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Abstract

We consider the problem of tiling large rectangles using smaller rectangles with the prescribed dimensions 4×6 and 5×7. Problem B-3 on the 1991 William Lowell Putnam Examination asked "Does there exist a natural number L such that if m and n are integers greater than L, then an $m \times n$ rectangle may be expressed as a union of 4×6 and 5×7 rectangles, any two intersect at most along their boundaries?" Narayan and Schwenk showed in 2002 that all rectangles with length and width at least 34 can be partitioned into 4×6 and 5×7 rectangles. We investigate necessary and sufficient conditions for an $m \times n$ rectangle to be tiled with 4×6 and 5×7 rectangles. Ashley et al. answered this question for all but 37 cases. We use an integer linear programming approach to eliminate all but five of these cases.

1 Introduction

Problem B-3 on the 1991 William Lowell Putnam Examination asked: "Does there exist a natural number L, such that if m and n are integers greater than L, then an $m \times n$ rectangle may be expressed as a union of 4×6 and 5×7 rectangles, any two of which intersect at most along their boundaries?". The answer is "yes" and a solution appeared in the American Mathematical Monthly (Klosinski, Alexanderson, and Larson 1992). The published solution printed is an existential proof and shows that any $m \times n$ rectangle can be tiled provided that $L \geq 2214$. Narayan and Schwenk later reduced this bound to $L = 33$ and showed that this new bound is best possible.

However the above results focus primarily on sufficient conditions for a rectangle to be tiled. In this paper we explore necessary and sufficient conditions for an $m \times n$ rectangle to be tiled with 4×6 and 5×7 rectangles. The results mentioned above show that every rectangle with $m \geq 33$ and $n \geq 34$ can be tiled and the 33×33 square cannot be tiled. However there is still the question of which rectangles with one dimension less than 33 can be tiled. In this paper we seek to answer this question. We use an array of methods from combinatorics and integer linear programming to determine whether or not a rectangle can be tiled.

We extend the base of known results to include necessary conditions. The overall goal is to determine whether or not $m \times n$ rectangles with one dimension less than 33 can be tiled.

We investigate necessary and sufficient conditions for an $m \times n$ rectangle to be tiled with 4×6 and 5×7 rectangles. Ashley et al. answered this question for all but 37 cases. Of these 37 cases we use an integer linear program (ILP) to evaluate 36 of the 37 rectangles. The 22×46 rectangle has not yet been solved in addition to the 36 rectangles that were evaluated using an ILP.

17×31	17×66	21×26	22×31	21×34	22×27	22×29	22×31	22×32
22×33	22×39	22×41	22×43	22×53	23×23	23×25	23×32	23×43
23×44	23×45	23×46	23×48	23×50	23×52	23×65	23×67	23×69
23×71	23×73	25×29	26×27	26×33	26×37	27×38	27×41	27×43

Table 1: The 36 rectangles evaluated using an integer linear program

We use an integer linear programming approach to eliminate all but four of these 36 cases.

We will refer to the 4×6 and 5×7 rectangles as "tiles", and if a given rectangle can be expressed as a union of 4×6 and 5×7 tiles, any two of which intersect at most along their boundaries, we will simply say the rectangle can be "tiled" or is "tileable". Since tiles can be rotated we do not distinguish a 4×6 tile from a $6\times$ 4 tile or a 5×7 tile from a $7\times$ 5 tile. When convenient we will refer to a 4×6 tile as an "even" tile and a 5×7 tile as an "odd" tile. The tiling of rectangles has been studied in (Ashley et al.), (Chung, Gilbert, Graham, and van Lint 1982), (Golomb 1994), and (Robinson 1981). This problem is somewhat different in that only two tiles with prescribed dimensions can be used. We note that the areas of the two tiles are relatively prime, a necessary condition. If an integer $p > 1$ divides each area, a tiling of a square with side-length congruent to 1 mod p would not be possible.

We show a rectangle can be tiled by decomposing it into smaller rectangles which can easily be tiled. In searching for these smaller rectangles, it is useful to note that the area A of a rectangle that can be tiled satisfies the equation $A = 24x + 35y$ where x and y are non-negative integers. For example, tiling a 34×37 rectangle would require 32 even tiles and 14 odd tiles since $x = 32$ and $y = 14$ is the unique solution of the equation $34 \times 37 = 1258 = 24x + 35y$ where x and y are non-negative integers. The numbers of the two types of tiles provide a starting point in the search for a possible tiling. Of course if there are no solutions to the equation $A = 24x + 35y$ where x and y are non-negative integers, then a rectangle with area A surely cannot be tiled. We will also use a series of combinatorial arguments to show that rectangles can or cannot be tiled.

2 Preliminaries and Background

Lemma 1 *For natural numbers a and b if $gcd(a, b) = 1$, then every integer $n \geq (a-1)(b-1)$ can be written as a non-negative linear combination and $n = (a-1)(b-1) - 1$ cannot (Sylvester 1884).*

Lemma 1 Is important for finding a greatest lower bound for which all rectangles of that size or larger can be formed by compositions.

In many cases we can construct a large rectangle that can be tiled by combining two small rectangles that can be tiled. We discuss these constructions in our next two lemmas.

Lemma 2 *If an $m \times n_1$ rectangle and an $m \times n_2$ rectangle can be tiled, then any $m \times (an_1 + bn_2)$ rectangle can be tiled, where a and b are non-negative integers.*

Proof. We arrange the rectangles in a row of width m and a length that is a linear combination of n_1 and n_2 with non-negative integer coefficients a and b. ■

Lemma 3 *If an $m_1 \times n_1$ rectangle and an $m_2 \times n_2$ rectangle can be tiled then an $(m_1 + m_2) \times \text{lcm}(n_1, n_2)$ rectangle can be tiled.*

Proof. Arrange the $m_1 \times n_1$ and $m_2 \times n_2$ rectangles so that the width of the rectangle to be tiled is $m_1 + m_2$. Continue the tiling holding the width constant and extending the length. We see that the right side edges of the two rectangles will first line up when the length equals $\text{lcm}(n_1, n_2)$. ■

2.1 Pinwheel Tilings

Many of the rectangles formed using the constructions from Lemmas 1, 2, and 3 have "fault lines", meaning that the rectangle has a subrectangle with the same length or width as the original rectangle. However a tileable rectangle may not have this form. An example of such a tiling is shown in Figure 1.

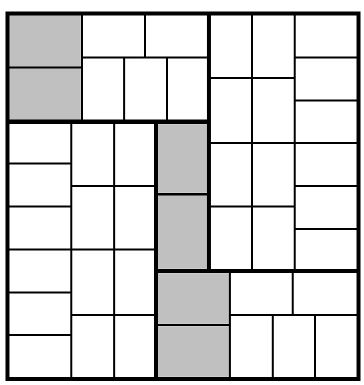

Figure 1: Pinwheel tiling of a 34×33 rectangle with 4×6 tiles in white and 5×7 tiles in grey.

We give the general form for a pinwheel tiling in Figure 2.

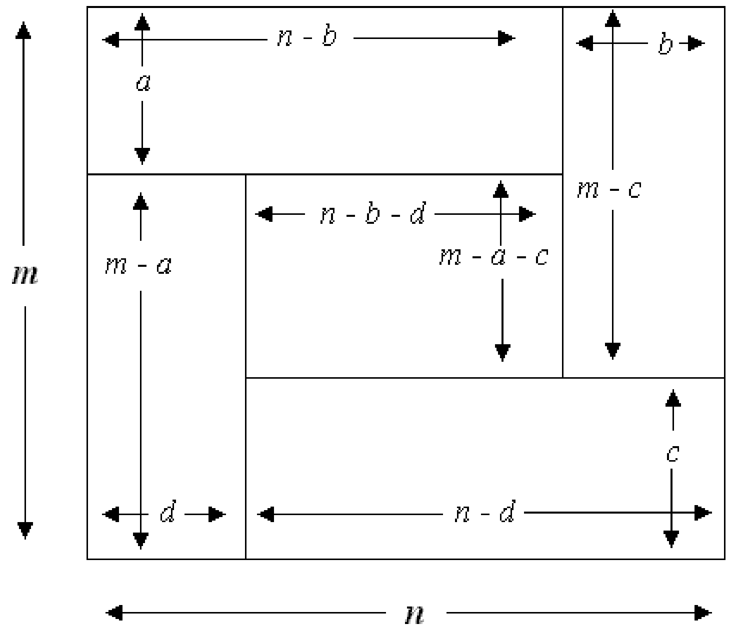

Figure 2: Pinwheel construction $P(a;b;c;d)$ for an $m \times n$ rectangle.

We denote such a "pinwheel" design by $P(a;b;c;d)$ where a, b, c and d are defined in Figure 2.

Lemma 4 (Pinwheel) *An $m \times n$ rectangle can be tiled, if there exist integers $a, b, c,$ and d such that the five rectangles $a \times (n-b)$, $(m-c) \times b$, $c \times (n-d)$, $(m-a) \times d$, $(m-a-c) \times (n-b-d)$ can be tiled.*

Proof. The proof follows immediately from Figure 2. ■

3 Sufficient Conditions

3.1 The General Store of Rectangles

The General Store is a framework for rectangle decompositions. In this section we create a list of tileable rectangles. In our list we will keep the width m fixed, and then give values of n for which the rectangle $m \times n$ can be tiled. The third column labeled "$n \geq$" contains the value where all rectangles of equal or greater length can be tiled. If the length n is less than the value in the third column or there is no value in the third column, the the second column gives a list of lengths that form tileable rectangles for a, b, c, d, e, f, and g are non-negative integers.

m	n	$n \geq$
4	$6a$	–
5	$7a$	–
6	$4a$	–
7	$5a$	–
8	$6a$	–
9	$42a$	–
10	$7a + 12b$	66
11	$28a + 30b$	–
12	$4a + 6b + 35c$	38
13	$20a + 42b$	–
14	$5a + 12b$	44
15	$7a + 30b$	174
16	$6a + 28b$	–
17	$14a + 18b + 27c + 35d + 37e$	67
18	$4a + 17b + 30c$	44
19	$10a + 23b + 26c + 29d + 35e + 42f$	68
20	$6a + 7b$	30
21	$5a + 18b + 23c + 28d + 42e + 44f$	40
22	$12a + 14b + 30c + 35d + 45e + 55f$	54
23	$19a + 21b + 28c + 30d + 34e + 36f + 42g$	74
24	$4a + 6b + 35c$	38
25	$7a + 18b + 20c + 30d$	34
26	$10a + 12b + 28d + 35e + 53f$	38
27	$14a + 17b + 25c + 30d + 33e + 36f$	47
28	$5a + 6b$	20
29	$14a + 30b + 33c + 34d + 35e + 41f + 45g$	28
30	$4a + 7b$	18
31	$10a + 28b + 35c + 42d$	20
32	$6a + 14b + 35c + 44e$	34

Table 2: The general store list

For the cases where m is small there are few configurations of the tiling of an $m \times n$ rectangle using 4×6 and 5×7 tiles. The cases where $4 \leq m \leq 7$ are trivial. The cases where $8 \leq m \leq 16$ are easy to show. We consider different combinations of $4, 5, 6$, and 7 that add up to m. For each combination we investigate the least common multiple of the resulting lengths. Since the width is small, it forces very limited configurations (in some cases unique).

When $m \geq 17$ the configurations become more complicated and more possibilities for n arise. These are discussed below. In each case, we either explicitly describe the tiling, or give a decomposition into subrectangles that can be easily tiled. For completeness, we include the details.

3.1.1 m=17

We will show that any $17 \times n$ rectangle can be tiled for any $n = 14a + 18b + 27c + 35d + 37e$ where a, b, c, d, e, f, and g are non-negative integers.

- A 17×14 rectangle can be tiled be joining a 5×14 rectangle with a 12×14 rectangle.
- A 17×18 rectangle is shown in Figure 3.

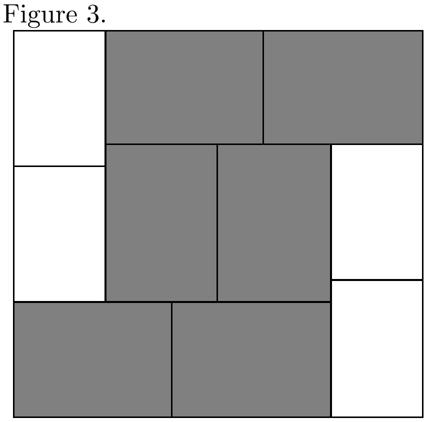

Figure 3: A 17 x 18 rectangle with 4 x 6 tiles in white and 5 x 7 tiles in grey

- A 17×27 rectangle can be tiled using the pinwheel $P(5; 6; 5; 6)$.
- A 17×35 rectangle can be tiled by joining a 5×35 and a 12×35.
- A 17×37 rectangle can be tiled using the pinwheel $P(5; 16; 5; 16)$.

3.1.2 m=18

We will show that any $18 \times n$ rectangle can be tiled for any $n = 4a + 17b + 30c$ where a, b,and c are non-negative integers.. The case of 18×4 is trivial and the 18×17 was covered in 17×18.

- An 18×30 rectangle can be tiled by combining a 14×30 rectangle and a 4×30 rectangle.

3.1.3 m=19

We will show that any $19 \times n$ rectangle can be tiled for any $n = 10a + 23b + 26c + 29d + 35e + 42f$ where $a, b, c, d, e,$and f are non-negative integers.. A 19×10 rectangle can be constructed by joining a 7×10 rectangle and a 12×10 rectangle.

- A 19×23 rectangle can be tiled using the pinwheel $P(7; 8; 7; 8)$.
- A 19×26 rectangle can be tiled using the pinwheel $P(7; 12; 7; 12)$.
- A 19×29 rectangle is can be tiled using the pinwheel $P(7; 8; 7; 14)$.
- A 19×35 rectangle can be tiled by joining a 7×35 and a 12×35.
- A 19×42 rectangle can be tiled by joining a 9×42 and a 10×42.

3.1.4 m=20

The fact that any $20 \times n$ rectangle can be tiled for any $n = 6a + 7b$ where a and b are non-negative integers. follows directly from Lemma 2.

3.1.5 m=21

We will show that any $21 \times n$ rectangle can be tiled for any $n = 5a + 18b + 42c + 44d$ where $a, b, c,$and d are non-negative integers.. For completeness we include the details. The case of a 21×5 is trivial.

- A 21×18 rectangle can be constructed by joining a 4×18 rectangle and a 17×18 rectangle.
- A 21×42 rectangle can be constructed by joining a 9×42 rectangle and an 12×42 rectangle.
- A 21×44 rectangle can be constructed by joining a 6×44 rectangle and an 15×44 rectangle.

3.1.6 m=22

We will show that any $22 \times n$ rectangle can be tiled for any $12a + 14b + 30c + 35d + 45e + 55f$ where $a, b, c, d, e,$and f are non-negative integers..

- A 22×12 rectangle can be constructed by joining three 6×12 rectangles and a 4×12 rectangle.
- A 22×14 rectangle can be constructed by joining a 12×14 rectangle and a 10×14 rectangle.
- A 22×30 rectangle can be constructed by joining a 4×30 rectangle and a 18×30 rectangle.
- A 22×35 rectangle can be constructed by joining a 12×35 rectangle and a 10×35 rectangle.
- A 22×45 rectangle can be constructed by joining a 12×45 rectangle, a 10×45 rectangle and a 10×21 rectangle.
- A 22×55 rectangle can be constructed by joining a 12×55 rectangle and a 10×55 rectangle.

3.1.7 m=23

We will show that any $23 \times n$ rectangle can be tiled for any $19a + 21b + 28c + 30d + 34e + 36f$ where $a, b, c, d, e,$and f are non-negative integers..

- A 23×19 rectangle was covered above in 19×23
- A 23×21 rectangle can be constructed by joining a 18×21 and a 5×21
- A 23×28 rectangle can be constructed by joining an 18×28 and a 5×28
- A 23×30 rectangle can be constructed by joining an 11×30 and a 12×30
- A 23×34 rectangle can be constructed using a pinwheel, $P(7; 14; 11; 6)$.
- A 23×36 rectangle can be constructed by joining a 19×36 and a 4×36.

3.1.8 m=24

The cases involving 24×4 and 24×6 are trivial. The 24×35 rectangle can be formed by joining two 12×35 rectangles.

3.1.9 m=25

$n = 7a + 18b + 20c + 30d$

25×7 is trivial

$25 \times 18 = (17 \times 18) + (8 \times 18)$

$25 \times 20 = (18 \times 20) + (7 \times 20)$

$25 \times 30 = (7 \times 30) + (18 \times 30)$

3.1.10 m=26

$n = 10a + 12b + 28d + 35e + 53f$

$26 \times 10 = (14 \times 10) + (12 \times 10)$

$26 \times 12 = (14 \times 12) + (12 \times 12)$

$26 \times 28 = (10 \times 28) + (16 \times 28)$

$26 \times 35 = (12 \times 35) + (14 \times 35)$

$26 \times 53 = (12 \times 53) + (14 \times 53)$

3.1.11 m=27

$n = 14a + 17b + 30c + 25d + 33e + 36f$

$27 \times 14 = (14 \times 12) + (14 \times 15)$

27×17 was covered already in 17×27

$27 \times 25 = (7 \times 25) + (20 \times 25)$

$27 \times 30 = (7 \times 30) + (20 \times 30)$

$27 \times 33 = 2(10 \times 19) + 2(17 \times 14) + (5 \times 7)$

$27 \times 36 = (19 \times 36) + (8 \times 36)$

3.1.12 m=28

$n = 5a + 6b$

Any $28 \times n$ rectangle can be tiled for any $n = 5a + 6b$ where a,and b are non-negative integers. follows directly from Lemma 2.

3.1.13 m=29

$n = 14a + 30b + 33c + 34d + 35e + 41f + 45g$

$29 \times 14 = 14 \times 29$

$29 \times 30 = (30 \times 8) + (30 \times 21)$

$29 \times 33 = (19 \times 10) + (19 \times 23) + (10 \times 33)$

$29 \times 35 = (10 \times 35) + (19 \times 35)$

$29 \times 41 = (10 \times 21) + (21 \times 15) + 14 \times 28) + (13 \times 20) + (6 \times 4)$

$29 \times 45 = (10 \times 45) + (19 \times 45)$

3.1.14 m=30

$n = 4a + 7b$

Any $30 \times n$ rectangle can be tiled for any $n = 4a + 7b$ where a,and b are non-negative integers. follows directly from Lemma 2.

3.1.15 m=31

$n = 10a + 28b + 35c + 42d$

$31 \times 10 = (7 \times 10) + (24 \times 10)$

$31 \times 28 = (11 \times 28) + (20 \times 28)$

$31 \times 35 = (12 \times 35) + (19 \times 35)$

$31 \times 42 = (12 \times 42) + (19 \times 42)$

3.1.16 m=32

$n = 6a + 14b + 35c + 51d$

32×6 is trivial

$32 \times 14 = (12 \times 14) + (20 \times 14)$

$32 \times 35 = (12 \times 35) + (20 \times 35)$

$32 \times 51 = (12 \times 51) + (20 \times 51)$

We use Lemma 1 to establish a lower bound for n. For example when $m = 10$, $n = 7a+12b$, the application of Lemma 1 gives us that any $m \times n$ rectangle can be tiled for any $n \geq (7-1)(12-1) = 66$. Table 2 contains rectangles that are known to be tileable either by fault-line decompositions, pinwheel formulations, or by physical construction. As a consequence of Table 2 we can reach a definitive answer on whether or not a rectangle can be tiled in all but a mere 37 cases. Complete details are given in Appendix B.

4 Necessary Conditions

4.1 Decomposition Methods

We present another method for showing that a rectangle cannot be tiled. The idea is that "impossible - possible" = "possible". We formally state this in our next lemma.

Lemma 5 *If a $m \times n$ rectangle cannot be tiled, and a $m \times n_1$ rectangle can be tiled, then a $m \times (n - n_1)$ rectangle cannot be tiled.*

Proof. We prove this by contradiction. If a $m \times (n - n_1)$ rectangle could be tiled, we could join it to a tiling of a $m \times n_1$ to form a tiling for a $m \times n$ rectangle. This is impossible. ■

4.2 Coloring Arguments

We invoke a series of coloring arguments to show that certain rectangles can not be tiled. In each of the cases we will either use a row coloring or a column coloring. We record the number of cells corresponding to each color. Then we consider the same coloring imposed on the individual tiles. The sum of the color quantities over the individual tiles must equal the color quantities of the original rectangle. We use this method to give a series of rectangles that can not be tiled.

We give an example of a row-coloring that uses 5 colors. Other row and column colorings were done in a similar fashion.

The number of different tiles would have to satisfy the color constraints. That is, the number of cells of each color in the set of tiles must equal the number of cells of each color in the rectangle.

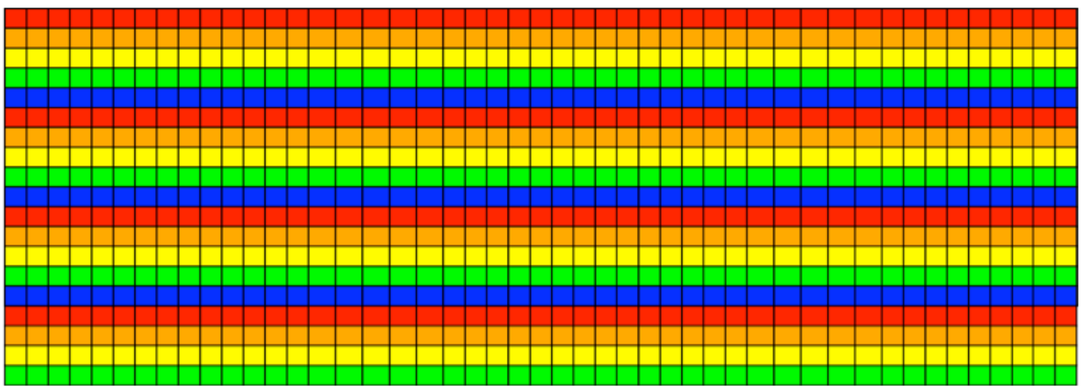

Figure 4: 5 row coloring of the 19×47rectangle

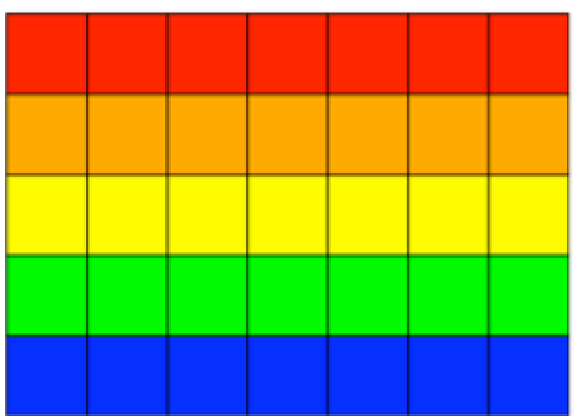

Figure 5: Horizontal odd tile with a 5 row coloring labeled $\langle 7,7,7,7,7\rangle$

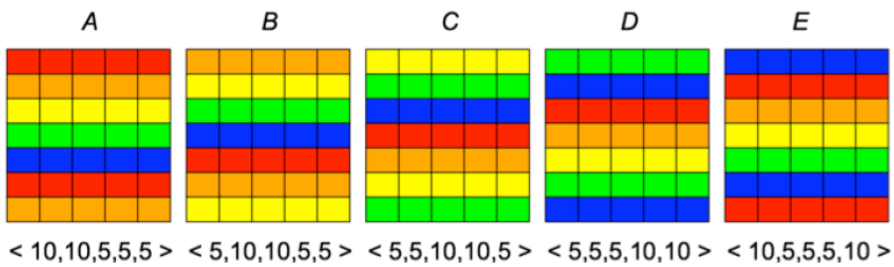

Figure 6: Possible expressions of the vertical odd tiles

Theorem 6 *The 25×31 rectangle cannot be tiled using 4×6 and 5×7 tiles.*

Proof. There is a unique way of expressing the area of this rectangle as a linear combination of the areas using nonnegative integer coefficients: $25 \times 31 = 775 = 5(35) + 25(24)$. This means that a tiling must use 5 odd tiles and 25 even tiles. Let H denote the number of odd tiles with their side of length 7 positioned horizontally and let V denote the number of tiles with their side of length 7 positioned vertically. Thus $H + V = 5$.

We note that if $H \leq 2$ then at least 2 columns will not contain an odd tile. This would be impossible since an odd dimension column can not be filled using only even dimension tiles. Hence $H \geq 3$. We are left with the following three cases: $H = 5$, $V = 0$; $H = 4$, $V = 1$; and $H = 3$, $V = 2$. We consider each of these cases separately.

♦ $H = 5, V = 0$. Since each odd tile covers 7 of the rectangles 31 columns there must be a column that intersects with more than one odd tile. Since the width of the rectangle is also odd it must be that each column intersects with an odd number of odd tiles. Hence there must be a column that contains at least three odd tiles. We note that for three odd tiles to overlap in a single column, the three tiles can cover at most 13 columns. This would leave 2 tiles that must be used to cover the remaining 18 tiles which is impossible.

♦ $H = 4, V = 1$. Color the rows so that for $0 \leq i \leq 24$, row $i+1$ gets color $1 + (i \bmod 5)$ for distinct colors $1, 2, 3, 4$, and 5. There will be 125 cells colored for each of the colors $1, 2, 3, 4$ and 5. We represent the number of cells for each color in the rectangle by the color vector $\langle 155, 155, 155, 155, 155 \rangle$. We will proceed with a parity (odd / even) argument where o will represent and odd number and e will represent an even number. We convert the vector $\langle 155, 155, 155, 155, 155 \rangle$ to $\langle o, o, o, o, o \rangle$.

Since each horizontal tile covers 7 cells of each color, we can represent its contribution by the color vector $\langle 7, 7, 7, 7, 7 \rangle$. Subtracting the contributions of the four horizontal tiles from $\langle o, o, o, o, o \rangle$ leaves the color vector $\langle o, o, o, o, o \rangle$. Each vertical tile will contribute 10 of two colors and 5 of three other colors. Subtracting the contribution of one vertical tile from $\langle o, o, o, o, o \rangle$ will leave a color vector containing two odd numbers. Since this color vector can not be expressed using only even tiles, this case is impossible.

♦ $H = 3, V = 2$. Color the rows so that for $0 \leq i \leq 24$, row $i+1$ gets color $1+(i \bmod 7)$ for distinct colors $1, 2, 3, 4, 5, 6$ and 7. There will be 124 cells colored for each of the colors $1, 2, 3, 4$ and 93 cells colored for each of the colors $5, 6$, and 7. We represent the number of cells for each color in the rectangle by the color vector $\langle 124, 124, 124, 124, 93, 93, 93 \rangle$. We will proceed with a parity (odd / even) argument where o will represent and odd number and e will represent an even number. We convert the vector $\langle 124, 124, 124, 124, 93, 93, 93 \rangle$ to $\langle e, e, e, e, o, o, o \rangle$. Since each vertical tile covers 5 cells of each color, its contribution can be represented by the color vector $\langle 5, 5, 5, 5, 5, 5, 5 \rangle$. Subtracting the contributions of the two vertical tiles from $\langle e, e, e, e, o, o, o \rangle$ leaves the color vector $\langle e, e, e, e, o, o, o \rangle$.

Each horizontal tile covers 7 cells for each of 5 different colors, and 0 cells for each of the two remaining colors. For a horizontal tile there are seven possible color vectors depending on the position in which a tile is placed. These configurations are defined in the table below:

$A = \langle o, o, o, o, o, e, e \rangle$
$B = \langle e, o, o, o, o, o, e \rangle$
$C = \langle e, e, o, o, o, o, o \rangle$
$D = \langle o, e, e, o, o, o, o \rangle$
$E = \langle o, o, e, e, o, o, o \rangle$
$F = \langle o, o, o, e, e, o, o \rangle$
$G = \langle o, o, o, o, e, e, o \rangle$

Table 3: Tile rotations for seven coloring

The only configurations that can satisfy $\langle \_, \_, \_, \_, o, o, o \rangle$ are $(A + F) = \langle e, e, e, o, o, o, o \rangle$ or $(B + G) = \langle o, e, e, e, o, o, o \rangle$. Subtracting either of these from $\langle e, e, e, e, o, o, o \rangle$ will leave one odd color in the vector. Since this color vector can not be expressed using only even tiles, this case is impossible.

Since all of the cases are impossible, the tiling of a 25×31 rectangle using 4×6 and 5×7 rectangles is therefore impossible. ■

Theorem 7 *A* 17×23 *rectangle can not be tiled using* 4×6 *and* 5×7 *tiles.*

Proof. There is a unique way of expressing the area of this rectangle as a linear combination of the areas using nonnegative integer coefficients: $17 \times 23 = 391 = 5(35) + 9(24)$. This means that a tiling must use 5 odd tiles and 9 even tiles. Let H denote the number of odd tiles with their side of length 7 positioned horizontally and let V denote the number of tiles with their side of length 7 positioned vertically. Thus $H + V = 5$.

Color the rows so that for $0 \leq i \leq 16$, row $i + 1$ gets color $1 + (i \bmod 5)$ for distinct colors $1, 2, 3, 4$, and 5. This yields the color vector $\langle 92, 92, 69, 69, 69 \rangle$. We will proceed with a parity (odd / even) argument where o will represent and odd number and e will represent an even number. We convert the vector $\langle 92, 92, 69, 69, 69 \rangle$ to $\langle e, e, o, o, o \rangle$. We note that each horizontal tile corresponds to the color vector $\langle 7, 7, 7, 7, 7 \rangle$ or $\langle o, o, o, o, o \rangle$ and each vertical corresponds to one of the rotations in Table 4. We first consider the cases where $H = 5$ or $H = 3$.

$A = \langle o, o, o, e, e \rangle$
$B = \langle o, o, e, e, o \rangle$
$C = \langle o, e, e, o, o \rangle$
$D = \langle e, e, o, o, o \rangle$
$E = \langle e, o, o, o, e \rangle$

Table 4: Tile rotations for five colorings

♦ $H = 5, V = 0$. Subtracting five copies of $\langle 7, 7, 7, 7, 7 \rangle$ from the original color vector leaves $\langle o, o, e, e, e \rangle$, which can not be expressed using the contributions of only even tiles. Hence this case is impossible.

♦ $H = 3, V = 2$. Subtracting three copies of $\langle 7, 7, 7, 7, 7 \rangle$ from the original color vector leaves $\langle o, o, e, e, e \rangle$. The only tile combinations that will satisfy, $\langle o, o, \_, \_, \_ \rangle$ are $(A+D) = \langle o, o, e, o, o \rangle$, $(B+D) = \langle o, o, o, o, e \rangle$, $(C + E) = \langle o, o, o, e, o \rangle$ .. Since this color vector cannot be expressed using only even tiles, this case is impossible.

Next we color the rows so that for $0 \leq i \leq 16$, row $i + 1$ gets color $1 + (i \bmod 7)$ for distinct colors $1, 2, 3, 4, 5, 6$ and 7. This yields the color vector $\langle 69, 69, 69, 46, 46, 46, 46 \rangle$. We will proceed with a parity (odd / even) argument where o will represent and odd number and e will represent an even number. We convert the vector $\langle 69, 69, 69, 46, 46, 46, 46 \rangle$ to $\langle o, o, o, e, e, e, e \rangle$. We note that each vertical tile corresponds to the color vector $\langle 5, 5, 5, 5, 5, 5, 5 \rangle$ or $\langle o, o, o, o, o, o, o \rangle$ and each horizontal tile covers 7 cells for each of 5 different colors, and 0 cells for each of the two remaining colors. For a horizontal tile there are seven possible color vectors depending on the position in which a tile is placed. These configurations are defined in the table 3. We consider the cases where $H = 0, H = 1$, and $H = 2$ below.

♦ $H = 0, V = 5$. Subtracting five copies of $\langle 5, 5, 5, 5, 5, 5, 5 \rangle$ from $\langle o, o, o, e, e, e, e \rangle$ leaves a color vector that is not all even numbers, which can not be expressed using only even tiles.

♦ $H = 1, V = 4$. Subtracting four copies of $\langle 5, 5, 5, 5, 5, 5, 5 \rangle$ from $\langle o, o, o, e, e, e, e \rangle$ leaves $\langle o, o, o, e, e, e, e \rangle$. Subtracting any rotation of $\langle 7, 7, 7, 7, 7, 0, 0 \rangle$ from Table 3 is guaranteed to leave at least one odd number in the resulting color vector.

♦ $H = 2, V = 3$. Since the length and width of the rectangle are both odd, each row and each column must intersect an odd number of odd tiles. Since there are only 17 rows, each column can contain at most one vertical tile. Hence the three vertical tiles cannot intersect any of the same columns. Since each row and each column must intersect an odd number of odd tiles and the vertical tiles, cover 15 columns, the two horizontal tiles must intersect the exact same set of columns. This leaves at least one column that does not intersect any odd tiles. Hence a tiling with this configuration is impossible

♦ $H = 4$,$V = 1$. We first note that since $4(7) + 1(5) > 23$ there must be some column containing more than one odd tile. We note that since the length of the rectangle is odd, each column must contain an odd number of odd tiles. The only manner in which three odd tiles can fit into a single column of dimension 17, is to have two horizontal tiles and one vertical tile. This arrangement is guaranteed to leave a gap of 7 cells on side of the vertical tile. This gap can only be filled with another vertical tile. Since we have only one vertical tile, this case is impossible. ■

Theorem 8 *A* 19×48 *rectangle can not be tiled.*

Proof. There are two ways of expressing the area of this rectangle as a linear combination of the areas using nonnegative integer coefficients: $19 \times 48 = 912 = 38(24) + 0(35) = 3(24) + 24(35)$. The first combination would mean that a tiling would only contain even tiles, which is impossible since the width is odd. Hence we will consider the other combination of 3 even tiles and 24 odd tiles.

We first consider the case where at least one of the even tiles is arranged vertically. If a column were to intersect exactly one even tile placed vertically, this would leave 13 cells to be covered by odd tiles, which is impossible. A column can not intersect a vertical tile and a horizontal tile, since this would leave 9 cells to be covered by odd tiles, which again is impossible. If a column intersected one vertical even tile and two horizontal tiles, then a neighboring column of the vertical tile would have 11 cells to be covered by odd tiles, which is impossible.

After removing the above cases, we are left with one arrangement where two of the even tiles are stacked to form a 12×4 rectangle, and the other tile is positioned horizontally in a different set of columns. We observe that it is impossible to place the 4×6 rectangle along a side of the rectangle surrounded by only odd tiles. Hence the 4×6 rectangle must lie in the interior of the rectangle. Since the width of the rectangle is 19 there must be five cells between the 4×6 tile and one vertical border and ten cells between the 4×6 and the other vertical border. Without loss of generality we will assume that the 4×6 tile is closer to the bottom edge than the top. Since there are five cells below the even tile and ten cells above the even tile, it must be that the tiles placed above and below the even time must be odd tiles positioned horizontally. Note that the tile immediately above the 4×6 tile must be a 5×7 tile, which must extend beyond one of the edges of the 4×6 tile. This creates a gap of 9 cells which must be filled using only odd tiles, which is impossible.

We next consider the cases where all three of the even tiles are placed horizontally. Each even tile must be five cells from one border and ten cells from the other border. Hence we are left with two configurations. In the first arrangement all three even tiles occupy the exact same set of four rows. In the second arrangement two even tiles occupy the same set of rows and the third tile occupies a different set of rows located five cells away from the first set of rows. Both of these cases can be handled using the method from the above paragraph. A 5×7 tile must be positioned along side one of the 4×6 tiles so that it overhangs, leaving a gap of 9 cells that must be filled using only odd tiles. This completes the proof. ■

Theorem 9 *A 19×67 rectangle can not be tiled.*

Proof. There are two ways of expressing the area of this rectangle as a linear combination of the areas using nonnegative integer coefficients: $19 \times 67 = 1273 = 37(24) + 11(35) = 2(24) + 35(35)$.

We first consider the first combination of areas. We note that each column must contain at least one odd tile. Hence $H \geq 6$. Next, we color the rows so that for $0 \leq i \leq 18$, row $i + 1$ gets color $1 + (i \bmod 5)$ for distinct colors $1, 2, 3, 4$, and 5. This yields the color vector $\langle 268, 268, 268, 268, 201 \rangle$. We will proceed with a parity (odd / even) argument where o will represent and odd number and e will represent an even number.

We convert the vector $\langle 268, 268, 268, 268, 201\rangle$ to $\langle e, e, e, e, o\rangle$. We note that each horizontal tile corresponds to the color vector $\langle 7, 7, 7, 7, 7\rangle$ or $\langle o, o, o, o, o\rangle$ and each vertical corresponds to one of the rotations in Table 4. We consider separate cases depending on the value of H.

♦ $H = 11, V = 0$. Subtracting 11 copies of $\langle 7, 7, 7, 7, 7\rangle$ from the original color vector leaves an odd number.

♦ $H = 10, V = 1$. Subtracting 10 copies of $\langle 7, 7, 7, 7, 7\rangle$ from the original color vector leaves $\langle e, e, e, e, o\rangle$. Subtracting any rotation from Table 4 will always leave at least one odd number in the resulting color vector.

♦ $H = 8, V = 3$. Subtracting 8 copies of $\langle 7, 7, 7, 7, 7\rangle$ from the original color vector leaves $X = \langle e, e, e, e, o\rangle$.

Since the last in the first position is odd, any decomposition must include one of the vectors $B = \langle o, o, e, e, o\rangle$, $C = \langle o, e, e, o, o\rangle$, or $D = \langle e, e, o, o, o\rangle$. This leaves $X - B = \langle o, o, e, e, e\rangle$, $X - C = \langle o, e, e, o, e\rangle$, and $X - D = \langle e, e, o, o, e\rangle$. Subtracting any vertical odd tile from any of these three vectors will not result in a vector with total of three odd numbers, all appearing consecutively. This the subtraction of any additional vertical odd tile will not result in a color vector with only even numbers.

♦ $H = 9, V = 2$. Next color the columns so that for $0 \leq i \leq 66$, row $i + 1$ gets color $1 + (i \bmod 7)$ for distinct colors $1, 2, 3, 4,\ 5, 6$ and 7. This yields the color vector $\langle 171, 171, 171, 152, 152, 152, 152\rangle$. Each horizontal tile corresponds to the color vector $\langle 5, 5, 5, 5, 5, 5, 5\rangle$ and each horizontal tile corresponds to some rotation from Table 3. Subtracting the contributions of the nine horizontal tiles we are left with $\langle 126, 126, 126, 107, 107, 107, 107\rangle$. Subtracting any vector corresponding to an odd tile does not result in a vector containing only three odd numbers all appearing in consecutive positions. Hence the subtraction of one more tile will not result in a vector with all even numbers.

♦ $H = 7, V = 4$. Since $7(7) + 4(5) = 69 > 67$. There must be at least one column that intersects with more than one odd tile. Since each column must contain an odd number of odd tiles, there must be a column containing at least three odd tiles. If a column contains three vertical odd tiles then these three tiles cover at most 9 nine columns.

Next we consider a tiling with two even tiles and 35 odd tiles. We first will show that neither of the tiles may be placed horizontally. If one or two even tiles are placed horizontally along a border then the tiles immediately above this even tile, must be odd tiles that are placed horizontally. These tiles will overlap along one edge of the even tiles and create a gap of four along an edge, which can not be filled using only odd tiles. Hence it must be that the tiles are placed in the interior. We note that if the even tiles do not share a common row then they must be separated by five rows. If this is the case then $19 - 4 - 4 - 5 = 6$ rows are left to be filled with odd tiles, which is impossible. If two even tiles appear in a 4×12 stack then there must be five rows one side of the stack and ten rows on the other. Without loss of generality assume that the 5 rows are located below the stack. Then the tiles immediately above the stack must be odd tiles placed horizontally. These tiles will extend beyond the stack in one direction creating a gap of 9. This gap can not be filled using only odd tiles.

We next consider the case where one or both of the even tiles are placed vertically. If the tiles do not overlap completely we are left with a gap of 13 rows that can not be filled using only odd tiles. Hence the even tiles must appear in a 12×4 stack. Again we note that since both dimensions are odd, there must be an odd number of odd tiles intersecting every row and every column. Since $19-12 = 7$ and $19 = 2(7)+1(5)$ there are 63 columns that intersect with two vertical tiles and 4 columns that intersect with one vertical tile. This creates a total of $4(1) + 63(2) = 130$ intersections. Since each vertical tile intersects 5 columns, a tiling must use $130/5 = 26$ vertical tiles and hence 9 horizontal tiles. The even stack leaves 12 rows each with 63 cells and 7 rows with 62 cells. We note that $63 = 4(7)+7(5) = 9(7)+0(5)$ and $62 = 6(7)+4(5) = 1(7)+11(5)$. Looking at the rows we see that there are 12 rows that must intersect with at least 4 horizontal odd tiles and seven rows that must intersect with at least one odd horizontal tile. This gives a total of $55/5 = 11$ horizontal tiles, which contradicts the quantity obtained above.

Finally we consider the case where $H = 6$ and $V = 5$. We note that $6(7) + 5(5) = 67$. Hence each column intersects exactly one odd tile. Consequently one odd tile must lie along the left or right side of the rectangle. Without loss of generality, assume it lies on the left side. Then the columns to the immediate right of the placed odd tile must intersect another odd tile. This tile must occupy the same set of rows as the first tile, or we will have a gap of either 5 or 7 between the second tile and the left side that can not be filled using only even tiles. ■

Theorem 10 *A* $15 \times n$ *rectangle can only be tiled if* $n = 7a + 30b$ *where* a *and* b *are non-negative integers.*

Proof. We consider three cases based on the form of the left border of a $15 \times n$ rectangle.

$Case(i)$. The left border is made up of three 5×7 tiles. This results in a 15×7 rectangle.

$Case(ii)$.The left border includes one 4×6 tile, one 5×7 tile and one 6×4 tile. We are then forced to place another 6×4 tile adjacent to the other 6×4 tile. This forces a second 4×6 tile to be adjacent to the first 4×6 tile.This forces a second 5×7 tile to be adjacent to the first 5×7 tile. At each stage one of the three tiles is forced. The right sides of the rectangle will not line up until we get to $\text{lcm}(4, 6, 7) = 42$.

$Case(iii)$.The left border includes two 4×6 tiles and one 7×5 tile. We are forced to place another 7×5 tile adjacent to the other 7×5 tile. This forces two 4×6 tiles to be adjacent to the two 4×6 tiles.This forces a third 5×7 tile to be adjacent to the second 5×7 tile. At each stage one of the three tiles is forced. The right sides will not line up until we get to $\text{lcm}(5, 6) = 30$. ■

Theorem 11 *The* 17×22 *rectangle cannot be tiled.*

Proof. The area of the 17×22 rectangle is 374 and the equation $374 = 24a + 34b$ has the unique solution $a = 1$ and $b = 10$. The single 4×6 tile can either be placed along a border or on the interior. We consider these as two separate cases. If the 4×6 tile is placed along a border it forces a second 4x6 tile to be adjacent to it. Hence the 4×6 tile must be placed on the interior, as illustrated in the figures below. We first consider the case where the even tile lies horizontally (See Figure 7). All of the remaining tiles must be odd. Hence a + $b = 16$. Since 16 cannot be expressed as a linear combination of 5 and 7 with non-negative coefficients , this configuration is impossible. Next we consider the case where the even tile lies vertically (See Figure 8). This would imply that $a + b = 18$. Since 18 cannot be expressed as a linear combination of 5 and 7 with non-negative coefficients, this configuration is also impossible. Hence the 17×22 rectangle cannot be tiled.

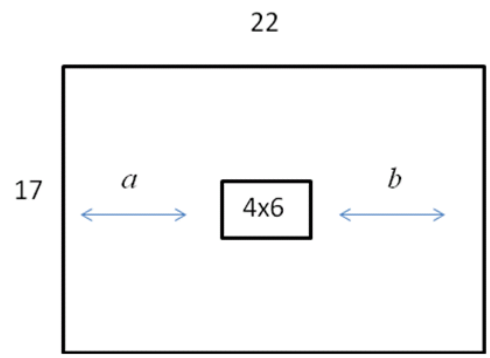

Figure 7: 17×22 rectangle with 4×6 orientation

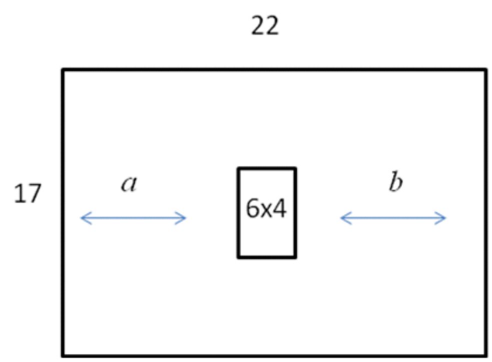

Figure 8: 17×22 rectangle with 6×4 oriantation

■

5 An integer linear program formulation

To assist with the computations, we designed a series of integer linear programs so that these cases could be investigated using computational methods.

A linear program can be written as

$$\max\ cx$$
$$Ax \leq b$$

where A is a matrix and b, c, and x are vectors, and each row of $Ax \leq b$ is a linear constraint on the vector x of variables. If the vector x satisfies all of the constraints then it is called a feasible solution. An integer linear program (ILP) requires that each x_i be an integer.

An unusual property of our formulation is that we are not concerned with the objective function, we simply want to know if at least one feasible solution exists. Our strategy will be to show that a rectangle cannot be tiled by showing that no feasible solution exists after passing the possible solutions through multiple integer linear programs in series.

The 4 and 5 color tiler for the 25×31 case are shown below

The 4 color tiler.

$\max \sum_{i=1}^{13} x_i$

subject to

$$14x_1 + 7x_2 + 7x_3 + 7x_4 + 10x_5 + 5x_6 + 10x_7 + 10x_8 + 6x_9 + 8x_{10} + 4x_{11} + 4x_{12} + 8x_{13} = 217$$

$$7x_1 + 14x_2 + 7x_3 + 7x_4 + 10x_5 + 10x_6 + 5x_7 + 10x_8 + 6x_9 + 8x_{10} + 8x_{11} + 4x_{12} + 4x_{13} = 186$$

$$7x_1 + 7x_2 + 14x_3 + 7x_4 + 10x_5 + 10x_6 + 10x_7 + 5x_8 + 6x_9 + 4x_{10} + 8x_{10} + 8x_{12} + 4x_{13} = 186$$

$$7x_1 + 7x_2 + 7x_3 + 14x_4 + 5x_5 + 10x_6 + 10x_7 + 10x_8 + 6x_9 + 4x_{10} + 4x_{11} + 8x_{12} + 8x_{13} = 186$$

$$35\left(\sum_{i=1}^{8} x_i\right) + 24\left(\sum_{i=9}^{13} x_i\right) = 25 \cdot 31$$

$x_i \geq 0$, integer

The 5 color tiler

$\max \sum_{i=1}^{16} x_i$

subject to

$$7x_1+10x_2+5x_3+5x_4+5x_5+10x_6+6x_7+0x_8+6x_9+6x_{10}+6x_{11}+8x_{12}+4x_{13}+4x_{14}+4x_{15}+4x_{16} = 155$$

$$7x_1+10x_2+10x_3+5x_4+5x_5+5x_6+6x_7+6x_8+0x_9+6x_{10}+6x_{11}+4x_{12}+8x_{13}+4x_{14}+4x_{15}+4x_{16}=155$$

$$7x_1+5x_2+10x_3+10x_4+5x_5+5x_6+6x_7+6x_8+6x_9+0x_{10}+6x_{11}+4x_{12}+4x_{13}+8x_{14}+4x_{15}+4x_{16}=155$$

$$7x_1+5x_2+5x_3+10x_4+10x_5+5x_6+6x_7+6x_8+6x_9+6x_{10}+0x_{11}+4x_{12}+4x_{13}+4x_{14}+8x_{15}+4x_{16}=155$$

$$7x_1+5x_2+5x_3+5x_4+10x_5+10x_6+0x_7+6x_8+6x_9+6x_{10}+6x_{11}+4x_{12}+4x_{13}+4x_{14}+4x_{15}+8x_{16}=155$$

$$35\left(\sum_{i=1}^{6} x_i\right)+24\left(\sum_{i=7}^{16} x_i\right)=25\cdot 31$$

$x_i \geq 0$, integer

Our integer linear programs were investigated using a software package, *What's Best* by *Lindo* where input could be given using an *Excel* spreadsheet.

Our constraints included a color constraint, an area constraint, and then other lower bounds on the number of even and odd tiles. An example of our ILP for a row 5-coloring is given below.

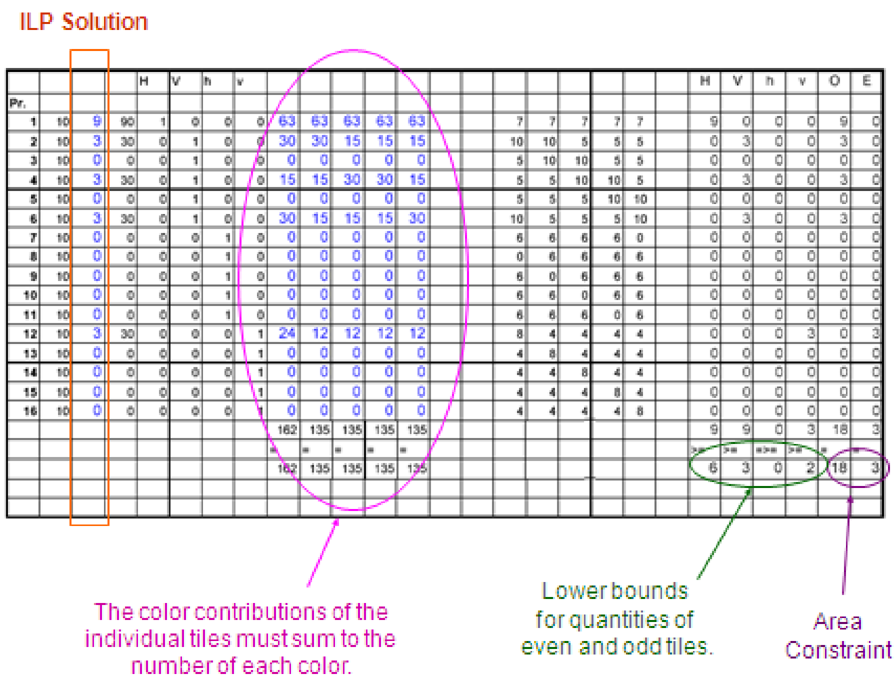

Figure 9: Integer Linear Program for a 5 row coloring

We explore coloring arguments using an integer linear programming approach.

The existing methods were insufficient for determining whether or not the remaining 36 rectangles could be tiled. We created additional tiling formulas in order to run each rectangle through more than one row or column coloring. If a rectangle can be tiled then it must work under all colorings. By running each rectangle through multiple colorings we were able to eliminate all but 4 of the 36 rectangles.

Feasible solution refers to the only possibilities left after the current tiler was used.

17 × 31

Feasible solutions with 7row tiler

H	V	h	v	Rejected by
10	3	0	3	5 row tiler

This is shown in a graphic organizer below. Here we can see that H represents 5×7 tiles in their horizontal orientation, V represents 5×7 tiles in their vertical orientation, h represents 4×6 tiles in their horizontal orientation, and v represents 4×6 tiles in their vertical position.

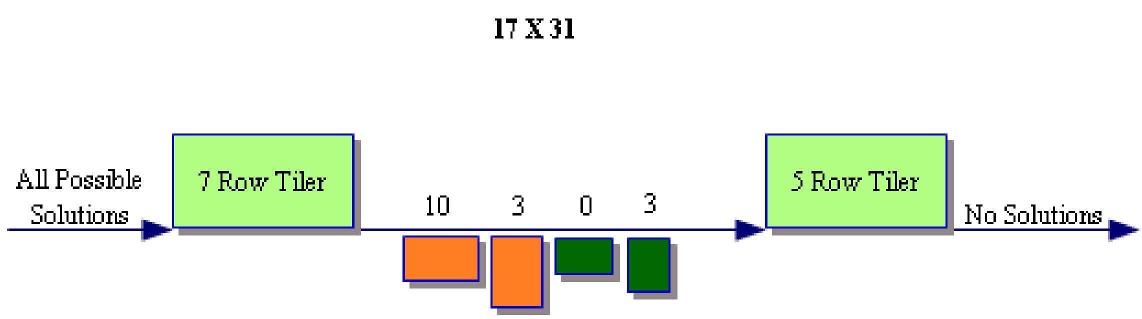

Figure 10: 17×31 rectangle passing through mutliple tilings constraining odd and even tiles.

17x31

Pass through 7 row tiler on Odd

$10, 3$

None pass through 5 row tiler.

This can be shown in a graphic organizer as seen below. Note that when it says "on Odd" it is referring to the Odd ($5x7$) tiles. In the Diagram below you will see that the odd tiles are the only ones being constrained. "on Even" will be done in a similar fashion.

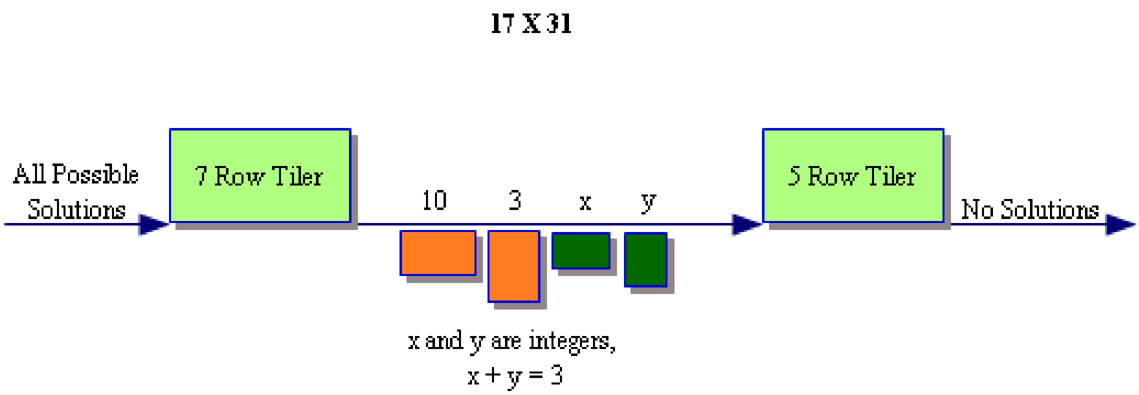

Figure 11: 17×31 rectangle passing through multiple tilings constraining only the odd tiles.

23 × 32

Pass through 4 row tiler on Odd

6, 2

3, 5

1, 7

And pass through 6 row tiler

6, 2

3, 5

And pass through 7 row tiler

6, 2

Feasible solutions with 7 row tiler

H	V	h	v	Rejected by
6	2	18	1	4 row tiler
6	2	17	2	4 row tiler
6	2	16	3	4 row tiler
6	2	15	4	4 row tiler
6	2	14	5	4 row tiler
6	2	12	7	4 row tiler
6	2	11	8	6 row tiler
6	2	10	9	4 row tiler
6	2	9	10	6 column tiler
6	2	8	11	4 row tiler
6	2	7	12	6 row tiler
6	2	6	13	4 row tiler
6	2	5	14	5 row tiler

This can be seen in the graphic organizer below.

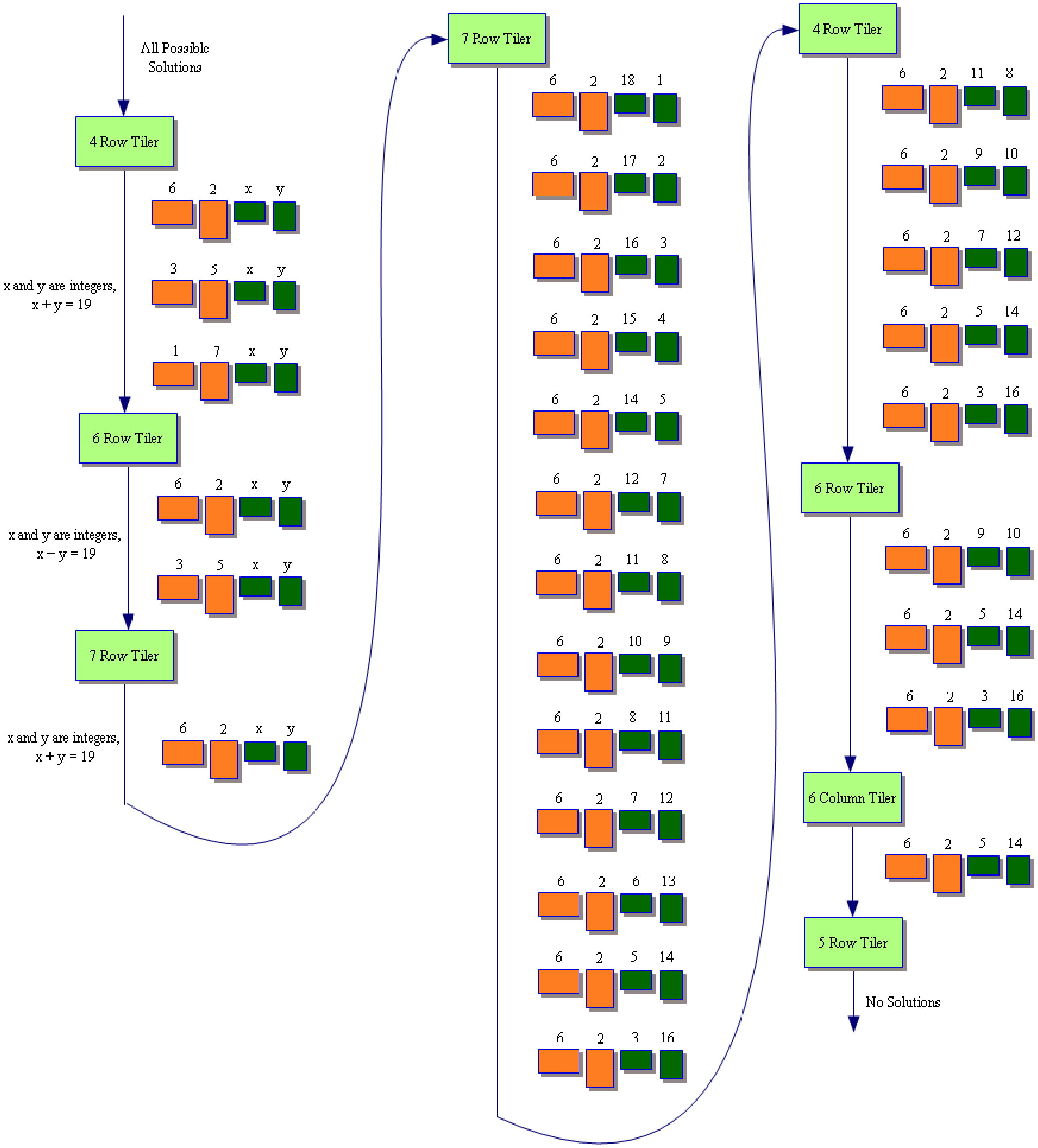

Figure 12: 23 × 32 rectangle passing through multiple tilings, first constraining only odds, then constraining both.

5.1 Applications

Tiling rectangles has several applications in the area of packaging. Rectangular truck trailers and train cars make for prime examples. Tiling can show an optimal way to pack one of these vehicles or if the vehicle can be packed completely full. By expanding the tiling process to a third dimension it becomes easy to see if there is an optimal way to load a pallet with product.

Figure 13: Semi Trailer

Figure 14: Boxcar

Figure 15: Shipping containers

Figure 16: Palletized boxes

There may also be applications to the paper industry where giant roles are cut down to make sheets of paper. If the paper will be used for photography stock, tiling will be able to tell if there will be scraps remaining or if all the paper can be used.

5.2 Conclusion

Through the combining of multiple colorings with integer linear programming, we know that 32 of the 36 remaining rectangles do not have an integer solution. The only cases remaining to be solved are the 23×46, 23×48, 23×65, and 23×69 rectangle. Of all the possible arrangements of the tiles the only possible combinations are listed below for each rectangle.

23×46 Rectangle

H	V	h	v
10	12	3	9

23×48 Rectangle

H	V	h	v
12	12	9	2
12	12	2	9

23×65 Rectangle

H	V	h	v
17	12	11	9
16	13	12	8
16	13	11	9
16	13	8	12
15	14	6	14
14	15	14	6
13	16	11	9
12	17	9	11

23×69 Rectangle

H	V	h	v
17	16	9	9
13	20	12	6

These particular rectangles were difficult for the integer linear program because the number of Odd tiles and Even tiles are both great. With very few possible combinations left to test, it would be possible to eliminate further cases using larger tilers.

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Appendix A

17 × 31
Pass through 17×31 7 row tiler on Odd
$10, 3$
None pass through 17×31 5 row tiler.

17 × 66
Pass through 17×66 7 row tiler on Odd
$27, 3$
$24, 6$
$20, 10$
None pass through 17×66 5 row tiler.

21 × **26**
Pass through 21×26 4 row tiler on Odd
$5, 1$
$3, 3$
None of these pass through the 21×26 5 row tiler

21 × 31
Pass through 21×31 4 row tiler on Odd
$7, 2$
$5, 4$
$3, 6$
None of these pass through the 21×31 6 row tiler

21 × 34
Pass through 21×34 4 row tiler on Odd
$2, 4$
None of these pass through the 21×34 6 row tiler

22 × 27
Pass through 22×27 4 row tiler Odd
$6, 0$
$4, 2$
$2, 4$
$0, 6$
and pass through 22×27 5 row tiler
$2, 4$
$0, 6$
And pass through 22×27 6 row tiler
$2, 4$
None pass through the 22×27 7 row tiler

22 × 29

Pass through 22×29 7 row tiler on Odd.
$10, 0$
$9, 1$
$7, 3$
$4, 6$
$3, 7$
None pass through 22×29 6 row tiler.

22 × 31
Feasible solutions with 22×31 6 row tiler

H	V	h	v	Rejected by
10	4	8	0	22×31 5 row tiler
6	8	8	0	22×31 5 row tiler
6	8	6	2	22×31 7 row tiler
4	10	8	0	22×31 7 row tiler
2	12	7	1	22×31 5 row tiler
2	12	4	4	22×31 7 row tiler

22 × 32
Feasible solutions with 22×32 7 row tiler

H	V	h	v	Rejected by
16	0	3	3	22×32 5 row tiler
16	0	1	5	22×32 5 row tiler
13	3	6	0	22×32 4 row tiler
12	4	4	2	22×32 5 row tiler
12	4	0	6	22×32 6 row tiler
6	10	6	0	22×32 6 row tiler

22 × 33
Feasible solutions with 22×33 7 row tiler

H	V	h	v	Rejected by
16	2	4	0	22×33 6 row tiler
15	3	2	2	22×33 6 row tiler
9	9	4	0	22×33 6 row tiler

22 × 39
Pass through 22×39 4 row tiler on Odd
$6, 0$
$4, 2$
$2, 4$
$0, 6$
And pass through 22×39 5 row tiler
$2, 4$
$0, 6$
And pass through 22×39 6 row tiler
$2, 4$

None pass through the 22×39 7 row tiler

22×41
Pass through 4 row tiler on Odd
$10, 0$
$8, 2$
$6, 4$
$4, 6$
$2, 8$
10.0
And pass through 5 row tiler
$6, 4$
$4, 6$
$2, 8$
$0, 10$
And pass through 6 row tiler
$2, 8$
None pass through 7 row tiler

22×43
Pass through 22×43 4 row tiler on Odds
$14, 0$
$12, 2$
$10, 4$
$8, 6$
$7, 7$
$6, 8$
$5, 9$
$4, 10$
$2, 12$
And pass through 22×43 5 row tiler
$8, 6$
$7, 7$
$6, 8$
$5, 9$
$4, 10$
$2, 12$
And pass through 22×43 7 row tiler
$8, 6$
$7, 7$
$6, 8$
$5, 9$
$4, 10$
Feasible solutions with 22×43 6 row tiler

H	V	h	v	Rejected by
8	6	18	1	22×43 4 row tiler
8	6	15	4	22×43 6 column tiler
8	6	14	5	22×43 5 row tiler
8	6	12	7	22×43 5 row tiler
7	7	18	1	22×43 4 row tiler
7	7	15	4	22×43 4 row tiler
7	7	12	7	22×43 4 row tiler
6	8	19	0	22×43 4 row tiler
6	8	17	2	22×43 4 row tiler
6	8	16	3	22×43 5 row tiler
6	8	14	5	22×43 7 row tiler
6	8	13	6	22×43 5 row tiler
6	8	11	8	22×43 7 row tiler
6	8	10	9	22×43 5 row tiler
6	8	8	11	22×43 7 row tiler
5	9	17	2	22×43 4 row tiler
5	9	14	5	22×43 4 row tiler
5	9	11	8	22×43 4 row tiler
4	10	19	0	22×43 4 row tiler
4	10	16	3	22×43 7 row tiler
4	10	13	6	22×43 7 row tiler
4	10	10	9	22×43 7 row tiler

22×53

Pass through 22×53 4 row tiler on Odds
$10, 0$
$8, 2$
$6, 4$
$4, 6$
$2, 8$
$0, 10$
And pass through 22×53 5 row tiler
$6, 4$
$4, 6$
$2, 8$
$0, 10$
And pass through 22×53 6 row tiler
$2, 8$
None pass through the 22×53 7 row tiler

23×23

Pass through 23×23 5 row tiler on Odd.
$7, 4$
$3, 8$

$2, 9$
$1, 10$
Feasible solutions with 23×23 5 row tiler.

H	V	h	v	Rejected by
7	4	6	0	23×23 4 row tiler
3	8	5	1	23×23 4 row tiler
3	8	1	5	23×23 7 row tiler
2	9	2	4	23×23 4 row tiler
1	10	3	3	23×23 4 row tiler

23×25
Pass through 23×25 5 row tiler on Even
$5, 0$
$4, 1$
$3, 2$
$2, 3$
$0, 5$
Pass through 23×25 7 row tiler
$3, 2$
Feasible solutions with 23×25 7 row tiler.

H	V	h	v	Rejected by
8	5	3	2	23×25 5 row tiler

23×32
Pass through 23×32 4 row tiler on Odd
$6, 2$
$3, 5$
$1, 7$
And pass through 23×32 6 row tiler
$6, 2$
$3, 5$
And pass through 23×32 7 row tiler
$6, 2$
Feasible solutions with 23×32 7 row tiler

H	V	h	v	Rejected by
6	2	18	1	23×32 5 row tiler
6	2	17	2	23×32 5 row tiler
6	2	16	3	23×32 4 row tiler
6	2	15	4	23×32 4 row tiler
6	2	14	5	23×32 4 row tiler
6	2	12	7	23×32 4 row tiler
6	2	11	8	23×32 6 row tiler
6	2	10	9	23×32 4 row tiler
6	2	9	10	23×32 6 column tiler
6	2	8	11	23×32 4 row tiler
6	2	7	12	23×32 6 row tiler
6	2	6	13	23×32 4 row tiler
6	2	5	14	23×32 5 row tiler
6	2	3	16	23×32 6 column tiler

23 × 43
Pass through 23×43 4 row tiler on Odd
$4, 3$
None pass through 23×43 7 row tiler

23 × 44
Pass through 23×44 4 row tiler on Odd
$19, 1$
$17, 3$
$15, 5$
$14, 6$
$13, 7$
$12, 8$
$11, 9$
$10, 10$
$9, 11$
$8, 12$
$7, 13$
$6, 14$
$5, 15$
$4, 16$
$3, 17$
$2, 18$
And pass through 23×44 5 row tiler
$12, 8$
$11, 9$
$10, 10$
$8, 12$
$7, 13$

$6, 14$
$5, 15$
$4, 16$
$3, 17$
$2, 18$
And pass through 23×44 6 row tiler
$12, 8$
$11, 9$
$10, 10$
$8, 12$
$7, 13$
$6, 14$
$4, 16$
$3, 17$
$2, 18$
And pass through 23×44 7 row tiler
$12, 8$
$10, 10$
$8, 12$
$6, 14$
Feasible solutions with 23×44 4 row tiler

H	V	h	v	Rejected by
12	8	10	3	23×44 7 row tiler
12	8	9	4	23×44 5 row tiler
12	8	8	5	23×44 5 row tiler
12	8	7	6	23×44 6 column tiler
12	8	6	7	23×44 6 column tiler
12	8	5	8	23×44 5 row tiler
12	8	4	9	23×44 5 row tiler
12	8	3	10	23×44 7 row tiler
12	8	2	11	23×44 5 row tiler
12	8	1	12	23×44 5 row tiler
12	8	0	13	23×44 6 row tiler
10	10	13	0	23×44 5 row tiler
10	10	12	1	23×44 6 column tiler
10	10	11	2	23×44 5 row tiler
10	10	10	3	23×44 5 row tiler
10	10	9	4	23×44 7 row tiler
10	10	8	5	23×44 7 row tiler
10	10	7	6	23×44 5 row tiler
10	10	6	7	23×44 5 row tiler
10	10	5	8	23×44 7 row tiler
10	10	4	9	23×44 5 row tiler
10	10	3	10	23×44 5 row tiler
10	10	2	11	23×44 7 row tiler
10	10	1	12	23×44 5 row tiler
10	10	0	13	23×44 5 row tiler
8	12	11	2	23×44 6 column tiler
8	12	10	3	23×44 5 row tiler
8	12	9	4	23×44 7 row tiler
8	12	8	5	23×44 5 row tiler

8	12	7	6	23×44 6 column tiler
8	12	6	7	23×44 6 row tiler
8	12	5	8	23×44 7 row tiler
8	12	4	9	23×44 6 row tiler
8	12	3	10	23×44 5 row tiler
8	12	2	11	23×44 6 row tiler
8	12	1	12	23×44 5 row tiler
8	12	0	13	23×44 5 row tiler
6	14	13	0	23×44 7 row tiler
6	14	12	1	23×44 5 row tiler
6	14	11	2	23×44 6 column tiler
6	14	10	3	23×44 5 row tiler
6	14	9	4	23×44 6 column tiler
6	14	8	5	23×44 7 row tiler
6	14	7	6	23×44 7 row tiler
6	14	6	7	23×44 7 row tiler
6	14	5	8	23×44 5 row tiler
6	14	4	9	23×44 7 row tiler
6	14	3	10	23×44 5 row tiler
6	14	2	11	23×44 5 row tiler
6	14	1	12	23×44 7 row tiler
6	14	0	13	23×44 5 row tiler

23×45

Pass through 23×45 4 row tiler on Odd

$7, 2$

$5, 4$

$0, 9$

And pass through 23×45 5 row tiler

$5, 4$

$0, 9$

And pass through 23×45 6 row tiler

$5, 4$

Feasible with 23×45 7 row tiler

H	V	h	v	Rejected by
5	4	30	0	23×45 4 row tiler
5	4	27	3	23×45 4 row tiler
5	4	26	4	23×45 4 row tiler
5	4	24	6	23×45 4 row tiler
5	4	22	8	23×45 4 row tiler
5	4	21	9	23×45 4 row tiler
5	4	20	10	23×45 6 row tiler
5	4	18	12	23×45 4 column tiler
5	4	16	14	23×45 6 row tiler
5	4	15	15	23×45 5 row tiler
5	4	14	16	23×45 6 row tiler
5	4	12	18	23×45 4 column tiler
5	4	10	20	23×45 6 row tiler
5	4	8	22	23×45 6 row tiler
5	4	6	24	23×45 4 column tiler

23×46

Pass through 23×46 4 row tiler on Odd
$20, 2$
$18, 4$
$17, 5$
$16, 6$
$15, 7$
$14, 8$
$13, 9$
$12, 10$
$11, 11$
$10, 12$
$9, 13$
$8, 14$
$7, 15$
$6, 16$
$5, 17$
$4, 18$
$3, 19$
$2, 20$
And pass through 23×46 5 row tiler
$14, 8$
$10, 12$
$9, 13$
$8, 14$
$6, 16$
$5, 17$
$4, 18$

3, 19
2, 20
And pass through 23×46 7 row tiler
14, 8
10, 12
8, 14
6, 16
Feasible with 23×46 7 row tiler

H	V	h	v	Rejected by
14	8	11	1	23×46 4 row tiler
14	8	9	3	23×46 4 row tiler
14	8	6	6	23×46 5 row tiler
10	12	10	2	23×46 5 row tiler
10	12	8	4	23×46 5 row tiler
10	12	6	6	23×46 6 column tiler
10	12	3	9	
8	14	11	1	23×46 4 row tiler
8	14	8	4	23×46 6 column tiler
6	16	7	5	23×46 5 row tiler

23×48
Pass through 23×48 4 row tiler on Odd
21, 3
20, 4
19, 5
18, 6
17, 7
16, 8
15, 9
14, 10
13, 11
12, 12
11, 13
10, 14
9, 15
8, 16
7, 17
6, 18
5, 19
4, 20
3, 21
1, 23
And pass through 23×48 5 row tiler
12, 12
8, 16

$7, 17$
$6, 18$
$4, 20$
$3, 21$
$1, 23$
And pass through 23×48 7 row tiler
$12, 12$
$8, 16$
$6, 18$
Feasible with 23×48 7 row tiler

H	V	h	v	Rejected by
12	12	9	2	
12	12	5	6	23×48 7 column tiler
12	12	2	9	
8	16	8	3	23×48 7 column tiler
8	16	6	5	23×48 6 column tiler
6	18	11	0	23×48 5 row tiler

23×50(2 5×7 and 45 4×6)
None pass through the 23×50 4 row tiler on Odd

23×50(26 5×7 and 10 4×6)
Pass through 23×50 4 row tiler on Odd
$23, 3$
$22, 4$
$21, 5$
$20, 6$
$19, 7$
$18, 8$
$17, 9$
$16, 10$
$15, 11$
$14, 12$
$13, 13$
$12, 14$
$11, 15$
$10, 16$
$9, 17$
$8, 18$
$7, 19$
$6, 20$
$5, 21$
$4, 22$
$2, 24$
$0, 26$
And pass through 23×50 5 row tiler

$10, 16$
$6, 20$
$5, 21$
$4, 22$
$2, 24$
$0, 26$
And pass through 23×50 7 row tiler
$10, 16$
Feasible with 23×50 7 row tiler

H	V	h	v	Rejected by
10	16	8	2	23×50 6 column tiler

23×52(4 5×7 and 44 4×6)
None pass through the 23×52 4 row tiler on Odd

23×52(28 5×7 and 9 4×6)
Pass through 23×52 7 row tiler on Odds
$28, 0$
$26, 2$
$25, 3$
$22, 6$
$18, 10$
$15, 13$
$8, 20$
And pass through 23×52 4 row tiler
$26, 2$
$22, 6$
$18, 10$
$15, 13$
$8, 20$
And pass through 23×52 5 row tiler
$8, 20$
Feasible with 23×52 7 row tiler

H	V	h	v	Rejected by
8	20	8	1	23×52 5 row tiler

23×65(5 5×7 and 55 4×6)
None pass the 23×65 4 row tiler on Odd

23×65(29 5×7 and 20 4×6)
Pass through 23×65 4 row tiler on Odd
$28, 1$
$26, 3$
$24, 5$
$23, 6$

$22, 7$
$21, 8$
$20, 9$
$19, 10$
$18, 11$
$17, 12$
$16, 13$
$15, 14$
$14, 15$
$13, 16$
$12, 17$
$11, 18$
$10, 19$
$9, 20$
$8, 21$
$7, 22$
$6, 23$
$5, 24$
$4, 25$
$2, 27$
$0, 29$
And pass through 23×65 5 row tiler
$21, 8$
$20, 9$
$19, 10$
$17, 12$
$16, 13$
$15, 14$
$14, 15$
$13, 16$
$12, 17$
$11, 18$
$10, 19$
$9, 20$
$8, 21$
$7, 22$
$6, 23$
$5, 24$
$4, 25$
$2, 27$
$0, 29$
And pass through 23×65 7 row tiler
$21, 8$
$20, 9$
$19, 10$
$17, 12$

$16, 13$
$15, 14$
$14, 15$
$13, 16$
$12, 17$
$11, 18$
$9, 20$
Feasible with 23×65 7 row tiler

H	V	h	v	Rejected by
21	8	19	1	23×65 4 row tiler
21	8	16	4	23×65 5 row tiler
21	8	15	5	23×65 6 column tiler
21	8	14	6	23×65 5 row tiler
21	8	13	7	23×65 5 row tiler
21	8	12	8	23×65 7 column tiler
21	8	11	9	23×65 5 row tiler
21	8	10	10	23×65 5 row tiler
21	8	9	11	23×65 5 row tiler
21	8	8	12	23×65 5 row tiler
21	8	7	13	23×65 5 row tiler
21	8	6	14	23×65 5 row tiler
21	8	5	15	23×65 5 row tiler
21	8	2	18	23×65 5 row tiler
20	9	20	0	23×65 4 row tiler
20	9	19	1	23×65 4 row tiler
20	9	17	3	23×65 4 row tiler
20	9	16	4	23×65 4 row tiler
20	9	14	6	23×65 5 row tiler
20	9	13	7	23×65 7 column tiler
20	9	12	8	23×65 5 row tiler
20	9	11	9	23×65 5 row tiler
20	9	10	10	23×65 7 column tiler
20	9	9	11	23×65 5 row tiler
20	9	8	12	23×65 5 row tiler
20	9	7	13	23×65 5 row tiler
20	9	6	14	23×65 5 row tiler
20	9	5	15	23×65 5 row tiler
20	9	4	16	23×65 5 row tiler

20	9	3	17	23×65 5 row tiler
19	10	19	1	23×65 4 column tiler
19	10	18	2	23×65 5 row tiler
19	10	17	3	23×65 6 column tiler
19	10	15	5	23×65 5 row tiler
19	10	14	6	23×65 7 column tiler
19	10	12	8	23×65 5 row tiler
19	10	11	9	23×65 7 column tiler
19	10	9	11	23×65 5 row tiler
17	12	15	5	23×65 5 row tiler
17	12	14	6	23×65 7 column tiler
17	12	13	7	23×65 7 column tiler
17	12	12	8	23×65 5 row tiler
17	12	11	9	
17	12	10	10	23×65 7 column tiler
17	12	9	11	23×65 5 row tiler
17	12	8	12	23×65 6 column tiler
17	12	7	13	23×65 7 column tiler
17	12	6	14	23×65 5 row tiler
17	12	4	16	23×65 7 column tiler
16	13	18	2	23×65 5 row tiler
16	13	17	3	23×65 4 row tiler
16	13	16	4	23×65 5 row tiler
16	13	14	6	23×65 6 column tiler
16	13	13	7	23×65 5 row tiler
16	13	12	8	
16	13	11	9	23×65 7 column tiler
16	13	10	10	23×65 5 row tiler
16	13	9	11	
16	13	8	12	

16	13	7	13	23×65 5 row tiler
16	13	5	15	23×65 7 column tiler
15	14	18	2	23×65 5 column tiler
15	14	16	4	23×65 7 column tiler
15	14	15	5	23×65 5 column tiler
15	14	14	6	23×65 5 row tiler
15	14	12	8	23×65 5 column tiler
15	14	11	9	23×65 5 row tiler
15	14	9	11	23×65 7 column tiler
15	14	8	12	23×65 5 row tiler
15	14	6	14	
14	15	19	1	23×65 4 row tiler
14	15	16	4	23×65 5 column tiler
14	15	14	6	
14	15	13	7	23×65 5 column tiler
14	15	12	8	23×65 5 row tiler
14	15	10	10	23×65 7 column tiler
13	16	20	0	23×65 4 row tiler
13	16	17	3	23×65 5 column tiler
13	16	14	6	23×65 5 column tiler
13	16	12	8	23×65 5 column tiler
13	16	11	9	
13	16	10	10	23×65 5 column tiler
13	16	8	12	
12	17	15	5	23×65 5 row tiler
12	17	9	11	
11	18	16	4	23×65 5 row tiler
11	18	15	5	23×65 5 column tiler
11	18	13	7	23×65 5 column tiler
9	20	18	2	23×65 5 row tiler
9	20	12	8	23×65 5 column tiler

23×67(7 5×7 and 54 4×6)
None pass through the 23×67 4 row tiler on Odd

23×67(31 5×7 and 19 4×6)
Pass through 23×67 4 row tiler on Odd
29, 2
27, 4
26, 5
25, 6
24, 7
23, 8
22, 9

$21, 10$
$20, 11$
$19, 12$
$18, 13$
$17, 14$
$16, 15$
$15, 16$
$14, 17$
$13, 18$
$12, 19$
$11, 20$
$10, 21$
$9, 22$
$8, 23$
$7, 24$
$6, 25$
$5, 26$
$4, 27$
$3, 28$
$1, 30$
And pass through 23×67 5 row tiler
$23, 8$
$19, 12$
$18, 13$
$17, 14$
$15, 16$
$14, 17$
$13, 18$
$12, 19$
$11, 20$
$10, 21$
$9, 22$
$8, 23$
$7, 24$
$6, 25$
$5, 26$
$4, 27$
$3, 28$
$1, 30$
And pass through 23×67 7 row tiler
$23, 8$
$19, 12$
$18, 13$
$17, 14$
$15, 16$
$14, 17$

$13, 18$
$12, 19$
$11, 20$
Feasible with 23×67 5 row tiler

H	V	h	v	Rejected by
23	8	17	2	23×67 5 column tiler
23	8	14	5	23×67 5 column tiler
19	12	18	1	23×67 4 row tiler
19	12	16	3	23×67 4 row tiler
19	12	13	6	23×67 5 column tiler
19	12	12	7	23×67 6 column tiler
19	12	9	10	23×67 6 column tiler
19	12	6	13	23×67 6 column tiler
18	13	19	0	23×67 7 row tiler
18	13	17	2	23×67 5 column tiler
18	13	14	5	23×67 7 row tiler
18	13	13	6	23×67 6 column tiler
18	13	10	9	23×67 6 column tiler
18	13	7	12	23×67 6 column tiler
18	13	4	15	23×67 6 column tiler
17	14	18	1	23×67 5 column tiler
17	14	15	4	23×67 5 column tiler
17	14	14	5	23×67 6 column tiler
17	14	11	8	23×67 6 column tiler
17	14	8	11	23×67 6 column tiler

17	14	5	14	23×67 6 column tiler
15	16	17	2	23×67 7 row tiler
15	16	15	4	23×67 7 row tiler
15	16	13	6	23×67 7 row tiler
15	16	12	7	23×67 6 row tiler
15	16	11	8	23×67 5 column tiler
15	16	10	9	23×67 6 column tiler
15	16	8	11	23×67 7 row tiler
15	16	7	12	23×67 6 column tiler
15	16	5	14	23×67 7 row tiler
15	16	4	15	23×67 6 column tiler
15	16	2	17	23×67 7 row tiler
15	16	1	18	23×67 6 column tiler
14	17	18	1	23×67 7 row tiler
14	17	16	3	23×67 5 column tiler
14	17	14	5	23×67 6 column tiler
14	17	13	6	23×67 7 row tiler
14	17	12	7	23×67 6 column tiler
14	17	11	8	23×67 7 row tiler
14	17	9	10	23×67 6 column tiler
14	17	8	11	23×67 6 column tiler
14	17	6	13	23×67 7 row tiler
14	17	5	14	23×67 6 column tiler
14	17	3	16	23×67 7 row tiler
14	17	2	17	23×67 5 column tiler
13	18	19	0	23×67 4 row tiler
13	18	17	2	23×67 5 column tiler
13	18	15	4	23×67 6 column tiler

13	18	14	5	23×67 6 row tiler
13	18	13	6	23×67 7 row tiler
13	18	12	7	23×67 6 column tiler
13	18	10	9	23×67 6 column tiler
13	18	9	10	23×67 7 row tiler
13	18	7	12	23×67 7 row tiler
13	18	6	13	23×67 6 column tiler
13	18	4	15	23×67 7 row tiler
13	18	3	16	23×67 6 column tiler
12	19	18	1	23×67 7 column tiler
12	19	16	3	23×67 6 column tiler
12	19	15	4	23×67 7 row tiler
12	19	14	5	23×67 7 column tiler
12	19	13	6	23×67 6 column tiler
12	19	11	8	23×67 7 row tiler
12	19	10	9	23×67 7 row tiler
12	19	8	11	23×67 6 column tiler
12	19	7	12	23×67 6 column tiler
12	19	5	14	23×67 7 row tiler
12	19	4	15	23×67 6 column tiler
11	20	19	0	23×67 5 column tiler
11	20	17	2	23×67 6 column tiler
11	20	16	3	23×67 7 row tiler
11	20	15	4	23×67 7 row tiler
11	20	14	5	23×67 7 row tiler
11	20	12	7	23×67 7 row tiler
11	20	11	8	23×67 6 column tiler
11	20	10	9	23×67 7 row tiler
11	20	9	10	23×67 7 row tiler
11	20	8	11	23×67 6 column tiler
11	20	7	12	23×67 7 row tiler
11	20	6	13	23×67 7 row tiler
11	20	5	14	23×67 7 row tiler
11	20	4	15	23×67 7 row tiler
11	20	3	16	23×67 7 column tiler
11	20	1	18	23×67 7 row tiler
11	20	0	19	23×67 7 row tiler

23×69(9 5×7 and 53 4×6)
None pass through the $23x69$ 4 row tiler

23x69(33 5×7 and 18 4×6)
Pass through $23x69$ 7 row tiler on Odd

$33, 0$
$32, 1$
$31, 2$
$30, 3$
$29, 4$
$28, 5$
$27, 6$
$26, 7$
$25, 8$
$24, 9$
$23, 10$
$22, 11$
$21, 12$
$20, 13$
$18, 15$
$17, 16$
$16, 17$
$15, 18$
$14, 19$
$13, 20$
$11, 22$
And pass through $23x69$ 4 row tiler
$30, 3$
$29, 4$
$28, 5$
$27, 6$
$26, 7$
$25, 8$
$24, 9$
$23, 10$
$22, 11$
$21, 12$
$20, 13$
$18, 15$
$17, 16$
$16, 17$
$15, 18$
$14, 19$
$13, 20$
$11, 22$
And pass through $23x69$ 5 row tiler
$21, 12$
$17, 16$
$16, 17$
$15, 18$
$13, 20$

$11, 22$

Feasible with $23x69$ 5 row tiler

H	V	h	v	Rejected by
21	12	18	0	$23x69$ 7 row tiler
21	12	14	4	$23x69$ 7 column tiler
21	12	11	7	$23x69$ 7 column tiler
21	12	8	10	$23x69$ 7 column tiler
17	16	17	1	$23x69$ 7 row tiler
17	16	15	3	$23x69$ 6 column tiler
17	16	13	5	$23x69$ 7 column tiler
17	16	12	6	$23x69$ 7 column tiler
17	16	10	8	$23x69$ 7 row tiler
17	16	9	9	
17	16	7	11	$23x69$ 7 row tiler
17	16	6	12	$23x69$ 7 column tiler
17	16	3	15	$23x69$ 6 row tiler
17	16	0	18	$23x69$ 7 row tiler
16	17	18	0	$23x69$ 7 row tiler
16	17	16	2	$23x69$ 6 column tiler
16	17	14	4	$23x69$ 7 row tiler
16	17	13	5	$23x69$ 6 column tiler
16	17	11	7	$23x69$ 7 column tiler
16	17	10	8	$23x69$ 6 column tiler
16	17	8	10	$23x69$ 7 row tiler
16	17	7	11	$23x69$ 6 column tiler
16	17	4	14	$23x69$ 6 row tiler
16	17	1	17	$23x69$ 6 row tiler
15	18	17	1	$23x69$ 7 row tiler
15	18	15	3	$23x69$ 7 row tiler
15	18	14	4	$23x69$ 6 column tiler
15	18	12	6	$23x69$ 7 row tiler
15	18	11	7	$23x69$ 7 column tiler

13	20	16	2	$23x69$ 7 row tiler
13	20	14	4	$23x69$ 7 column tiler
13	20	12	6	
13	20	11	7	$23x69$ 7 row tiler
13	20	10	8	$23x69$ 6 column tiler
13	20	9	9	$23x69$ 7 row tiler
13	20	8	10	$23x69$ 7 row tiler
13	20	7	11	$23x69$ 7 row tiler
13	20	6	12	$23x69$ 7 row tiler
13	20	5	13	$23x69$ 7 row tiler
13	20	4	14	$23x69$ 7 row tiler
13	20	2	16	$23x69$ 7 row tiler
11	22	18	0	$23x69$ 7 row tiler
11	22	16	2	$23x69$ 7 row tiler
11	22	14	4	$23x69$ 7 row tiler
11	22	13	5	$23x69$ 7 row tiler
11	22	12	6	$23x69$ 7 row tiler
11	22	11	7	$23x69$ 7 row tiler
11	22	10	8	$23x69$ 7 row tiler
11	22	9	9	$23x69$ 7 row tiler
11	22	8	10	$23x69$ 7 row tiler
11	22	7	11	$23x69$ 7 row tiler
11	22	6	12	$23x69$ 7 row tiler
11	22	4	14	$23x69$ 7 row tiler
11	22	1	17	$23x69$ 6 row tiler

23×71(11 5×7 and 52 4×6)
Pass 23×71 4 row tiler on Odd
$8, 3$
None pass through 23×71 4 column tiler

23×71 (35 5×7 and 17 4×6)
Pass through 23×71 4 row tiler on Odd
$32, 3$
$31, 4$
$30, 5$
$29, 6$
$28, 7$
$27, 8$
$26, 9$
$25, 10$
$24, 11$
$23, 12$
$22, 13$

$21, 14$
$20, 15$
$19, 16$
$18, 17$
$17, 18$
$16, 19$
$15, 20$
$14, 21$
$13, 22$
$12, 23$
$11, 24$
$10, 25$
$9, 26$
$8, 27$
$7, 28$
$6, 29$
$5, 30$
$4, 31$
$3, 32$
$2, 33$
And pass through 23×71 5 row tiler
$19, 16$
$15, 20$
$14, 21$
$13, 22$
$11, 24$
$10, 25$
$9, 26$
$8, 27$
$7, 28$
$6, 29$
$5, 30$
$4, 31$
$3, 32$
$2, 33$
And pass through 23×71 7 row tiler
$19, 16$
$15, 20$
$14, 21$
$13, 22$
$11, 24$
Feasible with 23×71 7 row tiler

H	V	h	v	Rejected by
19	16	15	2	23×71 7 column tiler
19	16	12	5	23×71 7 column tiler
19	16	11	6	23×71 7 column tiler
19	16	8	9	23×71 7 column tiler
15	20	12	5	23×71 7 column tiler
15	20	11	6	23×71 7 column tiler
15	20	9	8	23×71 7 column tiler
14	21	16	1	23×71 7 column tiler
14	21	10	7	23×71 7 column tiler
13	22	17	0	23×71 5 row tiler
13	22	14	3	23×71 7 column tiler
11	24	13	4	23×71 5 column tiler

23×73 (37 5×7 and 16 4×6)

Pass through 23×73 4 row tiler on Odd

$35, 2$
$33, 4$
$32, 5$
$31, 6$
$30, 7$
$29, 8$
$28, 9$
$27, 10$
$26, 11$
$25, 12$
$24, 13$
$23, 14$
$22, 15$
$21, 16$
$20, 17$
$19, 18$
$18, 19$
$17, 20$
$16, 21$
$15, 22$
$14, 23$
$13, 24$
$12, 25$
$11, 26$
$10, 27$
$9, 28$
$8, 29$
$7, 30$
$6, 31$

$5, 32$
$4, 33$
$3, 34$
$1, 36$
And pass through 23×73 5 row tiler
$17, 20$
$13, 24$
$12, 25$
$11, 26$
$9, 28$
$8, 29$
$7, 30$
$6, 31$
$5, 32$
$4, 33$
$3, 34$
$1, 36$
And pass through 23×73 7 row tiler
$17, 20$
$13, 24$
Feasible with 23×73 7 row tiler

H	V	h	v	Rejected by
17	20	15	1	23×73 5 row tiler
17	20	11	5	23×73 5 column tiler
17	20	8	8	23×73 5 column tiler
13	24	14	2	23×73 5 row tiler
13	24	12	4	23×73 5 row tiler

23×73 (13 5×7 and 51 4×6)
Pass through 23×73 4 row tiler on Odd
$11, 2$
$9, 4$
$4, 9$
And pass through 23×73 5 row tiler
$9, 4$
$4, 9$
And pass through 23×73 7 row tiler
$9, 4$
None pass through 23×73 7 column tiler

25×29
Pass through 25×29 4 row tiler on Odd
$4, 3$
$2, 5$
And pass through 25×29 5 row tiler

$2, 5$
None pass through the 25×29 6 row tiler

25×31
Pass through 25×31 4 row tiler on Odd
$3, 2$
None pass through 25×31 7 row tiler

26×27
Pass through 26×27 4 row tiler on Odd
$18, 0$
$16, 2$
$14, 4$
$12, 6$
$10, 8$
$9, 9$
$8, 10$
$6, 12$
$5, 13$
$4, 14$
$2, 16$
$0, 18$
And pass through 26×27 5 row tiler
$9, 9$
$4, 14$
And pass through 26×27 6 row tiler
$4, 14$
Feasible with 26×27 6 row tiler

H	V	h	v	Rejected by
4	14	2	1	26×27 4 row tiler

26×33
Pass through 26×33 4 row tiler on Odd
$6, 0$
$4, 2$
$2, 4$
$0, 6$
And pass through 26×33 5 row tiler
$4, 2$
$2, 4$
$0, 6$
None pass through 26×33 7 row tiler

26×37
Pass through 26×37 4 row tiler on Odd

$22, 0$
$20, 2$
$18, 4$
$16, 6$
$15, 7$
$14, 8$
$13, 9$
$12, 10$
$11, 11$
$10, 12$
$9, 13$
$8, 14$
$7, 15$
$6, 16$
$5, 17$
$4, 18$
$2, 20$
$0, 22$
And pass through 26×37 5 row tiler
$15, 7$
$14, 8$
$13, 9$
$11, 11$
$10, 12$
$9, 13$
$8, 14$
$7, 15$
$6, 16$
$5, 17$
$4, 18$
$2, 20$
$0, 22$
And pass through 26×37 6 row tiler
$14, 8$
$11, 11$
$10, 12$
$8, 14$
$7, 15$
$6, 16$
$5, 17$
$4, 18$
$2, 20$
$0, 22$
And pass through 26×37 7 row tiler
$14, 8$
$11, 11$

$10, 12$
$7, 15$
$5, 17$
Feasible with 26×37 7 row tiler

H	V	h	v	Rejected by
14	8	8	0	26×37 5 row tiler
14	8	2	6	26×37 5 row tiler
11	11	2	6	26×37 5 row tiler
11	11	1	7	26×37 6 row tiler
10	12	6	2	26×37 5 row tiler
10	12	0	8	26×37 5 row tiler
7	15	8	0	26×37 4 row tiler
7	15	2	6	26×37 6 row tiler
5	17	4	4	26×37 4 row tiler
5	17	3	5	26×37 5 row tiler

27 × 38
Pass through 27×38 4 row tiler on Odd
$4, 2$
None pass through the 27×38 6 row tiler

27 × 41
Pass through 27×41 4 row tiler on Odd
$5, 4$
$3, 6$
None pass through the 27×41 6 row tiler

27 × 43
Pass through 27×43 4 row tiler on Odd
$24, 3$
$23, 4$
$22, 5$
$21, 6$
$20, 7$
$19, 8$
$18, 9$
$17, 10$
$16, 11$
$15, 12$
$14, 13$
$13, 14$
$12, 15$
$11, 16$
$10, 17$
$9, 18$

$8, 19$
$7, 20$
$6, 21$
$5, 22$
$4, 23$
$3, 24$
$1, 26$
And pass through 27×43 5 row tiler
$22, 5$
$20, 7$
$18, 9$
$17, 10$
$16, 11$
$15, 12$
$14, 13$
$13, 14$
$12, 15$
$11, 16$
$10, 17$
$9, 18$
$8, 19$
$7, 20$
$6, 21$
$5, 22$
$4, 23$
$3, 24$
$1, 26$
And pass through 27×43 6 row tiler
$17, 10$
$16, 11$
$15, 12$
$14, 13$
$13, 14$
$12, 15$
$11, 16$
$10, 17$
$9, 18$
$8, 19$
$6, 21$
$4, 23$
And pass through 27×43 7 row tiler
$14, 13$
$12, 15$
$11, 16$
$4, 23$
And pass through the 27×43 7 column tiler

$14, 13$
Feasible with 7 column tiler

H	V	h	v	Rejected by
14	13	5	4	27×43 7 row tiler
14	13	2	7	27×43 7 row tiler

27×46
Pass through 27×46 4 row tiler on Odd
$28, 2$
$26, 4$
$25, 5$
$24, 6$
$23, 7$
$22, 8$
$21, 9$
$20, 10$
$19, 11$
$18, 12$
$17, 13$
$16, 14$
$15, 15$
$14, 16$
$13, 17$
$12, 18$
$11, 19$
$10, 20$
$9, 21$
$8, 22$
$7, 23$
$6, 24$
$5, 25$
$4, 26$
$3, 27$
$2, 28$
And pass through 27×46 5 row tiler
$24, 6$
$22, 8$
$20, 10$
$19, 11$
$18, 12$
$17, 13$
$16, 14$
$15, 15$
$14, 16$
$13, 17$

$12, 18$
$11, 19$
10.20
$9, 21$
$8, 22$
$7, 23$
$6, 24$
$5, 25$
$4, 26$
$3, 27$
$2, 28$
And pass through 27×46 6 row tiler
$19, 11$
$18, 12$
$17, 13$
$16, 14$
$15, 15$
$14, 16$
$13, 17$
$12, 18$
$11, 19$
$10, 20$
$9, 21$
$7, 23$
$6, 24$
$5, 25$
$4, 26$
$3, 27$
$2, 28$
And pass through 27×46 7 row tiler
$18, 12$
$16, 14$
$15, 15$
And pass through 27×46 7 column tiler
$16, 14$
$15, 15$
Feasible with 7 column tiler

H	V	h	v	Rejected by
16	14	7	1	27×46 7 row tiler
16	14	5	3	27×46 5 row tiler
15	15	8	0	27×46 4 row tiler
15	15	5	3	27×46 6 row tiler
15	15	1	7	27×46 5 row tiler

Appendix B

10	x	n	possible?	why yes or no
10	x	10	n	area
10	x	11	n	area
10	x	12	y	general store
10	x	13	n	area
10	x	14	y	general store
10	x	15	n	area
10	x	16	n	area
10	x	17	n	area
10	x	18	n	area
10	x	19	y	general store
10	x	20	n	area
10	x	21	y	general store
10	x	22	n	area
10	x	23	n	area
10	x	24	y	general store
10	x	25	n	area
10	x	26	y	general store
10	x	27	n	area
10	x	28	y	general store
10	x	29	n	area
10	x	30	n	area
10	x	31	y	general store
10	x	32	n	area
10	x	33	y	general store
10	x	34	n	area
10	x	35	y	general store
10	x	36	y	general store
10	x	37	n	area
10	x	38	y	general store
10	x	39	n	area
10	x	40	y	general store
10	x	41	n	area
10	x	42	y	general store
10	x	43	y	general store
10	x	44	n	area
10	x	45	y	general store
10	x	46	n	area
10	x	47	y	general store
10	x	48	y	general store
10	x	49	y	general store
10	x	50	y	general store
10	x	51	n	area
10	x	52	y	general store
10	x	53	n	area
10	x	54	y	general store
10	x	55	y	general store
10	x	56	y	general store
10	x	57	y	general store
10	x	58	n	area
10	x	59	y	general store
10	x	60	y	general store
10	x	61	y	general store
10	x	62	y	general store
10	x	63	y	general store
10	x	64	y	general store
10	x	65	n	area
10	x	66	y	general store

12	x	n	possible	why yes or no	
12	x	12	y	general store	
12	x	13	n	area	
12	x	14	y	general store	
12	x	15	n	area	
12	x	16	y	general store	
12	x	17	n	area	
12	x	18	y	general store	
12	x	19	n	area	
12	x	20	y	general store	
12	x	21	n	area	
12	x	22	y	general store	
12	x	23	n	area	
12	x	24	y	general store	
12	x	25	n	area	
12	x	26	y	general store	
12	x	27	n	area	
12	x	28	y	general store	
12	x	29	n	area	
12	x	30	y	general store	
12	x	31	n	area	
12	x	32	y	general store	
12	x	33	n	area	
12	x	34	y	general store	
12	x	35	y	general store	
12	x	36	y	general store	
12	x	37	n	disproved - only 1 4x6 tile*	
12	x	38	y	general store	

* The single even tile cannot be on the boundary. Then use a pinwheel (negative) proof.

14	x	number	possible?	why yes or no
14	x	14	n	area
14	x	15	y	general store
14	x	16	n	area
14	x	17	y	general store
14	x	18	n	area
14	x	19	n	area
14	x	20	y	general store
14	x	21	n	area
14	x	22	y	general store
14	x	23	n	area
14	x	24	y	general store
14	x	25	y	general store
14	x	26	n	area
14	x	27	y	general store
14	x	28	n	area
14	x	29	y	general store
14	x	30	y	general store
14	x	31	n	area
14	x	32	y	general store
14	x	33	n	area
14	x	34	y	general store
14	x	35	y	general store
14	x	36	y	general store
14	x	37	y	general store
14	x	38	n	area
14	x	39	y	general store
14	x	40	y	general store
14	x	41	y	general store
14	x	42	y	general store
14	x	43	n	area
14	x	44	y	general store

17	x	number	possible?	why yes or no
17	x	17	n	area
17	x	18	y	Figure 3
17	x	19	n	only 1 5x7
17	x	20	n	area
17	x	21	n	area
17	x	22	n	Theorem 11
17	x	23	n	coloring argument
17	x	24	n	all 4x6's can't fill odd dimension
17	x	25	n	area
17	x	26	n	area
17	x	27	y	general store
17	x	28	y	general store
17	x	29	n	area
17	x	30	n	area
17	x	31	n	Tiler
17	x	32	y	17 x 14 + 17 x 18
17	x	33	n	too few 5x7's (3) - too many columns of 17
17	x	34	n	area
17	x	35	y	general store
17	x	36	y	general store
17	x	37	y	general store
17	x	38	n	too few 5x7's (2) - too many columns of 17
17	x	39	n	area
17	x	40	y	10 x 40 + 7 x 40
17	x	41	y	17 x 14 and 17 x 27
17	x	42	y	17 x 28 and 17 x 14
17	x	43	n	only 1 5x7
17	x	44	n	proof on paper dr. narayan gave out
17	x	45	y	rectangles/pinwheel?
17	x	46	y	17 x 28 + 17 x 18
17	x	47	n	too few 5x7's
17	x	48	n	all 4x6's can't fill odd dimension
17	x	49	y	14+35
17	x	50	y	14+36
17	x	51	y	14+37
17	x	52	n	too few 5x7's
17	x	53	y	pinwheel
17	x	54	y	27+27
17	x	55	y	14+14+27
17	x	56	y	14+14+14+14
17	x	57	n	too few 5x7's OR only 1 4x6
17	x	58	y	17 x 40 + 17 x 18
17	x	59	y	17 x 41 + 17 x 18
17	x	60	y	general store
17	x	61	n	too few 5x7's
17	x	62	y	general store
17	x	63	y	general store
17	x	64	y	general store
17	x	65	y	14+14+37
17	x	66	n	Tilers
17	x	67	y	pinwheel
17	x	68	y	14+27+27
17	x	69	y	14+14+14+27
17	x	70	y	general store
17	x	71	y	17 x 35 and 17 x 36
17	x	72	y	general store
17	x	73	y	17 x 36 and 17 x 37
17	x	74	y	general store
17	x	75	y	pinwheel
17	x	76	y	14+27+35
17	x	77	y	14+14+14+35
17	x	78	y	14+14+14+36
17	x	79	y	14+14+14+37
17	x	80	y	pinwheel

18	x	number	possible?	why yes or no
18	x	18	n	area
18	x	19	n	area
18	x	20	y	a=5
18	x	21	y	a=1,b=1
18	x	22	n	area
18	x	23	n	area
18	x	24	y	a=6
18	x	25	y	a=2,b=1
18	x	26	n	too many columns with only 5x7s
18	x	27	n	area
18	x	28	y	a=7
18	x	29	y	a=3,b=1
18	x	30	y	c=1
18	x	31	n	area
18	x	32	y	a=8
18	x	33	y	a=4,b=1
18	x	34	y	a=1,c=1
18	x	35	n	too many columns (all) with only 5x7s
18	x	36	y	a=9
18	x	37	y	a=5,b=1
18	x	38	y	a=2,c=1
18	x	39	n	too many columns with only 5x7s
18	x	40	y	a=10
18	x	41	y	a=6,b=1
18	x	42	y	a=3,c=1
18	x	43	y	too many columns with only 5x7s

19	x	number	possible?	why yes or no
19	x	19	n	area
19	x	20	y	general store
19	x	21	n	area
19	x	22	n	area
19	x	23	y	P(7;12;7;12)
19	x	24	n	impossible - possible = impossible
19	x	25	n	area
19	x	26	y	pinwheel
19	x	27	n	too few 5x7's
19	x	28	n	area
19	x	29	y	pinwheel
19	x	30	y	general store
19	x	31	n	area
19	x	32	n	too few 4x6's
19	x	33	y	general store
19	x	34	n	impossible - possible = impossible
19	x	35	y	general store
19	x	36	y	pinwheel
19	x	37	n	area
19	x	38	n	area
19	x	39	y	pinwheel
19	x	40	y	general store
19	x	41	n	area
19	x	42	y	general store
19	x	43	y	general store
19	x	44	n	impossible - possible = impossible
19	x	45	y	general store
19	x	46	y	general store
19	x	47	n	impossible - possible = impossible
19	x	48	n	coloring argument
19	x	49	y	general store
19	x	50	y	general store
19	x	51	n	only 1 4x6
19	x	52	y	general store
19	x	53	y	general store
19	x	54	n	too few 5x7's
19	x	55	y	general store
19	x	56	y	general store
19	x	57	n	impossible - possible = impossible
19	x	58	y	general store
19	x	59	y	general store
19	x	60	y	general store
19	x	61	y	general store
19	x	62	y	general store
19	x	63	y	general store
19	x	64	y	19 x 35 + 19 x 29
19	x	65	y	general store
19	x	66	y	general store
19	x	67	n	coloring argument

20	x	number	possible?	why yes or no
20	x	20	y	general store
20	x	21	y	general store
20	x	22	n	area
20	x	23	n	area
20	x	24	y	general store
20	x	25	y	general store
20	x	26	y	general store
20	x	27	y	general store
20	x	28	y	general store
20	x	29	n	area
20	x	30	y	general store

21	**x**	**number**	**possible?**	**why yes or no**
21	x	21	n	only 3 5x7's
21	x	22	n	area
21	x	23	y	21 x 18 + 21 x 5
21	x	24	n	all 4x6's
21	x	25	y	general store
21	x	26	n	Tilers
21	x	27	n	area
21	x	28	y	general store
21	x	29	n	only 3 5x7's
21	x	30	y	general store
21	x	31	n	Tilers
21	x	32	n	subtraction: 32 x 27 - 32 x 6
21	x	33	y	general store
21	x	34	n	Tilers
21	x	35	y	general store
21	x	36	y	21 x 18 + 21 x 18
21	x	37	n	only 3 5x7's
21	x	38	y	general store
21	x	39	n	coloring argument
21	x	40	y	general store
21	x	41	y	21 x 18 + 21 x 23
21	x	42	y	general store
21	x	43	y	general store
21	x	44	y	pinwheel
21	x	45	y	general store
21	x	46	y	21 x 18 + 21 x 28
21	x	47	y	general store
21	x	48	y	general store
21	x	49	y	21 x 44 + 21 x 5
21	x	50	y	general store
21	x	51	y	21 x 46 + 21 x 5
21	x	52	y	general store
21	x	53	y	general store
21	x	54	y	21 x 44 + 21 x 10
21	x	55	y	general store
21	x	56	y	general store
21	x	57	y	general store
21	x	58	y	general store
21	x	59	y	21 x 54 + 21 x 5
21	x	60	y	general store
21	x	61	y	general store
21	x	62	y	general store
21	x	63	y	general store
21	x	64	y	21 x 54 + 21 x 10
21	x	65	y	general store
21	x	66	y	general store
21	x	67	y	general store
21	x	68	y	general store
21	x	69	y	21 x 64 + 21 x 5
21	x	70	y	general store

22	x	number	possible?	why yes or no
22	x	22	n	area
22	x	23	n	area
22	x	24	y	general store
22	x	25	n	only 2 5x7's
22	x	26	y	general store
22	x	27	n	Tilers
22	x	28	y	general store
22	x	29	n	Tilers
22	x	30	y	general store
22	x	31	n	Tilers
22	x	32	n	Tilers
22	x	33	n	Tilers
22	x	34	n	only 2 4x6's - must be together, doesn't work
22	x	35	y	general store
22	x	36	y	general store
22	x	37	n	only 2 5x7's
22	x	38	y	general store
22	x	39	n	Tilers
22	x	40	y	general store
22	x	41	n	Tilers
22	x	42	y	general store
22	x	43	n	Tilers
22	x	44	y	general store
22	x	45	y	pinwheel
22	x	46	?	
22	x	47	y	22 x 35 + 22 x 12
22	x	48	y	general store
22	x	49	y	general store
22	x	50	y	general store
22	x	51	n	coloring argument
22	x	52	y	general store
22	x	53	n	Tilers
22	x	54	y	general store

23	x	number	possible?	why yes or no
23	x	23	n	Tilers
23	x	24	n	area
23	x	25	n	Tilers
23	x	26	n	area
23	x	27	n	only 4 4x6's
23	x	28	y	general store
23	x	29	n	only 3 4x6's
23	x	30	y	general store
23	x	31	n	only 2 4x6's
23	x	32	n	Tilers
23	x	33	n	only 1 4x6's
23	x	34	y	pinwheel
23	x	35	n	area
23	x	36	y	general store
23	x	37	n	area
23	x	38	y	general store
23	x	39	n	area
23	x	40	y	7x40, 5x28, 11x28, 16x12
23	x	41	n	area
23	x	42	y	general store
23	x	43	n	Tilers
23	x	44	n	Tilers
23	x	45	n	Tilers
23	x	46	?	
23	x	47	y	general store
23	x	48	?	
23	x	49	y	general store
23	x	50	n	Tilers
23	x	51	y	23 x 21 + 23 x 30
23	x	52	n	Tilers
23	x	53	y	23 x 19 and 23 x 34
23	x	54	n	too few 4x6's
23	x	55	y	general store
23	x	56	y	general store
23	x	57	y	general store
23	x	58	y	general store
23	x	59	y	general store
23	x	60	y	general store
23	x	61	y	general store
23	x	62	y	pinwheel
23	x	63	y	general store
23	x	64	y	pinwheel
23	x	65	?	
23	x	66	y	general store
23	x	67	n	Tilers
23	x	68	y	a=1,b=1,c=1
23	x	69	?	
23	x	70	y	general store
23	x	71	n	Tilers
23	x	72	y	general store
23	x	73	n	Tilers
23	x	74	y	general store

24	x	number	possible?	why yes or no
24	x	24	y	general store
24	x	25	n	only with all 4x6's and it's got an odd dimension so it's disproved
24	x	26	y	general store
24	x	27	n	only with all 4x6's and it's got an odd dimension so it's disproved
24	x	28	y	general store
24	x	29	n	only with all 4x6's and it's got an odd dimension so it's disproved
24	x	30	y	general store
24	x	31	n	only with all 4x6's and it's got an odd dimension so it's disproved
24	x	32	y	general store
24	x	33	n	only with all 4x6's and it's got an odd dimension so it's disproved
24	x	34	y	general store
24	x	35	y	general store
24	x	36	y	general store
24	x	37	n	disproved (too few 4x6's)
24	x	38	y	general store

25	x	n	possible	why yes or no
25	x	25	y	general store
25	x	26	n	area
25	x	27	y	general store
25	x	28	y	general store
25	x	29	n	Tilers
25	x	30	y	general store
25	x	31	n	coloring argument
25	x	32	y	25 x 25 + 25 x 7
25	x	33	n	too few 5x7's
25	x	34	y	general store
25	x	35	y	general store
25	x	36	y	19 x 36 and 6 x 36
25	x	37	y	general store
25	x	38	y	25 x 18 + 25 x 20
25	x	39	y	25 x 32 + 25 x 7
25	x	40	y	general store

26	x	number	possible?	why yes or no
26	x	26	n	area
26	x	27	n	Tilers
26	x	28	y	general store
26	x	29	y	26 x 10 + 26 x 19
26	x	30	y	general store
26	x	31	y	26 x 12 + 26 x 19
26	x	32	y	general store
26	x	33	n	Tilers
26	x	34	y	general store
26	x	35	y	general store
26	x	36	y	general store
26	x	37	n	Tilers
26	x	38	y	general store
26	x	39	y	26 x 19 + 26 x 20
26	x	40	y	general store
26	x	41	y	pinwheel
26	x	42	y	general store
26	x	43	y	26 x 19 + 26 x 24
26	x	44	y	general store

27	x	number	possible?	why yes or no
27	x	27	n	area
27	x	28	y	general store
27	x	29	n	area
27	x	30	y	general store
27	x	31	y	general store
27	x	32	n	32 x 33 is impossible - 32 x 6 = impossible
27	x	33	y	pinwheel
27	x	34	y	general store
27	x	35	y	general store
27	x	36	y	general store
27	x	37	y	general store
27	x	38	n	Tilers
27	x	39	y	general store
27	x	40	y	general store
27	x	41	n	Tilers
27	x	42	y	general store
27	x	43	n	Tilers
27	x	44	y	general store
27	x	45	y	general store
27	x	46	n	Tilers
27	x	47	y	general store

28's bound is 20. Therefore anything above 20 can be done and we're only looking at 28 and above.
These therefore can all be done.
The areas under 28 will show up in their own lists.

As in 28, the bound for 30 is lower than 30 (18)
so the ones we're looking at (30 and above) can all be done