Bispectral reconstruction of speckle-degraded images

Song Jin
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by

Song Jin

A Thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Electrical Engineering

Approved by:

2/2/92

Professor M. Raghuveer, Thesis Advisor

Professor S. Dianat

Dr. D. Newman

Professor R. Unnikrishnan, Department Head

Department of Electrical Engineering
College of Engineering
Rochester Institute of Technology
Rochester, New York
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Abstract

The bispectrum of a signal has useful properties such as being zero for a Gaussian random process, retaining both phase and magnitude information of the Fourier transform of a signal, and being insensitive to linear motion. It has found applications in a wide variety of fields. The use of these properties for reducing speckle in coherent imaging systems was investigated. It was found that the bispectrum could be used to restore speckle-degraded images.

Coherent speckle noise is modeled as a multiplicative noise process. By using a logarithmic transformation, this speckle noise is converted to a signal independent, additive process which is close to Gaussian when an integrating aperture is used. Bispectral reconstruction of speckle-degraded images is performed on such logarithmically transformed images when we have independent multiple snapshots.
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Chapter 1

ABOUT THE RESEARCH

This research, entitled "bispectral reconstruction of multidimensional digital signals", is supported by National Science Foundation under grant MIP-9809701.

Triple correlations and bispectra have been found to be useful in areas such as optical processing, geophysics, astronomy, oceanography, biomedicine, plasma physics, etc [1, 2, 3]. Our research is focused on two dimensional bispectral technique and its applications in speckle reduction.

When an object is illuminated by a coherent source of radiation, and the object has a surface structure that is rough on the order of a wavelength of the incident radiation, a speckle pattern results. A fully developed speckle pattern appears chaotic and disorganized. In the image plane, the presence of speckle is seen as a collection of spots superimposed on the actual object. Thus, where image detail is important, speckle can be considered as noise that degrades an image.

Our concern here is the minimization of speckle effects when we already
have a digitized speckled image. We investigate the situation where we have a moving (translating) object illuminated by a coherent source and we have several images (snapshots or frames) taken at different stages of translation. In this case the maximum likelihood estimate of the image is obtained from ensemble averaging the multiple frames [4]. Perfect registration of each image with the other is required so that the resultant, averaged image is not blurred. This concern often precludes using ensemble averaging due to the difficulty in registering images exactly. However, we will exploit the shift-invariant property of the bispectrum and average in the bispectral domain so that we do not have to align the image. The bispectrum, defined as the Fourier transform of the triple-correlation function, is insensitive to linear phase shift and to certain types of additive noise such as Gaussian noise. We propose using bispectral reconstruction after taking the logarithm to make the multiplicative, signal dependent speckle noise additive and signal independent. With increasing size of the finite aperture, the speckle noise transformed this way is close to Gaussian. As we show, the bispectrum can then be used to reconstruct the speckled image. Experimental verification is provided through computer simulations.

Chapter 2 presents a review of the bispectrum technique, including the definition and properties of the bispectrum, the bispectrum estimation methods and the reconstruction methods.

Chapter 3 presents an overview of the background of the speckle, including the mathematical model and some of the existing speckle reduction method.

Chapter 4 presents the bispectral reconstruction of speckle degraded im-
ages, including the multiplicative noise model, homomorphic transformation, Gaussian approximation, bispectral reconstruction and computer simulation results.

Chapter 5 draws conclusions of the research and discusses the possibilities for further research.
2.1 Definitions

2.1.1 Cumulants and Higher Order Spectra

Higher order spectra are defined in terms of cumulants and therefore are also called cumulant spectra. Given a set of $n$ real random variables $x_1, x_2, \ldots, x_n$, their joint cumulants of order $r = k_1 + k_2 + \ldots + k_n$ are defined as

$$C_{k_1 \ldots k_n} = \{j_{i_1 j_2} \cdot \ldots \cdot j_{i_1 j_2} \}$$

$$= \lim_{n \to \infty} \left( \frac{1}{n!} \right)^r \left( \frac{\partial^{2r} \ln \varphi}{\partial u_1 \partial u_2 \ldots \partial u_n} \right)_{u_1 = u_2 = \ldots = u_n = 0}$$

where

$$\varphi(u_1, u_2, \ldots, u_n) = \exp \left\{ \sum_{i_1, i_2, \ldots, i_n} \lambda_{i_1} x_{i_1} u_{i_1} + \sum_{i_1, i_2} \gamma_{i_1 i_2} x_{i_1} x_{i_2} u_{i_1} u_{i_2} + \sum_{i_1, i_2, i_3} \delta_{i_1 i_2 i_3} x_{i_1} x_{i_2} x_{i_3} u_{i_1} u_{i_2} u_{i_3} + \ldots \right\}$$

and $\lambda_{i_1}, \gamma_{i_1 i_2}, \delta_{i_1 i_2 i_3}, \ldots$ are the parameters of the distribution.
is their joint characteristic function. Let us note that the joint moments of order \( r \) of the same set of random variables are given by

\[
m_{k_1...k_n} = E\{x_1^{k_1}x_2^{k_2}...x_n^{k_n}\} \\
= (-j)^r \frac{\partial^r \Phi(\omega_1, \omega_2, ..., \omega_n)}{\partial \omega_1^{k_1} \partial \omega_2^{k_2} ... \partial \omega_n^{k_n}} \bigg|_{\omega_1=\omega_2=...=\omega_n=0}
\]

Hence, the joint cumulants can be expressed in terms of the joint moments of the random variables. For example, if \( m_{1...0} = E\{x_1\} = 0 \), then

\[
C_{1...0} = 0 \\
C_{2...0} = m_{2...0} = E\{x_1^2\} \\
C_{3...0} = m_{3...0} = E\{x_1^3\} \\
C_{4...0} = m_{4...0} - 3c_{2...0} \\
= E\{x_1^4\} - 3m_{2...0}
\]

By taking \( X(k), k = 0, \pm 1, \pm 2, ... \) to be real stationary random process with zero mean, \( E\{X(k)\} = 0 \), then the moment sequences of the process are related to its cumulants as follows:

Autocorrelation sequence:

\[
E\{X(k)X(k + \tau_1)\} = m_2(\tau_1) \\
= c_2(\tau_1)
\]

Third-order moment or cumulant sequence:

\[
E\{X(k)X(k + \tau_1)X(k + \tau_2)\} = m_2(\tau_1, \tau_2) \\
= c_2(\tau_1, \tau_2)
\]
Fourth-order moment sequence:

\[ E\{X(k)X(k + \tau_1)X(k + \tau_2)X(k + \tau_3)\} = m_2(\tau_1, \tau_2, \tau_3) \]
\[ = c_2(\tau_1, \tau_2, \tau_3) + c_2(\tau_1)c_2(\tau_3 - \tau_2) \]
\[ = c_2(\tau_2)c_2(\tau_3 - \tau_1) + c_2(\tau_3)c_2(\tau_2 - \tau_1) \]

etc.

While the third-order moments and third-order cumulants are identical, this is not true for the fourth-order statistics. In order to generate the fourth-order cumulant sequence, we need to know the fourth-order moment and autocorrelation sequences.

The \( N \)th order spectrum \( C(\omega_1, \omega_2, ..., \omega_{N-1}) \) of the process \( \{X(k)\} \) is defined as the Fourier transform of its \( N \)th-order cumulant sequence \( C_N(\tau_1, \tau_2, ..., \tau_{N-1}), \) i.e.,

\[ C(\omega_1, \omega_2, ..., \omega_{N-1}) = \sum_{\tau_1 = -\infty}^{+\infty} ... \sum_{\tau_{N-1} = -\infty}^{+\infty} C_N(\tau_1, \tau_2, ..., \tau_{N-1}) \exp j(\omega_1\tau_1 + ... + \omega_n\tau_n) \]  \( (2.4) \)

In general, \( C_N(\omega_1, \omega_2, ..., \omega_{N-1}) \) is complex and a sufficient condition for its existence in that moment \( C_N(\tau_1, \tau_2, ..., \tau_{N-1}) \) is absolutely summable. The notion of considering a spectral representation for a cumulant sequence as shown in Eq. (2.4) (cumulant spectrum) is acknowledged to be due to Kolmogorov [5]. It should be noted that the term polyspectrum is due to Tukey [6] whereas the term higher order spectrum is due to Brillinger and Rosenblatt [7, 8].
The power spectrum, bispectrum, and trispectrum are special cases of the \( N \)th-order spectrum defined by Eq. (2.4), i.e.,

**Power Spectrum:** \( N = 2 \) (Wiener-Kinchine.):

\[
C(\omega_1) = \sum_{\tau_1 = -\infty}^{+\infty} c_2(\tau_1) \exp\{-j(\omega_1 \tau_1)\} \quad (2.5)
\]

**Bispectrum:** \( N = 3 \)

\[
C(\omega_1, \omega_2) = \sum_{\tau_1 = -\infty}^{+\infty} c_2(\tau_1) \sum_{\tau_2 = -\infty}^{+\infty} c_3(\tau_1, \tau_2) \exp\{-j(\omega_1 \tau_1 + \omega_1 \tau_2)\} \quad (2.6)
\]

**Trispectrum:** \( N = 4 \)

\[
C(\omega_1, \omega_2, \omega_3) = \sum_{\tau_1 = -\infty}^{+\infty} \sum_{\tau_2 = -\infty}^{+\infty} \sum_{\tau_3 = -\infty}^{+\infty} c_4(\tau_1, \tau_2, \tau_3) \exp\{-j(\omega_1 \tau_1 + \omega_2 \tau_2 + \omega_3 \tau_3)\} \quad (2.7)
\]

At this point, a natural question that arises is why the \( N \)th-order spectrum (or polyspectrum) is defined as the Fourier transform of the cumulant rather than of the moment sequence of \( \{X(k)\} \). The reason is twofold: a) if \( \{X(k)\} \) is a stationary Gaussian random process, then all its \( N \)th-order moments for \( N \geq 3 \) do not provide any additional information pertaining to the process. It is, therefore, better to have a function does so since higher order \( (N \geq 3) \) cumulants are zero for Gaussian process; b) if the random variables \( \{x_1, ..., x_n\} \) can be divided into any two or more groups which are statistically independent, their \( N \)th-order cumulants are identically zero [5]. Hence, cumulant spectra provide a suitable measure of statistical dependence. Finally, Brillinger [5] points out that ergodicity requirements are met more easily with cumulants than moments.
2.1.2 Definition of the Bispectrum

Let \( \{X(k)\} \) be a real, discrete, zero-mean stationary process with power spectrum \( P(\omega) \), defined as

\[
P(\omega) = \sum_{r=-\infty}^{+\infty} r(\tau) \exp[-j(\omega \tau)], \quad |\omega| < \pi
\]  

(2.8)

where

\[
r(\tau) = E\{X(k)X(k + \tau)\}
\]  

(2.9)

is its autocorrelation sequence. If \( R(m, n) \) denotes the third moment sequence of \( \{X(k)\} \), i.e.,

\[
R(m, n) = E\{X(k)X(k + m)X(k + n)\}
\]  

(2.10)

then its bispectrum is defined as

\[
B(\omega_1, \omega_2) = \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} R(m, n) \exp\{-j(\omega_1 m + \omega_2 n)\}
\]  

(2.11)

Since the third-order moments and cumulants are identical, the bispectrum is a third-order cumulant spectrum.

For a real, deterministic, discrete-time sequence \( \{X(n)\} \), we can define its bispectrum \( B(\omega_1, \omega_2) \) as

\[
B_x(\omega_1, \omega_2) = X(\omega_1)X(\omega_2)X^*(\omega_1 + \omega_2)
\]  

(2.12)

where \( X(\omega) \) is the Fourier transform of \( x(n) \):

\[
X(\omega) = \sum_{n=-\infty}^{\infty} X(n) \exp(-j\omega n) = |X(\omega)| \exp[j\phi(\omega)]
\]  

(2.13)
Actually it can be easily proved that the above two definitions of the bispectrum are equivalent:

From Eq. (2.10), we have

$$R(m, n) = E\{X(k)X(k + m)X(k + n)\}$$

$$= \sum_{k=-\infty}^{+\infty} [X(k)X(k + m)X(k + n)]$$

substitute this into Eq. (2.11), we have

$$B(\omega_1, \omega_2)$$

$$= \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} R(m, n) \exp[-j(\omega_1 m + \omega_2 n)]$$

$$= \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \left\{ \sum_{k=-\infty}^{+\infty} [X(k)X(k + m)X(k + n)] \right\} \exp[-j(\omega_1 m + \omega_2 n)]$$

$$= \left\{ \sum_{k=-\infty}^{+\infty} X(k) \exp[-j(\omega_1 + \omega_2)k] \right\}$$

$$\left\{ \sum_{m=-\infty}^{+\infty} X(k + m) \exp[-j(k + m)\omega_1] \right\}$$

$$\left\{ \sum_{n=-\infty}^{+\infty} X(k + n) \exp[-j(k + n)\omega_2] \right\}$$

$$= X(\omega_1)X(\omega_2)X^*(\omega_1 + \omega_2)$$

Directly from Eq. (2.12) we have

$$|B_X(\omega_1, \omega_2)| = |X(\omega_1)||X(\omega_2)||X(\omega_1 + \omega_2)|$$

$$\psi_X(\omega_1, \omega_2) = \phi(\omega_1) + \phi(\omega_2) - \phi(\omega_1 + \omega_2)$$

where $\psi_X(\omega_1, \omega_2)$ is the bispectrum phase.
Important symmetry conditions follow from the above definitions. From Eq. (2.8) and Eq. (2.9) we have

\[ r(\tau) = r(-\tau) \quad (2.16) \]

\[ P(\omega) = P(-\omega) \quad (2.17) \]

\[ P(\omega) > 0 \quad (\text{real, nonnegative function}). \quad (2.18) \]

From Eq. (2.10) it follows that the third moments obey the symmetry properties

\[ R(m, n) = R(n, m) \quad (2.19) \]

\[ = R(-n, m - n) \quad (2.20) \]

\[ = R(n - m, -m) \quad (2.21) \]

\[ = R(m - n, -n) \quad (2.22) \]

\[ = R(-m, n - m) \quad (2.23) \]

As a consequence, knowing the third moments in any one of the six sectors would enable us to find the entire third moment sequence. These sectors include their boundaries so that, for example, one of the sectors is an infinite wedge bounded by the lines \( m = 0 \), and \( m = n; m, n \geq 0 \).

### 2.2 Properties of the Bispectrum

#### 2.2.1 Basic Properties

From the definition of the bispectrum in Eq. (2.11) and the properties of the third moments in Eq. (2.23), it follows that
(a) $B(\omega_1, \omega_2)$ is generally complex, i.e., it has magnitude and phase

$$B(\omega_1, \omega_2) = |B(\omega_1, \omega_2)| \exp[j \phi_B(\omega_1, \omega_2)] \quad (2.24)$$

(b) $B(\omega_1, \omega_2)$ is doubly periodic with period $2\pi$

$$B(\omega_1, \omega_2) = B(\omega_1 + 2\pi, \omega_2 + 2\pi) \quad (2.25)$$

(c)

$$B(\omega_1, \omega_2) = B(\omega_2, \omega_1) = B^*(-\omega_2, -\omega_1)$$
$$= B^*(-\omega_1, -\omega_2) = B(-\omega_1 - \omega_2, \omega_2)$$
$$= B(\omega_1, -\omega_1 - \omega_2) = B(-\omega_1 - \omega_2, \omega_1)$$
$$= B(\omega_2, -\omega_1 - \omega_2) = B(-\omega_1 - \omega_2, \omega_1)$$

2.2.2 Additional Properties

Additional Properties of the bispectrum make it very attractive in practical applications. We outline them briefly below, more details can be found in [1].

1. Gaussian Processes: For a stationary zero-mean Gaussian process, its third-moment is zero and therefore its bispectrum is identically zero.

2. Linear Phase Shift: Given $\{X(k)\}$ with power spectrum $P_x(\omega)$ and bispectrum $B_x(\omega_1, \omega_2)$, the process $Y(k) = X(k - N)$, where $N$ is a constant integer, has power spectrum $P_y(\omega) = P_x(\omega)$ and bispectrum
\[ B_v(\omega_1, \omega_2) = B_2(\omega_1, \omega_2), \] i.e., the second and third order moments suppress linear phase information. However, the power spectrum (autocorrelation) suppresses all phase information, but the bispectrum (third-moment sequence) does not.

3. **Non-Gaussian White Noise**: If \( \{W(k)\} \) is a stationary non-Gaussian process with

\[
E\{W(k)\} = 0
\]

\[
E\{W(k)W(k + \tau)\} = Q.\delta(\tau)
\]

and

\[
E\{W(k)W(k + \tau)W(k + \rho)\} = \beta\delta(\tau, \rho)
\]

its power spectrum and bispectrum are both flat, i.e., \( P(\omega) = Q \) and \( B(\omega_1, \omega_2) = \beta \).

4. **Quadratic Phase Coupling**: In some practical situations, two harmonic components of a process interact with each other and thus there is contribution to the power at their sum and/or difference frequencies. Such phenomenon gives rise to certain phase relations called quadratic phase coupling. The power spectrum suppresses all phase relations thus it cannot provide the answer. The bispectrum is capable of detecting and quantifying phase coupling.

5. **Non-Gaussian Process into a Linear Filter**
Let \( \{X(k)\} \) be a zero-mean stationary non-Gaussian process with power spectrum \( P_X(\omega) \) and bispectrum \( B_X(\omega_1, \omega_2) \). Put

\[
Y(k) = \sum_{i=-\infty}^{+\infty} h(k - i)X(i)
\]

where

\[
h(k) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} H(\omega) \exp(j\omega k)d\omega.
\]

(2.26)

(2.27)

Then the power spectrum of \( \{Y(k)\} \) is

\[
P_Y(\omega) = |H(\omega)|^2 P_X(\omega)
\]

and the bispectrum is

\[
B_Y(\omega_1, \omega_2) = H(\omega_1)H(\omega_2)H^*(\omega_1 + \omega_2)B_X(\omega_1, \omega_2)
\]

If

\[
H(\omega) = |H(\omega)| \exp\{j\phi(\omega)\}
\]

then

\[
B_Y(\omega_1, \omega_2) = |H(\omega_1)||H(\omega_2)||H(\omega_1 + \omega_2)||B_X(\omega_1, \omega_2)|
\]

and

\[
\psi(\omega_1, \omega_2) = \phi(\omega_1) + \phi(\omega_2) - \phi(\omega_1 + \omega_2) + \psi_X(\omega_1, \omega_2)
\]

(2.28)

For the special case where \( \{X(k)\} \) is non-Gaussian white, then,

\[
|B_X(\omega_1, \omega_2)| = \beta \quad \psi(\omega_1, \omega_2) = 0
\]

(2.29)

If \( \{X(k)\} \) is stationary Gaussian,

\[
B_X(\omega_1, \omega_2) = 0
\]
and hence

\[ B_Y(\omega_1, \omega_2) = 0 \]

6. Minimum-, Maximum-, and Mixed-Phase Linear Filters: Unlike the autocorrelation or power spectrum, the bispectrum or third-moment sequence retains phase information (except for the linear phase component), therefore, it is able to identify non-minimum (mixed) phase systems or sequences.

7. Gaussian Process into a Nonlinear Device: Passing a zero-mean, stationary Gaussian sequence to a system, if the system is linear, the bispectrum of the output is identically zero. However, if the system is a nonlinear system, (for example Nth-order Volterra processor), a nonzero bispectrum of the output will exit. The nonlinearity can be identified by this mean [5].

8. Poisson Triggered Process: Consider a process of the form

\[ X(k) = \sum_m h(k - T_m) \tag{2.30} \]

where ..., \( T_{-1}, T_0, T_1, T_2, \) ... are the times of events of a Poisson process, with \( E\{T_{m+1} - T_m\} = \mu. \) Assuming that \( H(k) \) is related to \( H(\omega) \) as shown in Eq. (2.27), the power spectrum and bispectrum of the process are

\[ P_X(\omega) = \frac{1}{\mu} |H(\omega)|^2 \]

\[ B_X(\omega_1, \omega_2) = \frac{1}{\mu} H(\omega_2)H(\omega_2)H^*(\omega_1 + \omega_2) \]
Let us note that

\[ P_X(\omega) = B_X(\omega, 0)/H(0) = B_X(0, \omega)/H(0) \]

When the process \( \{X(k)\} \) of Eq. (2.30) acts as an additive noise to a Gaussian random process \( \{Y(k)\}, \{X(k)\} \) are independent, then the power spectrum and bispectrum are

\[ P_Z(\omega) = P_Y(\omega) + P_X(\omega) \]

\[ B_Z(\omega_1, \omega_2) = B_X(\omega_1, \omega_2) \]

### 2.3 Bispectrum Estimators

The problem met in practice is that given a finite set of observations, how to estimate the bispectrum of the process. Basically there are two approaches to estimate the bispectrum. One is conventional ("Fourier type") approach, the other is parametric approach, which is based on autoregressive (AR), moving average (MA), and ARMA models. A good summary of parametric approach can be found in [1].

The conventional methods may be classified into two classes, namely, indirect class and direct class. The advantages of the conventional bispectrum estimator are that the approximations are straightforward, and that the fast Fourier transform (FFT) can be used. However, there are limitations on statistical variance of the estimate. Even though FFT is used, large amount of computation time and memory are still needed.
2.3.1 Indirect Class of Conventional Bispectrum Estimator

Let \( \{X(1), X(2), ..., X(N)\} \) be the given data set. We have the following procedures for estimating the bispectrum.

1. Segment the data into \( K \) records of \( M \) samples each, i.e., \( N = KM \).

2. Subtract the average value of each record.

3. Assuming that \( \{X_i(k), k = 0, 1, ..., M - 1\} \) is the data set per segment \( i = 1, 2, ..., K \), obtain an estimate of the third-moment sequence

\[
r_i(m, n) = \frac{1}{M} \sum_{l=s_1}^{s_2} X_i(l) X_i(l + m) X_i(l + n) \tag{2.31}
\]

where

\[
i = 1, 2, ..., K
\]
\[
s_1 = \max(0, -m, -n)
\]
\[
s_2 = \min(M - 1, M - 1 - m, M - 1 - n).
\]

4. Average \( r_i(m, n) \) over all segments

\[
\hat{R}(m, n) = \frac{1}{K} \sum_{i=1}^{K} r_i(m, n)
\]

5. Generate the bispectrum estimate

\[
\hat{B}_{IN}(\omega_1, \omega_2) = \sum_{m=-L}^{L} \sum_{n=-L}^{L} \hat{R}(m, n) W(m, n) \exp\{-j(\omega_1 m + \omega_2 n)\} \tag{2.32}
\]

where \( L < M - 1 \) and \( W(m, n) \) is a two-dimensional window function.

The window function should satisfy the following constrains:
a) $W(m, n) = W(n, m) = W(-m, n - m) = W(m - n, -n)$;

b) $W(m, n) = 0$ outside the region of support of $\hat{R}(m, n)$;

c) $W(0, 0) = 1$;

d) $W(\omega_1, \omega_2) > 0$ for all $(\omega_1, \omega_2)$.

A class of functions which satisfies the above constraints for $W(m, n)$ is the following:

$$W(m, n) = d(m)d(n)d(n - m)$$  \hspace{1cm} (2.33)

where

$$d(m) = d(-m)$$
$$d(m) = 0, \quad m > L$$
$$d(0) = 1$$
$$D(\omega) \geq 0, \quad \text{for all } \omega.$$ 

The following are three windows that satisfy the above constraints:

a) Optimum window (minimum bispectrum bias):

$$d_o(m) = \begin{cases} \frac{1}{\pi} \sin \frac{\pi m}{L} + \left(1 - \frac{|m|}{L}\right) \cos \frac{\pi m}{L}, & |m| \leq L \\ 0, & |m| > L \end{cases}$$

b) Parzen window:

$$d_p(m) = \begin{cases} 1 - 6 \left(\frac{|m|}{L}\right)^2 + 6 \left(\frac{|m|}{L}\right)^3, & |m| \leq L/2 \\ 2 \left(1 - \frac{|m|}{L}\right)^3, & L/2 \leq |m| \leq L \\ 0, & |m| > L \end{cases}$$
Uniform window in frequency domain:

\[
W_U(\omega_1, \omega_2) = \begin{cases} 
\frac{3}{4} \left( \frac{\pi}{\Omega_0} \right), & |\omega| < \Omega_0 = a_0/L \\
0, & |\omega| > \Omega_0 = a_0/L 
\end{cases}
\]

where \( |\omega| = \max[|\omega_1|, |\omega_2|, |\omega_1 + \omega_2|] \) and \( a_0 \) a constant parameter.

### 2.3.2 Direct Class of Conventional Bispectrum Estimators

From the definition of the bispectrum in Eq. (2.12) we have the following bispectrum estimation method.

Given \( K \) independent records \( X_i(n); i = 1, 2, \cdots, K \) of \( \{X(n)\} \) (or one can follow the segmentation procedure described above to segment the data into \( K \) records), the bispectrum can be estimated in the following procedure:

1. Subtract the mean value of each record.

2. Calculate the Fourier transform of each record:

\[
X_i(\omega) = \sum_{n=-\infty}^{\infty} X_i(n) \exp(-j\omega n) \quad (2.34)
\]

Of course, given the sampling rate, FFT can be used.

3. The bispectrum \( B_X(\omega_1, \omega_2) \) is estimated as

\[
\hat{B}_X(\omega_1, \omega_2) = \frac{1}{K} \sum_{i=1}^{K} X_i(\omega_1)X_i(\omega_2)X_i^*(\omega_1 + \omega_2) \quad (2.35)
\]
2.4 Bispectrum Reconstruction

It is evident from the properties of bispectrum that both the magnitude and phase information is preserved in the bispectrum. This allows one to recover the data sequence from its bispectrum uniquely, except for the linear phase factor (lost due the shift-invariance property). In the past, a good amount of work has been done in this regard and some techniques have been developed to achieve this inverse transformation from the bispectral domain to the data domain [3, 2, 9, 10, 11, 12]. One of the techniques is the least square approach. However, the amount of computation of least squares approaches is vary larger and thus it is difficult to use them in practical applications, especially for multidimensional cases. Therefore in our application in laser speckle reduction, a two dimensional recursive approach for magnitude reconstruction suggested by Raghuveer and Dianat [13] is used. The author and Raghuveer have developed a two dimensional recursive method for phase reconstruction. Both methods are described in the following. Noted that the one dimensional version is simply a special case of the two dimensional one, and thus can be easily derived from the two dimensional approaches.

2.4.1 Phase Reconstruction

Given a real two dimensional discrete sequence $x(n_1, n_2)$, its bispectrum is defined as

$$B_x(\omega_1, \omega_2; \lambda_1, \lambda_2) = X(\omega_1, \omega_2)X(\lambda_1, \lambda_2)X^*(\omega_1 + \lambda_1; \omega_2 + \lambda_2)$$ (2.36)
where $X(\omega_1, \omega_2)$ is the two dimensional Fourier transform of $x(n_1, n_2)$:

$$X(\omega_1, \omega_2) = \sum_{n_1 = -\infty}^\infty \sum_{n_2 = -\infty}^\infty x(n_1, n_2) \exp(-j\omega_1 n_1) \exp(-j\omega_2 n_2)$$

$$= |X(\omega_1, \omega_2)| \exp[j\phi(\omega_1, \omega_2)]$$

The Fourier phase $\phi(\omega_1, \omega_2)$ and the phase of the bispectrum have the following relationship:

$$\psi(\omega_1, \omega_2; \lambda_1, \lambda_2) = \phi(\omega_1, \omega_2) + \phi(\lambda_1, \lambda_2) - \phi(\omega_1 + \lambda_1; \omega_2 + \lambda_2) \quad (2.37)$$

Our intention here is to compute the Fourier phase $\phi$ from the phase of the bispectrum $\psi$. Digitizing the frequency range $(0, 2\pi)$ into $N$ equally spaced values, we get the discrete form of Eq. (2.37) as

$$\psi(k_1, k_2; l_1, l_2) = \phi(k_1, k_2) + \phi(l_1, l_2) - \phi(k_1 + l_1, k_2 + l_2) \quad (2.38)$$

where $k_1, k_2, l_1, l_2 = 0, 1, 2, ..., N - 1$.

Let $k_1 = 0, \ k_2 = 1, \ l_1 = 0, \ l_2 = l$ in Eq. (2.38), we obtain

$$\psi(0, 1; 0, l) = \phi(0, 1) + \phi(0, l) - \phi(0, l + 1) \quad (2.39)$$

Summing Eq. (2.39) from $l = 1$ to $l = N/2 - 1$ yields

$$\sum_{l=1}^{N/2-1} \psi(0, 1; 0, l) = \frac{N}{2} \phi(0, 1) - \phi(0, N/2) \quad (2.40)$$

hence

$$\phi(0, 1) = \frac{2}{N} \left[ \sum_{l=1}^{N/2-1} \psi(0, 1; 0, l) + \phi(0, N/2) \right] \quad (2.41)$$

Let $k_1 = 0, \ k_2 = N/2, \ l_1 = N/2, \ l_2 = N/2$ in Eq. (2.38); we obtain

$$\psi(0, N/2; N/2, N/2) = \phi(0, N/2) + \phi(N/2, N/2) - \phi(N/2, 0) \quad (2.42)$$
Now letting \( k_1 = 0, \ k_2 = N/2, \ l_1 = N/2, \ l_2 = 0 \) in Eq. (2.38), we obtain

\[
\psi(0, N/2; N/2, 0) = \phi(0, N/2) + \phi(N/2, 0) - \phi(N/2, N/2)
\]

Summing the above two equations, we have

\[
\phi(0, N/2) = \frac{1}{2} [\psi(0, N/2; N/2, N/2) + \psi(0, N/2; N/2, 0)]
\]

\( \phi(0, N/2) \) can be computed from Eq. (2.44) and consequently \( \phi(0, 1) \) can be obtained by Eq. (2.41). All values of \( \phi(0, l), \ l = 2, ..., N/2 - 1 \) can be computed recursively with the following equation derived from Eq. (2.39)

\[
\phi(0, l + 1) = \phi(0, 1) + \phi(0, l) - \psi(0, 1; 0, l)
\]

We have obtained the values of \( \phi(0, l), \ l = 1, 2, ..., N/2 \). For a real sequence, since the Fourier transform phase has the following symmetry:

\[
\phi \left( \frac{N}{2} + i, \frac{N}{2} + j \right) = -\phi \left( \frac{N}{2} - i, \frac{N}{2} - j \right)
\]

where \( i, j = 1, 2, ..., N/2 - 1 \).

we can compute \( \phi(0, l) \) for \( l = N/2 + 1, \ldots, N - 1 \) as

\[
\phi \left( 0, \frac{N}{2} + l \right) = -\phi \left( 0, \frac{N}{2} - l \right)
\]

where \( l = 1, 2, ..., N/2 - 1 \). By using Eq. (2.47), all values of \( \phi(0, l), l = 1, 2, ..., N - 1 \) can be obtained.

By following similar steps, we can also compute \( \phi(k, 0), \ k = 1, ..., N - 1 \) from the following equations

\[
\phi(N/2, 0) = \frac{1}{2} [\psi(N/2, 0; N/2, N/2) + \psi(N/2, 0; 0, N/2)]
\]
\[
\phi(1, 0) = \frac{2}{N} \left[ \sum_{l=1}^{N/2-1} \psi(1, 0; k, 0) + \phi(N/2, 0) \right] \quad (2.49)
\]

\[
\phi(k + 1, 0) = \phi(1, 0) + \phi(l, 0) - \psi(1, 0; k, 0) \quad (2.50)
\]

From the values of \( \phi(0, l) \), \( l = 1, \ldots, N - 1 \) and \( \phi(k, 0) \), \( k = 1, \ldots, N - 1 \), we can compute other values of \( \phi(m, n) \) recursively.

Let \( k_1 = 0 \), \( k_2 = 1 \), \( l_1 = m \), \( l_2 = n \) in Eq. (2.38). Then,

\[
\psi(0, 1; m, n) = \phi(0, 1) + \phi(m, n) - \phi(m, n + 1) \quad (2.51)
\]

hence

\[
\phi(m, n + 1) = \phi(0, 1) + \phi(m, n) - \psi(0, 1; m, n) \quad (2.52)
\]

Let \( m = 1, 2, \ldots, N/2 - 1 \), and for each \( m \), let \( n = 0, 1, \ldots, N - 2 \) in Eq. (2.52), the values of \( \phi(m, n); m = 1, \ldots, N/2 - 1; n = 1, \ldots, N - 1 \) can be computed recursively.

Now use Eq. (2.46) again by letting \( i = 0, 1, \ldots, N/2 - 1 \), and for each \( i \), \( j = -(N/2 - 1), \ldots, (N/2 - 1) \), the values of \( \phi(m, n); m = N/2 - 1, \ldots, N - 1; n = 0, 1, \ldots, N - 1 \) can also be obtained.

Notice that the only values that have not been computed above are \( \phi(0, 0) \) and \( \phi(N/2, N/2) \), and they can be computed as follows.

Letting \( k_1 = k_2 = l_1 = l_2 = 0 \) in Eq. (2.38) yields

\[
\phi(0, 0) = \psi(0, 0; 0, 0) \quad (2.53)
\]

Let \( k_1 = k_2 = l_1 = l_2 = N/2 \) in Eq. (2.38); it can be easily shown that

\[
\phi(N/2, N/2) = \frac{1}{2} [\psi(N/2, N/2; N/2, N/2) + \psi(0, 0; 0, 0)] \quad (2.54)
\]
Now all values of the Fourier phase \( \phi(i,j); i = 0, ..., n - 1; j = 0, ..., N - 1 \) have been reconstructed from the bispectral phase \( \psi(k_1,l_1;k_2,l_2) \).

### 2.4.2 Magnitude Reconstruction

From Eq. (2.36) we have the magnitude relation:

\[
|B_x(\omega_1,\omega_2;\lambda_1,\lambda_2)| = |X(\omega_1,\omega_2)||X(\lambda_1,\lambda_2)||X(\omega_1 + \lambda_1;\omega_2 + \lambda_2)| \quad (2.55)
\]

Let \( \omega_1 = \lambda_1, \omega_2 = \lambda_2 \) we obtain:

\[
|B_x(\omega_1,\omega_2;\omega_1,\omega_2)| = |X(\omega_1,\omega_2)|^2|X(2\omega_1 2\omega_2)| \quad (2.56)
\]

Taking the natural logarithm of both side, we have

\[
2\tilde{X}(\omega_1,\omega_2) + \tilde{X}(2\omega_1,2\omega_2) = \tilde{B}(\omega_1,\omega_2) \quad (2.57)
\]

where

\[
\tilde{X}(\omega_1,\omega_2) = \ln |X(\omega_1,\omega_2)|
\]

\[
\tilde{B}(\omega_1,\omega_2) = \ln |B_x(\omega_1,\omega_2;\omega_1,\omega_2)|
\]

The discrete form of Eq. (2.57) is

\[
2\tilde{X}(n_1,n_2) + \tilde{X}(2n_1,2n_2) = \tilde{B}(n_1,n_2) \quad (2.58)
\]

Denoting the \( N \times N \) point DFTs of \( \tilde{X}(n_1,n_2) \) and \( \tilde{B}(n_1,n_2) \) by \( \mathcal{X}(n_1,n_2) \) and \( \mathcal{B}(n_1,n_2) \), respectively, we have from Eq. (2.58)

\[
2 \sum_{k_1=0}^{N-1} \sum_{k_2=0}^{N-1} \mathcal{X}(k_1,k_2) \exp \left[ j \frac{2\pi}{N} (n_1 k_1 + n_2 k_2) \right] \\
+ \sum_{k_1=0}^{N-1} \sum_{k_2=0}^{N-1} \mathcal{X}(k_1,k_2) \exp \left[ j \frac{4\pi}{N} (n_1 k_1 + n_2 k_2) \right] \\
= \sum_{k_1=0}^{N-1} \sum_{k_2=0}^{N-1} \mathcal{B}(k_1,k_2) \exp \left[ j \frac{2\pi}{N} (n_1 k_1 + n_2 k_2) \right] \quad (2.59)
\]
We now develop an approach to recover $\mathcal{X}(k_1, k_2)$ from $B(k_1, k_2)$, $k_1, k_2 = 0, 1, ..., N - 1$, where $N$ is a pure power of 2. The technique is based on equating like coefficients on both side of Eq. (2.59).

1. If either $k_1$ or $k_2$ is odd, then

$$\mathcal{X}(k_1, k_2) = \frac{1}{2} B(k_1, k_2)$$

2. If either $k_1/2$ or $k_2/2$ is odd, then

$$\mathcal{X}(k_1, k_2) = \frac{1}{2} B(k_1, k_2) - \frac{1}{2} \mathcal{X} \left( \frac{k_1}{2}, \frac{k_2}{2} \right)$$

$$- \frac{1}{2} \mathcal{X} \left( \frac{N}{2} + \frac{k_1}{2}, \frac{k_2}{2} \right)$$

$$- \frac{1}{2} \mathcal{X} \left( \frac{N}{2} + \frac{k_1}{2}, \frac{N}{2} + \frac{k_2}{2} \right)$$

$$- \frac{1}{2} \mathcal{X} \left( \frac{N}{2} + \frac{k_1}{2}, \frac{k_2}{2} \right)$$

3. Repeat the previous step successively for cases in which either $k_1/2^l$ or $k_2/2^l$ is odd for $l = 2, 3, ..., \log_2(N) - 1$.

4. Finally

$$\mathcal{X}(0, 0) = \frac{1}{3} \left[ B(0, 0) - \mathcal{X} \left( 0, \frac{N}{2} \right) - \mathcal{X} \left( \frac{N}{2}, 0 \right) - \mathcal{X} \left( \frac{N}{2}, \frac{N}{2} \right) \right]$$

5. Reconstruct the magnitude as

$$|X(k_1, k_2)| = \exp \{ IDFT [\mathcal{X}(k_1, k_2)] \}$$
2.5 Computer Simulations

In this section, several examples are given through computer simulations to show some of the important properties of the bispectrum. The fast recursive approaches described in section 2.4 and their one dimensional versions are used in the reconstruction from the bispectrum.

2.5.1 Example 1: Bispectrum and Basic Properties

Given a signal shown in Fig. 2.1, the magnitude and phase of its Fourier transform are shown in Fig. 2.2 and Fig. 2.3. The three dimensional plots of the magnitude and phase of its bispectrum are shown in Fig. 2.4 and Fig. 2.5. The two dimensional contour plots of the magnitude and phase of its bispectrum are shown in Fig. 2.6 and Fig. 2.7. The symmetry properties of the bispectrum can be easily seen in these plots.

2.5.2 Example 2: Linear Shift

The linear shift version of the above signal is shown in Fig. 2.8. The other features are shown in Fig. 2.8 to Fig. 2.14, similar to example 1. One can see that the phase of the Fourier transform has changed. The magnitude of the Fourier transform did not change. Notice that both the magnitude and the phase of the bispectrum did not change.
Figure 2.1: One dimensional signal
Figure 2.2: The magnitude of the Fourier transform of the one dimensional signal
Figure 2.3: The phase of the Fourier transform of the one dimensional signal
Figure 2.4: Three dimensional plot of the magnitude of the bispectrum
Figure 2.5: Three dimensional plot of the phase of the bispectrum
Figure 2.6: Two dimensional contour plot of the magnitude of the bispectrum
Figure 2.7: Two dimensional contour plot of the phase of the bispectrum
Figure 2.8: The shifted version of the one dimension signal in Example 1
Figure 2.9: The magnitude of the Fourier transform of the shifted signal
Figure 2.10: The phase of the Fourier transform of the shifted signal
Figure 2.11: Three dimensional plot of the magnitude of the bispectrum of the shifted signal
Figure 2.12: Three dimensional plot of the phase of the bispectrum of the shifted signal
Figure 2.13: The two dimensional contour plot of the magnitude of the bispectrum of the shifted signal
Figure 2.14: The two dimensional contour plot of the phase of the bispectrum of the shifted signal
2.5.3 Example 3: Gaussian Noise Reduction

Let the signal shown in Fig. 2.1 be corrupted by Gaussian noise with $SNR = -7db$. The $SNR$ is calculated as

$$SNR = 10 \log \left( \frac{\sum_{n=0}^{N-1} x^2(n)}{\sum_{n=0}^{N} w^2(n)} \right)$$

where $x(n)$ is the signal, $w(n)$ is the Gaussian noise and $N$ is the length of the signal. The noisy signals are shown in Fig. 2.15. The bispectral reconstruction after 10 runs, 30 runs, 100 runs and 2000 runs are shown in Fig. 2.16 through Fig. 2.19.

It is easily seen that the variance of the reconstruction is approaching 0 with the increase of the number of runs. The variance vs. the number of runs is show in Fig. 2.20.

2.5.4 Example 4: Two Dimensional Case

A two dimensional grey level image of $256 \times 256$ pixels is shown in Fig. 2.21, the noisy version (additive gaussian noise) with $SNR = 0.5db$ is shown in Fig. 2.22. The $SNR$ was calculated as

$$SNR = 10 \log \left( \frac{\sum_{m=0}^{N-1} \sum_{n=0}^{N-1} x^2(m, n)}{\sum_{m=0}^{N-1} \sum_{n=0}^{N} w^2(m, n)} \right)$$

where $x(m, n)$ is the original image of size $N \times N$ pixels and $w(m, n)$ is the Gaussian noise.

Reconstruction after 5 runs, 20 runs and 100 runs are shown in Fig. 2.23 through Fig. 2.25 (these reconstructed images have been manually aligned to remove the linear shift effect caused by the bispectrum reconstruction).
Figure 2.15: The noisy version of the signal in Example 1 corrupted by a Gaussian process with signal to noise ratio $SNR = -7db$
Figure 2.16: The bispectral reconstruction by averaging 10 realizations in the bispectrum domain
Figure 2.17: The bispectral reconstruction by averaging 30 realizations in the bispectrum domain
Figure 2.18: The bispectral reconstruction by averaging 100 realizations in the bispectrum domain
Figure 2.19: The bispectral reconstruction by averaging 2000 realizations in the bispectrum domain
Figure 2.20: The plot of the variance of the bispectral reconstruction from Gaussian vs. the number of realizations used in averaging
Figure 2.21: The original image “Camera Man” of size $256 \times 256$
Figure 2.22: The noisy version of the image "Camera man" corrupted by an additive Gaussian noise with signal to noise ratio $SNR = 0.5db$
Figure 2.23: The reconstructed image from 5 realizations
Figure 2.24: The reconstructed image from 20 realizations
Figure 2.25: The reconstructed image from 100 realizations
2.6 Conclusions

In this chapter, a review of the bispectrum estimation technique is given, including the definition and the properties of the bispectrum, the estimation of the bispectrum and the reconstruction method from the bispectrum. Examples through computer simulations are given in both one dimensional and two dimensional cases.
Chapter 3

SPECKLE

3.1 Introduction

Operation of the first cw HeNe laser in 1960 revealed an unexpected phenomenon: objects viewed in highly coherent light have a granular appearance. The detailed structure of this granularity has no obvious relationship to the macroscopic properties of the illuminated object, but appears chaotic and disorganized.

The origin of this phenomenon is due to the interference of the dephased but coherent wavelets. The surfaces of most materials are extremely rough on the scale of an optical wavelength. When nearly monochromatic light is reflected from such a surface, the optical wave at any moderately distant point consists of many coherent components or wavelet, each arising from a different microscopic element of the surface. The distances travelled by these wavelets may differ by several or many wavelength if the surface is
truly rough. This phenomenon of interference of the dephased but coherent wavelets resulting in the granular pattern of intensity is called “speckle”.

The random interference phenomenon underlying laser speckle can occur not only in laser illumination, but also in many other imagery, such as radar astronomy, synthetic aperture radar and acoustical imagery, ultrasound, x-ray scattering, electron scattering, microwaves. In addition, statistical phenomena entirely analogous to laser speckle are found in radio-wave propagation, temporal statistics of incoherent light, theory of narrow-band electrical noise and even in the general theory of spectral analysis of random process. As a consequence of all these parallels and analogies, the term “speckle” has taken a far more general meaning than could have been envisioned when it was first introduced in the 1960’s.

Studies related to the occurrence of speckle phenomenon are not new. Dainty [14] gave a brief history of observations of speckle patterns, beginning with Newton’s attempt to explain star twinkling. Exner, in late 19th century, sketched a speckle pattern created by a candle seen through fogged class. Dehaas, in the early 20th century, actually photographed such a phenomenon accelerated.

What is important is how speckle degrades an image. As is discussed by Kozma and Christensen [15], speckle increases the size of the minimum resolution patch obtained with a given aperture as compared to the same size aperture using incoherent illumination. This reduction in resolution makes feature identification difficult. In the image plane, the presence of speckle is seen as a collection of spots superimposed on the original object, therefore, speckle can be seen as a noise which degrades the original image. Therefore
speckle reduction is important in many applications mentioned above.

In this chapter, an overview of the statistics of the speckle and some of the existing speckle reduction methods are presented. It should be noted that while the context is generally that of laser speckle patterns, the results derived apply equally to virtually any coherent random-interference phenomenon, provided the basic underlying statistical assumptions are satisfied.

### 3.2 Statistics of the Speckle

To properly model speckle images, it is necessary to review the underlying physics and mathematics behind speckle theory. Since speckle is random, it lends itself quite well to statistical analysis and the literature is very complete regarding this aspect of speckle. In this section, we assume that the waves of concern are perfectly monochromatic and the speckle is perfectly polarized.

#### 3.2.1 From the Electromagnetic Field

Any representation of the electric field vector must obey Maxwell’s equation:

\[
\nabla^2 E - \mu \varepsilon \frac{\partial^2 E}{\partial t^2} = 0
\]

(3.1)

where:

- \(E\) is the electric field intensity vector \(E(x, y, z; t)\);
- \(\mu\) is the permeability;
- \(\varepsilon\) is the dielectric constant;
- \(\nabla^2\) is the Laplacian operator: \(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\).
One solution to the above wave equation is

\[ E(x, y, z, t) = A(x, y, z) \exp\{j2\nu t\} \]

where \( \nu \) is the mean frequency, \( A(x, y, z) \) is the amplitude function:

\[ A(x, y, z) = |A(x, y, z)| \exp[i\theta(x, y, z)] \]

The intensity is

\[ I(x, y, z) = \lim_{T \to \infty} \int_{-T/2}^{+T/2} |E(x, y, z; t)|^2 dt = |A(x, y, z)|^2 \]

The amplitude of the electric field at a given observation point consists of a multitude of dephase contributions from different scattering regions of the rough surface. Thus the amplitude \( A(x, y, z) \) is represented as a sum of many elementary phasor contributions:

\[ A(x, y, z) = \sum_{k=1}^{N} \frac{1}{\sqrt{N}} a_k(x, y, z) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} |a_k| e^{i\phi_k} \]

We assume that the elementary phasors have the following statistical properties:

1. The amplitude \( a_k \) and the phase \( \phi_k \) are statistically independent of each other and of the amplitudes and phases of all other elementary phasors.

2. The phase \( \phi_k \) are uniformly distributed on \((-\pi, \pi)\), i.e., the surface is rough compared to a wavelength.

The derivations in the rest part of this section are based on these assumptions. It should be noted that these assumptions hold in many practical applications.
3.2.2 Statistics of the Complex Amplitude

Attention is now focused on the real and imaginary parts of the resultant field,

\[ A^{(r)} = \text{Re}\{A\} = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} |a_k| \cos \phi_k \]  \hspace{1cm} (3.2)

\[ A^{(i)} = \text{Im}\{A\} = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} |a_k| \sin \phi_k \]  \hspace{1cm} (3.3)

The average values of \( A^{(r)} \) and \( A^{(i)} \) over an ensemble of macroscopically similar but microscopically different rough surfaces are

\[ < A^{(r)} > = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} < |a_k| > < \cos \phi_k > = 0 \]  \hspace{1cm} (3.4)

\[ < A^{(i)} > = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} < |a_k| > < \sin \phi_k > = 0 \]  \hspace{1cm} (3.5)

Similarly we have

\[ < [A^{(r)}]^2 > = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \sum_{m=1}^{N} < |a_k||a_m| > < \cos \phi_k \cos \phi_m > \]  \hspace{1cm} (3.6)

\[ = \sum_{k=1}^{N} \sum_{m=1}^{N} \frac{< |a_k|^2 >}{2} \]

\[ < [A^{(i)}]^2 > = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \sum_{m=1}^{N} < |a_k||a_m| > < \sin \phi_k \sin \phi_m > \]  \hspace{1cm} (3.7)

\[ = \sum_{k=1}^{N} \sum_{m=1}^{N} \frac{< |a_k|^2 >}{2} \]

\[ < A^{(r)} A^{(i)} > = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \sum_{m=1}^{N} < |a_k||a_m| > < \cos \phi_k \sin \phi_m > = 0 \]  \hspace{1cm} (3.8)
where we have used the fact that for independent and uniformly distributed phases,

\[
< \cos \phi_k \cos \phi_m > = \begin{cases} 
\frac{1}{2} & k = m \\
0 & k \neq m 
\end{cases} \quad (3.9)
\]

\[
< \cos \phi_k \sin \phi_m > = 0 \quad (3.10)
\]

Thus we see that the real and imaginary parts of the complex field have zero means, identical variances, and are uncorrelated.

Now we suppose, as is generally the case in practice, that the number \( N \) of elementary phasor contributions is extremely large. Thus the real and imaginary parts of the field are expressed by Eq. (3.2) and Eq. (3.3) as sums of a very large number of independent random variables. It follows from the central limit theorem that, as \( N \to \infty \), \( A^{(r)} \) and \( A^{(i)} \) are asymptotically Gaussian. Coupling this fact with the results of Eq. (3.4) through Eq. (3.8), the joint probability density function of the real and imaginary parts of the field is found to asymptotically approach

\[
p_{r,i}(A^{(r)}, A^{(i)}) = \frac{1}{2\pi \sigma^2} \exp \left\{ - \frac{[A^{(r)}]^2 + [A^{(i)}]^2}{2\sigma^2} \right\} \quad (3.11)
\]

Where

\[
\sigma^2 = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \frac{<|a_k|^2>}{2} \quad (3.12)
\]

Such a density function is commonly known as a circular Gaussian density function, since contours of constant probability density are circles in the complex plane. The three dimensional plot and two dimensional contour of a circular Gaussian density function with \( \sigma^2 = 1 \) are shown in Fig. 3.1 and Fig. 3.2.
Figure 3.1: Three dimensional plot of a circular Gaussian density function with $\sigma^2 = 1$
Figure 3.2: Two dimensional contour plot of a circular Gaussian density function with $\sigma^2 = 1$
3.2.3 Statistics of Intensity

For most experiments in the optical region of the spectrum, it is the intensity of the wave that is directly measured. Accordingly, from the known statistics of the complex amplitude, we wish to find the corresponding statistical properties of the intensity in a polarized speckle pattern.

The intensity $I$ and phase $\theta$ of the resultant field are related to the real and imaginary parts of the complex amplitude by the transformation

$$A^{(r)} = \sqrt{I} \cos \theta$$

$$A^{(i)} = \sqrt{I} \sin \theta$$

or equivalently by

$$I = [A^{(r)}]^2 + [A^{(i)}]^2$$

$$\theta = \tan^{-1} \frac{A^{(r)}}{A^{(i)}}$$

The joint density function is

$$p_{I,\theta}(I, \theta) = p_{r,i}(\sqrt{I} \cos \theta, \sqrt{I} \cos \theta)||J||$$

where $||J||$ is the Jacobian of the transformation

$$||J|| = \left| \begin{array}{cc} \frac{\partial A^{(r)}}{\partial I} & \frac{\partial A^{(r)}}{\partial \theta} \\ \frac{\partial A^{(i)}}{\partial I} & \frac{\partial A^{(i)}}{\partial \theta} \end{array} \right| = \frac{1}{2}$$

$||...||$ symbolizes the modulus of the determinant. Substituting Eq. (3.11) in Eq. (3.17), we find

$$p_{I,\theta}(I, \theta) = \left\{ \begin{array}{ll} \frac{1}{4\pi \sigma^2} \exp\left(-\frac{I}{2\sigma^2}\right) & I \geq 0, \ -\pi \leq \theta < \pi \\ 0 & \text{otherwise} \end{array} \right.$$
The marginal probability density function of the intensity alone is

\[ p_I(I) = \begin{cases} \frac{1}{2\sigma^2} \exp\left(-\frac{I}{2\sigma^2}\right) & I \geq 0 \\ 0 & \text{otherwise} \end{cases} \] (3.20)

And the marginal probability density function of the phase is

\[ p_{\theta}(\theta) = \begin{cases} \frac{1}{2\pi} & -\pi \leq \theta < \pi \\ 0 & \text{otherwise} \end{cases} \] (3.21)

It is readily shown that the mean value of the intensity is

\[ <I> = 2\sigma^2 \]

and the variance of the intensity is

\[ \sigma_I^2 = <I^2> - <I>^2 = <I^2> \]

Thus the standard deviation \( \sigma_I \) of a polarized speckle pattern is equal to the mean intensity.

The PDF (probability density function) of the intensity of the polarized speckle pattern in Eq. (3.20) can be also written as

\[ p_I(I) = \frac{1}{<I>} \exp\left(-\frac{I}{<I>}\right) \] (3.22)

for \( I \geq 0 \).

The curve of the PDF of the intensity expressed by Eq. (3.22) is shown in Fig. 3.3, which known as negative exponential.

### 3.2.4 Statistics of Integrated Speckle Patterns

In the experimental measurement of the intensity in a speckle pattern, the detector aperture must be of finite size. Hence the measured intensity is always a some what smoothed or integrated version of the ideal point-intensity,
Figure 3.3: The PDF curve of the intensity of the polarized speckle pattern
and the statistics of the measured speckle will be somewhat different from the ideal statistics developed above.

If a speckle pattern with intensity $I(x, y)$ falls upon a detector with extended aperture, the measured intensity can be expressed in the form

$$I_0 = \frac{1}{S} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(x, y) I(x, y) \, dx \, dy$$

where $\varphi(x, y)$ is a real and positive weighting function and

$$S = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(x, y) \, dx \, dy$$

In the event that the detector has uniform response over a finite aperture, the weighting function has the simple form

$$\varphi(x, y) = \begin{cases} 0 & \text{in the aperture} \\ 1 & \text{outside the aperture} \end{cases}$$

and $S$ is the area of that aperture.

The mean of the integrated speckle pattern is

$$< I_0 > = \frac{1}{S} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(x, y) I(x, y) \, dx \, dy = < I >$$

Other statistics take complicated forms and thus cannot be utilized in practical applications. Thus approximated forms of the statistics of integrated speckle patterns with good accuracy is needed.

The approximation is made by sampling the smoothly varying speckle intensity $I(x, y)$ on the measurement area. Here we assume that the imaging system has a resolution cell which is small compared to the spatial detail in the object or the roughness of the surface [16]. An equivalent condition is
that the imaging system bandwidth be greater than the signal bandwidth. We further assume that the intensity of the speckle in a sampling cell is approximately constant.

Under these assumptions, Eq. (3.23) takes the form

$$ I_0 = \frac{1}{S} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(x, y) I(x, y) dx dy = \frac{1}{S} \sum_{k=1}^{m} \varphi_k I_k $$

(3.27)

where $\varphi_k$ is the value of $\varphi(x, y)$ in the $k$th cell and $I_k$ is the constant intensity assumed in that cell and $m$ is the number of cells.

The characteristic function of each $I_k$ is taken to be that of polarized speckle at a single point,

$$ M_k(i, \nu) = \frac{1}{1 - i\nu < I >} $$

(3.28)

It follows that the characteristic function of the integrated speckle is given by

$$ M_k(i, \nu) = \prod_{k=1}^{m} \frac{1}{1 - i\nu \varphi_k / S < I >} $$

(3.29)

For the simple detection aperture in Eq. (3.25), we have

$$ \varphi_k / S = \frac{1}{m} \text{ for all } k $$

(3.30)

and the characteristic function becomes

$$ M_k(i, \nu) = \left[ \frac{1}{1 - i\nu < I >} \right]^m $$

(3.31)

The corresponding probability density function is the gamma density function,

$$ p_{I_0}(I_0) = \frac{(\frac{m}{< I >})^m I_0^{m-1} \exp(-\frac{I_0}{< I >})}{\Gamma(m)} $$

(3.32)
Figure 3.4: The plots of the PDF curve of the integrated speckle.
where $\Gamma(m)$ is a gamma function of argument $m$, and for $I \leq 0$. Fig. 3.4 shows plots for several values of the parameter $m$.

Notice that when $m = 1$, the measurement aperture shrinks towards zero size, i.e. there is no integration, the density function approaches the negative exponential function valid for speckle intensity at a point expressed in Eq. (3.22). As $m \to \infty$, there are infinite number of speckle patterns integrated together, the gamma density function can be shown to approach a Gaussian density function. By virtue of the central limit theorem, this is indeed the exact density function in the limit of $m \to \infty$. Thus the gamma density function is highly accurate for small $m$ and very large $m$. For intermediate values of $m$, we expect the approximate density function to depart from the true density some what.

### 3.3 Speckle Reduction

Although there are several optical methods for speckle reduction, our concern here is the reduction of the speckle when we already have a digitized speckled image. Guenther, et al. [17] have one of the earliest papers listing several different digital filters that could be applied to reduce speckle. They described the basic ensemble averaging technique (where several images are averaged together to reconstruct the image) as well as the spatial processing technique using an averaging window which moves across the image plane spatially. Two simple, digital, non-linear filters (square-root and squaring) are also described. Jain and Christensen [18] review similar technique as well as a homomorphic Wiener filter for digital speckle removal. Lim and Nawab
[4] compare other digital techniques for speckle reduction such as low-pass filtering in the frequency (Fourier) and density (logarithmic) domains as well as a method they called the "short space spectral subtraction image restoration technique." They discussed briefly the homomorphic approach to image restoration using a filter built specifically for this method but do not describe this filter in any detail, though they did discuss the benefits of using the homomorphic approach. Sadjadi [19] reviewed the before-mentioned averaging techniques, a median filter, local statistical filters, an adaptive filter, and a sigma filter and compared these with the homomorphic approach to speckle reduction. The homomorphic approach allows additive-noise reduction techniques to be applied to multiplicative noise conditions. A better description of a local statistics filter used in homomorphic processing is given by Arsenault and Levesque [20]. A general description of signal-dependent noise is presented in this article also.

Fienup [21] performed a comparison of the phase-retrieval algorithms and some aspects of these algorithms are used to restore speckled images [22]. Cederquist, et al. [23], used a phase retrieval algorithm for far field, computer-generated speckle. These algorithms are useful because phase information is lost when spectral processing a recorded image using the power spectral density (PDF). Recovering the phase helps one reconstruct the image uniquely.

Kuan, et al. [24], derived an adaptive restoration filter for speckled images. This article also brings up some important points. One key point is the difference between a noisy object and a noisy imaging system. A noisy object occurs when an incident coherent wavelet is scattered by the object
such that random interferences occur among the dephased but coherent reflected waves. This is the speckle we are investigating. The noisy imaging system results from a randomly variable transmitting medium, such as the atmosphere, and this is the case for stellar speckle.

Recently, Marathay, et al. [25], used computer-simulated speckle patterns and applied third and fourth order intensity correlations to restore the image. The speckle patterns generated were not imaged but rather found in the pupil plane. Newman and Van Vracken [26] combine the bispectrum with photon-bias correction techniques [27] to recover multiple, shifted objects in a uniform background of photon noise (Poisson). The signal-to-noise ratio(SNR) of these objects and background noise was less than 1!

In next chapter, we will present an approach to reduce the coherent speckle by using bispectrum.
Chapter 4

Speckle Reduction Using Bispectrum

In this chapter, we will propose a speckle reduction approach using bispectrum reconstruction technique.

Our concern is the reduction of the speckle when we already have digitized speckled images. Specifically, we investigate the situation where we have a moving (linear shift only) object illuminated by a coherent source and we have several images taken at different stages of translation. In this case the maximum likelihood estimate of the image is obtained from ensemble averaging the multiple frames [4]. Perfect registration of each image with the other is required so that the resultant, averaged image is not blurred. This concern often precludes using ensemble averaging due to the difficulty in registering images exactly. However, we will exploit the shift-invariant property of the bispectrum and average in the bispectral domain so that we
do not have to align the image.

As was discussed in chapter 2, the bispectrum technique is very effective in removing additive Gaussian noise. However, the speckle noise is generally modeled as a multiplicative noise in practical applications [24, 28, 29, 30, 31, 19, 32]. We used a homomorphic transformation (logarithm transformation), to make the multiplicative, signal dependent speckle noise additive and signal independent. With increasing size of the finite aperture, the speckle noise transformed this way is close to Gaussian. As we show, the bispectrum can then be used to reconstruct the speckled image. Experimental verification is provided through computer simulations.

In this chapter, we will first discuss the multiplicative model, then we will show that by using a logarithm transform and the finite aperture, the signal dependant, multiplicative speckle noise will become signal independent, additive noise which is close to gaussian. The bispectral reconstruction method will be described, and finally, the computer simulation results and the conclusion will be given.

4.1 The Multiplicative Noise Model

Suppose a laser illuminates a composite object composed of a diffuser with complex amplitude transmittance \( d(x,y) \), in contact with a transparency containing an object \( t(x,y) \), see Fig. 4.1.

Suppose the resolution of the imaging system is \( R_{cell} \), which means that the imaging system will image the area \( R_{cell} \) in the object plane to a point \( P \) in the image plane.
Figure 4.1: Optical system \( d(x,y) \) and \( t(x,y) \) are, respectively, the complex transmittance of the diffuser and object transparency.

The intensity of point \( P \) in image plane is

\[
I(x_P, y_P) = |A(x_P, y_P)|^2 = \left| \int \int_{R_{\text{cell}}} t(x, y) d(x, y) h(x_P - x, y_P - y) \, dx \, dy \right|^2
\]

(4.1)

where \( h(x, y) \) is the amplitude impulse response of the imaging system (assuming the imaging system has unity magnification).

If the resolution area \( R_{\text{cell}} \) is smaller than the detail of the object \( t(x, y) \), the object \( t(x, y) \) is approximately constant within \( R_{\text{cell}} \), i.e.,

\[
t(x, y) \approx t_0(x_P, y_P) = \text{constant}
\]

(4.2)

then the intensity of the speckle degraded image on the image plane at point \( P \) becomes

\[
I_s(x_P, y_P) = |t_0(x_P, y_P)|^2 \left| \int \int_{R_{\text{cell}}} d(x, y) h(x_P - x, y_P - y) \, dx \, dy \right|^2
\]

(4.3)
Let $I_O(x_P, y_P)$ be the original image intensity of $t(x, y)$ at point $P$ in the image plane, then

$$I_O(x_P, y_P) = \left| \int \int_{R_{cell}} t(x, y) h(x_P - x, y_P - y) dx dy \right|^2$$  \hspace{1cm} (4.4)

Consider the assumption that $t(x, y)$ is constant within $R_{cell}$, Eq. (4.4) becomes

$$I_O(x_P, y_P) = \left| t_0(x_P, y_P) \right|^2 \left| \int \int_{R_{cell}} h(x_P - x, y_P - y) dx dy \right|^2$$  \hspace{1cm} (4.5)

Since $h(x, y)$ is the impulse response of the system, we have

$$\left| \int \int_{R_{cell}} h(x_P - x, y_P - y) dx dy \right|^2 = c$$  \hspace{1cm} (4.6)

where $c$ is a constant. Therefore

$$I_O(x_P, y_P) = c \left| t_0(x_P, y_P) \right|^2$$  \hspace{1cm} (4.7)

Let the intensity of the speckle noise at $P$ in the image plane be $I_N(x_P, y_P)$, then

$$I_N(x_P, y_P) = \left| \int \int_{R_{cell}} d(x, y) h(x_P - x, y_P - y) dx dy \right|^2$$  \hspace{1cm} (4.8)

Substitute Eq. (4.7) and Eq. (4.8) into Eq. (4.3), we obtain the intensity of the speckle degraded image $I_S(x_P, y_P)$ as

$$I_S(x_P, y_P) = \frac{1}{c} I_O(x_P, y_P) I_N(x_P, y_P)$$  \hspace{1cm} (4.9)

This multiplicative noise model has been used by many researchers (Kuan, et al. [24]; April and Arsenault [33, 34, 35]; Guenther, et al. [17]; Jain and Christensen [18]). It has also been proven to be a valid noise model for speckle in practical applications such as Synthetic Aperture Radar (SAR).
It must be pointed out that this model is just an approximate model. As was discussed above, in order for this multiplicative model to hold, the imaging system must be able to resolve the details of the object. The equivalent condition is the bandwidth of the imaging system is greater than the bandwidth of the image. Otherwise the multiplicative model will not be applicable. Lim and Nawab [4] made the assumption that the sampling of the speckled image is coarse enough so that all points in the degraded image will be considered independent. The sampling distance must be larger than the correlation length of the speckle at neighboring pixels to ensure a spatially uncorrelated random field in the image [19]. In the later part of this thesis, we assume that the imaging system can resolve all of the details of the object surface.

4.2 Homomorphic Transformation

As was discussed in chapter 2, additive Gaussian noise can be removed effectively by using bispectrum technique. However, the speckle noise is not additive Gaussian noise but a multiplicative noise. Homomorphic transformation can be used to transform the multiplicative, signal depend noise to signal independent, additive noise. Here a negative natural logarithm transformation is used.

Take natural logarithm of both side of Eq. (4.9), we have

\[ D_S(x_P, y_P) = D_O(x_P, y_P) + D_N(x_P, y_P) \]  \hspace{1cm} (4.10)

where
\[ D_S(x_p, y_P) = -\ln[I_S(x_p, y_P)] \]
\[ D_O(x_p, y_P) = -\ln[I_O(x_p, y_P)] \]
\[ D_N(x_p, y_P) = -\ln[I_N(x_p, y_P)] \]

Our concern here is the statistics of the noise term \( D_N \). In next section, will discuss the statistics of the logarithm transformed noise \( D_N \).

### 4.3 Statistics of the Logarithm Transformed Speckle Noise

First let us recall the PDF of the intensity of the integrated speckle noise with a finite aperture expressed in Eq. (3.32)

\[
p_{I_N}(I_N) = \frac{M^M}{\Gamma(M)I_0} \left( \frac{I_N}{I_0} \right)^{M-1} \exp\left( -\frac{MI_N}{I_0} \right)
\]  
(4.11)

where \( I_0 \) is the mean of the intensity of the speckle noise, i.e.

\[
I_0 = \langle I_N \rangle
\]  
(4.12)

and \( M \) is the effective number of speckles in the integrating aperture which is proportional to the size of the aperture. Noted that \( m \) in Eq. (3.32) is replaced by \( M \), since \( m \) is used as a space coordinate in this chapter.

After a logarithmic transformation \( D_N = -\ln(I_N) \), the PDF in Eq. (4.11) is transformed into

\[
f_{D_N}(D_N) = \left[\frac{M^M}{\Gamma(M)}\right] \exp\left[-M(D_N - D_0)\right] \exp\{-M \exp(D_N - D_0)\}
\]  
(4.13)

where \( D_0 = -\ln(I_0) \). From Eq. (4.13) we see that the only effect of \( D_0 \) is to shift the PDF on the \( D \) axis by a quantity equal to \( D_0 \), without changing the
shape of the probability distribution. The statistics of the speckle noise are therefore additive and signal independent. Since the statistics do not depend on the signal of interest $D_0$, we can let $D_0 = 0$ for simplicity, and Eq. (4.13) becomes

$$f_{D_N}(D_N) = \left[\frac{M^M}{\Gamma(M)}\right] \exp\{-M[D_N + \exp(-D_N)]\} \quad (4.14)$$

The plots of the PDF of $D_N$ are shown in Fig. 4.2 for different values of $M$. The reader might have noticed that with increasing of the value $M$, the PDF curve seems to be approaching Gaussian. In fact this is true and we will give the approximation in next section.

### 4.4 Gaussian Approximation

We now determine under what conditions Eq. (4.14) can be approximated by a normal probability distribution.

The series expansion of exponential is

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + ... \quad (4.15)$$

replace $\exp(-D_N)$ in Eq. (4.14) by the first four terms of its series expansion, we have

$$f_{D_N}(D_N) \approx \frac{M^M}{\Gamma(M)} \exp\left\{-M \left[ D_N + \left(1 - D_N + \frac{D_N^2}{2} - \frac{D_N^3}{6}\right)\right]\right\}$$

$$= \frac{M^M}{\Gamma(M)} \exp(-M) \exp\left[-\frac{MD_N^2}{2} \left(1 - \frac{D_N}{3}\right)\right] \quad (4.16)$$
Figure 4.2: The PDF curve of the logarithmically transformed speckle patterns
The logarithm of the multiplicative speckle noise $D_N \ll 3$, this yields approximately

$$f_{D_N}(D_N) \approx \frac{M^M \exp(-M)}{\Gamma(M)} \exp\left(-\frac{MD_N^2}{2}\right)$$  \hspace{1cm} (4.17)

If $M \gg 1$, which means the effective number of speckle pattern is large and this is true in practice, then

$$\Gamma(M) \approx (M - 1)!$$  \hspace{1cm} (4.18)

Eq. (4.17) becomes

$$f_{D_N}(D_N) \approx \frac{M^M \exp(-M)}{(M - 1)!} \exp\left(\frac{MD_N^2}{2}\right)$$  \hspace{1cm} (4.19)

using Stirling’s approximation for the factorial we have

$$f_{D_N}(D_N) \approx \left(\frac{M}{2\pi}\right)^{1/2} \exp\left(-\frac{MD_N^2}{2}\right)$$

$$= \frac{1}{\sqrt{2\pi (\frac{1}{M})}} \exp\left[-\frac{D_N^2}{2\left(\frac{1}{M}\right)}\right]$$  \hspace{1cm} (4.20)

The significance of this expression is that if the number of speckles $M$ in the aperture is large enough, the logarithmically transformed speckle noise is approximately Gaussian additive noise with a variance equal to $1/M$, which means the variance decreases with the increase of $M$. This expression also shows that for large $M$, the mean value $D_0$ of the density approaches the logarithm of the mean intensity $I_0$.

In fact even without logarithm transform the PDF of the intensity of the speckle itself approaches to Gaussian form as well, as was discussed in chapter 3. However, after the logarithm transform, the PDF approaches to Gaussian form much faster than the PDF of the intensity itself. The reader
Figure 4.3: The speckle-degraded image is first scanned with a detector having a given aperture size, and then is digitized with a digitizer.

may compare Fig. 4.2 with Fig. 3.3. After the logarithm transform, for values as small as \( M = 3 \), the Gaussian form approximation for \( D_N \) is seen to be relatively good.

4.5 Speckle Reduction Using Bispectra

If an image corrupted by speckle noise is to be processed by digital means, it must first be scanned with a detector having a given aperture size, and with a given raster frequency, then it must be digitized, see Fig. 4.3.

Consider a two dimensional digitized speckle-degraded image. From Eq. (4.9), it can be modeled as:

\[
g(m, n) = x(m, n)w(m, n);
\]

(4.21)
where \((m, n)\) are discrete coordinates, \(x(m, n)\) is the original object, i.e., the digitized original image whose value is the intensity of the original image in the image plane at coordinate \((m, n)\). \(w(m, n)\) is the image of speckle noise whose value is the intensity of the speckle noise in the image plane at coordinate \((m, n)\), and \(g(m, n)\) is the recorded speckle degraded image.

Taking logarithm on both sides of Eq. (4.21), we get

\[
\tilde{g}(m, n) = \tilde{x}(m, n) + \tilde{w}(m, n)
\]  

(4.22)

where

\[
\tilde{g}(m, n) = -\ln (g(m, n))
\]  

(4.23)

\[
\tilde{x}(m, n) = -\ln (x(m, n))
\]  

(4.24)

\[
\tilde{w}(m, n) = -\ln (w(m, n))
\]  

(4.25)

In the case of nonintegrated speckle, \(\tilde{w}\) has the following PDF:

\[
p_{\tilde{w}}(\omega) = \exp(-\omega - \exp(-\omega))
\]  

(4.26)

which is the negative exponential PDF of the speckle.

If an integrating aperture is used (it is assumed that the size of the aperture is small enough to retain image detail), then from Eq. (4.20) the PDF of \(\tilde{w}\) is approximately Gaussian.

Here we investigate the situation in which the object moving against a dark background and illuminated by a coherent source of radiation. Further, let the motion be strictly linear, i.e., there is no rotational component. If we now take independent snapshots of the object and digitize the corresponding image, we obtain images \(g_i(m, n)\), \(i = 1, 2, \ldots, I\) where \(I\) is the number of
snapshots and \( g_i(m,n) \) is the intensity of the \( i \)th image at pixel location \((m,n)\). We have, in accordance with Eq. (4.21),

\[
g_i(m,n) = x(m-m_i,n-n_i)w_i(m,n) \tag{4.27}
\]

where \( w_i(m,n) \) is speckle noise and \( x(m,n) \) is the intensity of the object. \((m_i,n_i)\) denotes the translation of the object in the \( i \)th snapshot.

Taking logarithms on both sides of Eq. (4.27), we obtain

\[
\tilde{g}_i(m,n) = \tilde{x}(m-m_i,n-n_i) + \tilde{w}_i(m,n) \tag{4.28}
\]

where \( \tilde{g}(m,n) = -\ln(g(m,n)) \) and so on.

As discussed in previous sections, if a finite aperture is used, the PDF of \( \tilde{w}_i(m,n) \) is close to Gaussian.

Important properties of the bispectrum are discussed briefly in chapter 2, some of which can be directly applied here. Two important properties of bispectrum, which are used in our bispectral speckle reduction approach, are given in more details in the following.

A. Linear Shift Invariance

Given a discrete-time sequence \( x(n) \), its bispectrum is

\[
B_x(\omega_1, \omega_2) = X(\omega_1)X(\omega_2)X^*(\omega_1 + \omega_2) \tag{4.29}
\]

where \( X(\omega) \) is the Fourier transform of \( x(n) \):

\[
X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\exp(-j\omega n) = |X(\omega)|\exp[j\phi(\omega)] \tag{4.30}
\]
Let \( z_1(n) \) be the shifted version of \( x(n) \), where \( z_1(n) = x(n+n_0) \). Then the Fourier transform of \( z_1(n) \) is

\[
X_1(\omega) = \sum_{n=-\infty}^{\infty} z_1(n) \exp(-j\omega n) \quad (4.31)
\]

\[
= \sum_{n=-\infty}^{\infty} x(n - n_0) \exp(-j\omega n)
\]

\[
= X(\omega) \exp(j\omega n_0)
\]

The bispectrum of \( z_1(n) \) is

\[
B_{z_1}(\omega_1, \omega_2) = X_1(\omega_1)X_1(\omega_2)X_1^*(\omega_1 + \omega_2) \quad (4.32)
\]

\[
= X(\omega_1) \exp(j\omega_1 n_0)X(\omega_2) \exp(j\omega_2 n_0)
\]

\[
X_1^*(\omega_1 + \omega_2) \exp[-j(\omega_1 + \omega_2) n_0]
\]

\[
= X(\omega_1)X(\omega_2)X^*(\omega_1 + \omega_2)
\]

\[
= B_x(\omega_1, \omega_2)
\]

It should be pointed out that Eq. (4.34) also holds for two dimensional cases. Let the bispectrum of the deterministic signal \( \tilde{x}(m - m_i, n - n_i) \) in Eq. (4.28) be \( B_{\tilde{x}_i} \), \( i = 1, 2, \ldots, I \). Then

\[
B_{\tilde{x}_1} = B_{\tilde{x}_2} = \ldots = B_{\tilde{x}_I} = B_{\tilde{x}} \quad (4.34)
\]

B. Insensitivity to Gaussian Noise

Let \( x(n) \) be a deterministic signal. Suppose we have the observation \( y(n) \):

\[
y(n) = x(n) + w(n) \quad (4.35)
\]
where \( w(n) \) is an additive stationary Gaussian form noise, which can be white or colored. Given \( I \) independent observations \( y_i(n); i = 1, 2, \cdots, I \) of \( y(n) \), the bispectrum of the signal \( B_x(\omega_1, \omega_2) \) is estimated as

\[
\hat{B}_x(\omega_1, \omega_2) = \frac{1}{I} \sum_{i=1}^{I} Y_i(\omega_1)Y_i(\omega_2)Y_i^{*}(\omega_1 + \omega_2) \quad (4.36)
\]

where \( Y_i(\omega) \) is the Fourier transform of the \( i \)th realization \( y_i(n) \).

\( \hat{B}_x(\omega_1, \omega_2) \) is a consistent estimate of \( B_x(\omega_1, \omega_2) \), the bispectrum of \( x(n) \), for all \( |\omega_1|, |\omega_2| \leq \pi \) in the bispectral plane except in the region \( S \) defined as [3]

\[
S = \{(\omega_1, \omega_2): \omega_1 = 0 \ or \ \omega_2 = 0 \ or \ \omega_1 + \omega_2 = 0\} \quad (4.37)
\]

It must pointed out however that the bispectrum of the sum of two signals is not necessarily equal to the sum to the bispectrum of each signal. Eq. (4.36) also holds for two dimensional cases. This estimator is actually the direct class of conventional bispectrum estimator discussed in chapter 2.

Eq. (4.28) represents the situation described by Eq. (4.35) in two dimensional case except for the linear shift. From Eq. (4.34) we see that each translated version of the original image has the same bispectrum. Therefore the bispectrum of an original image can be estimated from the multiple snapshots we recorded. Then we can reconstruct the original image from this estimated bispectrum by using various techniques. We list the detailed procedure in the following:
1. Record $I$ frames of speckle degraded images, the $i$th frame is denoted by $g_i(m,n)$.

2. Take negative natural logarithm of $g_i(m,n)$:

$$\tilde{g}_i(m,n) = -\ln [g_i(m,n)]$$  \hspace{1cm} (4.38)

3. Calculate the discrete Fourier transform of $\tilde{g}_i(m,n)$ by

$$\tilde{G}_i(k,l) = \sum_{m=0}^{M_i} \sum_{n=0}^{N_i} \tilde{g}_i(m,n) \exp \left[ -2\pi i \left( \frac{kn}{M} + \frac{ln}{N} \right) \right]$$  \hspace{1cm} (4.39)

where $M_i$ and $N_i$ are the dimension the images. Of course the two dimensional FFT can be used. In order to save computation when using FFT, the size of the images should be pure power of 2.

4. Estimate the bispectrum of the original image $\tilde{x}_i(m,n)$. Of course various bispectrum estimators can be used. Here the we use direct class of the conventional bispectrum estimator described in Eq. (4.36):

$$\hat{B}_g(k_1, k_2; l_1, l_2) = \frac{1}{I} \sum_{i=1}^{I} \tilde{G}_i^2(k_1, l_1) \tilde{G}_i^2(k_2, l_2) \tilde{G}_i^*(((k_1 + k_2)_M, (l_1 + l_2)_N)$$  \hspace{1cm} (4.40)

where $\tilde{G}_i^*$ is the complex conjugation of $G_i$, and $(k_1 + k_2)_M$ and $(l_1 + l_2)_N$ is the modulo $M$ of $k_1 + k_2$ and the modulo $N$ of $l_1 + l_2$, respectively.

In practice, not all values of $\hat{B}_g$ are needed depending on what reconstruction method is used. Usually only a portion of $\hat{B}_g$ is estimated. For example, in our approach, for magnitude reconstruction, $\hat{B}_g(k, k; l, l)$ is estimated, denoted by $\hat{B}_g(k, l)$

$$\hat{B}_g(k, l) = \frac{1}{I} \sum_{i=1}^{I} \tilde{G}_i^2(k, l) \tilde{G}_i^*((2k)_N, (2l)_N)$$  \hspace{1cm} (4.41)
where $k, l = 0, 1, ..., N - 1$, we used $N \times N$ square images.

5. Obtain the reconstructed discrete Fourier coefficients $\hat{X}(k, l)$ from the estimated bispectrum $\hat{B}_z$.

Again various methods can be used for the above reconstruction, as was discussed in chapter 2. We used a two dimensional recursive discrete Fourier transform approach for the magnitude reconstruction and a recursive approach for the phase reconstruction. Both methods were described in section 2.4.

6. Calculate the inverse discrete Fourier transform of $\hat{X}(k, l)$ by

$$
\hat{x}(m, n) = \frac{1}{MN} \sum_{k=0}^{M_1} \sum_{l=0}^{N_1} \hat{X}(k, l) \exp \left[ 2\pi i \left( \frac{kn}{M} + \frac{ln}{N} \right) \right]
$$

(4.42)

Again a two dimensional FFT can be used.

7. Take an exponential transform, i.e., the inverse transform of logarithm transform, and the original image is finally obtained:

$$
\hat{x}(m, n) = \exp \left[ -\hat{x}(m, n) \right]
$$

(4.43)

8. Manually align $x(m, n)$ to remove the effect of the linear shift introduced by the bispectrum reconstruction

4.6 Computer Simulation

Based on the descriptions by Goodman [14] and our previously stated model for speckle, the following procedure was adopted to simulate the speckled object.
Random deviates $A^{(r)}(m, n)$ and $A^{(i)}(m, n)$ were drawn from two independent gaussian processes with zero mean and unit variance. The circular Gaussian complex field of the speckle is formed as

$$A(m, n) = A^{(r)}(m, n) + jA^{(r)}(m, n)$$  \hspace{1cm} (4.44)

where $A(m, n)$ is the complex amplitude of the speckle noise.

Let $w(m, n)$ denote the intensity of the speckle noise in the image plane at coordinate $(m, n)$

$$w(m, n) = |A(m, n)|^2 = [A^{(r)}]^2 + [A^{(i)}]^2$$  \hspace{1cm} (4.45)

The unscanned and scanned speckled images $y(m, n)$ and $g(m, n)$ were respectively formed as

$$y(m, n) = x(m, n)w(m, n) = x(m, n)[A^{(r)}]^2 + [A^{(i)}]^2]$$  \hspace{1cm} (4.46)

and

$$g(m, n) = \sum_{i=-(N-1)/2}^{(N-1)/2} \sum_{k=-(N-1)/2}^{(N-1)/2} y(m - i, n - k)h(i, k)$$  \hspace{1cm} (4.47)

where $x(m, n)$ is the intensity of the original image at coordinate $(m, n)$, $h(m, n)$ represents a scanning aperture of a size $M = N \times N$ pixels and $N$ takes odd values.

In order to test the closeness of our approach to the theory, we first let the intensity of the image be one for all $(m, n)$, i.e. $x(m, n) = 1$. This leaves only the noise term in Eq. (4.46). The size of the image was $128 \times 128$ pixels. PDF curves of the simulated speckled image were generated from the unified and smoothed histograms as follows.
1. Find the maximum value $Max$ and minimum value $Min$ of the input data.

2. Divide the region $[Min, Max]$ evenly into $K$ small regions, the $i$th region is denoted by $[a_i, b_i]$, where

$$b_i - a_i = \frac{Max - Min}{K}$$

3. Compute the number of pixels whose value is in the $i$th region, denoted by $N_{\text{pixel}}(i)$. The plot of $N_{\text{pixel}}(i)$ vs. $[a_i, b_i]$ is the histogram of the input data.

4. Estimate the probability density by unifying the histogram:

$$p(i) = \frac{N_{\text{pixel}}(i)}{N_{\text{data}}(b_i - a_i)} = \frac{N_{\text{pixel}}(i) \cdot K}{N_{\text{data}}(Max - Min)}$$

where $N_{\text{data}}$ is the total number of the data.

5. In order to obtain a smooth PDF curve, averaging is taken among the neighborhood:

$$p_{\text{smooth}}(i) = \frac{1}{2Nb_h} \sum_{m=i-Nb_h}^{i+Nb_h} p(m)$$

6. The PDF curve is obtained from the plot of the smoothed and unified histogram, i.e., $p_{\text{smooth}}(i)$ vs. $(b_i - a_i)/2$.

The histogram of a Gaussian process with zero mean and variance of 1 is shown in Fig. 4.4. Its smoothed and unified histogram is shown in Fig. 4.5. The plot of the Gaussian PDF function

$$g(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$$
Figure 4.4: Histogram of a Gaussian process with zero mean and unit variance
Figure 4.5: Smoothed and unified histogram of a Gaussian process with zero mean and unit variance
Figure 4.6: Plot of the PDF of a Gaussian process with zero mean and unit variance
is shown in Fig. 4.6.

Fig. 4.7 shows the smoothed unified histograms of \( g(m, n) \) with different aperture sizes \( M = N \times N \). Fig. 4.8 shows the smoothed unified histograms of the logarithmically transformed version of the speckle noise \( f(m, n) = -\ln[g(m, n)] \). By comparing these curves with Fig. 3.3 and Fig. 4.2, we see that these histograms match the curves of the probability density functions very well, indicating that our simulation closely reflects the mathematical model.

In order to test the bispectral reconstruction approach, the 2-D image of a computer terminal shown in Fig. 4.9 was simulated as follows.

Let \( x_0(m, n) \) denote the intensity of the original image, i.e. the grey level of the pixel at coordinate \( (m, n) \), which takes integer values from 0 to 255. The size of the original image is \( 128 \times 128 \) pixels. It was translated (jittered) randomly, using two independent uniform distributions between 0 and 128 to provide the translation in the \( M \) and \( n \) coordinates, in a dark background of size \( 256 \times 256 \) pixels.

The speckled version of each snapshot was simulated as described in Eq. (4.46) Eq. (4.47). Magnitude reconstruction was done by using the recursive discrete Fourier transform approach. The phase reconstruction was done using the improved fast recursive algorithm. Both are described in section 2.4.

Fig. 4.9 shows the original object (original image) of size \( 128 \times 128 \) pixels. Fig. 4.10 shows the simulated speckle degraded image without a scanning aperture, or with a scanning aperture of a size \( M = 1 \) pixel. Fig. 4.11 shows the corresponding bispectral reconstruction from 100 jittered snapshots with-
Figure 4.7: Smoothed and unified histogram of speckle noise
Figure 4.8: Smoothed and unified histogram of log-speckle noise
out an aperture, or with a scanning aperture of a size $M = 1$ pixel. Fig. 4.12 shows the speckled object with an aperture of size $3 \times 3 (N = 3, M = 9)$ with $h(i,k) = 1$ for $-1 \leq i \leq 1, -1 \leq k \leq 1$ in (4.47). Fig. 4.13 shows the corresponding bispectral reconstruction from 100 jittered snapshots.

4.7 Conclusion

The reconstructed image in Fig. 4.11 is close to the original object. This corresponds to a case where the logarithmically transformed speckle noise has a non-gaussian distribution. It is therefore surprising that, even without an aperture, the object was still reconstructed from its speckled version with a small amount of residual speckle effects. We believe this can be attributed to the fact that the bispectrum of the logarithmically transformed speckle noise was small in magnitude relative to the bispectrum of the object.

In the second case where we simulated a scanning aperture, the logarithmically transformed noise is close to Gaussian and the bispectral reconstruction does indeed work well. The blurring is due to the integrating aperture. The results suggest that the bispectral technique has good potential in reducing the effects of speckle in digitized images of moving objects. The conclusion is based on a multiplicative model for speckle which is valid under circumstances discussed before. For reconstructing the Fourier transform phase from the bispectrum, many techniques have been developed for one-dimensional signals. The approach that we used is an extension of one such technique, namely, the Lii-Rosenblatt approach [10]. This has the advantage of being simple and fast while not requiring a complicated two-dimensional
Figure 4.9: Image of a computer terminal
Figure 4.10: The speckle degraded image of the computer terminal without aperture
Figure 4.11: The bispectral reconstructed image without aperture
Figure 4.12: The speckle degraded image of the computer terminal with an aperture of size $M = 3 \times 3$
Figure 4.13: The bispectral reconstructed image with an aperture of size $M = 3 \times 3$
phase-unwrapping procedure. A two-dimensional version of the least-squares phase reconstruction technique of Haniff [12] will likely be an attractive alternative in terms of minimizing the variance of the phase estimates.

4.8 Results of the Research

The research concentrated on studying of properties, algorithms for the reconstruction and applications of the bispectrum in two dimensional cases. The following work has been performed.

4.8.1 The Reconstruction Algorithm

The reconstruction algorithm research concentrated on a fast algorithm, i.e., discrete Fourier transform approach for magnitude reconstruction and a fast recursive algorithm for phase reconstruction. A fast recursive phase reconstruction algorithm was developed by the author and Raghuveer Extensive computer simulations were performed in order to study the practical feature of these algorithms, such as the speeds, the sensitivity to Gaussian noise and to other types of noises, etc. Some of important results were given in chapter 2.

4.8.2 The Speckle Reduction

The statistics of the speckle were studied. Large amount of computer simulations were exploited. The multiplicative model and for speckle and the Gaussian approximation were studied and simulated.
During the research, an 8-bit, $128 \times 128$ pixels grey level monochrome image was used. Two dimensional version of the bispectral reconstruction algorithm was used. Both theoretical and computer simulation results proved that the bispectral approach performs very well especially when there are motions involved.

It needs to be pointed out that there might be some doubts about the multiplicative model itself. The multiplicative model for speckle is indeed not a universal model [10]. In previous chapters, we have clearly depicted the conditions and assumption such that the speckle can be modeled as multiplicative. However, this multiplicative model has practically been used extensively in many applications such as synthetic aperture radar (SAR), see [24, 28, 29, 30, 31, 19, 32, 10].

### 4.9 Future Research

Future research topics include:

#### 4.9.1 Speckle Reduction

Studies of a real speckle system such as laser illumination system or synthetic aperture radar (SNR) would be interesting and important. The statistics of speckle can be studied and the multiplicative model can be verified, conditions for such a model to right can be studied. The bispectral approach can be tested for the real speckle.
4.9.2 The Bispectral Reconstruction Algorithms

As mentioned before, there are various bispectrum reconstruction approaches and each has its advantages and constraints. Our fast recursive approach, for example, save the amount of computation considerably. However, they only used a small portion of the information contained by the bispectrum. On the other hand, other approaches such as least square approach, used more information contained in the bispectrum, therefore, better result may be achieved. However, the computation would be increased considerably and the phase unwrapping has to be done, which is difficult, especially in two dimensional case. These restraints make is almost impossible to use theses approach in practical applications. Therefore better bispectral reconstruction algorithm is need especially for two dimensional case.
Bibliography


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