Design of passive suspension system with non-linear springs using b-spline collocation method

Rahul Jhaveri

Follow this and additional works at: http://scholarworks.rit.edu/theses

Recommended Citation

This Thesis is brought to you for free and open access by the Thesis/Dissertation Collections at RIT Scholar Works. It has been accepted for inclusion in Theses by an authorized administrator of RIT Scholar Works. For more information, please contact ritscholarworks@rit.edu.
DESIGN OF PASSIVE SUSPENSION SYSTEM WITH NON-LINEAR SPRINGS USING

B-SPLINE COLLOCATION METHOD

By

Rahul Jhaveri

Thesis work submitted in partial fulfillment of the requirement for the Degree of Master of

Science in Mechanical Engineering

Under the guidance of:

Dr. Hany Ghoneim – Thesis Adviser
Department of Mechanical Engineering

Approved by:

Dr. Marca Lam - Member of Committee
Department of Mechanical Engineering

Dr. Agamemnon Crassidis - Member of Committee
Department of Mechanical Engineering

Dr. Alan Nye – Representative of Department
Department of Mechanical Engineering

Department of Mechanical Engineering
Rochester Institute of Technology
Rochester NY 14623
October 2011
Thesis Release Permission Form

DESIGN OF PASSIVE SUSPENSION SYSTEM WITH NON-LINEAR SPRINGS USING B-SPLINE COLLOCATION METHOD

Rochester Institute of Technology

Kate Gleason College of Engineering

I, Rahul Jhaveri, hereby grant permission to the Wallace Memorial Library to reproduce my thesis in whole or part.

________________________________________________________________________
Rahul Jhaveri

________________________________________________________________________
Date
ACKNOWLEDGEMENTS

The following thesis, while an individual work, was inspired by several people’s insights. First, I must express my deepest gratitude to Dr. Hany Ghoneim. His supervision, advice, guidance, encouragement and unflinching support from the research’s beginning were invaluable. His patience as an advisor, his enthusiasm as a teacher, his promptness while reviewing my writing and his boundless passion for research are all qualities I will strive to emulate. Your example, Dr. Ghoneim, will remain with me and provide lasting inspiration.

I thank Dr. Marca Lam and Dr. Agamemnon Crassidis for taking time out of their busy schedules to serve on my thesis committee. They enhanced my knowledge with their questions and comments at various stages. I would also like to thank them for their excellent teaching skill which guided me and made my research possible. Very special thanks to Dr. Edward Hensel Jr. for his guidance and counseling, and the Department of Mechanical Engineering for providing the generous financial support which made it possible for me to pursue the degree and research. I would also like to mention Jason Magoon’s research which served as a foundation for my thesis. David Anderson’s contribution towards experimentation and manufacturing the prototype proved of utmost importance for verification of the analysis, and I am most appreciative of his efforts.

On a more personal note, I would also like to thank my friends Amish, Ankit, Dipti, Jayneel, Keyur, Kunal, Suraush, Vaibhav, Zalak and others for their friendship and support throughout my thesis journey. Finally and most importantly, I extend heartfelt gratitude to my mother, Mina Jhaveri and my grandfather Jagdish Jhaveri for their unconditional love, limitless support and patience while raising me. Every good quality I possess today, I owe to them.
ABSTRACT

The issue of vibration isolation challenges engineers’ designs across the engineering spectrum. From an engineering standpoint, vibration control impacts the fields of transportation, manufacturing, construction and mechanical design. Dynamic systems produce vibrations for various reasons i.e. rotating unbalanced masses (high speed turbines); inertia of reciprocating components (internal combustion engines); irregular rolling contact (automobiles) or induced eddy current (locomotives) vibrations. In most cases, vibrations cause only physical discomfort and/or loss of accuracy. But in extreme cases, transmitted forces may cause a body to undergo high amplitude resonant vibrations, leading to high cyclic stresses and imminent fatigue failure resulting in a catastrophic occurrence and possible loss of life. Therefore, isolating the vibration’s source from other system components becomes essential. Deploying a parallel under-damped spring-damper arrangement achieves this required isolation by suspending the component’s mass.

The frequency response function (FRF) of a second order under-damped suspension model suggests that for a given excitation frequency, suspensions with lower natural frequencies benefits vibration isolation. Lowering the natural frequency requires springs with low stiffness. Using soft springs is not always plausible as it significantly reduces the suspension’s load carrying capacity. Therefore, in order to improve vibration isolation, the initial displacement requires high stiffness suspension followed by low stiffness beyond the required load carrying capacity. This initial high stiffness enables the suspension to sustain high loads, whereas the soft-spring behavior improves the suspension’s vibration isolation.
Current research explores improving vibration isolation with a suspension system which uses non-linear spring stiffness. It proposes a suspension mechanism with compliant cantilevered beams used as springs. The suspension spring is mathematically modeled using the Euler-Bernoulli equation for bending of beams to create a non-linear governing equation. The resulting governing equation provides a numerical solution to develop a force versus deflection plot.

The analysis reveals that two distinct regions come under consideration when evaluating suspension: stiff-spring and soft-spring. The ensuing dynamic analysis leads to a frequency response function (FRF) of a spring-mass-damper system which emulates the suspension’s operating condition. It reveals that vibration isolation manifests significant improvement when the suspension operates in the soft-spring region as compared to a linear spring arrangement. A numerical technique called B-spline collocation approximates the non-linear governing equation’s solution. A prototype of the suspension system is manufactured and tested for static and dynamic characteristics. The analytical and experimental results are found to be in agreement.
# TABLE OF CONTENTS

CHAPTER 1: INTRODUCTION

1.1 MOTIVATION

1.2 LITERATURE REVIEW

1.1.1 SUSPENSION SYSTEM AND MATHEMATICAL MODEL

1.1.2 NUMERICAL SOLUTION OF THE MATHEMATICAL MODEL

1.3 CONCLUSION

CHAPTER 2: MATHEMATICAL MODELING OF SUSPENSION

2.1 DESCRIPTION OF GEOMETRY

2.2 FORCE ANALYSIS

2.3 EULER-BERNOULLI EQUATION

2.4 STATIC MATHEMATICAL MODEL

2.4.1 BOUNDARY CONDITIONS

2.4.2 TRANSFORMATION TO CARTESIAN COORDINATES

2.5 MODELING FOR DYNAMIC RESPONSE

2.5.1 FREQUENCY RESPONSE

2.6 CONCLUSION
CHAPTER 3: ANALYTICAL SOLUTION

3.1 SOLUTION TO THE STATIC MODEL

3.1.1 BACKGROUND

3.1.2 STEPWISE B-SPLINE COLLOCATION

3.1.3 CONVERGENCE TESTING

3.2 SOLUTION OF THE DYNAMIC MODEL

3.2.1 SIMULINK® MODEL

3.3 CONCLUSION

CHAPTER 4: RESULTS AND DISCUSSION

4.1 ANALYTICAL RESULTS

4.1.1 STATIC ANALYSIS

4.1.2 DYNAMIC ANALYSIS

4.2 EXPERIMENTAL SET-UP AND RESULTS

4.2.1 EXPERIMENTAL SET-UP

4.2.2 EXPERIMENTAL RESULTS

4.3 COMPARISONS AND DISCUSSION

4.3.1 STATIC COMPARISON

4.3.2 DYNAMIC COMPARISON
LIST OF FIGURES

Figure 1.1: Frequency response of second order linear under-damped system..........................3

Figure 1.2: Force versus deflection plot desired for vibration isolation.................................4

Figure 1.3: Comparison of frequency response for hard and soft springs...............................5

Figure 1.4: Class 1A-c Constant Force Compliant Mechanism............................................8

Figure 2.1: Schematic of the proposed suspension system.......................................................13

Figure 2.2: Free body diagram of the right half of the suspension system...............................14

Figure 2.3: Definition of variables s and θ............................................................................16

Figure 2.4: Cantilever beam loading diagram........................................................................17

Figure 2.5 Spring-mass-damper model..................................................................................21

Figure 2.6: Free body diagram of oscillating mass block........................................................22

Figure 3.1: Basis functions for the parametric range of 0 ≤ t ≤ 1.........................................27

Figure 3.2: SIMULINK® Model............................................................................................37

Figure 3.3: Sample stiffness versus displacement map used for simulation..........................39

Figure 4.1: Slope of the beam, $\theta$ (radians) along the length of the beam, x(mm)...............43

Figure 4.2: Convergence of $\tan \beta$ for different loads.......................................................44

Figure 4.3: Variation of angle $\beta$ as a function of the applied load.....................................45
LIST OF TABLES

Table 4.1: Constant parameters used in the simulation………………………………………42
Table 4.2: Different load-cases used for the dynamic simulation……………………………..49
Chapter 1: INTRODUCTION

Traditionally, engineers have attempted to develop sophisticated vibration isolating solutions for system components. Suspensions like linear passive suspensions, active suspensions and semi-active suspensions typically serve this purpose. Although effective, inherent shortcomings as far as performance or cost-effectiveness and practicality affect most of these isolators. The most common suspension system in use today is the passive suspension with linear springs (Ebrahimi, Bolandhemmat, Khamsee, & Golnaraghi, 2011).

This thesis explores the idea of achieving better vibration isolation by developing a passive non-linear spring suspension with variable stiffness. This approach uses the non-linear characteristics of the suspension to achieve better vibration isolation than traditional suspensions. The proposed suspension’s effectiveness is verified analytically and experimentally.

A mechanism made out of compliant cantilevered beams hinged to rigid links comprises the proposed suspension design which is modeled mathematically using the Euler-Bernoulli equation for bending of beams. A governing non-linear differential equation is developed, and appropriate boundary conditions are defined. Static analysis numerically solves the model’s governing equation which produces a force-deflection relationship for the suspension. The resulting non-linear suspension system is analyzed dynamically as a spring-mass-damper model which prompts development of the Frequency Response Function for use under base excitation in various test cases.

A numerical technique called the B-spline collocation method solves the non-linear governing equation. This method involves fitting a B-spline curve onto the solution of the governing differential equation, which results in a closed-form governing equation solution. The solution’s
relative error is controlled to the order of $10^{-8}\%$. A spring-mass-damper model is analyzed using the results from the curve-fit.

Development of a test prototype brings the suspension system into physical reality. The prototype is tested for static and dynamic characteristics. The static test determines an experimental force versus deflection curve. The dynamic test evaluates the prototype under base excitation, and develops an experimental frequency response plot. Comparison of the analytical and experimental results elicits suitable conclusions.

1.1 MOTIVATION

A frequency response function (FRF) plots the amplitude ratio of transmitted vibrations to excitation amplitude (also known as amplitude ratio) versus the excitation frequency’s ratio to the system’s natural frequency (also known as frequency ratio, $r$). A typical frequency response function (FRF) for a linear under-damped spring-mass-damper system under base excitation is shown in Figure 1.1. As seen from the figure, the amplitude ratio for light damping reaches a maximum point when the frequency ratio equals unity. At this point, the system is said to be in resonance. At resonance, the transmitted vibration amplitude is maximized, and is restricted only by the system’s damping effects.

As the frequency ratio moves beyond the resonant condition, the amplitude ratio rapidly subsides becoming less than unity for values of $r$ greater than $\sqrt{2}$ (Inman, 2007). To achieve vibration isolation, a dynamic mechanical system must avoid the resonance range during operation. For a linear spring-mass-damper system, the natural frequency $\omega_n$ is given by:
\[ \omega_n = \sqrt{\frac{k}{m}} \]

Where, \( k \) is the stiffness of the spring and \( m \) is the mass equivalent.

Vibration isolation requires operating in a range of \( r \) above \( \sqrt{2} \). This is achieved by reducing the system’s natural frequency, accomplished by designing softer springs. However, soft springs reduce load carrying capacity which, in turn, limits the suspension system’s load. Therefore, for better vibration isolation, the spring must behave as a stiff spring to a point where it can carry the design load. Once the load carrying range is passed, the spring softens up to operate in the low transmission region (\( r > \sqrt{2} \)) of the FRF. A spring that is non-linear in nature exhibits the stiffness required for the initial displacement, until the force produced is high enough to support the load. Once the load carrying capacity is reached, the spring softens up, bringing down the
system’s natural frequency. Figure 1.2 shows a desired force versus deflection plot for a suspension system:

![Force versus displacement for the proposed suspension](image)

*Figure 1.2: Force versus deflection plot desired for vibration isolation*

Apart from the stiff-spring and soft-spring regions, Figure 1.2 also identifies a transition region. The transition region assists in maintaining a physical continuity in shifting from hard to soft stiffness. To reinforce the soft spring’s benefits, Figure 1.3 shows the amplitude ratios plotted against excitation frequencies for stiff and soft-springs:
Figure 1.3 underlines the soft-spring’s benefits as opposed to hard-springs. The figure clearly illustrates that the transmitted vibration amount is much less in the soft-spring’s case as compared to the hard-spring for any given excitation frequency beyond $\sqrt{2}$ times the natural frequency. This characteristic inspires the non-linear suspension’s design. After a chapter devoted to the literature review undertaken for the study, successive chapters define the suspension structure and analyze the design.
1.2 LITERATURE REVIEW

This thesis’ primary objective is to develop and analyze a non-linear spring suspension system based on large deflections of cantilevered beams. The second goal involves demonstrating the effective use and simplicity of the B-spline collocation method to numerically solve non-linear solid mechanics problems.

The literature’s objectives are two-fold:

1. Identify suitable mechanisms and methods of mathematical modeling for the development of the proposed suspension system;
2. Identify simple and effective numerical techniques to solve the mathematical model.

1.2.1 SUSPENSION SYSTEM AND MATHEMATICAL MODEL

Boyle, Howell, Midha and Millar have contributed a significant amount of research on compliant mechanisms which demonstrate the non-linear force-displacement relationship. Numerous modeling methods have defined the compliant mechanism’s force-displacement relationship.

Compliant mechanisms are defined as mechanisms which gain some or all of their motion from the relative flexibility of their members rather than from rigid body joints alone (Midha, Howell, & Norton, 2000). The research by Boyle (Boyle C. L., 2001) analyzes a “Class 1A-d” configuration of a compliant mechanism as categorized by Howell et al (Howell, Compliant Mechanisms, 2001). Pseudo Rigid-Body Modeling (PRBM) is a widely used compliant mechanisms modeling technique. Boyle incorporates PRBM with Lagrange’s equation to develop the compliant mechanism’s dynamic equation (Boyle C. L., 2001). The experimental
and theoretical results indicate that the mechanism develops a nearly constant opposing force for a range of input displacements leading to a non-linear force-displacement relationship.

Howell and Midha (Howell & Midha, 1994) define a special compliant mechanism sub-category called Constant-Force Compliant Mechanisms (CFCM). A compliant slider mechanism, with flexible and rigid segment dimensions optimized such that the output force’s variation is minimized over a range of displacement, is called a Constant Force Compliant Mechanism (CFCM) (Howell, Compliant Mechanisms, 2001). Owing to their unique abilities and applications, CFCM’s are studied by a host of researchers led by Howell and Midha (Howell & Midha, 1995). CFCM’s are useful in applications requiring a constant force on a time-varying or irregular surface, such as grinding, welding, deburring and assembly (Evans & Howell, 1999). Due to its manufacturability and large range of motion (Boyle, Howell, Magleby, & Evans, 2003), and the favorable force-displacement relationship, a CFCM qualifies as an appropriate mechanism to develop the non-linear suspension system. The proposed suspension system is based on the Class 1A-c configuration CFCM. Figure 1.4 (Millar, Howell, & Leonard, 1996) shows the mentioned mechanism:
For a given input displacement $\Delta x$, the CFCM yields the same force $F$ over the full range of its designed deflection. However, the force required for the initial part of the displacement from the equilibrium state will be significantly higher as compared to the force required in the constant force region. This leads to a non-linear force-displacement relationship which is similar to the force-deflection relation requirement for the suspension (Figure 1.2). The consequent literature review reveals the rationale behind the non-linear behavior of the mechanism. The non-linearity of the force-displacement relationship is attributed to the geometric non-linearity introduced by the large deflections of cantilevered beams (Malatkar, 2003).

Development of the suspension system’s governing differential equation (discussed in Chapter 2), necessitates building a model to experience first-hand the CFCM’s cantilever beam’s large deflections. Some basic information on beam elasticity is found in Boresi and Richard (Boresi & Richard, 2003), Budynas (Budynas, 1999), Case & Chivler (Case & Chivler, 1972) and Den Hartog (Den Hartog, 1961). Whereas, Malatkar (Malatkar, 2003) provides extensive analytical details of cantilever beams undergoing large deformations using Hamilton’s Principle and Lagrange’s equation. An Euler-Bernoulli approach to model large deflection of beams is discussed by Teo et al (Teo, Chen, Yang, & Lin, 2010) and Wang et al (Wang, Chen, & Liao, 2008). Large cantilever beam deformations are also analyzed through a series of complex differentiation, integration and elliptical integrals by Frisch-Fay (Frisch-Fay, 1962). Dimitrivova (Dimitrivova, 2010) provides insights on dynamic analysis of beams on piecewise homogeneous foundation with moving loads. Ghoneim (Ghoneim, 2008) examines the dynamics of a hyperbolic (non-linear) composite coupling; and Karkoub et al (Ghoneim & Karkoub, 2000) analyze the effects of compliance on the dynamics of a four-bar mechanism. El-Saeidy and
Stitcher (El-Saeidy & Stitcher, 2010) study the dynamics of a bearing system under rotating unbalanced loads.

1.2.2 NUMERICAL SOLUTION OF THE MATHEMATICAL MODEL

The proposed suspension is modeled using the Euler-Bernoulli equation because of its simplicity and data availability. The literature review reveals that many numerical as well as analytical techniques are employed for solving non-linear solid mechanics problems. For this thesis, numerical techniques to solve the non-linear governing differential equation are used.

The fundamental insights on using numerical methods to solve differential equations are found in Chapra and Canale (Chapra & Canale, 2002). Kadalbajoo and Gupta (Kadalbajoo & Gupta, 2010) enumerate an exhaustive list of numerical techniques used for non-linear analysis of solid mechanics and fluid problems. Fairweather and Meade (Fairweather & Meade, 1989) present a summary of spline collocation methods used specifically for boundary value problems. Some prominent non-linear compliant mechanism analysis methods are, among others, the finite-element method (Chakraverty & Petyt), homotopy analysis method (Chen & Liu, 2010) and topological optimization (Meaders & Mattson, 2009). Boedo and Eshkabilov (Boedo & Eshkabilov, 2003) use the finite element method along with genetic algorithms to solve non-linear tribology problems. Li et al (Li, Fairweather, & Bialecki, 2000) discuss the application of orthogonal spline colocation method in relation to non-linear vibration problems.

Magoon (Magoon, 2010) discusses the application of B-spline collocation to solid mechanics problems and implementation through a symbolic MATLAB® code. Magoon also describes a stepwise approach of implementing the B-spline collocation method to solve a cantilever beam

The spline collocation methods transform the differential equations into easily solvable sparse algebraic equations (Shao & Liang, 2010). The B-spline collocation method in particular fits a B-spline curve onto the solution of the differential equation. The B-spline collocation method has significant advantages over other numerical techniques used for non-linear analysis. Some of these advantages are:

- The B-spline collocation method provides the differential equation’s solution a piecewise-continuous closed-form approximation (Magoon, 2010).
- The B-spline collocation method avoids integration, making its use more elegant and simple as compared to the Galerkin finite element methods (Johnson, 2005).
- The closed form solution in polynomial form allows the higher order differential to be easily defined.
- The B-spline curves are computationally more efficient as compared to Galerkin finite element methods (Botella, 2002).

In light of the above mentioned advantages, we choose the B-spline collocation method to solve the proposed suspension system’s governing equation. The properties of the B-spline curves are
detailed by Rogers and Adams (Rogers & Adams, 1990). Some of these properties are discussed in APPENDIX I.

1.3 CONCLUSION

The literature review provides a starting point for developing and analyzing the non-linear suspension system. The proposed suspension spring is based on a Constant Force Compliant Mechanism (CFCM, shown in Figure 1.4), because of its favorable force-displacement response proven from previous experimentation, large range of motion, and ease of manufacturing. The CFCM chosen is Class 1A-c. The suspension spring model is analyzed by developing the exact differential equation using the Euler Bernoulli equation. The governing equation developed is expected to be non-linear in nature due to the cantilever beams’ large deflection. The B-spline collocation method is chosen to approximate the solution of the governing equation. The B-spline collocation method is chosen based on the previously obtained results for similar simulations as well as its advantages over the traditional finite element and integration methods.
Chapter 2: MATHEMATICAL MODELING OF SUSPENSION

In order to improve vibration isolation, the suspension model must possess a non-linear force-displacement relationship. The suspension must develop a high stiffness gradient for small displacements, and a lower stiffness gradient for subsequent displacements. Based on the literature review, a Constant Force Compliant Mechanisms (CFCM’s) satisfy these requirements (Boyle, Howell, Magleby, & Evans, 2003). Therefore, the proposed suspension model is developed from Class 1A-c CFCM (Howell, Compliant Mechanisms, 2001). The first objective is to describe the suspension model’s geometric properties. The suspension derives its non-linearity from large deflection of cantilevered beams which are assumed to be Euler-Bernoulli style beams. The second objective deals with deriving the suspension model’s exact governing equation using the Euler-Bernoulli equation for bending of beams. The third objective, using Newton’s Second Law of Motion, is to model the suspension for dynamic analysis under base excitation.

2.1 DESCRIPTION OF GEOMETRY

The suspension model is a subtle modification of the Class 1A-c CFCM. These modifications are required to make the design compatible to experimentation and practical application. The suspension model replaces the original design’s slider (see Figure 1.4) with a rigid-link which allows loading of the suspension. Moreover, to achieve greater balance and stability during runtime, the suspension is designed to be a mirror image of the CFCM.
A schematic representation of the proposed suspension model is shown in Figure 2.1. The suspension consists of two compliant cantilevered beams called $S_1$ and $S_2$. The beams, which also act as suspension springs, are of equal length ($l_F$) and fixed rigidly to the base. The beams run parallel to each other and are uniform with a rectangular cross-section. The free-end of each spring is hinged to rigid links of length $l_R$. The other end of both the rigid links is hinged to a horizontal link on which the load is applied. The horizontal link doubles as a loading platform for the suspension load. The assembly is symmetric around a central axis. The springs and rigid links are anticipated to undergo symmetric displacements during run-time. The assumption of symmetry ensures that the suspension preserves the characteristic force-displacement relationship demonstrated by the CFCM.
2.2 FORCE ANALYSIS

To develop the proposed suspension’s mathematical model, the static forces developed in the suspension need analysis. Figure 2.2 shows the free body diagram of the right half of the suspension assembly under equilibrium conditions. Due to symmetry, the assembly’s left half will reflect the same dynamics.

![Free body diagram of the right half of the suspension system](image)

*Figure 2.2: Free body diagram of the right half of the suspension system*

The suspension is loaded with a vertically static load, denoted as $P$, which acts in the downward direction. The hinges’ presence prevents the transfer of moment at each hinged joint when applying the vertical load. The rigid link on the right-hand side rotates in a counter-clockwise
direction under the load’s influence. The rigid link’s inclination with respect to the vertical axis at equilibrium is denoted as angle $\beta$. The resultant force at the tip of the cantilevered beam acts along the angle $\beta$. This force can be resolved into two mutually perpendicular forces along the horizontal and vertical axes. The force acting along the horizontal axis is denoted as $H$. The vertically resolved force is the applied static load, $P$ itself. The combined action of the forces $H$ and $P$ causes the compliant cantilevered beam to undergo lateral deflection. The beams, which are assumed to be perfectly elastic, produce an equal and opposite restoring force upon deflection. The restoring force developed in the beam acts on the loading platform through the rigid links, thus, supporting the applied load.

2.3 EULER-BERNOULLI EQUATION

Due to the mechanism’s geometry, the cantilever beams are subjected to a bending and buckling load combination. Due to the load’s application, the beams undergo outward lateral deflection. The lateral deflection of the compliant beam can be modeled as the deflection of a cantilever beam with a point load at the free end.

The beam’s deflection can be mathematically modeled using the basic form of the Euler-Bernoulli equation to develop the governing equation. The Euler-Bernoulli law for bending of a beam states that: “The bending moment at a point on the beam is proportional to the change in curvature caused by the action of the load” (Magoon, 2010). Mathematically, for a beam of uniform cross-section with an area moment of inertia $I$, the Euler Bernoulli equation can be stated as (Wang, Chen, & Liao, 2008):
Here, $M$ is the bending moment at a point at a distance $s$ and $\frac{1}{R}$ is the curvature of the beam at that point. The Young’s modulus of the beam material is $E$, and $\theta$ is the slope of the beam at a point at a distance $s$ from the origin along the beam. The distance $s$ is the arc-length of the deflected beam from the fixed end. Figure 2.3 shows the variables $\theta$ and $s$:

\[
\frac{1}{R} = \frac{M}{EI} = \left(\frac{\partial \theta}{\partial s}\right)
\]  

(2.1)

While applying the Euler-Bernoulli law to the problem at hand, the following assumptions are made (Magoon, 2010):

1. The beam has a homogeneous composition. Therefore, $E$ is constant throughout the beam length.
2. The area of the beam normal to the application load is always constant.
3. The beam is assumed to be rigid in shear. In other words, the beam does not undergo progressive shear deformation with the application of bending moment.
2.4 STATIC MATHEMATICAL MODEL

As seen from Figure 2.2, the cantilevered spring is nothing but a beam subjected to a point load inclined at an angle $\beta$ to the vertical axis. The load is resolved into vertical and horizontal components as $P$ and $H$ respectively as shown in Figure 2.4. These forces produce individual bending moments on the spring. The bending moment causes the beams to undergo deflections. The response’s non-linearity is achieved by the large deflection of the cantilevered beams. Further discussions lead to the development of the suspension system’s governing equation.

Figure 2.4: Cantilever beam loading diagram

Figure 2.4 shows the deflected cantilever beam under for a load instance $P$. A coordinate system is defined with the $X$-axis in the horizontal direction and the $Y$-axis in the vertical direction. The
origin is at the fixed support. The distance of the beam’s tip from the origin is designated as "a" in the X-direction and "b" in the Y-direction. A point $A(x, y)$ along the length of the beam is considered. The tangent to the beam at point $A(x, y)$ is inclined at an angle $\theta$ with regards to the X-axis. The bending moment $M(x)$ at point $A(x, y)$ is given as:

$$M(x) = P(a - x) + H(b - y) \quad (2.2)$$

Using Euler-Bernoulli equation to introduce the variable $\theta$ into equation (2.2),

$$\frac{\partial \theta}{\partial s} = \frac{M(x)}{EI} \quad (2.3)$$

Substituting equation (2.2) into equation (2.3),

$$\frac{\partial \theta}{\partial s} = \frac{P(a - x) + H(b - y)}{EI} \quad (2.4)$$

In order to eliminate the constants $a$ and $b$, the partial derivative of Equation (2.4) is taken with respect to $s$ to give

$$\frac{\partial^2 \theta}{\partial s^2} = \frac{1}{EI} \left\{ -P \frac{\partial x}{\partial s} - H \frac{\partial y}{\partial s} \right\} \quad (2.5)$$

$\frac{\partial x}{\partial s}$ and $\frac{\partial y}{\partial s}$ in Equation (2.5) can be expresses in terms of $\theta$ as:

$$\frac{\partial x}{\partial s} = \cos \theta \quad (2.6)$$

$$\frac{\partial y}{\partial s} = \sin \theta \quad (2.7)$$

Substituting equations (2.6) and (2.7) in equation (2.5):
Using geometry, it can be deduced that,

\[ H = P\tan\beta \quad \text{and} \quad \tan\beta = \frac{a}{\sqrt{l^2 - a^2}}, \]

The angle \( \beta \) is the inclination of the rigid link with respect to the horizontal axis. Substituting \( H \) in Equation (2.8),

\[ or \quad \frac{\partial^2 \theta}{\partial s^2} + \frac{1}{EI}\{P\cos\theta + P\tan\beta\sin\theta\} = 0 \quad (2.8) \]

The governing equation can be written as:

\[ \theta'' + \frac{P}{EI}(\cos\theta + \tan\beta\sin\theta) = 0 \quad (2.9) \]

Where the double prime denotes second derivative of \( \theta \) with respect to \( s \).

### 2.4.1 BOUNDARY CONDITIONS

The governing equation formed is of the second order and therefore two boundary conditions required to solve the equation. They are:

1. Slope at fixed end:
   
   The slope of the beam at the fixed-end \( (\theta|_{s=0}) \) remains at an angle of \( \frac{\pi}{2} \) radians irrespective of the deflection of other points on the beam,
or $\theta|_{z=0} = \frac{\pi}{2}$

2. Bending moment at the free end:

The bending moment at the free end is zero due to the presence of the hinge.

Mathematically, the bending moment at any point is expressed as $\frac{1}{E_0} \frac{\partial \theta}{\partial s}$

or $\left. \frac{1}{E_0} \frac{\partial \theta}{\partial s} \right|_{s=t_F} = 0$

Since the beam experiences an axial component, the beam is bound to undergo buckling at low values of angle $\beta$. To keep the buckling effect from dominating the solution, the beam is assumed to have a small curvature in the outward (positive $X$) direction. The small initial outward arch initiates the deflection in the desired outward direction. A boundary condition such that the curvature at the tip of the beam equals $\frac{1}{R}$, is defined as follows:

$$\left. \frac{\partial \theta}{\partial s} \right|_{s=t_F} = -\frac{1}{R}$$

Here, $R$ is the initial radius of curvature at the tip of the beam. The value of $R$ is kept very large so that $\frac{1}{R} \approx 0$.

### 2.4.2 Transformation to Cartesian Coordinates

The governing equation obtained in the variable $\theta$, can be transformed into $x$ and $y$ coordinates by integrating equations (2.6) and (2.7) respectively over the length of the beam.

$$x = \int_0^{t_F} dx = \int_0^{t_F} \cos[\theta(s)] ds \quad (2.10)$$
\[ y = \int_0^{t_F} dy = \int_0^{t_F} \sin[\theta(s)] ds \] (2.11)

The Equation (2.9) represents the suspension model’s governing differential equation. The differential equation is found to be non-linear as a result of large deflections of the beams. The solution of the differential equation can be approximated by implementing the B-spline collocation method, which is explained in the next chapter. The following section discusses the development of the suspension system’s dynamic model.

### 2.5 MODELING FOR DYNAMIC RESPONSE:

The suspension system can be considered an equivalent of the spring-mass-damper model with a non-linear spring. In order to model the suspension for base excitation, consider the spring-mass-damper model shown in Figure 2.5.

![Spring-mass-damper model](image)

In Figure 2.5, the variable spring stiffness \( k \) represents the overall stiffness of the suspension model. Mass \( M \) is equivalent to force applied on the loading platform, and the damping effect is caused by the component’s internal damping. The damping is assumed to be viscous in nature,
and the model is assumed to be underdamped throughout the analysis. The model is subjected to sinusoidal base excitation $y(t)$ given by:

$$y(t) = A \sin(\omega t)$$

Here, $A$ is the amplitude of displacement and $\omega$ is the driving velocity of the applied displacement. The variation of the spring stiffness with respect to the displacement can be obtained from the force-displacement relationship of the suspension model. The dynamic analysis produces a frequency response function (FRF) for the base excitation of the non-linear suspension for different load-cases. The development of mathematical model is discussed in the following section.

### 2.5.1 FREQUENCY RESPONSE

Consider the free-body diagram of the mass block $M$ as shown in Figure 2.6. The block $M$ is acted upon by the forces generated by the spring and the damper. There is also an inertia force which acts in the opposite direction of the motion of the block.

![Free body diagram of oscillating mass block](image)
The instantaneous position of the block is denoted by $x$ with respect to an arbitrary axis. The total instantaneous force generated in the spring, $F_{Spring}$ is:

$$F_{Spring} = k(x - y)$$

Here, $k$ is the instantaneous spring constant. The value of $k$ is dependent on the amount deflection. The total damping force, $F_{Damper}$ is

$$F_{Damper} = c(\dot{x} - \dot{y})$$

Here, $\dot{x}$ and $\dot{y}$ represent the derivative of the respective variables with respect to time. The damping constant $c$ is dependent on the material properties of the beam and friction. It is assumed to be constant throughout the analysis. The inertial force $F_{Inertia}$ opposes the motion, and is always directed away from the direction of motion. By Newton’s Second Law, the inertial force is given as:

$$\sum F = M\ddot{x}$$

Here, $\ddot{x}$ represents the acceleration of the block. Using the conditions of equilibrium, summing up forces in the $x$-direction,

$$F_{Inertia} = -F_{Damper} - F_{Spring}$$

and

$$M\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0$$

or

$$M\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$$

or

$$M\ddot{x} + c\dot{x} + kx = cA\omega \cos(\omega t) + kA\sin(\omega t)$$
The above equation governs the response of the suspension system to base excitation with displacement amplitude $A$ and frequency $\omega$.

2.6 CONCLUSION

The suspension model based on a CFCM is defined geometrically in this chapter. The exact governing differential equation for the suspension, derived using the Euler-Bernoulli equation for beam bending, is found to be non-linear. The solution of the static model is approximated by using the B-spline collocation method to develop the force-displacement relationship. The equation of the suspension’s motion is obtained by modeling the suspension as an under-damped spring-mass-damper system with base excitation. The equation of motion must be solved to determine the suspension model’s frequency response function. Solutions for both models are presented in the following chapter. The B-spline collocation is discussed in detail in APPENDIX I.
3.1 SOLUTION TO THE STATIC MODEL

The suspension model’s governing equation (Equation (2.9)) is found to be non-linear due to the cantilever beam’s large deflection. Various researchers have solved similar non-linear equations by using different exact and approximate methods. A common perception about obtaining a non-linear differential equation’s exact solution is that it is an exhaustive and extremely complicated procedure. On the contrary, the numerical techniques are quite generic, have a straightforward approach, and provide approximations within small tolerances. The B-spline collocation method in particular suitably approximates the solution of Equation (2.9) owing to the flexibility provided by the B-spline curves, and the ability to produce piecewise polynomial approximations. This chapter describes the B-spline collocation method’s use to approximate the static suspension model’s solution. A detailed description of the B-spline curves’ properties and the collocation technique is found in APPENDIX I.

3.1.1 BACKGROUND

The governing equation (Equation 2.9) of the proposed suspension system given as:

\[ \theta'' + \frac{P}{EI}(\cos \theta + \tan \beta \sin \theta) = 0 \]

Here, \( \theta \) is subjected to the following boundary conditions:

1. Slope at the fixed end, \( \theta|_{s=0} = \frac{\pi}{2} \)
2. Initial radius of curvature at the tip, \( \frac{\partial \theta}{\partial s} \big|_{s=l_F} = -\frac{1}{R} \)

APPENDIX I contains a brief background of the B-spline collocation method, and might prove a useful reference at this juncture. Further discussions describe the stepwise application of the B-spline collocation method to the above mentioned equation, ultimately leading to the suspension model’s force versus deflection curve development.

### 3.1.2 STEP-WISE B-SPLINE COLLOCATION

The B-spline collocation method is implemented by developing a symbolic MATLAB® code. The following steps are coded and output at each step is recorded.

1. **Selection of knot vector:**

   Since the governing equation is of order two, the approximated closed form solution is assumed to be of the same degree. A B-spline curve of order three (degree two) or more is expected to produce a converging approximation. As the governing equation involves sine and cosine functions of the variable, higher order of B-spline curves is recommended in order to improve the flexibility of the solution and obtain a better fit. Thus, a normalized knot vector is chosen to produce a fourth order B-spline curve. Moreover, the problem’s physical attributes do not constitute significant discontinuities. Therefore, a continuous B-spline curve fit satisfies the approximation.

   In light of the above mentioned points, a fifth order continuous B-spline curve is fitted to the solution. The knot vector required for the curve-fit is given below:

   \[
   x = [0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1]
   \]
2. Calculating basis-functions:

As the knot-vector has five replicates and no intermediate values between zero and one, \( k = 5 \) and \( m = 0 \). There will be \( k = 5 \) normalized basis functions for the curve-fit, each of degree four. The five basis-functions will lead to the same number of position vectors \( B_i \ (i = 1, 2, \ldots, 5) \). The basis-functions are developed by deploying recursive function in symbolic form using MATLAB® as shown in APPENDIX II. The basis-functions are as shown below:

\[
\begin{align*}
N_{1,5}(t) &= (t - 1)^4 \\
N_{2,5}(t) &= -4t(t - 1)^3 \\
N_{3,5}(t) &= 6t^2(t - 1)^2 \\
N_{4,5}(t) &= -4t^3(t - 1) \\
N_{5,5}(t) &= t^4
\end{align*}
\]

Figure 3.1 shows the basis-function variation for \( t \) ranging from 0 to 1.

---

\textit{Figure 3.1: Basis functions for the parametric range of} \( 0 \leq t \leq 1 \)
3. Greville Abscissae:

The Greville Abscissae are required to evaluate internal control points. The Greville Abscissae transform the original abscissae (in terms of $\theta$) into the parametric abscissae (in terms of parameter $t$). The advantage of using Greville Abscissae is the ability to perform direct substitution into the B-spline function in terms of parameter $t$ while evaluating the intermediate points (Magoon, 2010).

The number of intermediate points depends upon the order of the differential equation being solved and the order of the B-spline curve to be fitted. In fact, the required number of points is the difference between the order of the B-spline curve and the given number of boundary conditions. In this case, the order of the curves selected is five, and there are two boundary conditions defined. This means the three intermediate points will require three corresponding Greville Abscissae in order to fit the curve.

The Greville Abscissae are calculated using the following equation (Magoon, 2010):

$$
t_i = \frac{1}{n} \{x_i + x_{i+1} + \cdots + x_{i+n-1}\}
$$

Here, $x_i$ is the $i^{th}$ element of the knot vector. The above equation produces replicates of the first and last values which are dropped off. In the case at hand, for five position vectors, $n = 5 - 1 = 4$. The Greville Abscissae are calculated as follows:

$$
t_1 = \frac{1}{4} \{x_1 + x_2 + x_3 + x_4\} = \frac{1}{4} \{0 + 0 + 0 + 0\} = 0
$$

$$
t_2 = \frac{1}{4} \{x_2 + x_3 + x_4 + x_5\} = \frac{1}{4} \{0 + 0 + 0 + 0\} = 0
$$

$$
t_3 = \frac{1}{4} \{x_3 + x_4 + x_5 + x_6\} = \frac{1}{4} \{0 + 0 + 0 + 1\} = \frac{1}{4}
$$

$$
t_4 = \frac{1}{4} \{x_4 + x_5 + x_6 + x_7\} = \frac{1}{4} \{0 + 0 + 1 + 1\} = \frac{1}{2}
$$
The replicate of 0 and 1 in the beginning and the end respectively is eliminated and the Greville Abscissae vector is given as:

\[ t = [0 \ 1/4 \ 1/2 \ 3/4 \ 1] \]

4. B-spline equations:

The B-spline equation for the fifth order B-spline curves is written as:

\[ P(t) = \sum_{i=1}^{5} B_i N_{i,5}(t) \quad 0 \leq t \leq 1 \]

In matrix form,

\[ P(t) = B N(t) \quad (3.1) \]

Here, the characters in bold represent matrices given as:

\[ B = [B_1 \ B_2 \ B_3 \ B_4 \ B_5] \]

\[ N(t) = \begin{bmatrix} N_{1,5} \\ N_{2,5} \\ N_{3,5} \\ N_{4,5} \\ N_{1,5} \end{bmatrix} = \begin{bmatrix} (t - 1)^4 \\ -4t(t - 1)^3 \\ 6t^2(t - 1)^2 \\ -4t^3(t - 1) \\ t^4 \end{bmatrix} \]

As the governing equation consists of second order differential of the variable \( \theta \), it would be required to have the second order differential of \( P(t) \) available for later substitution.

Differentiating Equation (3.1) with respect to \( t \),

\[ P'(t) = B N'(t) \quad (3.2) \]
Differentiating again with respect to $t$,

$$
P''(t) = B N''(t) \quad (3.3)$$

$$
N''(t) = \begin{bmatrix}
4(t - 1)^3 \\
-4(4t^3 - 9t^2 + 6t - 1) \\
12t(2t^2 - 3t + 1) \\
-4t^2(4t - 3) \\
4t^3
\end{bmatrix}
$$

5. Use of Boundary Conditions:

The next step is to incorporate the boundary conditions to evaluate the control points’ end ordinates. The use of Greville Abscissae ensures that the parametric coordinate $t$ is constrained to the Cartesian coordinate $s$. In order to use the boundary conditions in the $t$ domain, it is necessary to transform the boundary conditions from $s$ to $t$ domain. The use of parameter $t$ causes the changes in the variable’s domain from $0 \leq s \leq l_F$ to $0 \leq t \leq 1$. The transformation between $s$ and $t$ domain is accomplished simply by using the total length of the beam, $l_F$ as the scaling factor. For any point at a distance $s$ from the origin in the $s$-domain, the corresponding point in the $t$-domain can be given as follows:

$$
t = \frac{1}{l_F} s
$$

Moreover, the differential in the $t$-domain can be obtained by writing the above equation in differential form:

$$
dt = \frac{ds}{l_F}
$$
The first boundary condition \( \theta|_{s=0} = \frac{\pi}{2} \) can be stated in the \( t \)-domain as \( P(t)|_{t=0} = \frac{\pi}{2} \) and the second boundary condition \( \frac{\partial \theta}{\partial s}\bigg|_{s=t_F} = -\frac{1}{R} \) translates to \( \frac{1}{l_F} \frac{\partial}{\partial t} P(t)\bigg|_{t=1} = -\frac{1}{R} \).

Substituting the parametric B-spline equations into the boundary conditions:

a. Boundary condition # 1:

\[
P(t)|_{t=0} = BN(0) = \frac{\pi}{2}
\]

Substituting \( t = 0 \) in equation (3.1):

\[
\begin{bmatrix}
1 \\
0 \\
0 \\
0 
\end{bmatrix}
\begin{bmatrix}
B_1 \\
B_2 \\
B_3 \\
B_4 \\
B_5
\end{bmatrix}
= \frac{\pi}{2}
\]

\[
or \quad B_1 = \frac{\pi}{2}
\]

b. Boundary condition # 2:

\[
\frac{\partial}{\partial t} P(t)\bigg|_{t=1} = BN'(1) = 0
\]

Substituting \( t = 1 \) in equation (3.2):

\[
\begin{bmatrix}
0 \\
0 \\
0 \\
4 
\end{bmatrix}
\begin{bmatrix}
B_1 \\
B_2 \\
B_3 \\
B_4 \\
B_5
\end{bmatrix}
= \frac{l_F}{R}
\]

\[
or 4B_4 + 4B_5 = -\frac{l_F}{R}
\]

\[
or B_4 + B_5 = -\frac{l_F}{4R}
\]
6. Governing equation in parametric form:

In order to solve for the remaining position vectors, the governing equation is converted into parametric form by substituting $P(t)$ in place of $\theta(s)$ and changing the function domain from $0 \leq s \leq l_F$ to $0 \leq t \leq 1$. Therefore, the parametric form of the governing equation is given as:

$$
\frac{1}{l_F^2} \frac{\partial^2}{\partial t^2} P(t) + \frac{P}{EI} \{\cos[P(t)] + \tan\beta \sin[P(t)]\} = 0
$$

Substituting $P(t)$ and its derivatives in terms of $B$ and $N(t)$, using equations (3.1), (3.2) and (3.3),

$$
\frac{1}{l_F^2} \frac{\partial^2}{\partial t^2} BN(t) + \frac{P}{EI} \{\cos[BN(t)] + \tan\beta \sin[BN(t)]\} = 0
$$

or

$$
\frac{1}{l_F^2} BN''(t) + \frac{P}{EI} \{\cos[BN(t)] + \tan\beta \sin[BN(t)]\} = 0 \quad (3.4)
$$

7. Evaluate the intermediate control points:

The intermediate control points are evaluated by substituting Greville Abscissae in the governing equation’s parametric form. These equations, along with the equations obtained from the boundary conditions, will result in a system of five non-linear equations with five unknown variables in $B$.

Substituting $t = \frac{1}{4}$ in equation (3.4) and writing out in matrix form:

$$
\text{or} \quad \frac{1}{l_F^2} BN'' \left(\frac{1}{4}\right) + \frac{P}{EI} \left\{\cos \left[BN \left(\frac{1}{4}\right)\right] + \tan\beta \sin \left[BN \left(\frac{1}{4}\right)\right]\right\} = 0
$$
Similarly, substituting \( t = 0.5 \) in the parametric governing equation gives:

\[
\begin{split}
\frac{1}{l_F^2} \begin{bmatrix} B_1 & B_2 & B_3 & B_4 & B_5 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -6 \\ 0 \\ 3 \end{bmatrix} \\
+ \frac{P}{EI} \left\{ \cos \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \end{bmatrix} \begin{bmatrix} 0.3164 \\ 0.4219 \\ 0.2109 \\ 0.0469 \\ 0.0039 \end{bmatrix} \right\} \\
+ \tan \beta \sin \begin{bmatrix} B_1 & B_2 & B_3 & B_4 & B_5 \end{bmatrix} \begin{bmatrix} 0.3164 \\ 0.4219 \\ 0.2109 \\ 0.0469 \\ 0.0039 \end{bmatrix} = 0
\end{split}
\]
Finally, substituting $t = 0.75$ in the parametric governing equation gives:

\[
\frac{1}{l_F^2} \begin{bmatrix} B_1 & B_2 & B_3 & B_4 & B_5 \end{bmatrix} \begin{bmatrix} 0.75 \\ 3 \\ -1.5 \\ -9 \\ 6.75 \end{bmatrix} + \frac{P}{EI} \begin{pmatrix} \cos \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \end{bmatrix} & \begin{bmatrix} 0.0039 \\ 0.0469 \\ 0.2109 \\ 0.4219 \\ 0.3164 \end{bmatrix} \\ \tan \beta \sin \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \end{bmatrix} & \begin{bmatrix} 0.0039 \\ 0.0469 \\ 0.2109 \\ 0.4219 \\ 0.3164 \end{bmatrix} \end{pmatrix} = 0
\]

Therefore, the above three equations and the equations obtained from the boundary conditions form a full ranked system of non-linear equations. Since the equations are non-linear, they are solved by implementing an optimization sequence using the \texttt{fsolve()} function on MATLAB® for given values of $\beta$ and $P$. The evaluation of the position vectors $(B)$ leads to the development of a polynomial $P(t)$ which maps the slope $\theta$ of the beam along its length. The x and y coordinates are related to the slope $\theta$ as:

\[
\frac{dx}{ds} = \cos[\theta(s)]
\]
\[
\frac{dy}{ds} = \sin[\theta(s)]
\]

In terms of parametric variable $t$,

\[
\frac{1}{l_F} \frac{dx}{dt} = \cos[\theta(t)]
\]
\[
\frac{1}{l_F} \frac{dy}{dt} = \sin[\theta(t)]
\]

The Cartesian coordinates of any point on the beam can be calculated as:

\[
x = \int_0^x \cos[\theta(s)] \, ds = \int_0^t l_F \cos[\theta(t)] \, dt
\]

\[
y = \int_0^y \sin[\theta(s)] \, ds = \int_0^t l_F \sin[\theta(t)] \, dt
\]

In order to develop the suspension system’s force versus deflection curve system, it is necessary to know the free-end co-ordinates \((a, b)\) of the cantilevered beam. The end co-ordinates can be calculated by integrating the B-spline polynomial obtained from previous analysis from 0 to 1.

\[
a = \int_0^1 l_F \cos[\theta(t)] \, dt = \int_0^1 l_F \cos[B \mathbf{P}(t)] \, dt
\]

\[
b = \int_0^1 l_F \sin[\theta(t)] \, dt = \int_0^1 l_F \sin[B \mathbf{P}(t)] \, dt
\]

### 3.1.3 CONVERGENCE TESTING

The governing equation has the slope of the beam, \(\theta\), as the primary variable. Although the angle \(\beta\) appears to be constant at equilibrium, its value depends on the end co-ordinates of the cantilever beam, which are a function of \(\theta\). Therefore, angle \(\beta\) acquires an indirect dependence on \(\theta\). It becomes computationally exhaustive to solve for the variable \(\theta\) while angle \(\beta\) is explicitly expressed as a function of \(\theta\) in the governing equation. A more straightforward approach would be to introduce an iterative process to determine angle \(\beta\). The steps involved in testing the convergence of \(\beta\) to reach a stable value are described as follows:
1. Set initial guess of $\tan \beta$ as $\tan \beta_I = \frac{a_0}{\sqrt{l_R^2 - a_0^2}}, a_0 = R \left(1 - \cos \left(\frac{lf}{R} \right)\right)$

   Set initial error, $Error = 1$

2. Using $\tan \beta_I$, fit a B-spline curve by following the steps in Section 3.1.2

3. Calculate the end co-ordinates of the free end of the beam

4. From the curve-fit, calculate $\tan \beta = \frac{a}{\sqrt{l_R^2 - a^2}}$

5. Calculate actual error as $Error = \frac{|\tan \beta - \tan \beta_I|}{100}$

6. If $Error > 10^{-8}\%$, set $\tan \beta_I = \tan \beta$ and repeat the procedure from step 2.

7. Calculate the deflection from the equilibrium position for various loads

The above relative error control structure makes it possible to exert a control over the relative convergence of the B-spline approximation curves. A relative convergence error of $10^{-8}\%$ is employed in the curve-fit for the suspension model. The APPENDIX III shows the MATLAB® code used for implementing the error-control and convergence testing of the B-spline curve fits.

### 3.2 SOLUTION OF THE DYNAMIC MODEL

#### 3.2.1 SIMULINK® MODEL

As seen from the dynamic model equation, the time domain equation for sinusoidal excitation of the suspension is given as:

$$M \ddot{x} + c \dot{x} + kx = cA \omega \cos(\omega t) + kA \sin(\omega t)$$
The non-linearity is introduced in the equation of motion due to the dependence of the stiffness $k$ on the displacement of mass block $x$. The above equation of motion is solved by using a SIMULINK® model. The SIMULINK® model is as shown Figure 3.2:

![SIMULINK® model diagram]

**Figure 3.2: SIMULINK® model**

The SIMULINK® model seen here is similar to a second order linear spring-mass-damper model. The model simulates the system response for a base-excitation frequency sweep. The model is run for a finite time for each frequency step. The time duration of each simulation is divided into small time-steps. The response of the displacement of the mass block is recorded in a displacement-vector form for every frequency step. A MATLAB® code calculates the peak displacement from the displacement-vector. The peak displacement is rationalized by the amplitude of the base excitation $A$ which yields the transmissibility ratio. The transmissibility
ratio thus obtained is plotted against the excitation frequency to develop the frequency response function (FRF). The suspension model is run for various test configurations as discussed in the next chapter.

To incorporate the non-linearity, a reference table is employed. The reference table maps the stiffness value for a given displacement $x$. It interpolates to evaluate the stiffness in between consecutive data points. The stiffness vector is calculated from the analytical force-displacement data by using finite differentiation between two consecutive data points. Figure 3.3 shows a sample of the stiffness mapping used in the simulation:

![Stiffness versus displacement map](image)

*Figure 3.3: Sample stiffness versus displacement map used for simulation*
The application of load causes an initial deflection in the suspension. Therefore, in order to account for the suspension’s initial deflection, the displacement needs to be offset by an amount equal to the initial deflection while mapping the stiffness. Another reference table is used to determine the initial deflection which is added to the dynamic displacement $x$ before looking up the corresponding stiffness. The frequency response plot for the suspension prototype obtained from SIMULINK® is discussed in the following chapter.

3.3 CONCLUSION

A continuous fifth order B-spline curve is made to approximate the governing differential equation’s solution. The B-spline collocation method essentially reduces the non-linear governing equation down to a system of non-linear algebraic equations which can be solved relatively easily. The solution to these algebraic equations leads to the development of a fourth degree polynomial which approximates the deflected cantilever beam’s slope. The deflected beam’s free-end coordinates obtained from the solution are used to develop the suspension model’s force-deflection plot. The collocation’s relative convergence testing structure provides a method to keep the error of the curve-fit under control. The analysis is performed by developing a symbolic MATLAB® code. The approximated force-displacement data is used to perform dynamic simulation of the suspension under base excitation to develop the FRF on SIMULINK®. The results obtained from the curve-fit are discussed in the following chapter.
Chapter 4: RESULTS AND DISCUSSIONS

The proposed suspension spring’s mathematical model is developed using the Euler-Bernoulli equation for bending of beams and appropriate boundary conditions are defined. The governing equation is found to be non-linear, and the B-spline collocation method is identified as being suitable for approximating the equation’s solution. Implementing a generic B-spline collocation procedure using a symbolic MATLAB® code leads to a fourth degree polynomial fit of the solution along with a relative-convergence error of $10^{-8}$. The collocation leads to the development of the suspension model’s force-displacement plot. The spring-mass-damper model is used to develop the motion equation for base excitation of the suspension. The motion equation is solved using SIMULINK®, and utilizes the static analysis’ force-displacement data to generate a FRF. In order to validate the non-linear suspension’s proposed theory, a prototype is built in-house and tested to develop an experimental force-displacement plot and FRF. This chapter presents the analysis and experimental results and compares them. The experimental set-ups and the prototype properties are also discussed.

4.1 ANALYTICAL RESULTS

4.1.1 STATIC ANALYSIS

The static analysis implements the B-spline collocation steps described in the previous chapter through a symbolic MATLAB® code. The simulation returns a B-spline polynomial which approximates the cantilever spring slope along its length as the primary output. The polynomial is further exercised to develop the force-deflection relationship plot. This section presents the
analysis results in graph form and discusses the consequential logical deductions. The constant parameters used in the simulation (presented in Table 4.1) match the parameters in the manufactured prototype.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of flexible beams, $l_F$</td>
<td>133 mm</td>
</tr>
<tr>
<td>Cross-section of flexible beam, width × thickness</td>
<td>$35 \text{mm} \times 1.702 \text{mm} \times 2 \text{springs}$</td>
</tr>
<tr>
<td>Young’s Modulus of flexible beam, $E$</td>
<td>$9 \times 10^3 \text{N/mm}^2$</td>
</tr>
<tr>
<td>Length of rigid links, $l_R$</td>
<td>103 mm</td>
</tr>
</tbody>
</table>

*Table 4.1: Constant parameters used in the simulation*

The suspension is simulated for vertical loads ranging from no-load condition to 10$N$ in equally spaced steps of 0.5$N$. The initial result obtained from the B-spline approximation is the slope of the deflected cantilever beam ($\theta$) as a function of its arc-length ($s$) for various load-steps ($P$). The plotted function is shown in Figure 4.1. The plot reassures the concurrence of the approximation with the specified boundary conditions. As seen from the Figure 4.1, the slopes for all the load-steps at $s = 0$ have the boundary value of $\theta = \frac{\pi}{2}$ radians. Moreover, the bending moment at the free-end, that is $\left. \frac{\partial \theta}{\partial s} \right|_{s=l_f=133\text{mm}}$ is approximately zero, signified by the slope of the lines at extreme end. The variable spacing between the consecutive lines for uniform increments in the suspension load also indicates the non-linear nature of the compliant cantilever springs.
As discussed from chapter 2, angle $\beta$ is a critical parameter which determines the nature of the point load at the end of the cantilever beam by governing the components of the applied load acting at the tip. The convergence of angle $\beta$ is tested to limit the residual error in the curve-fit. For any given load-step, the value of $\tan \beta$ is calculated for each iteration and compared with the result from the previous iteration. This process is continued until the relative convergence error of the parameter is less than $10^{-8}\%$. Figure 4.2 shows the stabilization of $\tan \beta$ plotted against the number of iterations:
As seen from the Figure 4.2, $\tan \beta$ requires small number of iterations to stabilize for small loads whereas the number of iterations increases as the load increases. Moreover, as the load increases uniformly, the stabilized value of $\tan \beta$ increases non-uniformly, a trend which is similar to that observed in Figure 4.1. A significantly large increase in the stabilized value of $\tan \beta$ is observed beyond a certain load for every further increase in load, reinforcing the notion of non-linearity present in the system. Figure 4.3 shows the variation of the angle $\beta$ (stabilized values) as a function of the applied load.
Figure 4.3 clearly manifests a transition region between the suspension loads value of 7 and 8N. The rate of increase in the angle $\beta$ with respect to the load is very small for the initial loads and increases rapidly beyond the transition region. The curve-fit obtained from the stabilized value of the angle $\beta$ is used to obtain the deflection of the beam by integrating along the length of the beam. Figure 4.4 shows the beam under deflection at various load-steps:
Figure 4.4 shows the position of the cantilever beam with progressive application of load. The beams’ \( x \) and \( y \) coordinates are obtained from the curve-fitted B-spline polynomial by integrating the cosine and sine functions of the polynomial, respectively, along the length of the beam. As observed from the Figure 4.4, the beam undergoes small deflections when the applied force on the suspension is small. Initially, the deflection does not change significantly with the increasing load, which implies a stiff spring-like behavior. As the suspension load increases, the beam experiences large deflections, and the differences in deflections between two consecutive load steps becomes significantly large as the suspension enters the “soft-spring” region. A force-
deflection relationship plot is generated by using the positional coordinates of the tip of the beam. The force-displacement plot is shown in Figure 4.5:

![Force versus Deflection for suspension](image)

*Figure 4.5: Force-deflection relationship obtained from analysis*

Figure 4.5 shows the variation of opposing force produced in the suspension springs with respect to the displacement. The plot is developed by using the positional coordinates of the tip of the deflected cantilever beam and the angle $\beta$ to compute the suspension’s deflection from its mean position, whereas the force on the y-axis represents the static load on the suspension.
The figure clearly displays three distinct regions of tension – stiff, soft and transition. The spring acts as a “stiff-spring” for the initial part of the displacement. This is the region characterized by small deformation of the cantilevered springs. The opposing force developed in the suspension per unit displacement is high. As the load crosses a threshold value (in this case 8N), the beams are subjected to large deflections and a significantly small gradient (or stiffness) of the suspension is observed. The suspension enters and remains in the soft region of operation for all the loads beyond the threshold value. A small transition region is observed between the loads of 7N to 8N. During this phase, the suspension transits from being “hard” to become “soft” progressively with increasing displacement. The force gradient in this region is not constant, and experiences a significant drop with increasing load and sharp rise with a decrease in load on the suspension.

4.1.2 DYNAMIC ANALYSIS

The dynamic analysis is aimed at testing the suspension for sinusoidal base excitation, and generates FRF’s under various operating conditions. The FRF is developed by using the equation of motion for base excitation developed in the earlier section. The equation is stated as:

\[ M\ddot{x} + c\dot{x} + k(x)x = cA\omega \cos(\omega t) + k(x)A\sin(\omega t) \]

Here, \( M \) is the load on the loading platform, \( c \) the assumed damping constant for the suspension and \( k \) the variable stiffness of the spring. The displacement input is sinusoidal with amplitude \( A \) and frequency \( \omega \), applied at the base of the suspension. As the spring stiffness is a function of the displacement, the dynamic analysis requires the consideration of the spring’s variable nature. In order to incorporate the spring stiffness variability, a model is developed on SIMULINK®
which uses reference tables to determine the instantaneous spring stiffness during the simulation.

The tables use the force-displacement data available from the static analysis to determine the instantaneous spring stiffness during simulation.

The model is simulated for 10 seconds to obtain a base excitation frequency sweep of $0 - 15 \text{ rad/s}$ with a step of 0.01 rad/s and three cases of loading, one in each of the three regions - hard, soft and transition. The assumed value of $c$ is $0.2 \text{ Ns/mm}$, and is assumed to be constant throughout the simulation. The masses and input amplitude displacements used in the simulation for the three cases are tabulated below:

<table>
<thead>
<tr>
<th>Case #</th>
<th>Mode of operation</th>
<th>Mass, $M(\text{kg})$</th>
<th>Displacement amplitude, $A(\text{mm})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Soft region</td>
<td>0.9</td>
<td>10</td>
</tr>
<tr>
<td>II</td>
<td>Hard region</td>
<td>0.6</td>
<td>0.05</td>
</tr>
<tr>
<td>III</td>
<td>Transition region</td>
<td>0.75</td>
<td>1</td>
</tr>
</tbody>
</table>

*Table 4.2: Different load-cases used for the dynamic simulation*

The maximum amplitude of vibration of the mass is recorded and normalized by the input displacement amplitude for every frequency-step to provide the amplitude ratio. The amplitude ratios (also known as transmissibility ratios) are plotted against their respective frequencies to develop the FRF. Figure 4.6 shows the FRF of the suspension model developed from the simulation for all the three cases:
The FRF manifests distinct resonance peaks for each loading case. The resonance frequency is close to 0.25 \( rad/s \) (0.04Hz) when operating in soft-spring conditions, mentioned in Case I of Table 4.2. When operating in the hard-spring region (conditions in Case II), the resonance occurs around 2.25 \( rad/s \) (0.36Hz). The transition zone (Case III) manifests a minor and a major peak. As expected, the resonance peak while operating in the soft-spring region occurs at the lowest excitation frequency, whereas the resonance frequency of the hard-spring region is the highest. The transition region peaks between the soft and hard region. The FRF clearly shows considerable improvement in the suspension’s transmissibility ratio when operating in the soft-spring mode. The suspension’s obvious operation advantage is realized from the FRF since the resonance peak occurs at a much lower frequency as compared to the conventional linear spring suspension operating under similar load. This effect results in better base excitation isolation even at low frequencies. An interesting dynamic is the resonance peak’s shifting toward the y-
axis while operating in the soft-spring mode, which enables it to achieve the $r > \sqrt{2}$ (isolation condition) condition for much lower frequencies.

**4.2 EXPERIMENTAL SET-UP AND RESULTS**

**4.2.1 EXPERIMENTAL SET-UP**

An in-house suspension system prototype is manufactured for the purpose of performing the static and dynamic testing. It also provides experimental data required for analytical results comparison.

The prototype is manufactured using aluminum plate as the base and epoxy-based fiberglass strips as springs. Balsam wood makes up the rigid links. Hinges are steel push-pins, and are lightly lubricated to provide minimal friction. The base plate has bolt-on arrangements for mounting on the testing rigs and mechanical shaker. Figure 4.7 shows the prototype while mounted on the universal tensile testing machine during static testing. The materials are chosen to keep the mechanism’s self-weight as light as possible so that the link’s inertial effects do not significantly contribute to the dynamics. A pair of vertical guides is provided, as seen in Figure 4.7, ensuring that the loading platform remains horizontal to the load application. It is essential for the loading platform to remain horizontal to eliminate pitching and yawing. Consequently, symmetric, near-perfect bending similar to the analytical counterpart, for both springs can be achieved. The critical dimensions of the links are given in Table 4.1.
The static testing is performed on a universal testing machine. The objective is to develop the suspension system’s force versus deflection behavior. The base is attached to the tensile tester’s fixture, and the support is assumed to be rigid. The model’s axis of symmetry is made exactly collinear with the tester’s loading axis to ensure unbiased loading on the springs. The test is performed by applying a constant rate of vertical deflection to the spring and recording the opposing spring force at numerous stages. The tests are conducted with different deflection rates.

As mentioned earlier, the model is tested for static as well as dynamic characteristics. The dynamic test’s objective is to produce a frequency response function (FRF) for base excitation. The FRF is obtained for different dead weights in order to investigate the applied weight’s effect on the dynamic response of the suspension.

Figure 4.7: Suspension model-prototype (l), prototype undergoing load test (r).
To perform the dynamic test, the base is rigidly attached with bolts to the shaker diaphragm. An initial deflection is imposed on the springs by loading a mass-block onto the platform. Accelerometers are mounted at the base and on the mass. The shaker is aligned vertically. The shaker is excited with a sinusoidal input signal with constant amplitude and frequency varying from 10 – 110 Hz. The displacements recorded by the accelerometers are fed to a data acquisition (DAQ) system, which is then transferred into a computer for generating the frequency response charts. The software used for data analysis is OROS® OR763. Figure 4.8 shows a schematic of the dynamic experimental set-up.

![Figure 4.8: Schematic representation of the dynamic test set-up](image)

A0: Base Input accelerometer
A1: Response accelerometer
4.2.2 EXPERIMENTAL RESULTS

4.2.2.1 STATIC TEST

With the universal testing machine’s help, the suspension was tested with different fiberglass strips of the same dimensions at different loading rates which were 0.3 $\text{in/} \text{min}$, 0.6 $\text{in/} \text{min}$ and 1 $\text{in/} \text{min}$ from zero deflection under no load to displacements up to 4 $\text{in}$ from the no load reference point. The resistive force developed by the spring was recorded at small time intervals, and a force versus deflection plot was generated by feeding the data to MATLAB®. Figure 4.9 shows the results obtained from the experiment. In general, the diagram can be divided into three distinct regions.

![Force versus Deflection](image.png)

*Figure 4.9: Experimental force versus displacement characterization from load testing of the prototype*
1. Hard-spring region
2. Transition region
3. Soft-spring region

Hard-spring region:

As seen from the figure, the curve initially has a steep positive slope for the deflection. This implies high resistive forces for the displacement’s initial stage. The spring is perceived as behaving like a hard spring. The cantilever springs undergo small deflections. For the suspension system’s given configuration, the hard spring behavior lasts up to a displacement of around 0.25 in and a maximum resistive spring force of around 1.6 lb is noted. As the amount of displacement increases, the spring moves into a transition zone.

Transition region:

In the transition zone, which lasts from a displacement of 0.25 in to 1.0 in, a rapid slope reduction with increasing load is observed. This implies the transition of the suspension stiffness from hard to soft. This also marks the beginning of large deflections occurring in the cantilevered beams.

Soft region:

As the displacement increases beyond 1 in, due to the cantilever springs’ large deformations, large displacement of the suspension from its equilibrium position is observed without significant relative force increase. The curve’s slope flattens out further, and the resistive force recorded is almost constant (as expected from the CFCM configuration). The suspension springs enter the soft-mode of operation. As a result of the low stiffness, the suspension’s displacement transmissibility for the given load is very small as compared to a linear spring operating under
similar load. The dynamic analysis focuses on the spring operating frequency response in this particular operation mode.

4.2.2.2 DYNAMIC TESTS

With the mechanical shaker’s help, the suspension is tested for displacement transmissibility from base-excitation. The suspension operates in the “transition” and “soft-spring” stiffness regions as identified from the static analysis. The suspension prototype was subjected to a base-excitation sine sweep test.

The input signal’s amplitude is such that the mass-block’s forced vibrations remain in the “soft-spring” region for the entire range of frequencies. Accelerometers measure the vibrations at the base (input) and the mass-block (output). The DAQ system accumulates the data which is fed to the OROS® software. This action plots a FRF in terms of the transmitted noise against the excitation frequency. Figures 4.10 and 4.11 show the experiment’s resulting FRF’s.

![Figure 4.10: Experimental frequency response for base excitation in “soft-spring” region](image)

Figure 4.10: Experimental frequency response for base excitation in “soft-spring” region
Figure 4.10 shows the FRF obtained while operating in the soft-spring region. After the initial spike at a frequency of around 2Hz due to resonance, the noise level drops down below zero and continues reducing throughout the frequency sweep. This indicates a displacement transmissibility of less than unity indicating vibration isolation. The small peak is seen at 60Hz which can be attributed to the electrical noise generated by the surrounding equipment and lights operating at the supply frequency of 60Hz.

![Figure 4.10: Experimental frequency response plot for base excitation in “soft” region](image)

Figure 4.11: Experimental frequency response plot for base excitation in “transition” region

Figure 4.11 shows the FRF achieved while operating the suspension in the “transition” region. The figure clearly shows two distinct resonance peaks before the transmitted noise falls below the 0dB mark. This behavior is expected in the transition region due to variable spring stiffness which leads to multiple resonances (Malatkar, 2003). However, at frequencies higher than 20Hz, a high amount of noise is observed in the output. Further investigation determines that the noise is attributable to the facility’s faulty mechanical shaker equipment which produced an irregular input signal due to frequent coil over-heating. Therefore, the FRF’s latter part can be disregarded.
as the suspension prototype’s actual response. A comparison of the static and dynamic results leads to some interesting conclusions.

4.3 COMPARISONS AND DISCUSSIONS

4.3.1 STATIC COMPARISONS

In order to compare the static analytical solution to the experimental result, the respective force-displacement relationship plots are overlaid. The overlaid plot is shown in Figure 4.12:

Figure 4.12: Analytical and experimental comparison of force versus deflection characteristic

Figure 4.12 shows that as expected, the analytical solution and the experimental result manifest three distinct regions of operation. The “stiff-spring” region ranges from 0 – 1.5 lb load. A
transition region is observed for a range of $1.5 - 1.8lb$ load. The “soft-spring” region extends beyond a load of $1.8lb$. The analytical and experimental results are found to be in close agreement.

4.3.2 DYNAMIC COMPARISON

The dynamic analysis and experiments lead to the generation of respective FRF’s for the suspension operating in the soft-spring and the transition region. The experimental and analytical FRF’s can be compared for characteristic similarities.

Figure 4.6 shows the resonance occurring in the simulation at a very small frequency of excitation while the suspension operates in the soft region. A similar response is observable in the FRF generated from the experiment (Figure 4.10) for similar conditions of operation. As seen from the Figure 4.6 and 4.10, the amplitude ratio descents below $1 (0dB)$ beyond the resonance frequency, and remains there for the sweep’s duration. On the other hand, while operating in the transition region, the FRF’s manifest two distinct resonant peaks and the transmitted noise’s subsequent subsiding. The analytical (Figure 4.6) and experimental (Figure 4.11) display qualitative similarities.
Chapter 5: CONCLUSIONS AND DISCUSSION

As discussed in the introduction, this research had two objectives. First, researching and developing a passive non-linear suspension system which achieves better vibration isolation than its linear counterpart. Second, using the B-spline collocation numerical method, analyze the suspension system and compare the analytical and corresponding experimental results. Both objectives were achieved, giving future researchers of non-linear spring suspension systems a solid foundation upon which to build.

A Constant Force Compliant Mechanism’s (CFCM) Class 1-Ac configuration inspired the non-linear spring suspension design. The literature review prompted this choice based on the configuration’s favorable force-displacement response and economical manufacturing capability. Using the Euler-Bernoulli equation for bending of beams, the selected mechanism was mathematically modeled. Owing to the beams’ large deflections, the modeling led to a non-linear governing differential equation. The governing equation’s boundary conditions were appropriately defined.

To accomplish the second objective, static analysis using the B-spline collocation method as documented by Magoon (Magoon, 2010) was implemented. To approximate the governing equation’s solution, a continuous B-spline curve of fifth-order was used, and the boundary conditions were duly satisfied. An iterative relative-error control structure was deployed, and the relative convergence error of $10^{-8}\%$ was maintained throughout the analysis. Development of a symbolic MATLAB® code aided the B-spline collocation method’s implementation. The static analysis resulted in the development of the suspension system’s force-displacement relationship. The B-spline collocation method proved highly efficient in solving the non-linear solid
mechanics problem. The suspension was simulated as a spring-mass-damper system with base excitation, the spring being non-linear. Simulations were performed on SIMULINK® to generate frequency response functions for various test cases.

A prototype of the proposed suspension was manufactured, and tested statically and dynamically. The static test resulted in a non-linear force versus deflection curve of the suspension. The dynamic test resulted in a frequency response function for base excitation.

The analytical and experimental results were in agreement within experimental limits. The static results identify three distinct regions of spring stiffness for suspension tension: stiff, transition and soft. The suspension’s stiffness depends on its displacement from the mean position. The suspension behaves as a stiff spring for small displacements (small loads), however stiffness reduces significantly beyond a certain point of displacement (or load), and the suspension behaves as a soft spring. Between the stiff and soft behavior, a transition region is observed. Conversely, the dynamic results reveal that the suspension provides very low transmissibility ratios for a wide range of base excitation frequencies due to softening up of the suspension springs. The low transmissibility ratios create a superior degree of vibration isolation from the source.

In summary, this thesis lays a foundation for passive non-linear spring suspension design. Through experiments and analysis, the advantages of deploying non-linear suspension systems were presented. The mathematical modeling approach, step-wise numerical solution, the symbolic MATLAB® code and the reference list all provide future researches useful tools for developing an array of practical applications using non-linear suspensions.
APPENDIX I

B-SPLINE COLLOCATION METHOD

INTRODUCTION

The B-Spline Collocation method is a curve-fitting procedure to approximate the solutions of linear as well as non-linear boundary value problems. This method fits a piecewise B-spline curve to the differential equation to approximate the solution of the differential equation. The one dimensional method required in the scope of study approximates the ordinates of the B-spline curves which approximate the solution by solving a system of equations.

B-SPLINE CURVES

A B-spline curve is a piecewise, continuous parametric curve that can be modeled to approximate a solution to a mathematical problem. The B-spline function essentially consists of position vectors \( B_i \) (constants) and normalized basis functions denoted by \( N_{i,k}(t) \). A third element known as the knot vector is present in the basis function. Mathematically, the B-spline curve \( P(t) \) is defined as,

\[
P(t) = \sum_{i=1}^{n+1} B_i N_{i,k}(t)
\]
Here, the B-spline function fits a $k^{th}$ order polynomial onto a defining polygon of $n + 1$ vertices. The basis function $N_{i,k}(t)$ is a recursive function known as Cox-deBoor recursion. The $i^{th}$ basis function of order $k$ (degree $k - 1$) is written as:

$$N_{i,1}(t) = \begin{cases} 1 & \text{if } x_i \leq t < x_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$N_{i,k}(t) = \frac{(t - x_i)N_{i,k+1}(t)}{x_{i+k-1} - x_i} + \frac{(x_{i+k} - t)N_{i+1,k-1}(t)}{x_{i+k} - x_{i+1}}$$

Here, $x$ is called the knot vector and is essential to the development of the B-spline curves. The resolution of the curve is determined by the knot vectors. A knot vector consists of elements arranged in ascending order and can be classified in the following three categories:

1. **Uniform**: Evenly spaced elements in the knot vector.
2. **Open-uniform**: Equal number of repeating elements at the beginning and end. The number of repetitions being the order of the B-spline curves.
3. **Non-uniform**: Unequal and/or unevenly spaced elements.

As far as the scope of the thesis and computational efficiency is considered open-uniform knot vectors seem to be the ideal choice. Some of the important relations between knot vector and B-spline curves can be shown by the following example:

Consider a knot vector as $x = [0 \ 0 \ 0 \ 1/2 \ 1 \ 1 \ 1]$. The sum of the number of repeating elements ($k = 3$ in this case) and the number of intermediate points ($m = 1$) equals to the number of control point required to fit the B-spline curve whereas the number of repeating points by its own represents the order of the polynomial being fitted.

The order $k$ of the polynomial can be defined by either of the two ways:
1. By specifying the number of vertices, \( n + 1 \) in the polygon and relating the order as \( k = n + 1 \). The degree of a \( k^{th} \) order B-spline function is \( n = k - 1 \).

2. By changing the number of repeating elements in the knot vector.

An important fact to note here is that the number of intermediate points does not affect the order of the B-spline curve, but changes the number of control points required to evaluate the position vectors \( B_t \).

A recursive function is developed in MATLAB® which returns the normalized B-spline basis function in symbolic form. The following figure shows the basis-functions calculated for \( k = 5 \) and \( 0 \leq t \leq 1 \) using \( x = [0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1] \). The function is shown in Appendix II.

![Figure 3.1: Basis functions for the parametric range of \( 0 \leq t \leq 1 \)](image)
Some of the important properties of the B-spline curve are listed below:

1. A B-spline curve of order $k$ is a polynomial of degree $n - 1$ fitted in the interval $x_i \leq t < x_{i+1}$.

2. The sum of the B-spline basis functions for a given parametric value $t$ is 1.

3. Basis function, $\sum_{i=1}^{n+1} N_{i,k}(t)$ at a given value of parameter $t$ is 1.

4. The B-spline curve is continuous up to $k - 2$ derivatives over the entire interval.

5. The B-spline curve lies within the convex-hull of its defining polygon.

The B-spline Collocation method applied to a boundary value problem is discussed in the following steps:

1. Choose a normalized knot vector. The selection of the knot vector is critical because the elements of the knot vector decide the order and hence the degree of the B-spline curve fitted. The frequency of the repeated elements is the order of the curve $k$ and the number of polygon vertices required is $k + m$ ($m$ is the number of intermediate points present in the knot vector).

2. The basis-functions are calculated recursively and $k$ basis-functions are obtained.

3. The abscissa co-ordinates are calculated using the Greville Abscissae equation stated as below:

$$t_i = \frac{1}{n}\{x_i + x_{i+1} + \cdots + x_{i+n-1}\}$$

The benefit of using the Greville Abscissae is that it makes possible a direct substitution of $t$ (calculated from $x$) in the B-spline equation.

4. Calculate the B-spline curve equations and the derivatives. The $n^{th}$ derivative of the B-spline equation ($P^n$) with respect to $t$ is expressed as follows:
\[ P^{n'}(t) = \sum_{i=1}^{n+1} B_i N_i^{n'}(t) \]

5. Use boundary conditions to evaluate end ordinate values.

6. Substitute B-spline equation into the differential equation to be solved. This step reduces the differential equation into a parametric equation of variable \( t \) and unknown interior ordinates \( B_i \).

7. Calculate the remaining internal ordinates using the Greville abscissae.

The following examples describe the application of the B-spline collocation method to solve a boundary value problem.

**EXAMPLE 1**

The simple case of a cantilever beam subjected to a positive (upward) displacement of 5mm and a slope of 1 at the free end. With the origin at the fixed end, length ‘\( l \)’, and the deflection \( u(x) \) (positive in the upward direction), the boundary conditions are described below:

\[
\begin{align*}
  u(0) &= 0 & \text{Deflection at fixed end} \\
  u'(0) &= 0 & \text{Slope at fixed end} \\
  u'(l) &= 1 & \text{Slope at free end} \\
  u(l) &= 5 \times 10^{-3}m & \text{Deflection at free end}
\end{align*}
\]

For length \( l = 2m \), using the Euler-Bernoulli equation, the analytical solution is given as:

\[ u(x) = 0.2488x^3 - 0.4963x^2 \]
B-Spline Solution:

As noted from the analytical solution, the deflection of the beam is cubic in nature. Therefore, a B-Spline curve of fourth order (or third degree) can be very closely fitted to represent the actual solution.

At first, the solution is attempted to be approximated with a continuous B-Spline curve of fourth order and afterward, modified with a fifth order curve with one central point. The results of a continuous B-Spline fit and one with a central point are depicted in Figure 2.

![Figure 2: Deflection of beam with slope and deflection boundary conditions approximated by B-Spline Curves](image)

As noted from Figure 2, the solution to the beam, which is cubic in nature, can be very accurately approximated by the fourth order B-spline curve, as the fourth order produces third degree approximation curves. The fifth order approximation seems to be redundant here, but it may come into picture when the boundary conditions and the loading conditions force the
analytical solution to be of higher degree. Moreover, the central point in the fifth order solution provides a greater control over the approximation at the cost of computation time.

The significance of central points and higher orders can be seen in the next example where we introduce a dis-continuous step loading.

**EXAMPLE 2**

In this example, a B-spline curve is fit onto a cantilever beam which is loaded with a uniform load \( w \left( \frac{n}{m} \right) \) for \( a < x < l \). Let \( b = l - a \) and \( EI \) be the rigidity modulus of the beam.

The boundary conditions are defined as:

\[
EI \frac{d^3 u}{dx^3} \bigg|_{x=l} = 0 \quad \text{Shear force at free end}
\]

\[
EI \frac{d^2 u}{dx^2} \bigg|_{x=l} = 0 \quad \text{Bending moment at free end}
\]

\[
EI \frac{du}{dx} \bigg|_{x=0} = 0 \quad \text{Slope at fixed end}
\]

\[u(0) = 0 \quad \text{Deflection at fixed end}\]

For length \( l \), the analytical solution is given as:

\[
u(x) = \frac{1}{EI} \left\{ \frac{-w \phi(x-a)^4}{24} + \frac{wbx^3}{6} + \frac{wb^2x^2}{4} - \frac{wblx^2}{2} \right\}
\]

Where, \( \phi \) is a unit step function such that,
\[
\phi = \begin{cases} 
0 & 0 < x < a \\
1 & a < x < l 
\end{cases}
\]

B-Spline Solution:

Since the solution contains fourth order term, we can expect a fifth order B-Spline curve to duly satisfy the approximation. However, the presence of discontinuity in the loading cannot be accounted by a continuous curve and leads to singularity. Hence, we introduce a central point to account for the discontinuity.

![Figure 3: B-Spline approximation of discontinuous loading problem](image)

Form Figure 3, the approximate solution converges rapidly towards the analytical solution when an intermediate point it introduced to account for the discontinuity.

It would be worth noticing that despite of the fact that the fifth order approximation with one central point has one more unknown than the continuous approximation, the degree of the approximation curves still remains the same (four). This is the essence of the B-spline curves.
Another point worth noting:

- In the first example, the solution to the fourth order Bernoulli Equation is of the third order (forced to be so by the boundary conditions). Therefore, the B-spline ordinates are calculated simply on the basis of the four boundary conditions when a fourth-order continuous curve is deployed. In other words, the Greville abscissae are redundant.

The accuracy and efficiency of this method is dependent on the order of B-spline curves and number of intermediate control points which are preselected. As observed from the computing times and convergence graphs, the method becomes more accurate for higher order B-spline curves and greater number of intermediate control points but at the same time becoming computationally expensive.

The advantage of the B-spline method over other curve fitting methods is attributed to the fact that the accuracy of the curve fit can be improved without increasing the order (and hence the degree of the polynomial) of the B-spline curves. Increasing the order of the B-spline curve is computationally expensive. This increased accuracy is achieved by introducing intermediate control points between the boundaries. This is possible due to the fact that for a B-spline curve, the number of control points is independent of the order of the B-spline curves. The order only dictates the minimum number of point required. Additional control points can be added by introducing intermediate control points, which does not affect the order and hence the degree of the collocation curve.
APPENDIX II

MATLAB® code for calculation of basis-functions

```matlab
function [N] = basis2(i,j,r)

%SUMMARY: This function produces a vector with symbolic variable 't' which
%represents the basis function used in the B-spline collocation method
%through recursion.

global X t

if j == 1
    if X(i+1)<=min(r)|| X(i)>=max(r)
        N = 0;
    else
        N = 1;
    end
else
    if (X(i+j-1)-X(i)) == 0
        A = 0;
    else
        A = (((t-X(i)).*basis2(i,j-1,r))/(X(i+j-1)-X(i)));
    end
    if (X(i+j)-X(i+1)) == 0
        B = 0;
    else
        B = (((X(i+j)-t).*basis2(i+1,j-1,r))/(X(i+j)-X(i+1)));
    end
    N = vpa(A) + vpa(B);
    return;
end

Published with MATLAB® 7.8
```
APPENDIX III

MATLAB® Code:

Main function:

clear all
close all
clc
Pf = -l*[0:0.5:10];
% Pf = -l*[0:0.01:0.2];
tnB = zeros(length(Pf),1);
for i = 1:length(Pf)
    if i == 1
        x0 = ones(5,1);
    end
    [x(:,i),y(:,i),tnB(i),ll(i),Q] = main10_nonlin61(Pf(i),x0);
x0 = Q;
end

B-spline solver function:

function [aa,bb,tnB,ll,Q] = main10_nonlin61(Pf,x0)
global X t k m interval

syms t N

k = 5;%order(TO BE TAKEN FROM USER)
m = 0;%number of intermediate points (TO BE TAKEN FROM USER);
X = knot(k,m);%X is the normalized open-uniform knot vector of size 2k+m
vertices = k + m;%number of vertices
n = vertices-1;
interval = zeros(1,m+2);%intervals

for i = k:k+m+1
    interval(1,i-k+1) = X(i);
end
i_size = m+1;%number of intervals
NN = zeros(vertices,i_size);
NN = vpa(NN);
for a = 1:i_size
    r = [interval(1,a) interval(1,a+1)];%Defines the range for the particular
    basis function
    N = zeros(n+1,k);
    N = vpa(N);
    for i= 1:k+m
        for j = 1:k
            N(i,j) = basis2(i,j,r);
        end
    end
    NN(:,a) = N(:,k);
end
subs(NN,t,3/4)
subs(diff(diff(NN),t),t,3/4)
NNv = subs(NN,t,0:1/100:1);

%Greville Absicissae
nn = k-1;
g = length(X);
x = zeros(1,g-nn+1);
for i = 1:g-nn+1
  j = i;
  x(1,i) = sum(X(1,j:j+nn-1))/nn;
end

if x(1,1) == x(1,2)
  x = x(1,2:length(x));
end

if x(1,length(x)-1) == x(1,length(x))
  x = x(1,1:length(x)-1);
end

%__________________________________
syms B1 B2 B3 B4 B5 B6 B7 B8 B9 B10 B11 B12 B13 B14
BB = [B1 B2 B3 B4 B5 B6 B7 B8 B9 B10 B11 B12 B13 B14];
BB = BB(1:k+m);
P = BB*NN;
PP = sum(P(:,1));
Lf = 133;
R = Lf*25;
Lr = 103;
alfa = Lf/R;
scale = Lf;
E = 9e3;
B = 35;
thc = 1.702;
I = B*thc^3/12;

L = vpa(zeros(k+m,1));
%1. P(0) = pi/2
L(1,1) = (subs(PP,t,0)-pi/2);%CHANGE HERE
%2. P'(L) = -1/R
L(2,1) = subs(diff(PP,t)/scale,t,1)+1/R;
a0 = R*(1-cos(alfa));%initial guess
b0 = R*sin(alfa);
Errctc = 1.0e-8;
ErrHOP = 1;%Initial
tnBI = a0/sqrt(Lr*Lr-a0*a0);%Initial guess
istep = 0;
while ErrHOP>Errctc
  istep = istep+1;
  if istep == 100
    break
  end
  tnB = tnBI;
  M = diff(diff(PP,t),t)/scale^2 + Pf*(cos(PP)+tnB*sin(PP))/(E*I);
  xx = zeros(1,length(x)-2);
for i = 2:length(x)-1
    xx(i-1) = x(i);
end

for i = 3:m+k
    L(i,1) = subs(M,t,xx(i-2));
end
F = inline(L);
options = optimset('TolFun',1e-20);
fh = @(y) (F(y(1),y(2),y(3),y(4),y(5)));
Q = fsolve(fh,x0,options);

end

hold on
plot(0:Lf*1/100:Lf*1,thtv)
y1 = sqrt(Lr.*Lr-a0.*a0);
htI = b0+y1;
ht = sqrt(Lr*Lr-a*a)+b;
defln = htI-h;
aa = (Lf*1/100*cumtrapz(dx));
bb = (Lf*1/100*cumtrapz(dy));
ll = defln;
end
REFERENCES


