited for an efficient, but accurate analysis.

To achieve this goal, the number of sampling points, NGQ, was varied while setting the number of counterweights option, NC, to zero. This, in effect, enabled the optimization program to analyze the unbalanced linkage. For each trial, the rms values of the normalized shaking force, shaking moment and input torque were recorded. A plot of this data is shown in Figures 11, 12 and 13. As can be seen from those figures, both types of integration converge to the same solution, with the Gaussian quadrature being slightly more efficient. It appears that NGQ must be greater than 10 for an accurate analysis using Gaussian quadrature, while NGQ must be greater than 15 for the simple summation quadrature. From an efficiency standpoint, the Gaussian quadrature, as expected, is better (approximately 33%). Therefore, the remaining example problems will use Gaussian quadrature as the method of numerical integration, with the number of sampling points equal to 10.
FIGURE 11:
NORMALIZED RMS INPUT TORQUE VS. NUMBER OF SAMPLING POINTS

▲ ▲ GAUSSIAN QUADRATURE
□ □ SIMPLE SUMMATION
--- CONVERGENT SOLUTION

NUMBER OF SAMPLING POINTS
NORMALIZED RMS SHAKING MOMENT VS. NUMBER OF SAMPLING POINTS

\[ \text{GAUSSIAN QUADRATURE} \]
\[ \text{SIMPLE SUMMATION} \]
\[ \text{CONVERGENT SOLUTION} \]

NUMBER OF SAMPLING POINTS

0 2 4 6 8 10 12 14 16 18 20

0 2 4 6 8 10
In addition to the type and degree of integration to be used, the effect of changing the weighting factors, \( W_1 \) and \( W_2 \), is also determined. Ideally, these weighting factors can be varied to allow for the relative importance of each term in the objective function. However, these two terms are not independent, and the logical choice of the weighting parameters isn't always obvious.

To determine the effect of varying the weighting parameters of the objective function upon the optimization, the OPTBAL program was run several times with emphasis placed on different terms. As shown in Tables 2 and 3, the rms values of the normalized shaking force, shaking moment and input torque were recorded. Numbers in parenthesis indicate the percent improvement over the unbalanced case.

From Tables 2 and 3, it appears that varying the weighting factors only partially effected the optimization result. For trials 1.1, 1.2 and 1.3, the differences in the final optimized results are barely noticeable, while for trial 1.5, where \( W_1 \) is 0 and \( W_2 \) is 1, the results were logical; the input torque was reduced by a factor of approximately 3, and the shaking force increased over the normal optimization results. Therefore, the effect of varying the weighting factors seems to be limited to either placing emphasis on both terms of the objective function, or placing more emphasis on the input torque term.
Table 2: Influence of Weighting Factors on Counterweight Parameters

<table>
<thead>
<tr>
<th>Trial #</th>
<th>$W_1$</th>
<th>$W_2$</th>
<th>$\frac{l_{1c}}{\theta_{1c}}$ (in) / (rad)</th>
<th>$\frac{l_{2c}}{\theta_{2c}}$ (in) / (rad)</th>
<th>$\frac{l_{3c}}{\theta_{3c}}$ (in) / (rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - 0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0 / 0.0</td>
<td>0.0 / 0.0</td>
<td>0.0 / 0.0</td>
</tr>
<tr>
<td>1 - 1</td>
<td>1.0</td>
<td>0.0</td>
<td>1.465/3.147</td>
<td>1.285/4.305</td>
<td>0.021/3.199</td>
</tr>
<tr>
<td>1 - 2</td>
<td>0.75</td>
<td>0.25</td>
<td>1.480/3.148</td>
<td>1.313/4.309</td>
<td>0.023/4.941</td>
</tr>
<tr>
<td>1 - 3</td>
<td>0.50</td>
<td>0.50</td>
<td>1.511/3.148</td>
<td>1.365/4.317</td>
<td>0.003/5.560</td>
</tr>
<tr>
<td>1 - 4</td>
<td>0.25</td>
<td>0.75</td>
<td>1.581/3.147</td>
<td>1.468/4.331</td>
<td>0.078/0.926</td>
</tr>
<tr>
<td>1 - 5</td>
<td>0.0</td>
<td>1.0</td>
<td>2.913/3.359</td>
<td>1.468/4.345</td>
<td>0.140/6.097</td>
</tr>
</tbody>
</table>

Note: Maximum normalized input torque, $(T_{DRMS})_{\text{max}} = 5.0$

$NGQ = 10, \ NRMS = 1, \ NC = 3$

Table 3: Influence of Weighting Factors on the Shaking Force, Shaking Moment and Input Torque

<table>
<thead>
<tr>
<th>Trial #</th>
<th>$W_1$</th>
<th>$W_2$</th>
<th>Normalized RMS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Shaking Force</td>
</tr>
<tr>
<td>1 - 0</td>
<td>0.0</td>
<td>0.0</td>
<td>2.860 (0.0%)</td>
</tr>
<tr>
<td>1 - 1</td>
<td>1.0</td>
<td>0.0</td>
<td>0.371 (87%)</td>
</tr>
<tr>
<td>1 - 2</td>
<td>0.75</td>
<td>0.25</td>
<td>0.374 (87%)</td>
</tr>
<tr>
<td>1 - 3</td>
<td>0.50</td>
<td>0.50</td>
<td>0.383 (87%)</td>
</tr>
<tr>
<td>1 - 4</td>
<td>0.25</td>
<td>0.75</td>
<td>0.422 (85%)</td>
</tr>
<tr>
<td>1 - 5</td>
<td>0.0</td>
<td>1.0</td>
<td>2.388 (17%)</td>
</tr>
</tbody>
</table>

Note: Maximum normalized input torque $(T_{DRMS})_{\text{max}} = 5.0$

$NGQ = 10, \ NRMS = 1, \ NC = 3$
As a final consideration, the effects of varying the regional constraint on the rms input torque were investigated. The reason for showing this is to find out how well the input torque can be controlled using a regional constraint. This type of situation may occur in the design of a linkage where the input torque is specified before balancing.

As shown in Tables 4 and 5, the procedure for determining the effect of the torque regional constraint was to run the OPTBAL program for various limiting values of the input torque. For each trial, the normalized shaking force, shaking moment and input torque were noted. Also, an indication of whether any regional constraints were violated, was recorded.

Observing the results from Tables 4 and 5, it may be seen that three different situations occur from varying the input torque regional constraint: the regional constraint does not effect the final results, the regional constraint controls the input torque below its normal optimized value, the regional constraint is not obeyed. The first type of situation occurs in trials 1.6, 1.7 and 1.8, where the regional constraint does not effect the optimization results. This seems to occur if the specified rms input torque is greater than that obtained from a normal optimization, and is herein refered to as a relaxed constraint. The second type occurs in trials 1.9 and 1.10, where the input torque is actually reduced below its normal optimization value (ie with relaxed constraints). As
Table 4: Influence of Torque Regional Constraint on Counterweight Parameters

<table>
<thead>
<tr>
<th>Trial #</th>
<th>Constraint Violation</th>
<th>l₁c / Θ₁c (in) / (rad)</th>
<th>l₂c / Θ₂c (in) / (rad)</th>
<th>l₃c / Θ₃c (in) / (rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - 0</td>
<td>-</td>
<td>0.0 / 0.0</td>
<td>0.0 / 0.0</td>
<td>0.0 / 0.0</td>
</tr>
<tr>
<td>1 - 6</td>
<td>no</td>
<td>1.511/3.148</td>
<td>1.365/4.317</td>
<td>0.003/5.560</td>
</tr>
<tr>
<td>1 - 7</td>
<td>no</td>
<td>1.511/3.148</td>
<td>1.365/4.317</td>
<td>0.019/5.835</td>
</tr>
<tr>
<td>1 - 8</td>
<td>no</td>
<td>1.511/3.148</td>
<td>1.365/4.317</td>
<td>0.003/5.560</td>
</tr>
<tr>
<td>1 - 9</td>
<td>yes (1%)</td>
<td>1.818/3.150</td>
<td>1.510/4.609</td>
<td>0.003/6.281</td>
</tr>
<tr>
<td>1 - 10</td>
<td>yes (1%)</td>
<td>2.678/3.143</td>
<td>1.512/4.617</td>
<td>0.000/6.283</td>
</tr>
<tr>
<td>1 - 11</td>
<td>yes (34%)</td>
<td>2.894/3.146</td>
<td>1.504/4.609</td>
<td>0.000/6.283</td>
</tr>
</tbody>
</table>

Note: W₁ = W₂ = 0.5 , NGQ = 10, NRMS = 1, NC = 3

Table 5: Influence of Torque Regional Constraint on the Shaking Force, Shaking Moment and Input Torque

| Trial # | Max. Normalized RMS Input Torque | Normalized RMS |
|---------|----------------------------------|----------------|----------------|----------------|----------------|
|         |                                  | Shaking Force  | Shaking Moment | Input Torque  |
| 1 - 0   | -                                | 2.860 (0.0%)   | 5.721 (0.0%)   | 1.565 (0.0%)  |
| 1 - 6   | 5.00                             | 0.383 (87%)    | 1.459 (75%)    | 0.160 (90%)   |
| 1 - 7   | 0.50                             | 0.383 (87%)    | 1.459 (75%)    | 0.160 (90%)   |
| 1 - 8   | 0.20                             | 0.383 (87%)    | 1.459 (74%)    | 0.160 (90%)   |
| 1 - 9   | 0.10                             | 0.808 (72%)    | 1.206 (79%)    | 0.101 (94%)   |
| 1 - 10  | 0.05                             | 2.080 (27%)    | 0.604 (89%)    | 0.050 (97%)   |
| 1 - 11  | 0.01                             | 2.363 (17%)    | 0.508 (91%)    | 0.044 (97%)   |

Note: W₁ = W₂ = 0.5 , NGQ = 10, NRMS = 1, NC = 3
noted from the results, some regional constraints have been violated, but the violations are very small in comparison to the input torques value. The last type occurs when the constraints on the input torque are virtually impossible to meet, and the regional constraint is significantly violated. This occurred in trial 1.11 and is commonly referred to as a case of incompatible specifications (i.e. the specifications are impossible to meet).

The results of the second and third portions of example one appear to show that the amount of reduction of the input torque can be varied from its normal optimization value (i.e. where the weighting factors are equal and the regional constraints are relaxed). However, the price for a greater reduction of the input torque is an increased shaking force. For example, in trial 1.3, a normal optimization decreased the shaking force by approximately 87% and reduced the input torque by 90%. However, in trial 1.5, where more emphasis was placed on the input torque reduction, the input torque was decreased 97% and the shaking force reduced by only 17% over the unbalanced linkage. Therefore, trial 1.5 increased the shaking force 70% over the normal optimization results, while the input torque was only reduced an additional 7%. In otherwords, the price of putting emphasis on the input torque seems to adversely effect the other balancing parameters. Since the main objective of this thesis is to reduce the combined effects, the remainder of the example problems will use equal weighting factors and relaxed regional constraints.
6.2 Balancing of an Inline Four Bar Linkage

The objective of this example problem is to specify counterweights for the inline four bar linkage of example one, so that the combined effects of the shaking force, shaking moment, bearing forces and input torque are balanced. The purpose of this example problem is to show a practical procedure for applying the OPTBAL program. In addition, this example will provide comparisons to other balancing methods such as the method of complete force balancing, and complete force and moment balancing.

The balancing procedure consists of varying the number of counterweights to be added to the linkage. The OPTBAL program was run for each of the three possible counterweight configurations with the regional constraints relaxed, and the weighting factors selected equal to each other.

Results for each one of the trials are shown in tables 6, 7 and 8. Table 6 shows the specifications of the counterweights, while Tables 7 and 8 list the rms and maximum dynamic quantities for each trial respectively. Numbers in parenthesis in these tables indicate the percent improvement over the unbalanced case. Trials 1.0, 2.4 and 2.5 have been included in each of the three tables for comparison purposes. Trial 1.0 is the unbalanced linkage, while trials 2.4 and 2.5 were obtained from Berkoffs paper [5]. Trial 2.4 is the complete force balanced linkage, and trial 2.5 is the complete force and moment balanced linkage.
Table 6: Counterweight Parameters for Example 2 (1)

<table>
<thead>
<tr>
<th>Trial #</th>
<th>NC</th>
<th>$l_1c / \Theta_1c$ (in)/(rad)</th>
<th>$l_2c / \Theta_2c$ (in)/(rad)</th>
<th>$l_3c / \Theta_3c$ (in)/(rad)</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - 0</td>
<td>0</td>
<td>0.0 / 0.0</td>
<td>0.0 / 0.0</td>
<td>0.0 / 0.0</td>
<td>unbalanced</td>
</tr>
<tr>
<td>2 - 1</td>
<td>1</td>
<td>1.299/3.062</td>
<td>0.0 / 0.0</td>
<td>0.0 / 0.0</td>
<td>OPTBAL</td>
</tr>
<tr>
<td>2 - 2</td>
<td>2</td>
<td>1.299/3.062</td>
<td>0.0 / 0.0</td>
<td>0.036/0.277</td>
<td>OPTBAL</td>
</tr>
<tr>
<td>2 - 3</td>
<td>3</td>
<td>1.511/3.148</td>
<td>1.365/4.317</td>
<td>0.003/5.560</td>
<td>OPTBAL</td>
</tr>
<tr>
<td>2 - 4</td>
<td>2</td>
<td>0.717/3.142</td>
<td>0.0 / 0.0</td>
<td>1.079/3.142</td>
<td>Ref. <a href="2">5</a></td>
</tr>
<tr>
<td>2 - 5</td>
<td>2</td>
<td>0.717/3.142</td>
<td>0.0 / 0.0</td>
<td>1.079/3.142</td>
<td>Ref. <a href="3">5</a></td>
</tr>
</tbody>
</table>

Note: (1) OPTBAL options used: NRMS=1, NGQ=10, $W_1=W_2=0.5$, $(T_{\text{DRMS}})^{max}=5.0$.

(2) Force balanced.

(3) Force and moment balanced. In addition to two circular counterweights, geared inertia counterweights have been added, and the shape of the coupler link has been augmented.

As can be seen from the results tables, the linkages balanced using the OPTBAL program are superior to those of the unbalanced, completely force balanced, and completely force and moment balanced linkages. The ranges of all force and moment reductions for the linkages balanced by the OPTBAL program are on the order 75 % to 92 % over the unbalanced linkage. In contrast, the completely force balanced linkage eliminated the shaking force, but increased all other balancing quantities over the unbalanced linkage. The situation is even worse for the completely force and moment balanced linkage. Although both the
Table 7: Comparison of Normalized RMS Quantities for Example 2

<table>
<thead>
<tr>
<th>Kinetic Quantity</th>
<th>1 - 0</th>
<th>2 - 1</th>
<th>2 - 2</th>
<th>2 - 3</th>
<th>2 - 4</th>
<th>2 - 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>F * SH</td>
<td>2.860 (0.0%)</td>
<td>0.620 (78%)</td>
<td>0.620 (78%)</td>
<td>0.383 (87%)</td>
<td>0.0 (100%)</td>
<td>0.0 (100%)</td>
</tr>
<tr>
<td>M * SH/01</td>
<td>5.721 (0.0%)</td>
<td>1.440 (75%)</td>
<td>1.440 (75%)</td>
<td>1.459 (74%)</td>
<td>7.610 (-33%)</td>
<td>0.0 (100%)</td>
</tr>
<tr>
<td>T * D</td>
<td>1.565 (0.0%)</td>
<td>0.394 (75%)</td>
<td>0.394 (75%)</td>
<td>0.160 (90%)</td>
<td>2.467 (-58%)</td>
<td>6.154 (-293%)</td>
</tr>
<tr>
<td>F * 21</td>
<td>3.434 (0.0%)</td>
<td>0.864 (75%)</td>
<td>0.864 (75%)</td>
<td>1.197 (65%)</td>
<td>5.271 (-53%)</td>
<td>6.434 (-87%)</td>
</tr>
<tr>
<td>F * 32</td>
<td>2.380 (0.0%)</td>
<td>0.599 (75%)</td>
<td>0.599 (75%)</td>
<td>0.543 (77%)</td>
<td>4.022 (-69%)</td>
<td>5.066 (-113%)</td>
</tr>
<tr>
<td>F * 41</td>
<td>3.801 (0.0%)</td>
<td>0.664 (82%)</td>
<td>0.664 (82%)</td>
<td>0.297 (92%)</td>
<td>4.583 (-20%)</td>
<td>5.734 (-51%)</td>
</tr>
<tr>
<td>F * 43</td>
<td>2.299 (0.0%)</td>
<td>0.579 (75%)</td>
<td>0.579 (75%)</td>
<td>0.547 (76%)</td>
<td>4.583 (-99%)</td>
<td>5.734 (-149%)</td>
</tr>
</tbody>
</table>

Table 8: Comparison of Normalized Maximum Quantities for Example 2

<table>
<thead>
<tr>
<th>Kinetic Quantity</th>
<th>1 - 0</th>
<th>2 - 1</th>
<th>2 - 2</th>
<th>2 - 3</th>
<th>2 - 4</th>
<th>2 - 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>F * SH</td>
<td>5.572 (0.0%)</td>
<td>1.212 (78%)</td>
<td>1.212 (78%)</td>
<td>0.745 (87%)</td>
<td>0.0 (100%)</td>
<td>0.0 (100%)</td>
</tr>
<tr>
<td>M * SH/01</td>
<td>14.68 (0.0%)</td>
<td>3.695 (75%)</td>
<td>3.695 (75%)</td>
<td>3.124 (79%)</td>
<td>17.66 (-20%)</td>
<td>0.0 (100%)</td>
</tr>
<tr>
<td>T * D</td>
<td>4.485 (0.0%)</td>
<td>1.129 (75%)</td>
<td>1.129 (75%)</td>
<td>0.405 (91%)</td>
<td>7.092 (-58%)</td>
<td>17.73 (-295%)</td>
</tr>
<tr>
<td>F * 21</td>
<td>8.407 (0.0%)</td>
<td>2.116 (75%)</td>
<td>2.116 (75%)</td>
<td>1.850 (78%)</td>
<td>12.89 (-53%)</td>
<td>16.11 (-92%)</td>
</tr>
<tr>
<td>F * 32</td>
<td>6.463 (0.0%)</td>
<td>1.627 (76%)</td>
<td>1.627 (76%)</td>
<td>1.272 (81%)</td>
<td>10.85 (-63%)</td>
<td>13.54 (-104%)</td>
</tr>
<tr>
<td>F * 41</td>
<td>8.881 (0.0%)</td>
<td>1.387 (84%)</td>
<td>1.387 (84%)</td>
<td>0.793 (91%)</td>
<td>12.11 (-36%)</td>
<td>15.09 (-70%)</td>
</tr>
<tr>
<td>F * 43</td>
<td>6.093 (0.0%)</td>
<td>1.534 (75%)</td>
<td>1.534 (75%)</td>
<td>1.198 (80%)</td>
<td>12.11 (-99%)</td>
<td>15.09 (-148%)</td>
</tr>
</tbody>
</table>
shaking force and moment have been eliminated, the remainder of the balancing quantities have been increased from 70% to 295%. In addition, the completely force and moment balanced linkage requires large, geared, inertia counterweights to be added to the input and output shafts, as well as circular counterweights.
6.3 Balancing of a Four Bar Linkage with an Offline Coupler Link

The purpose of this example is to balance a linkage containing an offline mass distribution. The mechanism was taken from Tricamo and Lowen [40] and is shown in Figure 14. As in example two, the counterweights are assumed to be made out of steel with a density of 0.283 lb/in³, and have a constant thickness of 0.625 inches. The linkage is also assumed to operate at a constant input speed of 500 rpm (52.3 rad/s). Counterweights are circular and can be attached tangentially to joints O₁', O₂', or O₃'. Other specifications for the linkage are given in Table 9.

The procedure for balancing is the same as that in example 2; the number of counterweights are varied through a series of optimization trials. The data file for this problem must, however, take into account the offline placement of the center of mass of the coupler link (see appendix D). From these trials, the most practical scheme for balancing will be graphically compared to the unbalanced, and completely force balanced linkages.

The results of this example can be seen in Tables 10 and 11, or graphically in Figures 15 through 22. Specifications of the counterweights for each trial are given in Table 10,
Figure 14: Four Bar Linkage with an Offline Coupler

Table 9: Parameters of the Unbalanced Four Bar Linkage with an Offline Coupler Link

<table>
<thead>
<tr>
<th>Link #, i</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length, Li (in)</td>
<td>2.000</td>
<td>6.000</td>
<td>3.000</td>
<td>5.500</td>
</tr>
<tr>
<td>Mass, mi (lb)</td>
<td>0.197</td>
<td>0.528</td>
<td>0.268</td>
<td>-</td>
</tr>
<tr>
<td>Thickness (in)</td>
<td>0.250</td>
<td>0.250</td>
<td>0.250</td>
<td>-</td>
</tr>
<tr>
<td>Length to C.G., li (in)</td>
<td>1.000</td>
<td>3.027</td>
<td>1.500</td>
<td>-</td>
</tr>
<tr>
<td>Angle, αi (rad)</td>
<td>0.000</td>
<td>0.133</td>
<td>0.000</td>
<td>-</td>
</tr>
<tr>
<td>Mass Moment of Inertia about C.G. Location I^9 (lb - in^2)</td>
<td>0.246</td>
<td>2.323</td>
<td>0.752</td>
<td>-</td>
</tr>
</tbody>
</table>
Table 10: Counterweight Parameters for Example 3

<table>
<thead>
<tr>
<th>Trial</th>
<th>NC</th>
<th>$l_1c / \theta_1c$</th>
<th>$l_2c / \theta_2c$</th>
<th>$l_3c / \theta_3c$</th>
<th>Ref</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 - 1</td>
<td>0</td>
<td>0.0 / 0.0</td>
<td>0.0 / 0.0</td>
<td>0.0 / 0.0</td>
<td>Unbalanced</td>
</tr>
<tr>
<td>3 - 2</td>
<td>1</td>
<td>1.534 / 3.088</td>
<td>0.0 / 0.0</td>
<td>0.0 / 0.0</td>
<td>OPTBAL (1)</td>
</tr>
<tr>
<td>3 - 3</td>
<td>2</td>
<td>1.534 / 3.088</td>
<td>0.0 / 0.0</td>
<td>0.031 / 0.236</td>
<td>OPTBAL (1)</td>
</tr>
<tr>
<td>3 - 4</td>
<td>3</td>
<td>1.958 / 3.318</td>
<td>1.483 / 4.239</td>
<td>0.166 / 6.158</td>
<td>OPTBAL (1)</td>
</tr>
<tr>
<td>3 - 5</td>
<td>2</td>
<td>1.095 / 3.046</td>
<td>0.0 / 0.0</td>
<td>1.292 / 3.231</td>
<td>Ref [4] (2)</td>
</tr>
</tbody>
</table>

Note:
(1) OPTBAL options used: NRMS=1, NGQ=10, $W_1=W_2=0.5$
(TDRMS)max = 5.0
(2) Force Balanced

Table 11: Comparison of Normalized RMS Quantities for Example 3

<table>
<thead>
<tr>
<th>Kinetic Quantity</th>
<th>3 - 1</th>
<th>3 - 2</th>
<th>3 - 3</th>
<th>3 - 4</th>
<th>3 - 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F^*_{SH}$</td>
<td>3.917 (0.0%)</td>
<td>0.445 (89%)</td>
<td>0.445 (89%)</td>
<td>0.230 (94%)</td>
<td>0.0 (100%)</td>
</tr>
<tr>
<td>$M^*_{SH/O1}$</td>
<td>6.866 (0.0%)</td>
<td>0.900 (87%)</td>
<td>0.900 (87%)</td>
<td>0.732 (89%)</td>
<td>1.339 (80%)</td>
</tr>
<tr>
<td>$M^*_{SH/O3}$</td>
<td>5.427 (0.0%)</td>
<td>0.877 (84%)</td>
<td>0.877 (84%)</td>
<td>0.349 (94%)</td>
<td>1.338 (75%)</td>
</tr>
<tr>
<td>$T_D^*$</td>
<td>1.992 (0.0%)</td>
<td>0.261 (87%)</td>
<td>0.261 (87%)</td>
<td>0.106 (95%)</td>
<td>0.716 (64%)</td>
</tr>
<tr>
<td>$F^*_{21}$</td>
<td>5.568 (0.0%)</td>
<td>0.730 (87%)</td>
<td>0.730 (87%)</td>
<td>0.899 (84%)</td>
<td>1.975 (64%)</td>
</tr>
<tr>
<td>$F^*_{32}$</td>
<td>3.969 (0.0%)</td>
<td>0.520 (87%)</td>
<td>0.520 (87%)</td>
<td>0.369 (91%)</td>
<td>1.600 (60%)</td>
</tr>
<tr>
<td>$F^*_{41}$</td>
<td>5.889 (0.0%)</td>
<td>0.596 (90%)</td>
<td>0.596 (90%)</td>
<td>0.244 (96%)</td>
<td>1.820 (69%)</td>
</tr>
<tr>
<td>$F^*_{43}$</td>
<td>3.896 (0.0%)</td>
<td>0.511 (87%)</td>
<td>0.511 (87%)</td>
<td>0.372 (90%)</td>
<td>1.820 (53%)</td>
</tr>
</tbody>
</table>
while rms values for important dynamic balancing quantities are shown in Table 11. Trial 3.2 was balanced by the OPTBAL program with one counterweight attached to the crank. For this case, all the balancing quantities were reduced from 84 % to 90 % over the unbalanced linkage. The effect of placing additional counterweights on the coupler and follower links can be seen in trials 3.3 and 3.4. From these trials, it is evident that the effect of the additional counterweights are not significant; the reductions only increase 2% to 10% over the case of adding one counterweight to the crank. From a practical point of view, this small amount of reduction may not be worth the trouble of adding a counterweight to both the coupler and the follower links. Therefore, the case of adding one counterweight to the crank, trial 3.2, is the recommended solution to the balancing problem. A comparison of the important kinetic parameters for the recommended linkage design, unbalanced linkage and completely force balanced linkage are shown graphically in Figures 15 through 22. From these graphs, the relative amount of imbalance between the three linkages can be seen. More importantly, the graphs give a feeling for the rate of change of the kinetic quantities over one cycle of the linkage.
FIGURE 15: NORMALIZED FORCE IN JOINT 01 VERSUS CRANK ANGLE

- UNBALANCED
- FORCE BALANCED BY OPTIMAL USING ONE COUNTERWEIGHT

CRANK ANGLE (DEGREES)
FIGURE 17:
NORMALIZED FORCE IN JOINT 03 VERSUS CRANK ANGLE

- UNBALANCED
- FORCE BALANCED
- BALANCED BY OPTIMAL USING ONE COUNTERWEIGHT

CRANK ANGLE (DEGREES)
FIGURE 18: NORMALIZED FORCE IN JOINT 04 VERSUS CRANK ANGLE

- UNBALANCED FORCE
- BALANCED BY OPTIMAL BALANCING

CRANK ANGLE (DEGREES)

0 45 90 135 180 225 270 315 360
18 16 14 12 10 8 6 4 2 0

ZORERIJIINED FORCE IN JOINT 04
7.0 DISCUSSION OF RESULTS

The theory and development of a computer program to balance planar four bar linkages has been presented. The theory assumes the links to be rigid and have a finite length. As shown in the three example problems, the OPTBAL program is capable of partially balancing the combined effects of the shaking force, shaking moments, bearing forces and the input torque. Efficient determination of the kinetic quantities was accomplished using the "direct" approach as presented by Lee and Cheng [21]. Counterweight sizes and angular locations were found by adapting an existing optimization program developed by Johnson [18].

Although the developed program provides a superior method of balancing linkages with rigid bodies, the author feels that the method will not always ensure smooth operation of the mechanism. The first, and foremost problem with the method is the assumption of rigid links. This assumption makes the analysis insensitive to resonant frequencies. Therefore, a situation could exist where the linkage has been balanced from a rigid body point of view, yet it could operate at its natural frequency. In such a case, no matter how small the predicted kinetic quantities are, the linkage would not be likely to operate smoothly. A second problem may arise from the assumption of the links and the counterweights being in the same
plane. In many practical designs this may be difficult to achieve, and the user must make an engineering judgement based upon the particular geometry. The program also assumes that the brackets which connect the counterweights, have a negligible mass in comparison to the counterweight or link. Again, the validity of this assumption is problem dependent and can not be judged in general. The remainder of the assumptions, however, should closely approximate a real linkage if proper bearings are used, and the linkage operates at a relatively high speed.

For linkages which can be accurately modeled using the above assumptions, the method of balancing developed in this thesis is excellent. The degree of balance which can be achieved has been demonstrated in three example problems, whose results are discussed below.

In part one of example one, a comparison is shown between the two forms of numerical integration available in the program: Gaussian quadrature and simple summation quadrature. In analyzing the unbalanced linkage, Gaussian quadrature provided approximately 33% better efficiency in evaluating the rms shaking force, shaking moment and input torque for the same degree of accuracy. However, the simple summation method is still used to evaluate the final results, since it is capable of accurately predicting maximum quantities. In order to compare the two forms of quadrature, the number of sampling points was varied. For this variation, both techniques were shown to
converge to the same solutions. Thus, the optimum number of sampling points for an accurate, but efficient analysis was determined for each method.

With the results of the first part of example one, the optimization program could accurately analyze a linkage. However, a point of significance regarding the accuracy of the analysis should be mentioned. In many optimization problems, as discussed by Johnson [16], equations which provide approximate results may still yield an equally optimum solution. It has been this author's experience with the OPTBAL program that approximate equations (ie. with a small number of sampling points), produced good optimization results. But, when the optimum number of sampling points were used, and hence an accurate analysis was provided, the results of the optimization were even better. Therefore, the remaining example problems used the optimum number of sampling points in their analyses.

In part two of example one, the effects of varying the relative weighting factors, $W_1$ and $W_2$, on the optimization results were shown. The investigation produced two distinctive sets of results. First, with emphasis on both of the terms of the objective function (ie. $W_1=W_2=0.5$), or with emphasis specifically placed on the ground bearings force term ($W_1=1, W_2=0$) the results of the optimization were not significantly different. In both situations the shaking force and input torque were reduced by approximately the same amount. That is,
the shaking force was reduced by 87%, and the input torque was reduced by 89% over the unbalanced case. The second distinctive set of results occurred for the situation where emphasis was only placed on the input torque term (ie. $W_1=0$, $W_2=1$). In that case the normalized input torque was reduced by 97%, while the normalized shaking force was only reduced by 17% over the unbalanced case. From the percent reductions in the two cases, it is obvious that the flexibility of the objective function is limited to optimizing the combined effects, or just balancing the input torque. This lack of control could be due to the two terms of the objective function not being completely independent of each other. Lee and Cheng obtained similar results in the analysis of a different linkage. The only valuable conclusion from both of these investigations, is that the best overall balancing situation occurs with the weighting factors equal to each other.

Part three of example one showed the effects of varying the torque regional constraint on the optimization results. As can be seen from the results, significant changes occurred only when the maximum allowable torque was set lower than the torque for a normal optimization (ie. where the regional constraints are relaxed). In such cases, the regional constraint could be used to lower the input torque an additional 7% over the normal optimization. But beyond that point, large regional constraint violations occurred, which indicate that a case of incompatible specifications had been encountered. Interestingly, the amount
of additional reduction of the torque through regional constraint variations is approximately the same as that obtained by modifying the objective function.

In example two, the inline linkage of example one was balanced. Versatility of the balancing program was shown as the number of counterweights were varied. The optimization enabled all important dynamic reactions to be reduced from 75% to 91% over the unbalanced linkage. By comparison, the method of complete force balancing summarized by Berkoff [5], was shown to eliminate the shaking force, with all other kinetic quantities increasing from 36% to 99%. For the case where both the shaking force and moment are eliminated the situation is even worse, as the remaining kinetic parameters are increased from 51% to 293%. Therefore, it appears that the total elimination of balancing quantities, such as the shaking force, has an adverse effect on the remainder of the unbalanced quantities, such as the bearing forces.

In example three, a four bar linkage with an offline coupler link was balanced. The percent reductions over the unbalanced linkage are comparable to those of example 2. For the case of adding only one counterweight to the crank, all the kinetic parameters are reduced by at least 84% over the unbalanced linkage. By contrast, the force balanced linkage eliminated the shaking force, and decreased the remaining balancing terms a minimum of 53%. So, although the linkage
balanced by the OPTBAL program produced better overall results, the force balanced linkage did decrease all the kinetic parameters. Therefore, the method of completely force balancing a linkage does not always adversely effect the remaining balancing quantities, and seems to be problem dependent. Also evident from the results is the effect of adding additional counterweights to the linkage. For this particular example, the degree of balance for two and three added counterweights, was not significantly better than placing one counterweight on the crank. This implies that the most practical design would only use one counterweight attached to the crank.

The balancing results of example 3 were also presented graphically using plots of all the kinetic parameters versus the crank angle. As can be seen in those plots, the major differences between the balanced and unbalanced linkages, occur at a crank angle of approximately zero. This point is where the follower link reverses motion and its effect is reflected throughout the linkage. The degree of balance for this example can be measured by how well this fluctuation has been reduced. As with the evaluation of rms quantities, the OPTBAL program was significantly better at reducing the maximums of all the kinetic quantities than the method of complete force balancing.
8.0 CONCLUSIONS

1. A general computer program to partially balance the combined effects of the shaking force, shaking moments, bearing forces and input torque of a planar, rigid body, four bar linkage has been successfully developed.

2. The key assumption of the entire analysis is that the links are rigid. Therefore, the OPTBAL balancing program is not capable of avoiding natural frequencies where the amount of vibration would be significant.

3. The direct method of Lee and Cheng [21] provides an efficient method for the kinetic analysis without the need for matrix solutions. The direct approach combines the theories of Euler and Lagrange to provide a closed form solution to the kinetic equations.

4. The general optimization program developed by Johnson [18] is capable of solving complicated optimization problems, such as balancing linkages. The basic search process of the program couples a generalized conjugate gradient search direction technique with a univariate line search.

5. The assembled OPTBAL program as a whole, is versatile and useful in evaluating different balancing situations. As shown in the example problems, the number of counterweights, regional constraint parameters, type and degree of numerical quadrature could easily be changed.
6. Gaussian quadrature provides the most efficient means of numerical quadrature in the evaluation of the objective function and other dynamic rms quantities.

7. The objective function proposed by Lee and Cheng [21], gives a relatively good reflection of the true dynamic forces and couples in the linkage. However, variation of the weighting factors shows the objective function is not as flexible as it was first proposed since, minimization of individual parameters, such as the input torque, were only partially successful. Before any definite conclusions can be drawn on this point, a more thorough investigation is needed.

8. Proper choice of the torque regional constraint permitted minimization with emphasis on that parameter. Unfortunately, this caused some of the remaining reactions, such as the shaking force, to be reduced less than could occur without that regional constraint. The degree to which the input torque could be reduced with regional constraints is approximately the same as that obtained by varying the objective function.

9. The method of balancing high speed linkages using the OPTBAL program appears to be significantly better at reducing the combined effects of imbalance, than other closed form solution techniques, such as the method of complete force balancing.
9.0 RECOMMENDATIONS

1. A real linkage should be constructed and experimentally analyzed to verify the assumptions and results of this thesis.

2. An elastic body analysis should be performed on several linkages to determine when the rigid body assumption is valid. References [2], [41] and [42] show techniques for analyzing four bar linkages with nonrigid members.

3. The basic techniques of the method presented in this thesis should be extended to balance slider cranks, six bars and other practical linkages.

4. The computer program developed in this thesis could be made more effective by adding pre- and post-processors. Conceivably, the input data could be read in interactively, while the results could be plotted.

5. A more indepth investigation is needed to compare the OPTBAL program to other balancing techniques that concentrate on eliminating the combined imbalance effects, such as references [9], [33] and [40].

6. The OPTBAL computer program should be revised so that the kinematic equations are flexible enough to handle the general case. For a given set of four links, there are two possible configurations and, therefore, two possible kinematic solutions. Currently, the program is set up to provide one of these solutions with no flexibility to incorporate the other.
REFERENCES


APPENDIX A

11.0 PRINCIPLES OF ANGULAR MOMENTUM

Justification of the equations used for Euler's second law of motion are discussed below. First, a general derivation of Euler's second law about a fixed joint will be presented. Next, definitions of angular momentum for different types of relative and absolute motion will be shown. Finally, the general case of applying Euler's second law to a two point mass model link rotating a moving point will be given.

Consider the system of n point masses shown in Figure A1. Each of the n particles is rigidly connected together by internal forces, $f_{ij}$, and a general external load, $f_i$, is applied to each. The masses have a combined center of gravity location, C, and can be expressed in terms of either the XYZ fixed coordinate system, or the xyz body coordinate system. The position, velocity and acceleration of the general point "O" are assumed to be known. Absolute position of point O is given by $r^{' O O}$, and the relative position of point C with respect to point O is given by $r_{OC}$. 
The time derivative of the absolute angular momentum vector for the ith particle of the system of n masses, with respect to the fixed point O' may be given as:

\[
\dot{\mathbf{r}}_{o'i} = \mathbf{r}_{o'i} \times m_i \dot{\mathbf{r}}_{o'i}
\]

where,

- \( \mathbf{r}_{o'i} \) is the absolute position vector of the ith mass.
- \( m_i \) is the mass of the ith particle.
- \( \dot{\mathbf{r}}_{o'i} \) is the absolute acceleration vector of the ith point mass.

Separating the ith mass from the system, Euler's second law
of motion can be applied to a single particle.

\[
\dot{\mathbf{r}}_i \times \frac{\mathbf{F}}{-\mathbf{R}_i} = \dot{\mathbf{L}}_i
\]

(A2)

where,

\[
\frac{\mathbf{F}}{-\mathbf{R}_i}
\]

is the resultant force vector on the \(i\)th point mass due to both external (\(f_{ij}\)) and internal (\(f_{ii}\)) forces.

Thus, we can write equations similar to the above expression for each of the \(n\) point masses, \(m_i, i = 1, 2, 3, \ldots, i, \ldots, n\).

Or,

\[
\dot{\mathbf{r}}_1 \times \frac{\mathbf{F}}{-\mathbf{R}_1} + \dot{\mathbf{r}}_2 \times \frac{\mathbf{F}}{-\mathbf{R}_2} + \dot{\mathbf{r}}_3 \times \frac{\mathbf{F}}{-\mathbf{R}_3} + \ldots + \dot{\mathbf{r}}_i \times \frac{\mathbf{F}}{-\mathbf{R}_i} + \ldots + \dot{\mathbf{r}}_n \times \frac{\mathbf{F}}{-\mathbf{R}_n} = \dot{\mathbf{L}_n}
\]

(A3)

\[
\dot{\mathbf{r}}_1 \times \frac{\mathbf{F}}{-\mathbf{R}_1} + \dot{\mathbf{r}}_2 \times \frac{\mathbf{F}}{-\mathbf{R}_2} + \dot{\mathbf{r}}_3 \times \frac{\mathbf{F}}{-\mathbf{R}_3} + \ldots + \dot{\mathbf{r}}_i \times \frac{\mathbf{F}}{-\mathbf{R}_i} + \ldots + \dot{\mathbf{r}}_n \times \frac{\mathbf{F}}{-\mathbf{R}_n} = \dot{\mathbf{L}_n}.
\]

(A4)

\[
\dot{\mathbf{r}}_1 \times \frac{\mathbf{F}}{-\mathbf{R}_1} + \dot{\mathbf{r}}_2 \times \frac{\mathbf{F}}{-\mathbf{R}_2} + \dot{\mathbf{r}}_3 \times \frac{\mathbf{F}}{-\mathbf{R}_3} + \ldots + \dot{\mathbf{r}}_i \times \frac{\mathbf{F}}{-\mathbf{R}_i} + \ldots + \dot{\mathbf{r}}_n \times \frac{\mathbf{F}}{-\mathbf{R}_n} = \dot{\mathbf{L}_n}.
\]

(A5)

To find the time derivative of the angular momentum for the system of \(n\) particles, equations Al through A7 must be summed, or:

\[
\sum_{i=1}^{n} \dot{\mathbf{r}}_i \times \frac{\mathbf{F}}{-\mathbf{R}_i} = \sum_{i=1}^{n} \dot{\mathbf{L}_i}.
\]

(A8)
Where the internal force cross product terms conveniently cancel. Now, the cross product of the absolute position vector, \( \mathbf{r}'_o \), and the external force vector applied to the \( i \)th mass actually form a couple about point \( O' \).

\[
C_{O'R} = \sum_{i=1}^{n} \mathbf{r}'_o \times \mathbf{f}_j
\]  

(A9)

Combining equations A8 and A9, the time derivative of the absolute angular momentum can be expressed in terms of the couple about point \( O' \).

\[
\dot{C}_{O'R} = \sum_{i=1}^{n} \dot{A}'_o
\]

(A10)

Equation A10 is the most common form of Euler's second law, and can be used to relate couples and angular momentum about a fixed point \( O' \). However, there are many different variations of the law which stem from different definitions of the angular momentum. The author currently knows of six definitions for angular momentum, and these are listed in table A1.

For the derivation of Euler's second law for a two point mass model link which rotates about a moving point \( O \), the definition given by case 4 will be used. From table A1, the definition of angular momentum for case 4 is given as:

\[
A_{oREL} = \sum_{i=1}^{n} \mathbf{r}_o \times \mathbf{m}_i \times \mathbf{r}_o
\]

(A11)
Taking a time derivative of the above equation produces:

\[ \dot{\mathbf{A}}_{\text{REL}} = \sum_{i=1}^{n} \left( \mathbf{r}_{o_i} \times m_i \mathbf{\hat{r}}_{o_i} + \mathbf{r}_{o_i} \times m_i \mathbf{\ddot{r}}_{o_i} \right) \]  

(A12)

<table>
<thead>
<tr>
<th>Case #</th>
<th>Nomenclature</th>
<th>Type of motion</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Absolute</td>
</tr>
<tr>
<td>1</td>
<td>$\dot{\mathbf{A}}_o$</td>
<td>$\sum_{i=1}^{n} \mathbf{r}<em>{o_i} \times m_i \mathbf{\hat{r}}</em>{o_i}$</td>
</tr>
<tr>
<td>2</td>
<td>$\dot{\mathbf{A}}_{o'\text{REL}}$</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>$\ddot{\mathbf{A}}_o$</td>
<td>$\sum_{i=1}^{n} \mathbf{r}<em>{o_i} \times m_i \mathbf{\hat{r}}</em>{o_i}$</td>
</tr>
<tr>
<td>4</td>
<td>$\dot{\mathbf{A}}_{o\text{REL}}$</td>
<td>-</td>
</tr>
<tr>
<td>5</td>
<td>$\dot{\mathbf{A}}_c$</td>
<td>$\sum_{i=1}^{n} \mathbf{r}<em>{o_i} \times m_i \mathbf{\hat{r}}</em>{o_i}$</td>
</tr>
<tr>
<td>6</td>
<td>$\dot{\mathbf{A}}_{c\text{REL}}$</td>
<td>-</td>
</tr>
</tbody>
</table>

Where the first term can be eliminated since the cross product of two parallel vectors is a null vector. The relative position and acceleration vectors of the second term can be written in terms of absolute quantities.
\[ r_{0i} = r_{0'i} - r_{0'o} \]  \hspace{1cm} (A13)

Therefore,

\[ \dot{r}_{0i} = \dot{r}_{0'i} - \dot{r}_{0'o} \]  \hspace{1cm} (A14)

\[ \ddot{r}_{0i} = \ddot{r}_{0'i} - \ddot{r}_{0'o} \]  \hspace{1cm} (A15)

Substituting equation A15 into A12,

\[ \dot{A}_{oREL} = \sum_{i=1}^{n} \left[ r_{0i} \times m_i \left( \dot{r}_{0'i} - \dot{r}_{0'o} \right) \right] \]  \hspace{1cm} (A16)

or,

\[ \dot{A}_{oREL} = \sum_{i=1}^{n} \left[ r_{0i} \times m_i \dddot{r}_{0'i} - r_{0i} \times m_i \dddot{r}_{0'o} \right] \]  \hspace{1cm} (A17)

Where, the resultant couple about point O, is just the first term in equation A17.

\[ C_{oR} = \sum_{i=1}^{n} r_{0i} \times m_i \dddot{r}_{0'i} \]  \hspace{1cm} (A18)

Substituting A17, into equation A18, and solving for \( C_{oR'} \),

\[ C_{oR'} = \dot{A}_{oREL} + \sum_{i=1}^{n} \left[ r_{0i} \times m_i \dddot{r}_{0'o} \right] \]  \hspace{1cm} (A19)

Equation A19 is the result we desire to apply Euler's second law of motion to link 2, which rotates about a non-fixed
joint. Note that for the case where the moving joint coincides with the fixed joint (ie. \( r_{0'} \) is null), equation A19 reduces to the form of applying the law to a fixed joint, equation A10. Equation A19 also is the same result as obtained by Meriam (reference [29], page 191, equation 100).

At this point, "n" is usually allowed to go to infinity in equations A10 and A19, so that the expression may be applied to a real body. However, for the application of the two point mass model we simply let n equal 2. Summarizing, the two forms of Euler's second law applied to the two point mass model are given by:

\[
\begin{align*}
\sum_{i=1}^{2} r_{0_i} \times m_i \dddot{r}_{0_i} &= \sum_{i=1}^{2} \left[ r_{0_i} \times m_i \dddot{r}_{0_i} + \dddot{r}_{0_i} \times m_i \ddddot{r}_{0_i} \right] \\
C_{0R} &= \sum_{i=1}^{2} r_{0_i} \times m_i \dddot{r}_{0_i} + \dddot{r}_{0_i} \times m_i \ddddot{r}_{0_i} \\
\end{align*}
\]

(A20) (A21)

Where equation A20 can be applied to links which rotate about a frame joint (eg. links 1 and 3), and equation A21 can be applied to links which rotate about a general moving joint (eg. link 2).
APPENDIX B
12.0 SOLUTION OF THE TWO POINT MASS MODEL CONVERSION

The purpose of this section is to show the solution of four equations for the conversion from a real model to a two point mass model. As given in section 4.2, the four equations are:

\[ m_{i1} + m_{i2} = m_{iT} \]  \hspace{1cm} (B1)

\[ m_{i1} l_{i1} = m_{iT} l_{iT} \cos(\Theta_{iT} - \Theta_i) \]  \hspace{1cm} (B2)

\[ m_{i2} l_{i2} = m_{iT} l_{iT} \sin(\Theta_{iT} - \Theta_i) \]  \hspace{1cm} (B3)

\[ m_{i1} l_{i2}^2 + m_{i2} l_{i2}^2 = I_{iT} l_{iT} + m_{iT} l_{iT} \]  \hspace{1cm} (B4)

The four unknowns of the above set equations are the parameters of the two point mass model: \( m_{i1}, m_{i2}, l_{i1}, l_{i2} \). Solving equation B3 for \( m_{i2} \),

\[ m_{i2} = \frac{m_{iT} l_{iT} \sin(\Theta_{iT} - \Theta_i)}{l_{i2}} \]  \hspace{1cm} (B5)

Solving equation B1 for \( m_{i1} \),

\[ m_{i1} = m_{iT} - m_{i2} \]  \hspace{1cm} (B6)
Substituting B5 into the above equation allows $m_{i'i'}$ to be expressed solely in terms of $l_{i'z}$, and other known quantities.

$$m_{i'i'} = \frac{m_{i'i'} [l_{i'z} - l_{i'i'} \sin(\Theta_{i'i'} - \Theta_i)]}{l_{i'z}}$$

(Solution) (B7)

Solving equation B2 for $l_{i'i'}$,

$$l_{i'i'} = \frac{m_{i'i'} l_{i'i'} \cos(\Theta_{i'i'} - \Theta_i)}{m_{i'i'}}$$

(B8)

or, using the expression for $m_{i'i'}$ in equation B7, $l_{i'i'}$ becomes,

$$l_{i'i'} = \frac{l_{i'z} l_{i'i'} \cos(\Theta_{i'i'} - \Theta_i)}{l_{i'z} - l_{i'i'} \sin(\Theta_{i'i'} - \Theta_i)}$$

(B9)

Now, substitution of equations B5, B7 and B9 into equation B4 will eliminate all the unknown parameters except for $l_{i'z}$.

$$\left\{ \frac{m_{i'i'} [l_{i'z} - l_{i'i'} \sin(\Theta_{i'i'} - \Theta_i)] [l_{i'z} l_{i'i'} \cos(\Theta_{i'i'} - \Theta_i)]^2}{l_{i'z}^2} + \frac{[m_{i'i'} l_{i'i'} \sin(\Theta_{i'i'} - \Theta_i)]^2}{l_{i'z}^2} \right\} l_{i'z}^2 = I_{i'i'} + m_{i'i'} l_{i'i'}$$

(B10)

To simplify the above expression, let:

$$l_{i'i'} = l_{i'i'} \cos(\Theta_{i'i'} - \Theta_i)$$

(B11)
Substituting these definitions into equation B10, simplifying, and rearranging,

\[ [1_{i2T}]^2 - [2l_{i2T} + \frac{l_T}{m_{iT}}]l_{i2} + \left(\frac{l_T}{m_{iT}}l_{i2T} + l_{i2T}^2 + l_{i2T}^3\right) = 0 \]  

(B13)

Equation 13 is now of the form where the quadratic equation can be used to solve for \( l_{i2} \). Or,

\[ l_{i2} = \frac{-B \pm \sqrt{B - 4AC}}{2A} \]  

(B14)

where,

\[ A = l_{i2T} \]  

(B15)

\[ B = -[2l_{i2T} + \frac{l_T}{m_{iT}}] \]  

(B16)

\[ C = l_{i2T}^2 \frac{l_T}{m_{iT}} + l_{i2T}^2 + l_{i2T}^3 \]  

(B17)

Once \( l_{i2} \) is known using the above expression, the remainder of the parameters can be found using equations B5, B6 and B8 and a back substitution procedure. The final results may be summarized as:

\[ m_{i2} = m_{iT} l_{iT} \sin(\theta_{iT} - \theta_{i})/l_{i2} \]  

(B18)

\[ m_{i1} = m_{iT} - m_{i2} \]  

(B19)

\[ l_{i1} = m_{iT} l_{iT} \cos(\theta_{iT} - \theta_{i})/m_{i2} \]  

(B20)
APPENDIX C

13.0 SUMMARY OF NUMERICAL INTEGRATION

Determination of the objective function and root mean square quantities requires integration. Since these functions were too complicated to be expressed in closed form solution, methods of numerical quadrature were applied. Listed in the OPTBAL program, appendix D, are two options for numerical integration: Gaussian quadrature and simple summation quadrature. Offering two types of quadrature allows the program to take full advantage of the best features of each. For example, Gaussian quadrature is very efficient, while the simple summation technique is capable of accurately predicting maximums.

In general, numerical quadrature is a geometric method of approximating the area under the curve of a given function. If the integrand is chosen as the equation describing that curve in a given region, then the area will be representative of the numerical value of the definite integral. The procedure basically consists of evaluating the integrand at various sampling points within the region of interest. Once the function values are known, the area
is approximated using a series of rectangular, trapezoidal, or other shaped strips. For rectangular strips, the basic form of the area approximation is given by:

\[ \int_{a}^{b} f(x) \, dx = \sum_{i=1}^{n} h_i f(x_i) \]  

(C1)

where,

- \( h_i \) is the width of the \( i \)th strip
- \( f(x_i) \) is the height of the \( i \)th strip
- \( n \) is the number of strips
- \( x_i \) is the \( i \)th sampling point

Both the simple summation and Gaussian quadrature use rectangular strips to approximate the area. The main difference between the two methods is the manner in which the strip widths, \( h_i \), and the abscissa values, \( x_i \), are chosen.

The simple summation technique assumes the width of each strip to be equal, and in general may be given by:

\[ h_i = \frac{(b - a)}{n} \]  

(C2)

Once the number of sampling points, \( n \), is known, then the region defined by the limits of integration, \( a \) and \( b \), is divided up into \( n \) equally spaced sections. Thus, the distance between successive abscissa values is determined by \( n \), \( a \) and \( b \). If the function is periodic and the integration
is over one cycle of the function, choice of the starting abscissa value is not fixed. For convenience in programming, the arbitrary starting point in the OPTBAL program is $1$ degree of the crank angle, $\theta_1$. The general form of the abscissa values in the OPTBAL program for simple summation quadrature are given by:

$$x_i = 2\pi \left[ \frac{360(i - 1) + n}{360} \right] \text{ (rad)}$$

In contrast to the simple summation technique, Gaussian quadrature provides a much more efficient means of numerical integration. Instead of assuming equally spaced abscissa locations, Gaussian quadrature evaluates the function at predetermined optimum locations. Instead of summing each of the strip areas to get the total area, Optimum weighting factors are employed to multiply each area before it is summed. For these reasons, Gaussian quadrature requires the fewest number of sampling points to integrate a function of specific order. The theory of determining the optimum weighting factors and Gaussian abcissas may be found in Hildebrand [13]. Specific values of these optimum parameters for a range of two through twenty-four sampling points are shown here in table C1. Once the number of sampling points is known, then the width of each strip may be given as:
13.0 SUMMARY OF NUMERICAL INTEGRATION

\[ hi = W_i \frac{(b - a)}{2} \]  \hspace{1cm} (C4)

where,

\( W_i \) is the weighting factor of the \( i \)th sampling point given in Table C1.

Also, the abscissa values at which the integrand will be evaluated are given by:

\[ X_i = \left[ a + b + r_i \left( a - b \right) \right] \frac{1}{2} \]  \hspace{1cm} (C5)

where,

\( r_i \) is the optimum Gaussian abscissa value of the \( i \)th sampling point given in Table C1.

Once the numerical integration is done, then the root mean square (rms) quantities may be determined. The basic definition of the rms value of any function \( f \) is given by:

\[ f_{\text{RMS}} = \sqrt{\frac{1}{T} \int_{0}^{T} f^2(x) \, dx} \]  \hspace{1cm} (C6)

where \( T \) is the period of the function \( f(x) \). The above equation can be solved with either types of numerical quadrature using equations C2 or C3. For the simple summation quadrature this reduces to:

\[ f_{\text{RMS}} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} f^2 \left[ \frac{2\pi}{360} \left( 360 \, (i-1) + n \right) \right]} \]  \hspace{1cm} (C7)
and for Gaussian quadrature,

\[ f_{\text{RMS}} = \sqrt{(1/2) \sum_{i=1}^{n} w_i f^2 [\pi(1 - r_i)]} \]  

(C8)

The example problems presented in this thesis were for single degree of freedom linkages, hence they only need one input, \( \Theta_i \). For this reason the objective function and rms quantities were integrated with respect to crank angle, \( \Theta_i \). Further, the examples assumed the input link to rotate through a complete revolution per cycle. Thus, the period, \( T \), for these problems is \( 2\pi \) radians, and the abscissa values for integration correspond to values of the input crank, \( \Theta_i \).
### TABLE C1

**ABSCISSAS AND WEIGHT FACTORS FOR GAUSSIAN INTEGRATION**

\[ \int_{-1}^{1} f(x) dx = \sum_{i=1}^{n} w_i f(x_i) \]

<table>
<thead>
<tr>
<th>Abscissas, ( x_i ) (Zeros of Legendre Polynomials)</th>
<th>Weight Factors, ( w_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = 2 )</td>
<td>0.07735 02691 89262</td>
</tr>
<tr>
<td>( 0.0000 0000 0000 )</td>
<td>0.0000 0000 0000</td>
</tr>
<tr>
<td>( n = 3 )</td>
<td>0.77459 66962 41483</td>
</tr>
<tr>
<td>( 0.0000 0000 0000 )</td>
<td>0.83333 33333 33333</td>
</tr>
<tr>
<td>( n = 4 )</td>
<td>0.39999 38135 84296</td>
</tr>
<tr>
<td>( 0.0000 0000 0000 )</td>
<td>0.62724 51548 62546</td>
</tr>
<tr>
<td>( n = 5 )</td>
<td>0.61613 63113 49533</td>
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<tr>
<td>( 0.0000 0000 0000 )</td>
<td>0.34785 48451 37434</td>
</tr>
<tr>
<td>( n = 6 )</td>
<td>0.23861 91850 83197</td>
</tr>
<tr>
<td>( 0.0000 0000 0000 )</td>
<td>0.47951 37945 72691</td>
</tr>
<tr>
<td>( n = 7 )</td>
<td>0.66120 98646 62553</td>
</tr>
<tr>
<td>( 0.0000 0000 0000 )</td>
<td>0.36208 15780 48139</td>
</tr>
<tr>
<td>( n = 8 )</td>
<td>0.93246 95142 32152</td>
</tr>
<tr>
<td>( 0.0000 0000 0000 )</td>
<td>0.17132 44923 79170</td>
</tr>
</tbody>
</table>

OPTBAL is a computer program capable of balancing four bar linkages using counterweights. It combines a general optimization program developed by R. C. Johnson [18], and the theory of balancing the combined effects of input torque, shaking force, shaking moment, and bearing forces as defined by Lee and Cheng [21]. A description of the optimization search process, and balancing theory are presented in sections 4.0 and 5.0 of this thesis. The purpose of this appendix is to summarize the overall capabilities, limitations, and specifics of the input and output data. In addition, a complete listing of the program is provided.

Capabilities of the OPTBAL program include:

1. Analysis of four bar linkages with offline mass distributions.
2. Specification of regional constraints placed on rms balancing quantities: maximum bearing force, shaking force, maximum shaking moment, input torque.
3. Variation in the number of counterweights to
be attached.

4. Flexibility in the degree, and type of numerical integration used to evaluate rms quantities and the objective function.

5. Power to balance the combined effects of shaking force, shaking moment, input torque and bearing forces.

6. Flexibility in the objective function to allow for balancing individual terms such as the input torque.

With minor changes to the program, additional features may be incorporated:

1. Balancing with specific loads placed on the follower.
2. Analysis of a linkage with an accelerating crank.

Major limitations of the OPTBAL program can be summarized as:

1. The balancing theory assumes the links to be rigid bodies. Thus the program is incapable of predicting, and avoiding natural frequencies.
2. The program is not capable of analyzing links with an infinite length such as sliders.
3. The theory is based on the linkage and counterweights being coplanar.
4. Counterweights can only be attached to specific bearing joints.
5. The program is not capable of providing completely general kinematic solutions.
The FORTRAN program was designed for online use with a VAX/VMS 11/782 digital computer. Input and output to the program is accomplished using assigned data files. Input data is read in from a device assigned to for003, while three different types of output are sent to data files using devices for004, for006, for007. Output assigned to the for004 device is a complete listing of the optimization search process including final results. The output assigned to both for006, and for007 devices is identical and provides a summary of the final counterweight design, regional constraints, and important final quantities of balancing. The devices should be assigned so that for006 is the computer terminal, and for007 is an output data file. Examples of input, and output data files taken from example 3 are given in the remainder of this appendix while options available for the input are listed below.

Options available using the OPTBAL input data file:

Line 1,

- **kc** - number of calculated constants; not used enter 0.
- **kg** - parameter of optimization; not used enter 0.
- **md** - minimum number of random generation points used to determine an initial start point.
- **mn** - parameter of optimization; not used enter 0.
- **ms** - maximum number of random generation points used to determine an initial start point.
- **nd** - specifies the type of gradient technique used
in the optimization search.

\[ = 1 \text{ for method of steepest descent} \]
\[ = 2 \text{ for generalized conjugate gradient method} \]

\[ \text{ne} \] - number of equality constraints; not used enter 0.

\[ \text{nf} \] - number of final items specified in "final" subroutine.

\[ \text{ng} \] - number of given constants.

\[ \text{np} \] - print mode for the search output (for004 device).

\[ = 1 \text{ for minimal search information} \]
\[ = 2 \text{ for maximum search information} \]

\[ \text{nr} \] - number of regional constraints

\[ \text{nu} \] - parameter of optimization; not used enter 0.

\[ \text{nv} \] - number of variables for optimization.

\[ \text{nrms} \] - specifies type of quadrature used.

\[ = 1 \text{ for Gaussian quadrature} \]
\[ = 2 \text{ for simple summation quadrature} \]

\[ \text{nc} \] - number of counterweights to be applied.

\[ = 1 \text{ for one c.w. attached to the crank} \]
\[ (\text{ie. joint } O') \]
\[ = 2 \text{ for two c.w.'s attached to crank and follower} (\text{ie. joints } O', O') \]
\[ = 3 \text{ for three c.w.'s attached to crank, coupler, and follower} (\text{ie. joints } O', O', O') \]

\[ \text{ngq} \] - specifies the degree of quadrature ,or the number of sampling points used.

\[ \text{nseed} \] - specifies an initial seed for the random number generation
Line 2

\[ cn(i), \ i = 1,\nv \] - approximate lower limits of design variables.

Line 3

\[ cx(i), \ i = 1,\nv \] - approximate upper limits of design variables.

Line 4,5,6

\[ cg(i), \ i = 1,\ng \] - given constants, \ng in number.

see definitions block of subroutine analysis for further definition.

line 7

\[ cs(i), \ i = 1,\nv \] - user specified start point for optimization search. May only be used if \mn \ and \mx \ are equal to zero.
Example Data File for the OPTBAL Program assigned to device FOR003

DATA FILE FOR EXAMPLE # 3

0,0,0,64,0,512,2,0,7,29,2,19,0,6,1,2,10,.8
0.0,0.0,0.0,0.0,0.0,0.0,0.0
1.0,1.0,1.0,1.0,1.0,1.0
2.0,3.0,1.5,2.75,52.35988,0.0,0.5,0.5,0.0
1.0,3.0265,1.5,0.19730,0.52825,0.26808,0.24595,1.32292,0.751853,
0.0,0.13353,0.0,0.283,0.625,6.283185308,5.0
5.0,5.0,5.0,5.0

DESCRIPTION OF DATA FILE

KC, KG, MD, MN, MS, MX, ND, NE, NF, NG, NP, NR, NU, NV, NRMS, NC, NGQ, NSEED
CN(I), I=1,NV ! APPROX. LOWER LIMITS OF VARIABLES FOR SGSEARCH
CX(I), I=1,NV ! APPROX. UPPER LIMITS OF VARIABLES FOR SGSEARCH
CG(1), CG(2), CG(3), CG(4), CG(5), CG(6), CG(7), CG(8), CG(9)
CG(10), CG(11), CG(12), CG(13), CG(14), CG(15), CG(16), CG(17), CG(18)
CG(19), CG(20), CG(21), CG(22), CG(23), CG(24), CG(25)
CG(26), CG(27), CG(28), CG(29)
CS(I), I=1,NV ! NOT SHOWN ABOVE
Summary of Optimization Results from Devices FOR006 and FOR007

MODSER - P519RE FORTRAN VERSION FOLLOWS (6/6/84)
APPLICATION: BALANCING OF FOUR BAR MECHANISMS

OPTIMUM VARIABLES

<table>
<thead>
<tr>
<th>THTC1</th>
<th>THTC2</th>
<th>THTC3</th>
<th>LC1</th>
<th>LC2</th>
<th>LC3</th>
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<td>0.000</td>
<td>0.236</td>
<td>1.534</td>
<td>0.000</td>
<td>0.031</td>
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</table>

REGIONAL CONSTRAINT # 1 WAS OBEYED : 0.4914055E+00
REGIONAL CONSTRAINT # 2 WAS OBEYED : 0.0000000E+00
REGIONAL CONSTRAINT # 3 WAS OBEYED : 0.3759425E-01
REGIONAL CONSTRAINT # 4 WAS OBEYED : 0.3068319E+00
REGIONAL CONSTRAINT # 5 WAS OBEYED : 0.0000000E+00
REGIONAL CONSTRAINT # 6 WAS OBEYED : 0.6294738E-02
REGIONAL CONSTRAINT # 7 WAS OBEYED : 0.5085945E+00
REGIONAL CONSTRAINT # 8 WAS OBEYED : 0.1000000E+01
REGIONAL CONSTRAINT # 9 WAS OBEYED : 0.9624058E+00
REGIONAL CONSTRAINT # 10 WAS OBEYED : 0.6931681E+00
REGIONAL CONSTRAINT # 11 WAS OBEYED : 0.1000000E+01
REGIONAL CONSTRAINT # 12 WAS OBEYED : 0.9937053E+00
REGIONAL CONSTRAINT # 13 WAS OBEYED : 0.4555073E+01
REGIONAL CONSTRAINT # 14 WAS OBEYED : 0.4270540E+01
REGIONAL CONSTRAINT # 15 WAS OBEYED : 0.4100474E+01
REGIONAL CONSTRAINT # 16 WAS OBEYED : 0.4738155E+01
REGIONAL CONSTRAINT # 17 WAS OBEYED : 0.3644772E+01
REGIONAL CONSTRAINT # 18 WAS OBEYED : 0.1349408E+02
REGIONAL CONSTRAINT # 19 WAS OBEYED : 0.7858233E+01

RMS VALUES

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<th>F32</th>
<th>F41</th>
<th>F43</th>
<th>AO1D</th>
<th>AO3D</th>
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<td>0.510</td>
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<td>0.877</td>
<td>0.262</td>
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</table>

MAXIMUM VALUES

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<tr>
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<th>F21</th>
<th>F32</th>
<th>F41</th>
<th>F43</th>
<th>AO1D</th>
<th>AO3D</th>
<th>TD</th>
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</thead>
<tbody>
<tr>
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<td>1.845</td>
<td>1.849</td>
<td>2.959</td>
<td>1.375</td>
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</table>
Summary of the optimization Search Process from Device FOR004

EXAMPLE # 3 OUTPUT

MODSER - AOD FORTRAN VERSION (6/6/84)
APPLICATION : BALANCING OF FOUR BAR MECHANISMS

**INPUT DATA:**

\[
\begin{align*}
& AB, AE, AG, AL, AM, AW, AT, CA, CL, CF, JX, KK = \\
& 10.00000000000000 \quad 0.50000000000000 \quad 1000.00000000000000 \\
& 10000.00000000000000 \quad 8.00000000000000 \quad 20000.00000000000000 \\
& 2.00000000000000 \quad 0.85000000000000 \quad 0.25000000000000 \\
& 0.25000000000000 \quad 300 \quad 250 \\
& KC, KG, MD, MN, MS, MX, ND, NE, NF, NG, NP, NR, NU, NV = \\
& 0 \quad 0 \quad 0 \quad 64 \quad 0 \quad 512 \\
& 2 \quad 0 \quad 7 \quad 29 \quad 2 \quad 19 \\
& 0 \quad 6 \\
\end{align*}
\]

\[CN(1) \text{ FOR } I = 1, \ldots, NV \text{ ARE =}
\]

\[
\begin{align*}
& 0.0000000E+00 \quad 0.0000000E+00 \quad 0.0000000E+00 \quad 0.0000000E+00 \\
& 0.0000000E+00 \quad 0.0000000E+00
\end{align*}
\]

\[CX(1) \text{ FOR } I = 1, \ldots, NV \text{ ARE =}
\]

\[
\begin{align*}
& 0.1000000E+01 \quad 0.1000000E+01 \quad 0.1000000E+01 \quad 0.1000000E+01 \\
& 0.1000000E+01 \quad 0.1000000E+01
\end{align*}
\]

\[CG(1) \text{ FOR } IG = 1, \ldots, NG \text{ ARE =}
\]

\[
\begin{align*}
& 0.2000000E+01 \quad 0.3000000E+01 \quad 0.1500000E+01 \quad 0.2750000E+01 \\
& 0.5235988E+02 \quad 0.0000000E+00 \quad 0.5000000E+00 \quad 0.5000000E+00 \\
& 0.0000000E+00 \quad 0.1000000E+01 \quad 0.3026500E+01 \quad 0.1500000E+01 \\
& 0.1973000E+00 \quad 0.5282500E+00 \quad 0.2680800E+00 \quad 0.2459500E+00 \\
& 0.2322920E+01 \quad 0.7518530E+00 \quad 0.0000000E+00 \quad 0.1335300E+00 \\
& 0.0000000E+00 \quad 0.2830000E+00 \quad 0.6250000E+00 \quad 0.6283185E+01 \\
& 0.5000000E+01 \quad 0.5000000E+01 \quad 0.5000000E+01 \quad 0.5000000E+01 \\
& 0.5000000E+01
\end{align*}
\]

**INITIAL CALCULATIONS :**

\[
\begin{align*}
& AD, FD, FS, FX, FH, FM = \\
& 0.2449490E+00 \quad 0.1224745E-03 \quad 0.2449490E-04 \quad 0.1224745E-04 \\
& 0.1224745E+00 \quad 0.4000000E+04
\end{align*}
\]

**SG SEARCH GAVE FOLLOWING POINT:**

\[
\begin{align*}
& QG, PG, X : \quad 0.4322246032509261 \quad 0.0000000000000000E+00 \\
& 0.4322246032509261
\end{align*}
\]

\[VG(I) \text{ FOR } I=1, \ldots, NV \text{ ARE=}
\]
0.5338135E+00  0.0000000E+00  0.3265381E-01  0.2576904E+00
0.0000000E+00  0.5126953E-01

START OF MODSER SEARCH PROCESS:
BASE POINT PRINTOUTS FOLLOW:

KT,K,XB,QB = 1 1 0.4322246E+00 0.4322246E+00
VB(I) FOR I = 1,......,NV =
0.5338135E+00  0.0000000E+00  0.3265381E-01  0.2576904E+00
0.0000000E+00  0.5126953E-01

RB(IR) FOR IR = 1,.........,NR ARE =
0.5338135E+00  0.0000000E+00  0.3265381E-01  0.2576904E+00
0.0000000E+00  0.5126953E-01
0.9673462E+00  0.7423096E+00  0.1000000E+01  0.9487305E+00
0.4497246E+01  0.4024686E+01  0.3789716E+01  0.4659437E+01
0.2591001E+01  0.1349408E+02  0.6954521E+01

ID= 1
JT,FB,FE= 3 0.2449489742783178  6.1237243569579452E-02
DB= 6.9839021471182823E-02

KT,K,XB,QB = 2 2 0.3865183E+00 0.3865183E+00
VB(I) FOR I = 1,......,NV =
0.5172582E+00  0.0000000E+00  0.3275896E-01  0.3255278E+00
0.0000000E+00  0.5004963E-01

RB(IR) FOR IR = 1,.........,NR ARE =
0.5172582E+00  0.0000000E+00  0.3275896E-01  0.3255278E+00
0.0000000E+00  0.5004963E-01
0.9672410E+00  0.6744722E+00  0.1000000E+01  0.9499504E+00
0.4498318E+01  0.4345842E+01  0.4079986E+01  0.4771581E+01
0.3967681E+01  0.1349408E+02  0.7003176E+01

ID= 2
JT,FB,FE= 4 5.9363168250505400E-02  2.9681584125252700E-02
DB= 2.9658846517563092E-02

KT,K,XB,QB = 3 3 0.3823282E+00 0.3823282E+00
**VB(I) FOR I = 1,…….,NV =**

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**RB(IR) FOR IR = 1,…….,NR ARE =**

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**ID= 2**

**JT,FB,FE=**

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**DB= 1.3894205963529090E-02**

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**KT,K,XB,QB =**

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**VB(I) FOR I = 1,…….,NV =**

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**RB(IR) FOR IR = 1,…….,NR ARE =**

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**ID= 2**

**JT,FB,FE=**

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**DB= 4.6777076705050751E-03**

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**KT,K,XB,QB =**

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**VB(I) FOR I = 1,…….,NV =**

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**RB(IR) FOR IR = 1,…….,NR ARE =**

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<td>DB</td>
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**ME = 1**

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0.4909220E+00 0.0000000E+00 0.3604766E-01 0.3068665E+00
0.0000000E+00 0.2249507E-01

RB(IR) FOR IR = 1, ..., NR ARE =

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0.9639523E+00 0.6931335E+00 0.1000000E+01 0.9775049E+00
0.4556680E+00 0.4275151E+01 0.4103507E+01 0.4746918E+01
0.3641885E+01 0.1349408E+02 0.7735260E+01

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VB(I) FOR I = 1, ..., NV =

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0.0000000E+00 0.2118434E-01

RB(IR) FOR IR = 1, ..., NR ARE =

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0.9638095E+00 0.6928256E+00 0.1000000E+01 0.9788157E+00
0.4555900E+01 0.4276418E+01 0.4105120E+01 0.4747360E+01
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ID = 2

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**KT,K,XB,QB** = 16  7  0.3803075E+00  0.3803075E+00

**VB(I) FOR I = 1,......,NV =**

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**RB(IR) FOR IR = 1,...........,NR ARE =**

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**DB= 7.5301208576304786E-04**

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**RB(IR) FOR IR = 1,...........,NR ARE =**

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<tr>
<td>35 7 0.3803038E+00 0.3803038E+00</td>
<td>0.4914038E+00 0.0000000E+00 0.3755639E-01 0.3068707E+00</td>
<td>0.4914038E+00 0.0000000E+00 0.3755639E-01 0.3068707E+00</td>
</tr>
<tr>
<td></td>
<td>0.0000000E+00 0.6834011E-02</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.9624645E+00 0.6931704E+00 0.1000000E+01 0.9928811E+00</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.3644293E+01 0.1349408E+02 0.7855969E+01</td>
<td></td>
</tr>
</tbody>
</table>

**Additional Information:**
- **ID:** 2
- **JT, FB, FE:** 2
- **DB:** 3.4771213680814373E-04
- **KT, K, XB, QB:** 33
- **NV:** 5
- **VB(I) FOR I = 1,........,NV:**
  - 0.4914148E+00 0.0000000E+00 0.3751034E-01 0.3067822E+00
  - 0.0000000E+00 0.7455852E-02
- **RB(IR) FOR IR = 1,...........,NR ARE:**
  - 0.4914148E+00 0.0000000E+00 0.3751034E-01 0.3067822E+00
  - 0.0000000E+00 0.7455852E-02 0.5085852E+00 0.1000000E+01
  - 0.9624897E+00 0.6932178E+00 0.1000000E+01 0.9925441E+00
  - 0.4556675E+01 0.4274828E+01 0.4103380E+01 0.4746806E+01
  - 0.3643728E+01 0.1349408E+02 0.7854945E+01

**Additional Information:**
- **ID:** 2
- **JT, FB, FE:** 2
- **DB:** 2.9398327743749250E-04
- **KT, K, XB, QB:** 34
- **NV:** 6
- **VB(I) FOR I = 1,........,NV:**
  - 0.4913477E+00 0.0000000E+00 0.3753547E-01 0.3068296E+00
  - 0.0000000E+00 0.7118901E-02
- **RB(IR) FOR IR = 1,...........,NR ARE:**
  - 0.4913477E+00 0.0000000E+00 0.3753547E-01 0.3068296E+00
  - 0.0000000E+00 0.7118901E-02 0.5086523E+00 0.1000000E+01
  - 0.9624645E+00 0.6931704E+00 0.1000000E+01 0.9928811E+00
  - 0.4556631E+01 0.4275023E+01 0.4103622E+01 0.4746874E+01
  - 0.3644293E+01 0.1349408E+02 0.7855969E+01

**Additional Information:**
- **ID:** 2
- **JT, FB, FE:** 2
- **DB:** 2.9398327743749250E-04
- **KT, K, XB, QB:** 35
- **NV:** 7
- **VB(I) FOR I = 1,........,NV:**
  - 0.4914038E+00 0.0000000E+00 0.3755639E-01 0.3068707E+00
  - 0.0000000E+00 0.6834011E-02
- **RB(IR) FOR IR = 1,...........,NR ARE:**
  - 0.4914038E+00 0.0000000E+00 0.3755639E-01 0.3068707E+00
  - 0.0000000E+00 0.6834011E-02 0.5086523E+00 0.1000000E+01
  - 0.9624645E+00 0.6931704E+00 0.1000000E+01 0.9928811E+00
  - 0.4556631E+01 0.4275023E+01 0.4103622E+01 0.4746874E+01
  - 0.3644293E+01 0.1349408E+02 0.7855969E+01
<table>
<thead>
<tr>
<th>ID</th>
<th>JT,FB,FE</th>
<th>DB</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>2.4988578582186862E-04</td>
</tr>
<tr>
<td></td>
<td>2</td>
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</tr>
<tr>
<td></td>
<td>2</td>
<td>2.782038646030374E-04</td>
</tr>
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**KT,K,XB,QB** = 36 8 0.3803038E+00 0.3803038E+00

**VB(I)** FOR I = 1,......,NV =

<table>
<thead>
<tr>
<th>I</th>
<th>VB(I)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4914549E+00 0.0000000E+00 0.3757596E-01 0.3068267E+00</td>
</tr>
<tr>
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<td>0.0000000E+00 0.6564820E-02</td>
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</table>

**RB(IR)** FOR IR = 1,........,NR ARE =

<table>
<thead>
<tr>
<th>IR</th>
<th>RB(IR)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4914549E+00 0.0000000E+00 0.3757596E-01 0.3068267E+00</td>
</tr>
<tr>
<td></td>
<td>0.0000000E+00 0.6564820E-02</td>
</tr>
</tbody>
</table>

**ID** = 2

**JT,FB,FE** = 2 2.3647328491255318E-04 2.3647328491255318E-04

**DB** = 1.4477154122509679E-04

**KT,K,XB,QB** = 37 9 0.3803038E+00 0.3803038E+00

**VB(I)** FOR I = 1,......,NV =

<table>
<thead>
<tr>
<th>I</th>
<th>VB(I)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4914228E+00 0.0000000E+00 0.3758592E-01 0.3068071E+00</td>
</tr>
<tr>
<td></td>
<td>0.0000000E+00 0.6425377E-02</td>
</tr>
</tbody>
</table>

**RB(IR)** FOR IR = 1,........,NR ARE =

<table>
<thead>
<tr>
<th>IR</th>
<th>RB(IR)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4914228E+00 0.0000000E+00 0.3758592E-01 0.3068071E+00</td>
</tr>
<tr>
<td></td>
<td>0.0000000E+00 0.6425377E-02</td>
</tr>
</tbody>
</table>

**ID** = 2

**JT,FB,FE** = 2 1.2305581004133227E-04 1.2305581004133227E-04

**DB** = 6.5887444842649220E-05
KT,K,XB,QB = 38 10 0.3803038E+00 0.3803038E+00

VB(I) FOR I = 1, ...., NV =

0.4913991E+00 0.0000000E+00 0.3758987E-01 0.3068299E+00
0.0000000E+00 0.6368425E-02

RB(IR) FOR IR = 1, ......., NR ARE =

0.4913991E+00 0.0000000E+00 0.3758987E-01 0.3068299E+00
0.0000000E+00 0.6368425E-02 0.5086009E+00 0.1000000E+01
0.9624101E+00 0.6931701E+00 0.1000000E+01 0.9936316E+00
0.4556605E+01 0.4275024E+01 0.4103627E+01 0.4746874E+01
0.3644679E+01 0.1349408E+02 0.7858044E+01

ID= 2
JT,FB,FE= 2 5.6004328116251838E-05 5.6004328116251838E-05
DB= 2.9526784119060994E-05

KT,K,XB,QB = 39 11 0.3803038E+00 0.3803038E+00

VB(I) FOR I = 1, ...., NV =

0.4914052E+00 0.0000000E+00 0.3759165E-01 0.3068334E+00
0.0000000E+00 0.6339805E-02

RB(IR) FOR IR = 1, ......., NR ARE =

0.4914052E+00 0.0000000E+00 0.3759165E-01 0.3068334E+00
0.0000000E+00 0.6339805E-02 0.5085948E+00 0.1000000E+01
0.9624083E+00 0.6931666E+00 0.1000000E+01 0.9936602E+00
0.4556596E+01 0.4275038E+01 0.4103644E+01 0.4746879E+01
0.3644802E+01 0.1349408E+02 0.7858118E+01

ID= 2
JT,FB,FE= 3 2.5097766501201845E-05 5.0195533002403690E-05
DB= 2.6306105594259598E-05

KT,K,XB,QB = 40 12 0.3803038E+00 0.3803038E+00

VB(I) FOR I = 1, ...., NV =

0.4914099E+00 0.0000000E+00 0.3759317E-01 0.3068284E+00
0.0000000E+00 0.6314448E-02

RB(IR) FOR IR = 1, ......., NR ARE =

0.4914099E+00 0.0000000E+00 0.3759317E-01 0.3068284E+00
0.0000000E+00  0.6314448E-02  0.5085901E+00  0.1000000E+01
0.9624068E+00  0.6931716E+00  0.1000000E+01  0.9936856E+00
0.4556602E+01  0.4275018E+01  0.4103619E+01  0.4746872E+01
0.3644724E+01  0.1349408E+02  0.7858182E+01  0.9936856E+00

---

\[ \text{ID} = 2 \]

\[ \text{JT, FB, FE} = 2 \ 2.236018975512069 \times 10^{-05} \ 2.236018975512069 \times 10^{-05} \]

\[ \text{DB} = 1.1450402586579092 \times 10^{-05} \]

\[ \text{ME} = 3 \]

---

\[ \text{KT, XB, QB} = 41 \ 1 \ 0.3803038E+00 \ 0.3803038E+00 \]

\[ \text{VB(I) FOR I = 1, \ldots, NV} = \]

\[ \begin{array}{ccc}
0.4914085E+00 & 0.0000000E+00 & 0.3759382E-01 \\
0.0000000E+00 & 0.6303109E-02 \\
\end{array} \]

---

\[ \text{RB(IR) FOR IR = 1, \ldots, NR ARE} = \]

\[ \begin{array}{ccc}
0.4914085E+00 & 0.0000000E+00 & 0.3759382E-01 \\
0.0000000E+00 & 0.6303109E-02 \\
\end{array} \]

---

\[ \text{ID} = 2 \]

\[ \text{JT, FB, FE} = 3 \ 9.732842198592228 \times 10^{-06} \ 1.946568439718445 \times 10^{-05} \]

\[ \text{DB} = 9.846312490779583 \times 10^{-06} \]

\[ \text{ME} = 3 \]

---

\[ \text{CONVERGENCE ANTICIPATED, WITH AC = 1.772879434396054} \]

\[ \text{READY FOR TERMINATION TEST AT 755, AND FT = 1.1358961481767196 \times 10^{-04}} \]

---

\[ \text{TERMINATION TEST PASSED AT 755} \]

---

\[ \text{SOLUTION FOUND IS AT (K+1), NOW TRANSFERRED TO (K) STORAGE FOR PRINTOUT} \]

\[ \text{KT, XB, QB} = 41 \ 1 \ 0.3803038E+00 \ 0.3803038E+00 \]

\[ \text{VB(I) FOR I = 1, \ldots, NV} = \]

\[ \begin{array}{ccc}
0.4914055E+00 & 0.0000000E+00 & 0.3759425E-01 \\
0.0000000E+00 & 0.6294738E-02 \\
\end{array} \]

---

\[ \text{RB(IR) FOR IR = 1, \ldots, NR ARE} = \]

\[ \begin{array}{ccc}
0.4914055E+00 & 0.0000000E+00 & 0.3759425E-01 \\
0.0000000E+00 & 0.6294738E-02 \\
\end{array} \]
<table>
<thead>
<tr>
<th></th>
<th>GM= 1.0390750709209900E-04</th>
</tr>
</thead>
<tbody>
<tr>
<td>DOUBLE PRECISION VB(I) FOR I=1,...,NV ARE=</td>
<td></td>
</tr>
<tr>
<td>0.4914054669859E+00</td>
<td>0.000000000000000E+00</td>
</tr>
<tr>
<td>0.3068319346359E+00</td>
<td>0.000000000000000E+00</td>
</tr>
</tbody>
</table>

| FINAL ITEMS F(JF) FOR JF = 1,.......NF ARE= |
| 0.3820699E+00 | 0.3087592E+01 | 0.0000000E+00 | 0.2362116E+00 |
| 0.1534160E+01 | 0.0000000E+00 | 0.3147369E-01 |

**MF, MG, MR, NX** = 139 42 0 0
THIS IS THE FORTRAN VERSION OF THE MODSER PROGRAM "AOD." THIS VERSION (6/6/84) HAS BEEN MODIFIED FOR USE ON A VAX/VMS 11/782 DIGITAL COMPUTER.

DR. R. C. JOHNSON, PHD., P.E.

SUBROUTINES USED

1) ANALYSIS
   : CALCULATES THE VALUE OF THE PENALIZED OPTIMIZATION QUANTITY, REGIONAL CONSTRAINTS, AND EQUALITY CONSTRAINTS.

2) FBPRINT
   : PERFORMS A FULL BASEPOINT PRINTOUT.

3) GRADIENT
   : APPROXIMATES THE GRADIENT VECTOR, g, BY FINITE DIFFERENCES.

4) INPUTPO
   : GIVES A PRINTOUT OF THE INPUT DATA.

5) LSEARCH
   : CONTROL OF LINE SEARCH BY FDAP TECHNIQUE; ALSO PERFORMS CP CALCULATION BY SELF TUNING.

6) MBPRINT
   : PERFORMS A MINIMAL BASEPOINT PRINTOUT.

7) NEWBP
   : PROVIDES A NEWBASE POINT FOR THE SEARCH.

8) PRINT(L)
   : PROVIDES A LINE OF CHARACTERS IN THE PRINTOUT AS DETERMINED BY THE CHOICE OF L.

9) SDIR
   : DETERMINES THE SEARCH DIRECTION, s, USING EITHER MSD OR GCG TECHNIQUES.

10) SGSEARCH
    : RANDOM GENERATION OF A GOOD STARTING POINT.

11) STORINDEX
    : PROVIDES A STROAGE AND INDEXING OPERATION.

SUBFUNCTIONS USED

1) SGN(P)
   : RETURNS EITHER A "+1","-1", OR "0" DEPENDING UPON
THE SIGN OF P.

2) RND(SEED)
   : RETURNS A RANDOM NUMBER BASED UPON A USER CHOOSEN SEED.

************************************************************************

PARAMETER (MAXN = 32)
REAL * 8 C(MAXN), CD(MAXN), CG(MAXN), CN(MAXN), CS(MAXN), CX(MAXN),
   DN(MAXN), DP(MAXN), DV(MAXN), E(MAXN), EA(MAXN), EB(MAXN),
   F(MAXN), G(24), GA(3), GB(MAXN), GC(MAXN), GS(MAXN), II(10),
   MI(10), OA(MAXN), OD(MAXN), OE(MAXN), OF(MAXN), OG(MAXN),
   OI(MAXN), OK(MAXN), OS(MAXN), PI(10), R(MAXN), RA(MAXN),
   RB(MAXN), RR(19), RS(MAXN), S(MAXN), SA(MAXN), SB(MAXN),
   SC(MAXN), SD(MAXN), SE(MAXN), SF(MAXN), SK(MAXN), TB(MAXN),
   U(MAXN), UA(MAXN), UB(MAXN), V(MAXN), VA(MAXN), VB(MAXN),
   VD(MAXN), VE(MAXN), VF(MAXN), VG(MAXN), VIC(MAXN), YD(MAXN),
   YE(MAXN), YF(MAXN), YS(MAXN), ZZ(24), A01DRS, A01DMS, A03DRS,
   A03DMS, A, AB, AC, AD, AE, AG, AL, AM, AP, AW, AR, AT, B, BA, BB, CA,
   CL, CP, CF, DB, DC, DD, DL, DS, DT, FB, FD, FE, FH, FN, FP, FR, FS,
   F21MAX, F21RMS, F32MAX, F32RMS, F41MAX, F41RMS, F43MAX,
   F43RMS, FSHMAX, FSHRMS, FT, FX, GM, GN, GU, GV, GW, GX, GY, GZ, H,
   HA, HB, OU, OV, OW, P, PA, PB, PG, Q, QA, QB, QD, QF, QG, RM, SEED,
   TDMAX, TDRMS, W, WA, WB, WD, WE, WF, WK, X, XA, XB, XD, XE, XF, XG, XI,
   X3, YJ, YM
INTEGER * 4 N(MAXN), NA(MAXN), NB(MAXN), NC(MAXN), NS(MAXN), QUIT,
   &
   ID, IT, J, JT, JX, K, KA, KC, KE, KG, KK, KP, KT, KX, LZZ, MD,
   &
   ME, MF, MG, MN, MR, MS, MX, ND, NE, NF, NG, NN, NP, NR, NRMS, NU,
   &
   NV, NX
COMMON G, ZZ, RR, AP, FSHMAX, F21MAX, D32MAX, F41MAX, F43MAX, A01DMX,
   &
   A03DMX, TDMAX, FSHRMS, F21RMS, F32RMS, F41RMS, F43RMS, A01DRS,
   &
   A03DRS, TDRMS/A1/NRMS, NUC, NGQ, NR

WRITE(4, *)'MODSER - AOD FORTRAN VERSION (6/6/84)'
WRITE(4, *)'APPLICATION : BALANCING OF FOUR BAR MECHANISMS'
WRITE(6, *)'************************************************************************

WRITE(6,*)'APPLICATION : BALANCING OF FOUR BAR MECHANISMS'
WRITE(6,*)'************************************************************************

WRITE(6,*)'OUTPUT IS PROVIDED ON THE FOLLOWING :'
WRITE(6,*)'************************************************************************

WRITE(6,*)' FOR004 : AOD SEARCH PROCESS'
WRITE(6,*)' FOR006 : SUMMARY OF FINAL RESULTS'
WRITE(6,*)' FOR007 : SUMMARY OF FINAL RESULTS'
WRITE(6,*)' FOR009 - FOR017 IS FOR PLOTTING THE FINAL RESULTS'
WRITE(6,*)'************************************************************************

WRITE(6,*)' FOR009 : CRANK ANGLE FROM 1 - 360 DEGREES'
WRITE(6,*)' FOR010 : FORCE F21 FOR EACH CRANK ANGLE'
WRITE(6,*)' FOR011 : FORCE F32 FOR EACH CRANK ANGLE'
WRITE(6,*)' FOR012 : FORCE F41 FOR EACH CRANK ANGLE'
WRITE(6,*)' FOR013 : FORCE F43 FOR EACH CRANK ANGLE'
WRITE(6,*)' FOR014 : MOMENT MSH01 FOR EACH CRANK ANGLE'
WRITE(6,*)' FOR015 : MOMENT MSH03 FOR EACH CRANK ANGLE'
WRITE(6,*)' FOR016 : FORCE FSH FOR EACH CRANK ANGLE'
WRITE(6,*)' FOR017 : INPUT TORQUE FOR EACH CRANK ANGLE'
WRITE(7,*)' MODSER - P519RE FORTRAN VERSION follows (6/6/84)'
WRITE(9,*) ' THIS IS A DATA FILE FOR THTO'
WRITE(10,*) ' THIS IS A DATA FILE FOR FINAL F21'
WRITE(11,*) ' THIS IS A DATA FILE FOR FINAL F32'
WRITE(12,*) ' THIS IS A DATA FILE FOR FINAL F41'
WRITE(13,*) ' THIS IS A DATA FILE FOR FINAL F43'
WRITE(14,*) ' THIS IS A DATA FILE FOR FINAL -AO1D'
WRITE(15,*) ' THIS IS A DATA FILE FOR FINAL -AO3D'
WRITE(16,*) ' THIS IS A DATA FILE FOR FINAL FSH'
WRITE(17,*) ' THIS IS A DATA FILE FOR FINAL TD'
CALL PRINT(0)
WRITE(4,*) ' INPUT DATA:
AP = 4.0*ATAN(1.0)
AR = 57.29578
CP = 0.
KP = 0
QUIT = 0.
DATA AB, AE, AG, AL, AM, AW, AT, CA, CL, CF, JX, KX/10., .5, 1.E03, 1.E05, 8.,
& 2.E04, 2., .85, .25, .25, 300, 250/
READ(3,*) KC, KG, MD, MN, MS, MX, ND, NE, NF, NG, NR, NU, NV, NRMS,
& NUC, NGQ, SEED
READ(3,*) (CN(I), I=1, NV)
READ(3,*) (CX(I), I=1, NV)
IF (NG.NE.0) THEN
  READ(3,*) (CG(I), I=1, NG)
ENDIF
IF (MX.LE.0) THEN
  READ(3,*) (CS(I), I=1, NV)
ENDIF
IF (NRMS.EQ.1) THEN
  CALL PGAUSS(NGQ)
ENDIF

**
**
** SPECIFIED START POINT DATA (CASE OF NO SG) **
**
IF (MS.NE.0) THEN
  READ(3,*) (OS(IS), IS = 1, MS)
ENDIF
IF (KC.EQ.0) THEN
  GO TO 33
ENDIF
**
** USER PROGRAMS EQUATIONS FOR CALCULATED CONSTANTS C(IC), (IC=1 TO KC) **
**
CALL INPUTPO(MAXN,C,CN,CG,CS,CX,OS,AB,AE,AG,AL,AM,AW,AT, & CA,CL,CF,JX,KX,KC,KG,MD,MN,MS,MX,ND,NE, & NF,NG,NP,NR,NU,NV)
WRITE(4,*) 'INITIAL CALCULATIONS :
DO 53 I = 1,NV
   CD(I)=(CX(I)-CN(I))/AL
53 ENDDO
A = 0.0
DO 71 I = 1,NV
   A = A + (CX(I) - CN(I))**2
71 ENDDO
AD = SQRT(A) / AB
FD = AB * AD / AW
FS = FD / 5.
FX = FS / AT
FH = AD / 2.
IF (NE.EQ.0.AND.NR.EQ.0) THEN
   FM = 0.
ELSE
   FM = AW / AE / AB
ENDIF
WRITE(4,112) AD,FD,FS,FX,FH,FM
112 FORMAT(AD,FD,FS,FX,FH,FM = ',/.4(3X,E14.7))
CALL PRINT(1)
FS = 100.* FS
FX = 100.* FX
ME = 0
IF (MX.GT.0) THEN
ENDIF
140 DO 158 I = 1,NV
   VB(I) = CS(I)
   V(I) = VB(I)
158 ENDDO
159 IF (MS.EQ.0) THEN
    CALL ANALYSIS(MAXN,C,CG,E,OI,R,S,U,V,B,CP,FP,
    H,P,Q,X,XI,X3,W,N,KG,NE,NR,NU)
ELSE
    DO 188 IS = 1,MS
        OK(IS) = OS(IS)
        OI(IS) = OK(IS)
    END DO
    CALL ANALYSIS(MAXN,C,CG,E,OI,R,S,U,V,B,CP,FP,
    H,P,Q,X,XI,X3,W,N,KG,NE,NR,NU)
ENDIF
200 IF (NU.GT.0) THEN
    DO 203 IU = 1,NU
        UB(IU) = U(IU)
    END DO
203 ENDIF
215 IF (MS.EQ.0) THEN
    GO TO 245
ENDIF
232 IF (MS.EQ.0) THEN
    GO TO 245
ENDIF
245 IF (KG.GT.0) THEN
    DO 248 JG = 1,KG
        NB(JG) = N(JG)
    END DO
248 ENDIF
263 IF (NE.GT.0) THEN
    DO 263 IE = 1,NE
        EB(IE) = E(IE)
    END DO
263 ENDIF
278 IF (NR.GT.0) THEN
    DO 278 IR = 1,NR
        RB(IR) = R(IR)
    END DO
296 CALL PRINT(0)
WRITE(4,*) 'START OF MODSER SEARCH PROCESS:'
CALL PRINT(1)
WRITE(4,*) 'BASE POINT PRINTOUTS FOLLOW:'
320 CALL PRINT(2)
CALL PRINT(1)
322 IF (KT.EQ.1) THEN
    CALL FBPRINT(MAXN,EB,OK,RB,SK,UB,VB,NB,QB,XB,K,
    KG,KT,MS,NE,NR,NU,NV)
    GO TO 350
ENDIF
335 IF (NP.EQ.1) THEN
CALL MBPRINT(VB(1), QB, XB, KT)
ENDIF
IF (NP.EQ.2) THEN
CALL FBPRINT(MAXN, EB, OK, RB, SK, UB, VB, NB, QB, XB, & K, KG, KT, MS, NE, NR, NU, NV)
ENDIF
IF (QUIT.EQ.1) THEN
GO TO 1000
ENDIF
IF (QUIT.EQ.1) THEN
GO TO 1000
ENDIF
FE = FR
WRITE(4, *) 'JT, FB, FE=' , JT, FB, FE
XI = 0.0E0
DO 399 I = 1, NV
GX = VA(I) - VB(I)
XI = XI + GX * GX
ENDDO
X3 = SQRT(XI)
DB = X3
WRITE(4, *) 'DB=' , DB
IF (KA .EQ. 1) THEN
DL = DB
KA = 0
ENDIF
IF (DB.GT.FS) THEN
GO TO 500
ENDIF
IF (ME.EQ.3) THEN
GO TO 530
ENDIF
IF (ME.EQ.2) THEN
ME = 3
GO TO 530
ENDIF
IF (ME.EQ.0) THEN
ME = 1
FX = FX / 10.
FS = FS / 10.
GO TO 530
ENDIF
IF (ME.EQ.1) THEN
ME = 2
FX = FX / 10.
FS = FS / 10.

ENDIF

500 IF (KT.LT.KX) THEN
    GO TO 515
ENDIF

WRITE(4,*), 'KT=KX STOP REVIEW; BASE POINT PRINTOUT:'
CALL PRINT(1)
CALL PRINT(1)
CALL FBPRINT(MAXN, EB, OK, RB, SK, UB, VB, NB, QB, XB, K, KG, KT, MS, NE, &
            NR, NU, NV)
DO 510 M = 1, 3
    WRITE(4,*)
510 ENDDO

WRITE(6,*) 'ENTER "1" FOR NEXT LOOP, "2" TO END'
READ(5,*) LZZ
IF (LZZ.EQ.1) THEN
    GO TO 515
ELSE
    GO TO 1000
ENDIF

515 CALL STORINDEX(MAXN, EA, EB, GB, GC, OA, OK, RA, RB, SA, SB, SC, SK, UA, UB, &
             VA, VB, BA, BB, DB, DC, DD, HA, HB, QA, QB, PA, PB, WA, WB, XA, &
             XB, NA, NB, NC, KG, MS, NE, NR, NU, NV)
K = K + 1
KT = KT + 1
518 GO TO 320

530 WRITE(4,*), 'ME=', ME

531 IF (CP.EQ.0.) THEN
    GO TO 560
ENDIF

545 IF (FP.LT.FM) THEN
    GO TO 590
ENDIF

560 IF (K.EQ.1.AND.ME.EQ.3) THEN
    GO TO 680
ENDIF

575 K = 0
576 IF (CP.EQ.0.) THEN
    GO TO 500
ENDIF

590 FP = AM * FP
591 WRITE(4,*), 'PENALTY INCREASED AT 590, WITH FP=', FP
593 CALL PRINT(2)

605 DO 607 I = 1, NV
    V(I) = VA(I)
607 ENDDO

608 IF (MS.GT.0) THEN
    DO 611 IS = 1, MS
        OI(IS) = OA(IS)
611 ENDDO

609 CALL ANALYSIS(MAXN, C, CG, E, OI, R, S, U, V, B, CP, FP, H &
             P, Q, X, XI, X3, W, N, KG, NE, NR, NU)
BA = B
HA = H
WA = W
PA = P
XA = X

IF (NE.GT.0) THEN
   DO 638 IE = 1,NE
      EA(IE) = E(IE)
   ENDDO
ENDIF

IF (NR.GT.0) THEN
   DO 653 IR = 1,NR
      RA(IR) = R(IR)
   ENDDO
ENDIF

KA = 1
GO TO 500

KE = KE + 1
WRITE(4,*) 'TERMINATION TEST COUNT KE=',KE
CALL PRINT(2)

GX = DB + DC
IF (GX.GT.0.0E0) THEN
   GY = DC + DD
   AC = GY / GX
   ELSE
   AC = 1.E20
ENDIF

IF (AC .LE.1.) THEN
   WRITE(4,*) 'DIVERGENCE ANTICIPATED, WITH AC=',AC
   WRITE(4,*) 'WENT TO 575 FOR NEXT LOOP'
   CALL PRINT(2)
   GO TO 575
ENDIF

WRITE(4,*) 'CONVERGENCE ANTICIPATED, WITH AC=',AC
FT = (AC - 1.0) * 6.0 * FS
WRITE(4,*) 'READY FOR TERMINATION TEST AT 755, AND FT=',FT
IF (DB.GT.FT) THEN
   WRITE(4,*) 'TERMINATION TEST FAILED AT 755.'
   WRITE(4,*) 'WENT TO 575 FOR NEXT LOOP.'
   CALL PRINT(2)
   GO TO 575
ENDIF

WRITE(4,*) 'TERMINATION TEST PASSED AT 755'

CALL PRINT(0)

WRITE(4,*) 'SOLUTION FOUND IS AT (K+1), NOW TRANSFERRED TO (K)'
WRITE(4,*) 'STORAGE FOR PRINTOUT'
WRITE(4,*)

KK = 0
CALL STORINDEX(MAXN,EA,EB,GB,GC,OA,OK,RA,RB,SA,SB,SC,SK,UA,UB,
VA,VB,BA,BB,DB,DC,DD,HA,HB,QA,QB,PA,PB,WA,WB,XA,
XB,NA,NB,NC,KG,MS,NE,NR,NU,NV)

CALL GRADIENT(MAXN,C,CD,CG,DN,DP,E,GB,GI,R,S,SK,U,V,VI,VB,
& B,CP,FP,H,P,Q,X,XI,X3,W,N,KG,MG,MS,NE,NR,NU,NV)

XI = 0.0E0
DO 806 I = 1,NV
   XI = XI + GB(I) * GB(I)
ENDDO
X3=SQR(XI)
GM = X3
CALL FBPRINT(MAXN,EB,OK,RB,SK,UB,VB,NB,QB,XB,K,KG,KT,MS,NE, &
   NR,NU,NV)
WRITE(4,*)'GM=',GM
CALL PRINT(1)
WRITE(4,*)'DOUBLE PRECISION VB(I) FOR I=1,...,NV ARE='
WRITE(4,850) (VB(I),I=1,NV)
IF (NU.EQ.0) THEN
   GO TO 860
ENDIF
WRITE(4,*)'DOUBLE PRECISION UB(IU) FOR IU=1,...,NU ARE ='
WRITE(4,850) (UB(IU),IU=1,NU)
CALL PRINT(1)
DO 863 I =1,NV
   V(I) = VB(I)
ENDDO
IF (MS.GT.0) THEN
   DO 867 IS = 1,MS
      OI(IS) = OK(IS)
   ENDDO
ENDED
NGQ = 1
NRMS = 2
CALL ANALYSIS(MAXN,C,CG,E,OI,R,S,U,V,B,CP,FP,H &
   ,P,Q,X,XI,X3,W,N,KG,NE,NR,NU)
CALL FINAL(MAXN,C,CG,F,Q,U,V,KC,NF,NU,NV)
CALL PRINT(0)
WRITE(4,*)'MF,MG,MR,NX=',MF,MG,MR,NX
1000 STOP
END
SUBROUTINES
SUBROUTINE PRINT

SUBROUTINE PRINT(L)
IF (L.EQ.0.0) THEN
  WRITE(4,10)
  FORMAT(72('*'))
ENDIF
IF (L.EQ.1.0) THEN
  WRITE(4,20)
  FORMAT(72('-'))
ENDIF
IF (L.EQ.2.0) THEN
  WRITE(4,30)
  FORMAT(25('-'))
ENDIF
RETURN
END
**NEW BASE POINT SUBROUTINE**


REAL * 8 C(MAXN), CG(MAXN), E(MAXN), EB(MAXN), OI(MAXN), OK(MAXN),
  R(MAXN), RB(MAXN), S(MAXN), SK(MAXN), U(MAXN), UB(MAXN),
  V(MAXN), VB(MAXN), B, BB, CP, FP, H, HB, P, PB, Q, QB, W, WB, X, XI,
  X3, XB

INTEGER * 4 N(MAXN), NB(MAXN), KG, MS, NU, NE, NR, NV

1090 DO 1107 I = 1, NV
  V(I) = VB(I)
ENDDO

1107 IF (MS.GT.0) THEN
  DO 1111 IS = 1, MS
    OI(IS) = OK(IS)
  ENDIF

1111 ENDDO

CALL ANALYSIS(MAXN, C, CG, E, OI, R, S, U, V, B, CP, FP, H,
  P, Q, X, XI, X3, W, N, KG, NE, NR, NU)

QB = Q
BB = B
HB = H
WB = W
PB = P
XB = X

IF (NU.GT.0) THEN
  DO 1130 IU = 1, NU
    UB(IU) = U(IU)
  ENDDO

1130 IF (MS.GT.0) THEN
  DO 1138 IS = 1, MS
    SK(IS) = S(IS)
  ENDIF

1138 ENDIF

IF (NE.GT.0) THEN
  DO 1153 IE = 1, NE
    EB(IE) = E(IE)
  ENDIF

1153 ENDIF

IF (NR.GT.0) THEN
  DO 1168 IR = 1, NR
    RB(IR) = R(IR)
  ENDIF

1168 IF (KG.GT.0) THEN
  DO 1183 JG = 1, KG
    NB(JG) = N(JG)
  ENDIF

1183 ENDIF

RETURN

END
SUBROUTINE STORINDEX(MAXN, EA, EB, GB, GC, OA, OK, RA, RB, SA, SB, SC, SK, & UA, UB, VA, VB, BA, BB, DB, DC, DD, HA, HB, QA, QB, PA & PB, WA, WB, XB, NA, NB, NC, KG, MS, NE, NR, NU, NV)
REAL * 8 EA(MAXN), EB(MAXN), GB(MAXN), GC(MAXN), OA(MAXN), OK(MAXN), & RA(MAXN), RB(MAXN), SA(MAXN), SB(MAXN), SC(MAXN), SK(MAXN), & UA(MAXN), UB(MAXN), VA(MAXN), VB(MAXN), BA, BB, DB, DC, DD, HA, & HB, QA, QB, PA, PB, WA, WB, XB, NA, NB, NC, KG, MS, NE, NR, NU, NV
INTEGER * 4 NA(MAXN), NB(MAXN), NC(MAXN), KG, MS, NE, NR, NU, NV
DD = DC
DC = DB
DO 1230 I = 1, NV
   GC(I) = GB(I)
   SC(I) = SB(I)
1230 ENDDO
QB = QA
BB = BA
HB = HA
WB = WA
PB = PA
XB = XA
DO 1248 I = 1, NV
   VB(I) = VA(I)
1248 ENDDO
IF (MS.GT.0) THEN
   DO 1254 IS = 1, MS
      OK(IS) = OA(IS)
      SK(IS) = SA(IS)
1254 ENDDO
ENDIF
IF (NU.GT.0) THEN
   DO 1258 IU = 1, NU
      UB(IU) = UA(IU)
1258 ENDDO
ENDIF
IF (NE.GT.0) THEN
   DO 1273 IE = 1, NE
      EB(IE) = EA(IE)
1273 ENDDO
ENDIF
IF (NR.GT.0) THEN
   DO 1288 IR = 1, NR
      RB(IR) = RA(IR)
1288 ENDDO
ENDIF
IF (KG.GT.0) THEN
   DO 1304 JG = 1, KG
      NC(JG) = NB(JG)
      NB(JG) = NA(JG)
1304 ENDDO
ENDIF
RETURN
END
SUBROUTINE SGSEARCH(MAXN, C, CG, CN, CS, CX, E, OI, R, S, U, V, VG, AG, B, CP, 
& FP, H, P, PG, Q, QG, RM, SEED, X, XI, X3, XG, W, N, KG, 
& MN, MX, NE, NR, NU, NV)
REAL * 8 C(MAXN), CG(MAXN), CN(MAXN), CS(MAXN), CX(MAXN), E(MAXN), 
& OI(MAXN), R(MAXN), S(MAXN), U(MAXN), V(MAXN), VG(MAXN), AG, 
& B, CP, FP, H, P, PG, Q, QG, RM, SEED, X, XI, X3, XG, W
INTEGER * 4 N(MAXN), KG, KS, MN, MX, NE, NR, NU, NV
KS = 1
XG = 1.E19
CP = AG
FP = 1.
2014 DO 2017 I = 1, NV
   RM = RND(SEED)
   V(I) = CN(I) + RM * (CX(I) - CN(I))
   SEED = RM
2017 ENDDO
CALL ANALYSIS(MAXN, C, CG, E, OI, R, S, U, V, B, CP, FP, H 
& P, Q, X, XI, X3, W, N, KG, NE, NR, NU)
2030 IF (X.LT.XG) THEN
   XG = X
   QG = Q
   PG = P
   DO 2036 I = 1, NV
      VG(I) = V(I)
2036 ENDDO
ENDIF
IF (KS.LT.MN) THEN
   GO TO 2075
ENDIF
IF (PG.EQ.0.) THEN
   GO TO 2090
ENDIF
2075 IF (KS.LT.MX) THEN
   KS = KS + 1
   GO TO 2014
ENDIF
WRITE(4,*)
WRITE(4,*) 'SG SEARCH GAVE FOLLOWING POINT: '
WRITE(4,*)
WRITE(4,*) 'QG, PG, X: ', QG, PG, XG
WRITE(4,*)
WRITE(4,*) 'VG(I) FOR I=1,...,NV ARE: '
WRITE(4,2130)(VG(I), I=1, NV)
WRITE(4,*)
2130 FORMAT(/, 4(4X, E14.7))
2135 DO 2137 I = 1, NV
   CS(I) = VG(I)
2137 ENDDO
WRITE(6,*) 'STARTING POINT OBTAINED BY SG SEARCH'
WRITE(6,*) (VG(I), I=1, NV)
2138 RETURN
END
SEARCH DIRECTION SUBROUTINE

SUBROUTINE SDIR(MAXN,C,CD,CG,DN,DP,E,GB,GC,OI,R,S,SB,SC,SK,TB,U,
V,VU,VB,N,NB,NC,B,CP,DC,DT,FP,GX,GY,H,P,Q,X,XI,
X3,W,WK,ID,KA,KG,MR,MS,ND,NE,VR,QUIT)
REAL * 8 C(MAXN),CD(MAXN),CG(MAXN),DN(MAXN),DP(MAXN),E(MAXN),
& GB(MAXN),GC(MAXN),OI(MAXN),R(MAXN),S(MAXN),SB(MAXN),
& SC(MAXN),SK(MAXN),TB(MAXN),U(MAXN),V(MAXN),VU(MAXN),
& VB(MAXN),B,CP,DC,DT,FP,GX,GY,H,P,Q,X,XI,X3,W,WK
INTEGER * 4 N(MAXN),NB(MAXN),NC(MAXN),
& ID,KA,KG,MR,MS,ND,NE,
& VR,QUIT
CALL GRADIENT(MAXN,C,CD,CG,DN,DP,E,GB,OI,R,S,SK,U,V,VU,VB,
& B,CP,FP,H,P,Q,X,XI,X3,W,N,KG,MR,MS,NE,VR,QUIT)
IF (ND.EQ.1.OR.DC.EQ.0.) THEN
  GO TO 3100
ENDIF
IF (KA.GT.0) THEN
  GO TO 3100
ENDIF
IF (ND.EQ.2.OR.KG.EQ.0) THEN
  GO TO 3150
ENDIF
JG = 1
3040
IF (NB(JG).NE.NC(JG)) THEN
  GO TO 3090
ENDIF
IF (JG.EQ.KG) THEN
  GO TO 3150
ENDIF
JG = JG + 1
GO TO 3040
3090
NX = NX + 1
WRITE(4,*)'NX,JG=',NX,JG
3100
DO 3125 I = 1,NV
  SB(I) = - GB(I)
3125
ENDDO
3138
ID = 1
3139
GO TO 3200
3150
GX = 0.0E0
3151
GY = GX
3165
DO 3168 I = 1,NV
  GX = GX + (GB(I) - GC(I)) * GB(I)
  GY = GY + (GB(I) - GC(I)) * SC(I)
3168
ENDDO
3169
IF (GY.EQ.0.) THEN
  WRITE(4,*)'GY=0. AT 3165; GOTO 3100'
  GO TO 3100
ENDIF
WK = GX / GY
3181
DO 3183 I = 1,NV
  SB(I) = - GB(I) + WK * SC(I)
3183
ENDDO
3184
IF (ID = 1)
3200
WRITE(4,*)'ID=',ID
DT = 0.0E0
DO 3213 I = 1,NV
    DT = DT + GB(I) * SB(I)
ENDDO
IF (DT.LT.0.) THEN
    GO TO 3240
ENDIF
IF (DT.EQ.0.AND.ID.EQ.1) THEN
    WRITE(4,*)'STATIONARY POINT FOUND AT 3225,WITH ID,DT=' ,ID,DT
    RETURN
ENDIF
IF (ID.EQ.2) THEN
    MR = MR + 1
    GO TO 3100
ENDIF
WRITE(4,*)'FAILED DESCENT DIRECTION TEST AT 3240,WITH GX=' ,GX
WRITE(6,*)'FAILED DESCENT DIRECTION TEST AT 3240,WITH GX=' ,GX
QUIT = 1
RETURN
XI = 0.0E0
DO 3243 I = 1,NV
    XI = XI + SB(I) * SB(I)
ENDDO
X3 = SQRT(XI)
DO 3315 I = 1,NV
    IF (X3.GT.0.0E0) THEN
        TB(I) = SB(I) / X3
    ELSE
        TB(I) = 0.0E0
    ENDIF
ENDDO
RETURN
END
SUBROUTINE GRADIENT AT BASEPOINT VB(I)  

SUBROUTINE GRADIENT(MAXN,C,CD,CG,DN,DP,E,GB,OI,R,S,SK,U,V,VU,VB,
B,CP,FP,H,P,Q,X,XI,X3,W,N,KG,MG,MS,NE,NR,NU,NV)
REAL * 8 C(MAXN),CD(MAXN),CG(MAXN),DN(MAXN),DP(MAXN),E(MAXN),
& GB(MAXN),OI(MAXN),R(MAXN),S(MAXN),SK(MAXN),U(MAXN),
& V(MAXN),VU(MAXN),VB(MAXN),B,CP,FP,H,P,Q,X,XI,X3,W
INTEGER * 4 N(MAXN),KG,MG,MS,NE,NR,NU,NV
MG = MG + 1
DO 4000 I = 1,NV
  V(I) = VB(I)
4000  IF (MS.EQ.0) THEN
      GO TO 4010
ENDIF
DO 4005 IS = 1,MS
  OI(IS) = SK(IS)
4005  ENDDO
4010  DO 4070 I = 1,NV
  VU(I) = V(I)
  V(I) = VB(I) + CD(I)
  CALL ANALYSIS(MAXN,C,CG,E,OI,R,S,U,V,B,CP,FP,H
& ,P,Q,X,XI,X3,W,N,KG,NE,NR,NU)
  DP(I) = X
  V(I) = VB(I) - CD(I)
  CALL ANALYSIS(MAXN,C,CG,E,OI,R,S,U,V,B,CP,FP,H
& ,P,Q,X,XI,X3,W,N,KG,NE,NR,NU)
  DN(I) = X
  GB(I) = (DP(I) - DN(I)) / 2.0 / CD(I)
  V(I) = VU(I)
4070  ENDDO
4080 RETURN
END
SUBROUTINE LINE SEARCH WITH CP TUNING


REAL * 8 C(MAXN), CD(MAXN), CG(MAXN), DN(MAXN), DP(MAXN), DV(MAXN), E(MAXN), EA(MAXN), EB(MAXN), GB(MAXN), GC(MAXN), OA(MAXN), OD(MAXN), OE(MAXN), OF(MAXN), OG(MAXN), OI(MAXN), OK(MAXN), R(MAXN), RA(MAXN), RB(MAXN), S(MAXN), SA(MAXN), SB(MAXN), SC(MAXN), SD(MAXN), SE(MAXN), SF(MAXN), SK(MAXN), TB(MAXN), U(MAXN), UA(MAXN), UB(MAXN), V(MAXN), VA(MAXN), VB(MAXN), VD(MAXN), VE(MAXN), VF(MAXN), VU(MAXN), YD(MAXN), YE(MAXN), YF(MAXN), YS(MAXN), AD, B, BA, BB, CA, CL, CP, CF, DC, DL, DS, DT, FB, FH, FP, FR, FS, FX, GN, GU, GV, Gw, Gx, Gy, Gz, H, Ha, HB, Ou, OV, OW, P, PA, PB, Q, QA, QB, QD, QE, QF, W, WA, WB, WD, WE, WF, Wk, X, Xa, XB, XD, XE, XF, XI, X3, YM, YJ

INTEGER * 4 N(MAXN), NA(MAXN), NB(MAXN), NC(MAXN), ID, J, JX, JT, K, KA, KP, KG, KT, MF, MG, MR, MS, ND, NE, NN, NR, NT, NU, NV, NX, QUIT

5005 IF (DT.EQ.0. AND ID.EQ.1. AND KT.GT.1. AND CP.NE.0.) THEN
FR = 0.
FB = 0.
J = 0.
GO TO 5375
ENDIF

5010 IF (KT.EQ.1) THEN
FR = AD
GO TO 5040
ENDIF

5015 IF (KA.GT.0) THEN
FR = CL * DL
GO TO 5040
ENDIF

5020 IF (DC.EQ.0.) THEN
FR = FS / 2.
GO TO 5040
ENDIF

5030 IF (DC.LT.AD) THEN
FR = CA * DC
ELSE
FR = AD
ENDIF

5040 FB = FR
5050 DO 5060 I = 1, NV
    DV(I) = FR * TB(I)
5060 ENDDO
5100 JT = 0
    J = 0
    QD = QB
    WD = WB
XD = XB
DO 5145  I = 1,NV
   YD(I) = 0.0E0
   VD(I) = VB(I)
ENDDO
IF (MS.EQ.0) THEN
   GO TO 5150
ENDIF
DO 5148  IS = 1,MS
   OG(IS) = SK(IS)
   OD(IS) = OK(IS)
   SD(IS) = SK(IS)
ENDDO
5150
QF = QE
WF = WE
XF = XE
DO 5151  I = 1,NV
   YF(I) = YE(I)
   VF(I) = VE(I)
ENDDO
IF (MS.EQ.0) THEN
   GO TO 5195
ENDIF
DO 5180  IS = 1,MS
   OF(IS) = OE(IS)
   SF(IS) = SE(IS)
ENDDO
5183
QE = QD
WE = WD
XE = XD
DO 5201  I = 1,NV
   YE(I) = YD(I)
   VE(I) = VD(I)
ENDDO
IF (MS.EQ.0) THEN
   GO TO 5225
ENDIF
DO 5210  IS = 1,MS
   OE(IS) = OD(IS)
   SE(IS) = SD(IS)
ENDDO
5213
JT = JT + 1
5226
J = J + 1
DO 5244  I = 1,NV
   YD(I) = YE(I) + DV(I)
   VD(I) = VB(I) + YD(I)
   V(I) = VD(I)
ENDDO
IF (MS.EQ.0) THEN
   GO TO 5270
ENDIF
DO 5258  IS = 1,MS
   OD(IS) = OG(IS)
   OI(IS) = OD(IS)
ENDDO

IF (MS.EQ.0) THEN
  GO TO 5300
ENDIF

DO 5288 IS = 1,MS
  SD(IS) = S(IS)
  OG(IS) = SD(IS)
ENDO

MF = MF + 1
QD = Q
WD = W
XD = X

IF (CP.EQ.0.AND.WD.GE.WE.AND.WD.GT.0.) THEN
  GO TO 5945
ENDIF

NN = 0

IF (J.GE.2) THEN
  GO TO 5420
ENDIF

IF (XD.LE.XE) THEN
  MC = 1
  GO TO 5150
ENDIF

GZ = FX * CF

IF (FR.GT.GZ) THEN
  GO TO 5405
ENDIF

DO 5380 I = 1,NV
  VA(I) = VD(I)
  V(I) = VA(I)
ENDO

IF (MS.EQ.0) THEN
  GO TO 5555
ENDIF

DO 5393 IS = 1,MS
  OA(IS) = OD(IS)
  OI(IS) = OA(IS)
ENDDO

ENDO

GO TO 5555

J = 0
FR = CF * FR

DO 5409 I = 1,NV
  DV(I) = CF * DV(I)
ENDO

ENDO

GO TO 5225

IF (XD.LT.XE) THEN
  GO TO 5855
ENDIF

GX = XD + XF - 2.0 * XE

IF (GX.EQ.0) THEN
  DS = FR
ELSE
  DS = FR * (XD - XE) / GX
ENDIF
XI = 0.0E0

DO 5468 I = 1,NV
    XI = XI + YD(I) * YD(I)
ENDDO

X3 = SQRT(XI)

YJ = X3

YM = YJ - FR / 2.0 - DS

IF (YM.LT.0.0E0) THEN
    WRITE(4,*)'FAILED TEST AT 5480'
    GO TO 5795
ENDIF

DO 5499 I = 1,NV
    YS(I) = YM * TB(I)
    VA(I) = VB(I) + YS(I)
    V(I) = VA(I)
ENDDO

GN = DS + FR / 2.0
GU = FR * FR
GV = 2.0 * FR

IF (MS.EQ.0) THEN
    GO TO 5555
ENDIF

DO 5545 IS = 1,MS
    OU = OD(IS)
    OV = OB(IS)
    OW = OF(IS)
    GW = OV - OU
    GX = OW - OU
    GY = (4.0 * GW - GX) / GV
    GX = (GW - GY * FR) / GU
    GW = GN * (GX * GN + GY)
    GZ = GW + OU
    OI(IS) = GZ
ENDDO

CALL ANALYSIS(MAXN,C,CG,E,OI,R,S,U,V,B,CP,FP,H,&
P,Q,X3,W,N,KG,NE,NR,NU)

MF = MF + 1
QA = Q
BA = B
HA = H
WA = W
PA = P
XA = X

IF (NU.GT.0) THEN
    DO 5603 IU = 1,NU
        UA(IU) = U(IU)
    ENDDO
ENDIF

IF (MS.EQ.0) THEN
    GO TO 5645
ENDIF

DO 5637 IS = 1,MS
    OA(IS) = OI(IS)
    SA(IS) = S(IS)
ENDDO
IF (NE.GT.0) THEN
  DO 5651 IE = 1,NE
    EA(IE) = E(IE)
  ENDDO
ENDIF

IF (NR.GT.0) THEN
  DO 5666 IR = 1,NR
    RA(IR) = R(IR)
  ENDDO
ENDIF

IF (KG.GT.0) THEN
  DO 5681 JG = 1,KG
    NA(JG) = N(JG)
  ENDDO
ENDIF

IF (XA.GT.XB) THEN
  GO TO 5720
ENDIF
RETURN

WRITE(4,*)' FAILED FUNCTION DECREASE TEST AT 5690'

IF (FR.GT.GZ) THEN
  WRITE(4,*)' WENT TO 5825 AND 5405'
  GO TO 5825
ELSE
  RETURN
ENDIF

IF (FR.GT.GZ) THEN
  WRITE(4,*)' WENT TO 5823 FROM 5765'
  GO TO 5825
ENDIF

IF (ABS(YM).GT.GZ) THEN
  WRITE(4,*)' WENT TO 5825 FROM 5780'
  GO TO 5825
ENDIF

DO 5805 I = 1,NV
  VA(I) = VB(I)
  V(I) = VA(I)
ENDDO

IF (MS.EQ.0) THEN
  GO TO 5555
ENDIF

DO 5819 IS = 1,MS
  OA(IS) = OK(IS)
  OI(IS) = OA(IS)
ENDDO

GO TO 5555

QE = QF
WE = WF
XE =XF

DO 5831 I = 1,NV
  YE(I) = YF(I)
  VE(I) = VF(I)
ENDDO
IF (MS .GT. 0) THEN
  DO 5836 IS = 1,MS
    OE(IS) = OF(IS)
    SE(IS) = SF(IS)
  ENDDO
ENDIF
GO TO 5405
IF (J.GT.JX) THEN
  GO TO 5856
ELSE
  GO TO 5870
ENDIF
WRITE(6,*)'J>JX AT 5855;STOP REVIEW'
WRITE(6,*)'ENTER 1 TO CONTINUE , 2 TO HALT '
READ(5,*) LJX
IF (LJX.EQ.1) THEN
  WRITE(6,*)'ENTER NEW JX VALUE >',JX
  READ(5,*) JX
  GO TO 5855
ELSE
  QUIT = 1
  CALL PRINT(0)
  CALL FBPRINT(MAXN,EB,OK,RB,SK,UB,VB,NB,QB,XB,K,
                KG,KT,MS,NE,NR,NU,NV)
  RETURN
ENDIF
IF (FR.GT.FH) THEN
  GO TO 5150
ENDIF
IF (MC.LT.1) THEN
  MC = 1
  GO TO 5150
ENDIF
FR = 2.0 * FR
QE = QF
WE = WF
XE = XF
DO 5908 I = 1,NV
  DV(I) = 2.0 * DV(I)
  YE(I) = YF(I)
  VE(I) = VF(I)
ENDDO
IF (MS.EQ.0) THEN
  MC = 0
  GO TO 5930
ENDIF
DO 5918 IS = 1,MS
  OE(IS) = OF(IS)
  SE(IS) = SF(IS)
ENDDO
MC = 0
GO TO 5150
IF (KP.GT.0) THEN
  GO TO 5975
ENDIF
IF (QD.GT.QE) THEN
    NN = 0
    WRITE(4,5965)
    GO TO 5345
ENDIF

FORMAT(' IN PENALTY ZONE AT 5945 WITH WD > WE, BUT NO TUNE SINCE 
&QD > QE ')

IF (NN.EQ.0) THEN
    GO TO 6035
ENDIF

******************* AT FIRST PROBE EDGE OF PENALTY ZONE ************

IF (KP.EQ.0) THEN
    KP = 1
    GO TO 5150
ELSE
    KP = 2
    GO TO 6095
ENDIF

******************** CP CALCULATION ****************************

SEGMENT 6052: FOR CP SELF-TUNING PROCESS

WRITE(4,*),'START SEGMENT 6035 -'
WRITE(4,*)' AT EDGE OF PENALTY ZONE WITH WD>WE AND QD<QE'
DO 6052 I = 1,NV
    VB(I) = VE(I)
ENDDO

IF (MS.EQ.0) THEN
    GO TO 6080
ENDIF

DO 6067 IS = 1,MS
    OK(IS) = OE(IS)
ENDDO

CALL NEWBP(MAXN,C,CG,E,EB,OI,OK,R,RB,S,SK,U,UB,V,VB,B, BB, 
&CP,FP,H,HB,P, PB, Q, QB, W, WB, X, XI, X3, XB, N, NB, KG, MS 
&,NU,NE,NR,NV)

IF (QUIT.EQ.1) THEN
    RETURN
ENDIF

FB = FR
NN = 1
KP = 0
ND = NT
GO TO 5050

WRITE(4,*),'FOR CP CALCULATION AT 6095'

IF (WD.NE.WF) THEN
    CP = .75 * (QF - QD) / (WD - WF)
ENDIF
IF (CP.LT.0.0) THEN
    CP = (QF - QE) / (WE - WF)
ENDIF

6125 WRITE(4,*), 'AT (J-2), AND KP=0: QF,WF=', QF,WF
6140 WRITE(4,*), 'AT (J-2), AND KP=0: QF,WF=', QF,WF
6155 WRITE(6,*), 'AT (J-1), AND KP=1: QE,WE=', QE,WE
6160 WRITE(4,*), 'AT (J-1), AND KP=1: QE,WE=', QE,WE
6170 WRITE(4,*), 'REVIEW CALCULATED CP=', CP
WRITE(6,*), 'REVIEW CALCULATED CP=', CP
6185 DO 6187 M = 1,3
    WRITE(6,*), 'AT (J-2), AND KP=0: QF,WF=', QF,WF
6190 IF (LCP.EQ.1) THEN
    GO TO 6200
ENDIF
IF (LCP.EQ.2) THEN
    GO TO 6215
ENDIF

6200 WRITE(6,*), 'ENTER "1" TO CHANGE CP OR "2" TO RETAIN CP'
6215 WRITE(5,*), LCP
IF (LCP.EQ.1) THEN
    GO TO 6200
ENDIF
IF (LCP.EQ.2) THEN
    GO TO 6215
ENDIF

6220 CALL NEWBP(MAXN,C,CG,E,EB,OI,OK,R,RB,S,SK,U,UB,V,VB,B,BB,
& CP,FP,H,HB,P,PB,Q,QB,W,WB,X,XI,X3,XB,N,NB,KG,MS
& ,NU,NE,NR,NV)
6225 GO TO 6275
6230 DO 6235 IS = 1,MS
    OK(IS) = OE(IS)
6235 ENDDO
6240 CALL NEWBP(MAXN,C,CG,E,EB,OI,OK,R,RB,S,SK,U,UB,V,VB,B,BB,
& CP,FP,H,HB,P,PB,Q,QB,W,WB,X,XI,X3,XB,N,NB,KG,MS
& ,NU,NE,NR,NV)
6245 GO TO 6275
6250 DO 6255 IS = 1,MS
7270 IF (QUIT.EQ.1) THEN
        RETURN
ENDIF
FR = AD
NN = 2
ND = NT
FB = FR
GO TO 5050
END
SUBROUTINE INPUTPO(MAXN, C, CN, CG, CS, CX, OS, AB, AE, AG, AL, AM, AW, AT, &
CA, CL, CF, JX, KK, KC, KG, MD, MN, MS, MX, ND, NE, &
NF, NG, NP, NR, NU, NV)
REAL * 8 C(MAXN), CN(MAXN), CG(MAXN), CS(MAXN), CX(MAXN), OS(MAXN), &
AB, AE, AG, AL, AM, AW, AT, CA, CL, CF
INTEGER * 4 JX, KK, KC, KG, MD, MN, MS, MX, ND, NE, NF, NG, NP, NR, NU, NV
WRITE(4,*) 'AB, AE, AG, AL, AM, AW, AT, CA, CL, CF, JX, KK = '
WRITE(4,*) AB, AE, AG, AL, AM, AW, AT, CA, CL, CF, JX, KK
WRITE(4,*) 'KC, KG, MD, MN, MS, MX, ND, NE, NF, NG, NP, NR, NU, NV = '
WRITE(4,*) KC, KG, MD, MN, MS, MX, ND, NE, NF, NG, NP, NR, NU, NV
WRITE(4,*)
8010 FORMAT(/,4(4X,E14.7))
WRITE(4,*) 'CN(I) FOR I = 1, ........ NV ARE = '
WRITE(4,8010) (CN(I), I=1,NV)
WRITE(4,*)
WRITE(4,*) 'CX(I) FOR I = 1, ........ NV ARE = '
WRITE(4,8010) (CX(I), I=1,NV)
WRITE(4,*)
IF(NG.EQ.0) THEN
    GO TO 8090
ENDIF

8075 WRITE(4,*)'CG(IG) FOR IG = 1, ........ NG ARE = '
WRITE(4,8010) (CG(IG), IG=1,NG)
8090 IF(MX.GT.0) THEN
    GO TO 8150
ENDIF

8105 WRITE(4,*) 'SPECIFIED CS(I) FOR I = 1, ........ NV ARE ='
WRITE(4,8010) (CS(I), I=1,NV)
WRITE(4,*)
IF (MS.EQ.0) THEN
    GO TO 8150
ENDIF

8135 WRITE(4,*) 'SPECIFIED OS(IS) FOR IS = 1, ........ MS ARE ='
WRITE(4,8010) (OS(IS), IS=1,MS)
8150 IF(KC.EQ.0) THEN
    GO TO 8210
ENDIF

8165 WRITE(4,*)'CALCULATED CONSTANTS C(IC) FOR IC = 1, ...... KC ARE ='
WRITE(4,8010) (C(IC), IC=1,KC)
8210 CALL PRINT(1)
CALL PRINT(2)
RETURN
END
*  ***** MINIMAL BASEPOINT PRINTOUT  SUBROUTINE (OPTION , NP = 1)  *****  *

SUBROUTINE MBPRINT(VB,QB,XB,KT)
REAL * 8 VB,QB,XB
INTEGER * 4 KT
WRITE(4,10) KT,VB,XB,QB
10 FORMAT(' KT,VB(1),XB,QB = ',I3,3X,3(E14.7,3X),/)
CALL PRINT(1)
CALL PRINT(2)
RETURN
END

************************************************************************

*  **** FULL BASEPOINT PRINTOUT (OPTION , NP = 2) SUBROUTINE ************

SUBROUTINE FBPRINT(MAXN,EB,OK,RB,SK,UB,VB,NB,QB,XB,K,KG,KT,MS,NE &
& REAL * 8 EB(MAXN),OK(MAXN),RB(MAXN),SK(MAXN),UB(MAXN),VB(MAXN),
& QB,XB
INTEGER * 4 NB(MAXN),K,K,KG,KT,MS,NE,NR,NU,NV
WRITE(4,10) KT,K,XB,QB
10 FORMAT(' KT,K,XB,QB = ',2(I5),5X,2(E14.7,3X),/)
WRITE(4,*),' VB(I) FOR I = 1,.......,NV = '
WRITE(4,20) (VB(I),I=1,NV)
20 FORMAT(/,4(4X,E14.7))
WRITE(4,*)
IF(MS.EQ.0) THEN
  GO TO 30
ENDIF
WRITE(4,*),' OK(IS) FOR IS = 1,.......,MS ARE = '
WRITE(4,20) (OK(IS),IS=1,MS)
WRITE(4,*)
WRITE(4,*),' SK(IS) FOR IS = 1,.......,MS ARE = '
WRITE(4,*)(SK(IS),IS=1,MS)
30 IF (NU.LE.0) THEN
  GO TO 40
ENDIF
WRITE(4,*),' UB(IU) FOR IU = 1,.......,NU ARE = '
WRITE(4,20) (UB(IU),IU=1,NU)
WRITE(4,*)
40 IF (NE.LE.0) THEN
  GO TO 45
ENDIF
WRITE(4,*),' EB(IE) FOR IE = 1,.......,NE ARE = '
WRITE(4,20)(EB(IE),IE=1,NE)
WRITE(4,*)
45 IF(NR.LE.0) THEN
  GO TO 60
ENDIF
WRITE(4,*),' RB(IR) FOR IR = 1,...........,NR ARE = '
WRITE(4,20) (RB(IR),IR=1,NR)
WRITE(4,*)
60 IF (KG.LE.0) THEN
GO TO 90
ENDIF
WRITE(4,*), 'NB(JG) FOR JG = 1, .........., KG ARE = '
WRITE(4,20) (NB(JG), JG=1, KG)
WRITE(4,*)
90 CALL PRINT(1)
CALL PRINT(2)
RETURN
END
FUNCTION SGN(P)  
REAL * 8 SGN,P  
IF (P.GT.0) THEN  
    SGN = 1  
ENDIF  
IF (P.EQ.0) THEN  
    SGN = 0  
ENDIF  
IF (P.LT.0) THEN  
    SGN = -1  
ENDIF  
RETURN  
END

FUNCTION RND(SEED)  
REAL * 8 RND,PI,FRAC,SEED  
PI = 4.0*ATAN(1.0)  
BASE = (PI + SEED)**5  
FRAC = IFIX(BASE)  
RND = BASE - FRAC  
RETURN  
END
**SUBROUTINE ANALYSIS**

THIS SUBROUTINE CALCULATES THE VALUE OF THE OBJECTIVE FUNCTION, \( Q \), REGIONAL CONSTRAINTS, \( R \), AND EQUALITY CONSTRAINTS, \( E \) FOR A GIVEN BASEPOINT MATRIX, \( V \). THE FOLLOWING ANALYSIS IS FOR THE BALANCING OF AN INLINE FOUR BAR MECHANISM.

ANALYSIS WRITTEN BY; TIMOTHY C. HEWITT, AUGUST 1984

**VARIABLE IDENTIFICATION**

- **AOiD**: TIME DERIVATIVE OF THE TOTAL ANGULAR MOMENTUM OF THE MECHANISM WITH RESPECT TO THE JOINT \( O_i \).
- **AOiDMAX**: THE MAXIMUM AOiD IN ONE REVOLUTION OF THE INPUT LINK.
- **AP**: VALUE OF PI (3.14159).
- **AOiDRMS**: RMS VALUE OF AOiD.
- **AU**: DIFFERENCE IN THE LIMITS OF INTEGRATION.
- **C**: MATRIX OF CALCULATED CONSTANTS (NOT USED).
- **CP**: PENALTY FUNCTION TUNING PARAMETER.
- **CG**: MATRIX OF GIVEN CONSTANTS.
- **CG(1)**: LENGTH OF LINK 1 (IN).
- **CG(2)**: NORMALIZED LENGTH OF LINK 2.
- **CG(3)**: 
  - *: 
  - **CG(4)**: 
  - *: 
- **CG(5)**: ANGULAR VELOCITY OF THE INPUT LINK (RAD/S).
- **CG(6)**: ACCELERATION OF THE INPUT LINK (RAD/S/S).
- **CG(7)**: GROUND BEARING FORCE WEIGHTING FACTOR IN Q.
- **CG(8)**: INPUT TORQUE WEIGHTING FACTOR IN Q.
- **CG(9)**: DISTANCE FROM JOINT O3' TO EXTERNAL LOADING POINT APPLICATION ON LINK 3.
- **CG(10)**: DISTANCE FROM JOINT O1' TO REAL LINK 1 C.G. LOC (IN).
- **CG(11)**: 
  - *: 
  - **CG(12)**: 
  - *: 
- **CG(13)**: MASS OF REAL LINK 1 (LB).
- **CG(14)**: 
  - *: 
  - **CG(15)**: 
  - *: 
- **CG(16)**: MASS MOMENT OF INERTIA OF THE REAL LINK 1 (LB-IN2).
- **CG(17)**: 
  - *: 
  - **CG(18)**: 
  - *: 
- **CG(19)**: ANGLE FROM LINE O1'O2 TO REAL LINK 1 C.G. LOC. (RAD).
- **CG(20)**: 
  - *: 
  - **CG(21)**: 
  - *: 
- **CG(22)**: DENSITY OF THE COUNTERWEIGHTS ADDED (LB/IN3).
- **CG(23)**: THICKNESS OF THE COUNTERWEIGHTS ADDED (IN).
- **CG(24)**: MAXIMUM COUNTERWEIGHT ANGLE (I.E. 2*PI).
- **CG(25)**: MAXIMUM C.W. RADIUS (IN).
- **CG(26)**: MAXIMUM ALLOWABLE NORMALIZED RMS SHAKING FORCE.
- **CG(27)**: 
  - *: 
  - **CG(28)**: 
  - *: 
- **CG(29)**: 
  - *: 
  - **CTHTT**: COSINE OF THTT.
- **DS1**: BODY COORDINATES FOR THE "x" LOCATION OF THE COMBINED CENTER OF GRAVITY FOR THE \( i \)TH LINK (IN).
- **DS2**: BODY COORDINATES FOR THE "y" LOCATION OF THE COMBINED
CENTER OF GRAVITY FOR THE iTH LINK (IN).

E : MATRIX OF EQUALITY CONSTRAINTS (NOT USED).
FP : PENALTY FUNCTION FACTOR.
Fij : THE NORMALIZED BEARING FORCE BETWEEN LINKS i AND j.
FijX : "X" COMPONENT OF THE FORCE Fij.
FijY : "Y" COMPONENT OF THE FORCE Fij.
FijMAX : MAXIMUM Fij FORCE.
FijRMS : RMS Fij.
FSH : NORMALIZED SHAKING FORCE OF THE MECHANISM.
FSHMAX : MAXIMUM FSH
FSHRMS : RMS FSH.
FX : EXTERNAL X-FORCE APPLIED TO THE OUTPUT LINK (LINK 3).
FY : "  Y-   "  "  "  "  "  "  "  "  "  ".
G : ABSICCA OF THE SAMPLING POINT; USED IN GAUSSIAN QUADRATURE INTEGRATION.
IC : MASS MOMENT OF INERTIA OF THE iTH COUNTERWEIGHT ABOUT ITS C.G. (LB-IN2).
II : NORMALIZED MASS MOMENTS OF INERTIA MATRIX WHERE EACH INERTIA IS ABOUT THE iTH JOINT.
IT : MASS MOMENT OF INERTIA OF THE iTH COUNTERWEIGHT AND AND iTH REAL LINK ABOUT THEIR COMBINED C.G. (LB-IN2).
L1 : DIST. OF TWO POINT MASS MODEL TO MASS M1(IN).
L2 : "  "  "  "  "  "  "  "  "  "  "  "  "  M2(IN).
LC : RADIUS OF THE iTH COUNTERWEIGHT (IN).
LT : LENGTH FROM iTH JOINT TO THE COMBINED C.G. LOC.(IN).
M1 : NORMALIZED MASS MATRIX FOR THE TWO POINT MASS MODEL FOR THE iTH LINK.
M1 : MASS IN THE L1 DIR. FOR THE iTH LINK(LB).
M2 : "  "  "  L2  "  "  "  "  "(LB).
MC : MASS OF THE iTH COUNTERWEIGHT(LB).
MT : "  "  "  COMBINED LINK(LB).
N : NOT USED.
NC : NUMBER OF COUNTERWEIGHTS TO BE ADDED ( 1 < NC < 3 ).
NGQ : NUMBER OF SAMPLING POINTS FOR NUMERICAL INTEGRATION.
NE : "  "  EQUALITY CONSTRAINTS.
NR : NUMBER OF REGIONAL CONSTRAINTS.
NRMS : DETERMINES THE TYPE OF INTEGRATION USED.
NU : NUMBER OF VARIABLES IN MATRIX U (NOT USED).
OI : NOT USED.
P : PENALTY FUNCTION.
PI : NORMALIZED MASS DISTANCE PRODUCT OF THE TWO POINT MASS MODEL.
Q : OBJECTIVE FUNCTION.
R : REGIONAL CONSTRAINT MATRIX.
RMAXB : THE MAXIMUM RMS BEARING FORCE.
RMAXH : THE MAXIMUM RMS SHAKING MOMENT.
S : NOT USED.
STHTT : SIN OF THTT.
SUM : SUM OF "F"'S.
SUMAOiD : SUM OF AOiD'S ; USED TO CALCULATE AOiD RMS.
SUMFij : SUM OF Fij'S ; "  "  "  FijRMS.
SUMFSH : SUM OF FSH'S ; "  "  "  FSHRMS.
SUMTD : SUM OF TD'S ; "  "  "  TDRMS.
TD : NORMALIZED DRIVING (INPUT) TORQUE.
SUBROUTINE ANALYSIS(MAXN,C,CG,E,OI,R,S,U,V,B,CP,FP,H
&
P,Q,X,XI,X3,W,N,KG,NE,NR,NU)
REAL * 8 C(MAXN),CG(MAXN),E(MAXN),G(24),OI(MAXN),I(10),
&
MI(10),P(10),R(MAXN),RR(19),S(MAXN),U(MAXN),V(MAXN),
&
ZZ,A,A01D,A01DMX,A01DRS,A03D,A03DMX,A03DRS,AP,AW,
&
AV,AX,B,BR,CA,CP,CTHTT,DEN,DS1,DS2,F,F21,F21X,F21Y,
&
F21MAX,F21RMS,F32,F32X,F32Y,F32MAX,F32RMS,F41,F41X,
&
F41Y,F41MAX,F41RMS,F43,F43X,F43Y,F43MAX,F43RMS,FP,FSH,
&
FSMAX,FSHRMS,FY,H,IC,IT,LC,LT,L1,L2,M1,M2,MC,MT,P,
&
Q,RMAXB,RMAXX,SUM,SUMA01D,SUMA03D,SUMFSH,SUMF21,SUMF32,
&
SUMF41,SUMF43,SUMTD,STHTT,TA,TBB,TC,TD,TMAX,TDREMS,TE,
&
TF,TG,TH,TH1,TH2,TH3,TT,TT,C,TD,TMAX,TDREMS,TE,
&
TM,TO,TP,TQ,TR,TS,TT,TU,TV,TY,TZ,X,X1,X3,W
INTEGER * 4 N(MAXN),KG,NC,NE,NR,NU,NRMS
COMMON G,ZZ,RR,AP,FSHMAX,F21MAX,F32MAX,F41MAX,F43MAX,AP,
&
A01DMX,A03DMX,TMAX,FSHRMS,F21RMS,F32RMS,F41RMS,F43RMS,
&
A01DRS,A03DRS,TDREMS/A1/NRMS,NC,NGQ
TY = 0.0
TZ = 2.0*AP
AU = TZ - TY
IF (NC.EQ.0) THEN
   DO 3 I = 1,6
      V(I) = 0.0
   ENDDO
ENDIF
IF (NC.EQ.1) THEN
   V(2) = 0.0
   V(3) = 0.0
   V(5) = 0.0
   V(6) = 0.0
ENDIF
IF (NC.EQ.2) THEN  !CASE OF 2 COUNTERWEIGHTS
    V(2) = 0.0
    V(5) = 0.0
ENDIF

CONVERSION FROM COUNTERWEIGHT ANGLES AND RADII TO TWO POINT
MASS MODEL PARAMETERS

DO 100 I = 1,3
    LC = CG(25) * V(I + 3)
    THTC = CG(24) * V(I)
    MC = AP*CG(22)*CG(23)*LC**2
    IC = MC*LC**2/2.0
    MT = CG(I+12) + MC
    LT = SQRT((MC*LC)**2 + (CG(I+12)*CG(I+9))**2 + 2.0*MC*LC*CG(I+9)
        *CG(I+12)*COS(THTC - CG(I+18))/MT
    STHTT = (MC*LC*SIN(THTC) + CG(I+12)*CG(I+9)*SIN(CG(I+18)))/MT/LT
    CTHTT = (MC*LC*COS(THTC) + CG(I+12)*CG(I+9)*COS(CG(I+18)))/MT/LT

DETERMINATION OF THTT

IF (STHTT.GE.0.0.AND.CTHTT.GE.0.0) THEN
    THTT = ASIN(STHTT)
ENDIF
IF (STHTT.LT.0.0.AND.CTHTT.GE.0.0) THEN
    THTT = 2.0*AP + ASIN(STHTT)
ENDIF
IF (STHTT.GE.0.0.AND.CTHTT.LT.0.0) THEN
    THTT = ACOS(CTHTT)
ENDIF
IF (STHTT.LT.0.0.AND.CTHTT.LT.0.0) THEN
    THTT = 2.0*AP - ACOS(CTHTT)
ENDIF

IT = IC+CG(I+15)+MC*(LC**2 + LT**2 - 2.0*MC*LT*COS(THTC-THTT))
    +CG(I+12)*(CG(I+9)**2+LT**2-2.0*CG(I+9)*LT*COS(CG(I+18)-THTT))

DS1 = LT*COS(THTT)
DS2 = LT*SIN(THTT)

DETERMINATION OF L2 USING THE QUADRATIC FORMULA

A = DS2
BR = -(2.0*DS2**2 + IT/MT)
CA = IT*DS2/MT + DS1**2*DS2 + DS2**3
IF ((BR**2).LT.(4.0*A*CA)) THEN
    L2 = - BR/2.0/A
    GO TO 50
ENDIF

CASE OF DS2  >  DS1

IF (ABS(A).LT.ABS(DS1/100000.0)) THEN
    IF (LT.EQ.0.0) THEN
        WRITE(6,*)'LT = 0.0'
        M1 = MT
        M2 = 0.0
L1 = 0.0
L2 = 0.0
ELSE
L1 = (IT + MT * DS1 ** 2) / MT / DS1
M1 = MT * DS1 / L1
M2 = MT - M1
ENDIF

GO TO 60
ENDIF

L2 = (-BR + SQRT(BR ** 2 - 4.0 * A * CA)) / 2.0 / A
M2 = MT * DS2 / L2
M1 = MT - M2
L1 = MT * DS1 / M1

IF (I.EQ.1) THEN
DEN = M1 + M2
ENDIF
MI(I) = M1 / DEN
MI(I+3) = M2 / DEN
PI(I) = MI(I) * L1 / CG(1)
PI(I+3) = MI(I+3) * L2 / CG(1)
IF (LT.EQ.0.0) THEN
II(I) = IT
ELSE
II(I) = PI(I) * L1 / CG(1) + PI(I+3) * L2 / CG(1)
ENDIF
MI(I) = MI(I) + MI(I+3)
R(I+16) = BR ** 2 - 4.0 * A * CA ! REGIONAL CONSTRAINT TO

ENDDO

SUM = 0.0
SUMF21 = 0.0
SUMF32 = 0.0
SUMF41 = 0.0
SUMF43 = 0.0
SUMFSH = 0.0
SUMA01D = 0.0
SUMA03D = 0.0
SUMTD = 0.0
FSHMAX = 0.0
F21MAX = 0.0
F32MAX = 0.0
F41MAX = 0.0
F43MAX = 0.0
AO1DMX = 0.0
AO3DMX = 0.0
TDMAX = 0.0
IF (NRMS.EQ.1) THEN
M = NGQ
NUM = 1
ENDIF
IF (NRMS.EQ.2) THEN
M = 360
NUM = NGQ
ENDIF
DO 200 I = 1, M, NUM
KINEMATIC ANALYSIS

IF (NRMS.EQ.1) THEN
THT1 = (TY + TZ + G1*AU)/2.0
ELSE
THT1 = TZ * FLOAT(I)/360.0
ENDIF

TA = 1 + CG(2)**2-CG(3)**2+CG(4)**2-2.0*CG(4)*COS(THT1)
TBB = 2.0*CG(2)*(COS(THT1) - CG(4))
TC = 2.0*CG(2)*SIN(THT1)
THT2 = 2.0*ATAN((-TC + SQRT(TC**2 - TA**2 + TBB**2))/TA - TBB)

THT3 = ACOS((COS(THT1) + CG(2)*COS(THT2) - CG(4))/CG(3))
AV = THT1 - THT2
AX = THT1 - THT3
TK = (CG(2)*SIN(AV) - CG(4)*SIN(THT1))/(CG(2)*
& (CG(4)*SIN(THT2) + SIN(AV)))
TE = (CG(3)*SIN(AV) + CG(4)*SIN(THT1))/(CG(3)*
& (CG(4)*SIN(THT3) + SIN(AV)))
TF = SIN(AV) + CG(4)*SIN(THT2)
TG = SIN(AV) + CG(4)*SIN(THT3)
TKD = CG(5)*((CG(2)*(1.0 - TK)*COS(AV) - CG(4)*COS(THT1)
& )*TF-(CG(2)*SIN(AV) - CG(4)*SIN(THT1))*((1.0 - TK)
& *COS(AV) + CG(4)*TK*COS(THT2))/TF**2/CG(2)
TED = CG(5)*((CG(3)*(1.0 - TE)*COS(AV) + CG(4)*COS(THT1)
& )*TG-(CG(3)*SIN(AV) + CG(4)*SIN(THT1))*(1.0 - TE)
& *COS(AV) + CG(4)*TE*COS(THT3))/TG**2/CG(3)
TJ = CG(6)/CG(5)**2
TH = TKD/CG(5) + TK* TJ
TI = TED/CG(5) + TE*TJ

KINETIC ANALYSIS

A01D =TJ*(MI(2) + II(1) + PI(2)*COS(AV) + PI(5)*SIN(AV))
& + (1.0 - TK**2)*(-PI(2)*SIN(AV) + PI(5)*COS(AV))
& + TH*(II(2) + PI(2)*COS(AV) + PI(5)*SIN(AV))
& + TI*(II(3)+CG(4)*PI(3)*COS(THT3)-PI(6)*SIN(THT3))
& - TE**2*CG(4)*PI(3)*SIN(THT3) + PI(6)*COS(THT3))

A03D = TJ*(MI(2) + II(1) + PI(2)*COS(AV) + PI(5)*SIN(AV))
& -CG(4)*(PI(1)*COS(THT1)-PI(4)*SIN(THT1)+MI(2)*COS(AV))
& + (1.0-TK**2)*(-PI(2)*SIN(AV)+PI(5)*COS(AV))
& + CG(4)*(PI(1)*SIN(THT1)+PI(4)*COS(THT1)+MI(2)*SIN(AV))
& + TH*(II(2) + PI(2)*COS(AV) + PI(5)*SIN(AV))
& - CG(4)*(PI(2)*COS(THT2)-PI(5)*SIN(AV))
& +TK**2*CG(4)*(PI(2)*SIN(THT2) + PI(5)*COS(THT2)) + TI*II(3)
TT = TJ*(MI(2)+II(1)+TK*(PI(2)*COS(AV)+PI(5)*SIN(AV)))
& +TH*(II(2)+TK*PI(2)*COS(AV)+PI(5)*SIN(AV))+TI*II(3)*TE
& +TK*(1.0 - TK)*(-PI(2)*SIN(AV) + PI(5)*COS(AV))

TN = 0.0 ! BETA
TL = 0.0 ! LOADING TORQUE APPLIED TO OUTPUT LINK
FX = 0.0 ! EXTERNAL X FORCE " " "
FY = 0.0 ! EXTERNAL Y FORCE " " "
TD = TT - TE*(TL + CG(9)*(-FX*SIN(TN) + FY*COS(TN)))
F43Y=(A01D + FX*CG(9)*SIN(TN)-TD-TL-FY*CG(4) +
&
\[ F_{32Y} = -F_{43Y} - \pi(3)*(-T_i\cos(T_3) + T_2^2\sin(T_3)) \]
&
\[ -\pi(6)*(T_i\sin(T_3) + T_2^2\cos(T_3)) - F_Y \]
&
\[ F_{32X} = (F_{32Y}G(3)\cos(T_3) + T_L\text{-}I(3)*T_i\text{+}G(9)\cos(T_3) \]
&
\[ -F_XG(9)\sin(T_3)/(G(3)\sin(T_3)) \]
&
\[ F_{43X} = -F_{32X} - \pi(3)*(T_i\sin(T_3) + T_2^2\cos(T_3)) \]
&
\[ +\pi(6)*(-T_i\cos(T_3) + T_2^2\sin(T_3)) - F_X \]
&
\[ F_{41Y} = (-A_03\text{+}T_D\text{+}T_L\text{+}G(9)*G(9)\cos(T_3) - F_Y G(9)\cos(T_3) \]
&
\[ -F_{41Y} - \pi(1)*(-T_J\cos(T_1) + \sin(T_1)) \]
&
\[ -\pi(4)*(-T_J\cos(T_1) + \sin(T_1)) \]
&
\[ F_{21Y} = F_{41Y} - \pi(1)*(-T_J\cos(T_1) + \sin(T_1)) \]
&
\[ +\pi(4)*(-T_J\cos(T_1) + \sin(T_1)) \]
&
\[ F_{21X} = F_{32X} + \pi(2)*(T_H\sin(T_2) + T_K^2\cos(T_2)) \]
&
\[ +\pi(5)*(T_H\cos(T_2) - T_K^2\sin(T_2)) \]
&
\[ +M I(2)*(T_J\sin(T_1) + \cos(T_1)) \]
&
\[ F_{41X} = F_{21X} - \pi(1)*(T_J\sin(T_1) + \cos(T_1)) \]
&
\[ +\pi(4)*(-T_J\cos(T_1) + \sin(T_1)) \]
&
\[ F_{SH} = \sqrt{(F_{41X} + F_{43X} + F_X)^2 + (F_{41Y} + F_{43Y} + F_Y)^2} \]
&
\[ F_{21} = \sqrt{(F_{21X})^2 + (F_{21Y})^2} \]
&
\[ F_{32} = \sqrt{(F_{32X})^2 + (F_{32Y})^2} \]
&
\[ F_{41} = \sqrt{(F_{41X})^2 + (F_{41Y})^2} \]
&
\[ F_{43} = \sqrt{(F_{43X})^2 + (F_{43Y})^2} \]
&
\[ F = G(7)*\sqrt{(F_{41})^2 + (F_{43})^2} + G(8)*\text{ABS}(T_D) \]
&
\[ \text{IF} (\text{NRMS.EQ.}1) \text{ THEN} \]
&
\[ \text{SUM} = \text{SUM} + ZZ(I)*F \]
&
\[ \text{SUMF}_{SH} = \text{SUMF}_{SH} + ZZ(I)*F_{SH}^2 \]
&
\[ \text{SUMF}_{21} = \text{SUMF}_{21} + ZZ(I)*F_{21}^2 \]
&
\[ \text{SUMF}_{32} = \text{SUMF}_{32} + ZZ(I)*F_{32}^2 \]
&
\[ \text{SUMF}_{41} = \text{SUMF}_{41} + ZZ(I)*F_{41}^2 \]
&
\[ \text{SUMF}_{43} = \text{SUMF}_{43} + ZZ(I)*F_{43}^2 \]
&
\[ \text{SUMA}_01D = \text{SUMA}_01D + ZZ(I)*A_01D^2 \]
&
\[ \text{SUMA}_03D = \text{SUMA}_03D + ZZ(I)*A_03D^2 \]
&
\[ \text{SUMT}_{D} = \text{SUMT}_{D} + ZZ(I)*T_{D}^2 \]
&
\[ \text{ELSE} \]
&
\[ \text{SUM} = \text{SUM} + F \]
&
\[ \text{SUMF}_{SH} = \text{SUMF}_{SH} + F_{SH}^2 \]
&
\[ \text{SUMF}_{21} = \text{SUMF}_{21} + F_{21}^2 \]
&
\[ \text{SUMF}_{32} = \text{SUMF}_{32} + F_{32}^2 \]
&
\[ \text{SUMF}_{41} = \text{SUMF}_{41} + F_{41}^2 \]
&
\[ \text{SUMF}_{43} = \text{SUMF}_{43} + F_{43}^2 \]
&
\[ \text{SUMA}_01D = \text{SUMA}_01D + A_01D^2 \]
&
\[ \text{SUMA}_03D = \text{SUMA}_03D + A_03D^2 \]
&
\[ \text{SUMT}_{D} = \text{SUMT}_{D} + T_{D}^2 \]
&
\[ \text{ENDIF} \]
&
\[ \text{IF} (\text{NRMS.EQ.2}) \text{ THEN} \]
&
\[ \text{THTO} = 360*THT_1/2.0/\text{AP} \]
&
\[ \text{WRITE}(9,*), \text{THTO} \]
&
\[ \text{WRITE}(10,*), \text{F}_{21} \]
&
\[ \text{WRITE}(11,*), \text{F}_{32} \]
&
\[ \text{WRITE}(12,*), \text{F}_{41} \]
&
\[ \text{WRITE}(13,*), \text{F}_{43} \]
&
\[ \text{WRITE}(14,*), -A_01D \]
&
\[ \text{WRITE}(15,*), -A_03D \]
&
\[ \text{WRITE}(16,*), \text{F}_{SH} \]
&
\[ \text{WRITE}(17,*), \text{T}_{D} \]
&
\[ \text{ENDIF} \]
CALCULATION OF MAXIMUM VALUES

IF (NRMS.NE.1) THEN
  IF (FSH.GT.FSHMAX) THEN
    FSHMAX = FSH
  ENDIF
  IF (ABS(F21).GT.F21MAX) THEN
    F21MAX = ABS(F21)
  ENDIF
  IF (ABS(F32).GT.F32MAX) THEN
    F32MAX = ABS(F32)
  ENDIF
  IF (ABS(F41).GT.F41MAX) THEN
    F41MAX = ABS(F41)
  ENDIF
  IF (ABS(F43).GT.F43MAX) THEN
    F43MAX = ABS(F43)
  ENDIF
  IF (ABS(AOID).GT.AOIDMX) THEN
    AOIDMX = ABS(AOID)
  ENDIF
  IF (ABS(AO3D).GT.AO3DMX) THEN
    AO3DMX = ABS(AO3D)
  ENDIF
  IF (ABS(TD).GT.TDMAX) THEN
    TDMAX = ABS(TD)
  ENDIF
ENDIF
ENDDO

CALCULATION OF RMS VALUES

IF (NRMS.EQ.1) THEN
  Q = SUM/2.0
  FSHRMS = SQRT(SUMFSH/2.0)
  F21RMS = SQRT(SUMF21/2.0)
  F32RMS = SQRT(SUMF32/2.0)
  F41RMS = SQRT(SUMF41/2.0)
  F43RMS = SQRT(SUMF43/2.0)
  AO1DRS = SQRT(SUMAO1D/2.0)
  AO3DRS = SQRT(SUMAO3D/2.0)
  TDRMS = SQRT(SUMTD/2.0)
ELSE
  Q = SUM*FLOAT(NUM)/360.0
  FSHRMS = SQRT(SUMFSH*FLOAT(NUM)/360.0)
  F21RMS = SQRT(SUMF21*FLOAT(NUM)/360.0)
  F32RMS = SQRT(SUMF32*FLOAT(NUM)/360.0)
  F41RMS = SQRT(SUMF41*FLOAT(NUM)/360.0)
  F43RMS = SQRT(SUMF43*FLOAT(NUM)/360.0)
  AO1DRS = SQRT(SUMAO1D*FLOAT(NUM)/360.0)
  AO3DRS = SQRT(SUMAO3D*FLOAT(NUM)/360.0)
  TDRMS = SQRT(SUMTD*FLOAT(NUM)/360.0)
ENDIF
RMAXB = MAX(F21RMS,F32RMS,F41RMS,F43RMS)
REGIONAL CONSTRAINTS

RMAXH = MAX(AO1DRS,AO3DRS)
DO 250 I = 1,6
   R(I) = V(I)
   R(I + 6) = 1.0 - V(I)
250 ENDDO

R(13) = CG(26) - FSHRMS
R(14) = CG(27) - RMAXB
R(15) = CG(28) - RMAXH
R(16) = CG(29) - TDRMS
DO 260 I = 1,NR
   RR(I) = R(I) ! RR(I) IS COMMON
260 ENDDO

DETERMINATION OF THE PENALIZED OPTIMIZATION FUNCTION

B = 0.
H = 0.
IF(NE.EQ.0) THEN
   GO TO 300
ENDIF

DO 310 IE = 1,NE
   B = B + E(IE) * E(IE)
310 ENDDO

300 IF (NR.EQ.0) THEN
   GO TO 350
ENDIF

DO 320 IR = 1,NR
   IF (R(IR).LT.0.) THEN
      H = H + R(IR)*R(IR)
   ENDIF
320 ENDDO

350 W = B + H
P = CP * FP * W ! PENALTY FUNCTION
X = Q + P ! PENALIZED OPTIMIZATION QUANTITY
RETURN
END
**FINAL ITEMS SUBROUTINE**

This subroutine prints out all the final results for the optimization study.

**SUBROUTINES CALLED**

SUBROUTINE PRINT(I)
- Prints a row of characters defined by I.
  ie. If I = 1 then a row of "*" will be printed

**VARIABLES USED**

- AOiD : Time derivative of the total angular momentum of the mechanism with respect to the joint Oi.
- AOiDMAX : The maximum AOiD in one revolution of the input link.
- AP : Value of PI (3.14159).
- AOiDRMS : RMS value of AOiD.
- AU : Difference in the limits of integration.
- C : Matrix of calculated constants (not used).
- CG : Matrix of given constants.
- CG(24) : Maximum counterweight angle (i.e. 2*PI).
- CG(25) : Maximum C.W. radius (in).
- E : Matrix of equality constraints (not used).
- F : Final items matrix.
- Fij : The normalized bearing force between links i and j.
- FijX : "X" component of the force Fij.
- FijY : "Y" component of the force Fij.
- FijMAX : Maximum Fij force.
- FijRMS : RMS Fij.
- FSH : Shaking force of the mechanism.
- FSHMAX : Maximum shaking force.
- FSHRMS : RMS shaking force.
- G : Absolute value of the sampling point; used in Gaussian quadrature integration.
- LC : Radius of the ith counterweight (in).
- N : Not used.
- NC : Number of counterweights to be added (1 < NC < 3).
- NGQ : Number of sampling points for numerical integration.
- NE : "" equality constraints.
- NF : "" final items.
- NR : Number of regional constraints.
- NRMS : Determines the type of integration used.
- NU : Number of variables in matrix U (not used).
- OI : Not used.
- P : Penalty function.
- Q : Objective function.
- RR : Regional constraint matrix.
- RMAXB : The maximum RMS force.
- RMAXH : The maximum RMS shaking moment.
- TD : Normalized driving (input) torque.
- THTC : Angle between link i and the ith counterweight.
- U : Not used.
- V : Scaled variables; angles and radii of the counterweights.
ZZ : WEIGHTING COEFFICIENTS FOR GAUSSIAN QUADRATURE INTEGRATION.

*************** MAIN SUB-PROGRAM ***********************

SUBROUTINE FINAL(MAXN,C,CG,F,Q,U,V,KC,NF,NU,NV)
REAL * 8 C(MAXN),CG(MAXN),F(MAXN),G(24),RR(19),U(MAXN),V(MAXN),
& ZZ(24),Q,AP,FSHMAX,F21MAX,F32MAX,F41MAX,F43MAX,AO1DMX,
& AO3DMX,TDMAX,FSHRMS,F21RMS,F32RMS,F41RMS,F43RMS,AO1DRS,
& AO3DRS,TDRMS
INTEGER * 4 NC,NF,NU,NV,NRMS,NR
COMMON G,ZZ,RR,AP,FSHMAX,F21MAX,F32MAX,F41MAX,F43MAX,AO1DMX,
& AO3DMX,TDMAX,FSHRMS,F21RMS,F32RMS,F41RMS,F43RMS,
& AO1DRS,AO3DRS,TDRMS/A1/NRMS,NC,NGQ,NR

HERE USER PROGRAMS FINAL EQUATIONS OF INTEREST

F(1)= Q
F(2)=CG(24)*V(1)
F(3)=CG(24)*V(2)
F(4)=CG(24)*V(3)
F(5)=CG(25)*V(4)
F(6)=CG(25)*V(5)
F(7)=CG(25)*V(6)
WRITE(6,601)
WRITE(7,601)

601 FORMAT(//25X,' OPTIMUM VARIABLES ',//,8X,' THTC1',4X,' THTC2',
& 4X,' THTC3',5X,' LC1',6X,' LC2',6X,' LC3',//)
WRITE(6,602) (F(JF),JF=2,NF)
WRITE(7,602) (F(JF),JF=2,NF)

602 FORMAT(4X,6(4X,F6.3),//)
DO 100 I = 1,NR
  IF (RR(I).LT.0.0) THEN
    WRITE(6,603) I,RR(I)
    WRITE(7,603) I,RR(I)
    ELSE
    WRITE(6,604) I,RR(I)
    WRITE(7,604) I,RR(I)
ENDIF

100 ENDDO
WRITE(6,605)
WRITE(7,605)
WRITE(6,606)
WRITE(7,606)
WRITE(6,607) FSHRMS,F21RMS,F32RMS,F41RMS,F43RMS,
& AO1DRS,AO3DRS,TDRMS
WRITE(7,607) FSHRMS,F21RMS,F32RMS,F41RMS,F43RMS,
& AO1DRS,AO3DRS,TDRMS
WRITE(6,608)
WRITE(7,608)
WRITE(6,609)
WRITE(7,609)
WRITE(6,610) FSHMAX,F21MAX,F32MAX,F41MAX,F43MAX,
& AO1DMX,AO3DMX,TDMAX
WRITE(7,610) FSHMAX,F21MAX,F32MAX,F41MAX,F43MAX, 
& AO1DMX,AO3DMX,TDMAX
603 FORMAT(10X,' REGIONAL CONSTRAINT # ',I3,' WAS VIOLATED : ',E14.7)
604 FORMAT(10X,' REGIONAL CONSTRAINT # ',I3,' WAS OBEYED : ',E14.7)
605 FORMAT(///,25X,' RMS VALUES ','/)
606 FORMAT(2X,' FSH ','4X,' F21 ','4X,' F32 ','4X,' F41 ','4X,' F43 ','
& 4X,' AO1D ','3X,' AO3D ','3X,' TD ','/)
607 FORMAT(1X,8(F5.3,4X),///)
608 FORMAT(///,25X,' MAXIMUM VALUES ','/)
609 FORMAT(2X,' FSH ','4X,' F21 ','4X,' F32 ','4X,' F41 ','4X,' F43 ','
& 4X,' AO1D ','3X,' AO3D ','3X,' TD ','/)
610 FORMAT(1X,8(F6.3,3X),///)
*---------------------------------------------------------------------*
* CALL PRINT(1) ! PRINTS A ROW OF "*"
WRITE(4,*),' FINAL ITEMS F(JF) FOR JF = 1,.........,NF ARE = '
WRITE(4,611) (F(JF),JF=1,NF)
611 FORMAT(/,4(3X,E14.7))
RETURN
END
**GAUSS QUADRATURE INTEGRATION**

This subroutine provides double precision numbers for Gaussian quadrature integration. A maximum of 10 sampling points can be chosen.

**DEFINITION OF VARIABLES**

- **M**: # of sampling points
- **G**: Absicca value of the sampling point, M
- **ZZ**: The weighting coefficient associated with the sampling point, M

**MAIN PROGRAM**

```
SUBROUTINE PGAUSS(M)
REAL*8 G(24), ZZ(24)
COMMON G, ZZ
GO TO (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, &
     22, 23, 24), M
1  G(1) = 0.0
   ZZ(1) = 2.0
   RETURN
2  G(1) = 1.0/SQRT(3.0)
   G(2) = -G(1)
   ZZ(1) = 1.0
   ZZ(2) = 1.0
   RETURN
3  G(1) = SQRT(0.6)
   G(2) = 0.0
   G(3) = -G(1)
   ZZ(1) = 5./9.
   ZZ(2) = 8./9.
   ZZ(3) = ZZ(1)
   RETURN
4  G(1) = 0.861136311594053
   G(2) = 0.339981043584856
   G(3) = -G(2)
   G(4) = -G(1)
   ZZ(1) = 0.347854845137454
   ZZ(2) = 0.652145154862546
   ZZ(3) = ZZ(2)
   ZZ(4) = ZZ(1)
   RETURN
5  G(1) = 0.906179845938664
   G(2) = 0.53846931043584856
   G(3) = 0.0
   G(4) = -G(2)
   G(5) = -G(1)
   ZZ(1) = 0.236926885056189
   ZZ(2) = 0.478628670499366
   ZZ(3) = 0.568888888888889
   ZZ(4) = ZZ(2)
   ZZ(5) = ZZ(1)
```
\begin{align*}
G(1) &= 0.932469514203152 \\
G(2) &= 0.661209386466265 \\
G(3) &= 0.238619186083197 \\
G(4) &= -G(3) \\
G(5) &= -G(2) \\
G(6) &= -G(1) \\
ZZ(1) &= -0.171324492379170 \\
ZZ(2) &= -0.360761573048139 \\
ZZ(3) &= -0.467913934572691 \\
ZZ(4) &= -ZZ(3) \\
ZZ(5) &= -ZZ(2) \\
ZZ(6) &= -ZZ(1) \\
\text{RETURN} \\
G(1) &= 0.949107912342759 \\
G(2) &= 0.741531185599394 \\
G(3) &= 0.405845151377397 \\
G(4) &= 0.0 \\
G(5) &= -G(3) \\
G(6) &= -G(2) \\
G(7) &= -G(1) \\
ZZ(1) &= -0.129484966168870 \\
ZZ(2) &= 0.279705391489277 \\
ZZ(3) &= 0.381830050505119 \\
ZZ(4) &= 0.41795183673469 \\
ZZ(5) &= -ZZ(3) \\
ZZ(6) &= -ZZ(2) \\
ZZ(7) &= -ZZ(1) \\
\text{RETURN} \\
G(1) &= 0.960289856497536 \\
G(2) &= 0.796666477413627 \\
G(3) &= 0.525532409916329 \\
G(4) &= 0.18343641295650 \\
G(5) &= -G(4) \\
G(6) &= -G(3) \\
G(7) &= -G(2) \\
G(8) &= -G(1) \\
ZZ(1) &= 0.101228536290376 \\
ZZ(2) &= 0.222381034453374 \\
ZZ(3) &= 0.313706645877887 \\
ZZ(4) &= 0.362683783378362 \\
ZZ(5) &= -ZZ(4) \\
ZZ(6) &= -ZZ(3) \\
ZZ(7) &= -ZZ(2) \\
ZZ(8) &= -ZZ(1) \\
\text{RETURN} \\
G(1) &= 0.968160239507626 \\
G(2) &= 0.836031107326636 \\
G(3) &= 0.613371432700590 \\
G(4) &= 0.324234253423403809 \\
G(5) &= 0.0 \\
G(6) &= -G(4) \\
G(7) &= -G(3) \\
G(8) &= -G(2) \\
G(9) &= -G(1)
\end{align*}
ZZ(1) = 0.081274388361574
ZZ(2) = 0.180648160694857
ZZ(3) = 0.260610696402935
ZZ(4) = 0.312347077040003
ZZ(5) = 0.330239355001260

ZZ(6) = ZZ(4)
ZZ(7) = ZZ(3)
ZZ(8) = ZZ(4)
ZZ(9) = ZZ(1)

RETURN

10 G(1) = 0.973906528517172
G(2) = 0.865063366688985
G(3) = 0.679409568299024
G(4) = 0.433395394129247
G(5) = 0.148874338981631
G(6) = -G(5)
G(7) = -G(4)
G(8) = -G(3)
G(9) = -G(2)
G(10) = -G(1)

ZZ(1) = 0.066671344308688
ZZ(2) = 0.149451349150581
ZZ(3) = 0.219086362515982
ZZ(4) = 0.269266719309996
ZZ(5) = 0.295524224714753
ZZ(6) = ZZ(5)
ZZ(7) = ZZ(4)
ZZ(8) = ZZ(3)
ZZ(9) = ZZ(2)
ZZ(10) = ZZ(1)

RETURN

WRITE(6,*) ' NGQ = 11 NOT AVAILABLE '
RETURN

12 G(1) = 0.981560634246719
G(2) = 0.904117256370475
G(3) = 0.769902674194305
G(4) = 0.587317954286617
G(5) = 0.367831498998180
G(6) = 0.125233408511469
G(7) = -G(6)
G(8) = -G(5)
G(9) = -G(4)
G(10) = -G(3)
G(11) = -G(2)
G(12) = -G(1)

ZZ(1) = 0.047175336386512
ZZ(2) = 0.106939325995318
ZZ(3) = 0.160078328543346
ZZ(4) = 0.203167426723066
ZZ(5) = 0.233492536538355
ZZ(6) = 0.249147045813403
ZZ(7) = ZZ(6)
ZZ(8) = ZZ(5)
ZZ(9) = ZZ(4)
ZZ(10) = ZZ(3)
ZZ(11) = ZZ(2)
ZZ(12) = ZZ(1)
RETURN

13 WRITE(6,*) ' NGQ = 13 NOT AVAILABLE ' RETURN
14 WRITE(6,*) ' NGQ = 14 NOT AVAILABLE ' RETURN
15 WRITE(6,*) ' NGQ = 15 NOT AVAILABLE ' RETURN

16 G(1) = 0.989400934991649932596
G(2) = 0.944575023073232576078
G(3) = 0.865631202387831743880
G(4) = 0.755404408355003033895
G(5) = 0.617876244402643748447
G(6) = 0.458016777657227386342
G(7) = 0.281603550779258913230
G(8) = 0.095012509837637440185
G(9) = -G(8)
G(10) = -G(7)
G(11) = -G(6)
G(12) = -G(5)
G(13) = -G(4)
G(14) = -G(3)
G(15) = -G(2)
G(16) = -G(1)
ZZ(1) = 0.027152459411754094852
ZZ(2) = 0.062253523938647892863
ZZ(3) = 0.095158511682492784610
ZZ(4) = 0.124628971255533872052
ZZ(5) = 0.149595988816576732081
ZZ(6) = 0.169156519395002538189
ZZ(7) = 0.182603415044923588867
ZZ(8) = 0.189450610455068496285
ZZ(9) = ZZ(8)
ZZ(10) = ZZ(7)
ZZ(11) = ZZ(6)
ZZ(12) = ZZ(5)
ZZ(13) = ZZ(4)
ZZ(14) = ZZ(3)
ZZ(15) = ZZ(2)
ZZ(16) = ZZ(1)
RETURN

17 WRITE(6,*) ' NGQ = 17 NOT AVAILABLE ' RETURN
18 WRITE(6,*) ' NGQ = 18 NOT AVAILABLE ' RETURN
19 WRITE(6,*) ' NGQ = 19 NOT AVAILABLE ' RETURN

20 G(1) = 0.993128599185094924786
G(2) = 0.963971927277913791268
G(3) = 0.912234428251325905868
G(4) = 0.839116971822218823395
G(5) = 0.746331906460150792614
G(6) = 0.636053680726515025453
G(7) = 0.510867001958827098004
G(8) = 0.373706088715419560673
G(9) = 0.227785851141645078080
G(10) = 0.076526521133497333755
G(11) = -G(10)
G(12) = -G(9)
G(13) = -G(8)
G(14) = -G(7)
G(15) = -G(6)
G(16) = -G(5)
G(17) = -G(4)
G(18) = -G(3)
G(19) = -G(2)
G(20) = -G(1)

ZZ(1) = 0.017614007139152118312
ZZ(2) = 0.040601429800386941331
ZZ(3) = 0.062672048334109063570
ZZ(4) = 0.083276741576704748725
ZZ(5) = 0.101930119817240435037
ZZ(6) = 0.118194531961518417312
ZZ(7) = 0.131688638449176626898
ZZ(8) = 0.142096109318382051329
ZZ(9) = 0.149172986472603746788
ZZ(10) = 0.152753387130725850698
ZZ(11) = ZZ(10)
ZZ(12) = ZZ(9)
ZZ(13) = ZZ(8)
ZZ(14) = ZZ(7)
ZZ(15) = ZZ(6)
ZZ(16) = ZZ(5)
ZZ(17) = ZZ(4)
ZZ(18) = ZZ(3)
ZZ(19) = ZZ(2)
ZZ(20) = ZZ(1)

RETURN

21 WRITE(6,*) ' NGQ = 21 NOT AVAILABLE ' RETURN

22 WRITE(6,*) ' NGQ = 22 NOT AVAILABLE ' RETURN

23 WRITE(6,*) ' NGQ = 23 NOT AVAILABLE ' RETURN

24 G(1) = 0.995187219997021360180
G(2) = 0.974728555971309498198
G(3) = 0.938274552002732758524
G(4) = 0.886415527004401034213
G(5) = 0.820001985973902921954
G(6) = 0.740124191578554364244
G(7) = 0.648093651936975569252
G(8) = 0.545421471388839535658
G(9) = 0.433793507626045138487
G(10) = 0.315042679696163374387
G(11) = 0.191118867473616309159
G(12) = 0.064056892862605626085
G(13) = -G(12)
G(14) = -G(11)
G(15) = -G(10)
G(16) = - G(9)
G(17) = - G(8)
G(18) = - G(7)
G(19) = - G(6)
G(20) = - G(5)
G(21) = - G(4)
G(22) = - G(3)
G(23) = - G(2)
G(24) = - G(1)

ZZ(1) = 0.012341229799987199547
ZZ(2) = 0.028531388628933663181
ZZ(3) = 0.044277438817419806169
ZZ(4) = 0.059298584915436780746
ZZ(5) = 0.073346481411080305734
ZZ(6) = 0.086190161531953275917
ZZ(7) = 0.097618652104113888270
ZZ(8) = 0.107444270115965634763
ZZ(9) = 0.115505668053725601353
ZZ(10) = 0.121670472927803391204
ZZ(11) = 0.125837456346828296121
ZZ(12) = 0.127938195346752156794
ZZ(13) = ZZ(12)
ZZ(14) = ZZ(11)
ZZ(15) = ZZ(10)
ZZ(16) = ZZ(9)
ZZ(17) = ZZ(8)
ZZ(18) = ZZ(7)
ZZ(19) = ZZ(6)
ZZ(20) = ZZ(5)
ZZ(21) = ZZ(4)
ZZ(22) = ZZ(3)
ZZ(23) = ZZ(2)
ZZ(24) = ZZ(1)
RETURN
END