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Further Insight into Gravitational Recoil

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Further insight into gravitational recoil

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(Dated: March 13, 2008)

We test the accuracy of our recently proposed empirical formula to model the recoil velocity imparted to the merger remnant of spinning, unequal-mass black-hole binaries. We study three families of black-hole binary configurations, all with mass ratio \( q = 3/8 \) (to maximize the unequal-mass contribution to the kick) and spins aligned (or counter aligned) with the orbital angular momentum, two with spin configurations chosen to minimize the spin-induced tangential and radial accelerations of the trajectories respectively, and a third family where the trajectories are significantly altered by spin-orbit coupling. We find good agreement between the measured and predicted recoil velocities for the first two families, and reasonable agreement for the third. We also re-examine our original generic binary configuration that led to the discovery of extremely large spin-driven recoil velocities and inspired our empirical formula, and find reasonable agreement between the predicted and measured recoil speeds.

PACS numbers: 04.25.Dm, 04.25.Nx, 04.30.Db, 04.70.Bw

I. INTRODUCTION

Thanks to recent breakthroughs in the full non-linear numerical evolution of black-hole-binary spacetimes \[2, 3\], it is now possible to accurately simulate the merger process and examine its effects in this highly non-linear regime \[4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\]. Black-hole binaries will radiate between 2% and 8% of their total mass and up to 40% of their angular momenta, depending on the magnitude and direction of the spin components, during the merger \[6, 7, 8\]. In addition, the radiation of net linear momentum by a black-hole binary leads to the recoil of the final remnant hole \[19, 20, 21\] and the part of the trajectories respectively, and a third family where the trajectories are significantly altered by spin-orbit coupling. We find good agreement between the measured and predicted recoil velocities for the first two families, and reasonable agreement for the third. We also re-examine our original generic binary configuration that led to the discovery of extremely large spin-driven recoil velocities and inspired our empirical formula, and find reasonable agreement between the predicted and measured recoil speeds.

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I. INTRODUCTION

Thanks to recent breakthroughs in the full non-linear numerical evolution of black-hole-binary spacetimes \[1, 2, 3\], it is now possible to accurately simulate the merger process and examine its effects in this highly non-linear regime \[4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\]. Black-hole binaries will radiate between 2% and 8% of their total mass and up to 40% of their angular momenta, depending on the magnitude and direction of the spin components, during the merger \[6, 7, 8\]. In addition, the radiation of net linear momentum by a black-hole binary leads to the recoil of the final remnant hole \[19, 20, 21\]. The maximum possible recoil arises from unequal-masses or the spin components normal to the orbital plane. The maximum possible recoil results from equal-mass, maximally spinning holes with spins in the orbital plane. The maximum possible recoil velocity of 475 km s\(^{-1}\), mostly along the orbital angular momentum direction. It was thus found that the recoil normal to the orbital plane (due to spin components lying in the orbital plane) can be larger than the in-plane recoil originating from either the unequal-masses or the spin components normal to the orbital plane. The maximum possible recoil arises from equal-mass, maximally spinning holes with spins in the orbital plane and counter-aligned. This maximum recoil, which will be normal to the orbital plane, is nearly 4000 km s\(^{-1}\).

In \[28\] we introduced the following heuristic model for the gravitational recoil of a merging binary.

\[
\vec{V}_{\text{recoil}}(q, \vec{\alpha}_i) = v_m \hat{e}_1 + v_\perp (\cos(\xi) \hat{e}_1 + \sin(\xi) \hat{e}_2) + v_\parallel \hat{e}_3, \tag{1}
\]

where

\[
v_m = A \frac{q^2(1-q)}{(1+q)^3} \left( 1 + B \frac{q}{(1+q)^2} \right), \tag{2a}
\]

\[
v_\perp = H \frac{q^2}{(1+q)^3} \left( \alpha_2 - q \alpha_1 \right), \tag{2b}
\]

\[
v_\parallel = K \cos(\Theta - \Theta_0) \frac{q^2}{(1+q)^3} \left| \tilde{\alpha}_z - q \tilde{\alpha}_1 \right|, \tag{2c}
\]

Here \( A = 1.2 \times 10^4 \) km s\(^{-1}\) \[23\], \( B = -0.93 \) \[23\], \( K \) here we find \( H = (6.9 \pm 0.5) \times 10^3 \) km s\(^{-1}\), \( \tilde{\alpha}_i = \tilde{S}_i/m_i^2 \), \( \tilde{S}_i \)
and \( m_1 \) are the spin and mass of hole \( i \), \( q = m_1/m_2 \) is the mass ratio of the smaller to larger mass hole, the index \( \perp \) and \( \parallel \) refer to perpendicular and parallel to the orbital angular momentum respectively at the effective moment of the maximum generation of the recoil (around merger time), \( \hat{e}_1, \hat{e}_2 \) are orthogonal unit vectors in the orbital plane, and \( \xi \) measures the angle between the “unequal mass” and “spin” contributions to the recoil velocity in the orbital plane. The angle \( \Theta \) was defined as the angle between the in-plane component of \( \Delta \equiv (m_1 + m_2)/2 \) and \( \hat{S}_1 \) and the infall direction at merger. The form of Eq. (20) was proposed in [23, 40], while the form of Eqs. (20) and (22) was proposed in [28] based on the post-Newtonian expressions in [47]. In Ref. [48] we determined that \( K = (6.0 \pm 0.1) \times 10^4 \text{km s}^{-1} \). Although \( \xi \) may in general depend strongly on the configuration, the results of [30] and post-Newtonian calculations show that \( \xi \) is \( 90^\circ \) for headon collisions, and the results presented here indicate that \( \xi \approx 145^\circ \) for a wide range of quasi-circular configurations. A simplified version of Eq. (1) that models the magnitude of \( V_{\text{recoil}} \) was independently proposed in [32], and a simplified form of Eq. (1) for the equal-mass aligned spin case was proposed in [29].

Our heuristic formula (11) describing the recoil velocity of a black-hole binary remnant as a function of the parameters of the individual holes has been theoretically verified in several ways. In [48] the \( \cos \Theta \) dependence was established and was confirmed in [37] for binaries with larger initial separations. In Ref. [30] the decomposition into spin components perpendicular and parallel to the orbital plane was verified, and in [41] it was found that the quadratic-in-spin corrections to the in-plane recoil velocity are less than \( 20 \text{km s}^{-1} \).

Consistent and independent recoil velocity calculations have also been obtained for equal-mass binaries with spinning black holes that have spins aligned/counter-aligned with the orbital angular momentum [27, 29]. Recoils from the merger of non-precessing unequal mass black-hole binaries have been modeled in [32].

The net in-plane remnant recoil velocity arises both from the asymmetry due to unequal masses, which gives its \( z \to -z \) symmetric behavior, only contributes to recoil along the orbital plane, and the asymmetry produced by the black-hole spin component perpendicular to the orbital plane. Even if we can parametrize the contribution of each of these two components of the recoil in terms of only one angle, \( \xi \), the modeling of it appears in principle very complicated. \( \xi \) may depend on the mass ratio \( q \) of the holes, as well as their individual spins \( S_1^\perp \) and \( S_2^\perp \), but also on their orbital parameters such as initial coordinates and momenta, or initial separation and eccentricity. We clearly have to reduce the dimensionality of this parameter space as part of the modeling process. In order to do so, we shall choose a model for \( \xi \) that only depends on \( q \) and \( \Delta z \) for quasi-circular orbits. We then perform simulations to determine how accurately this reduced-parameter-space model for \( \xi \) reproduces the observed recoil velocities and find that \( \xi \approx 145^\circ \), independent of either \( q \) or \( \Delta z \).

The paper is organized as follows, in Sec. II we review the numerical techniques used for the evolution of the black-hole binaries and the analysis of the physical quantities extracted at their horizons. In Sec. III we review the post-Newtonian dynamics of binary systems in order to motivate our study of equivalent trajectories for unequal mass, nonspinning and spinning holes. We focus on four families of such configurations. In Sec. IV we give the initial data parameters for these families. The results of the evolution of those configurations are given in Sec. V where we also introduce a novel analysis of the trajectories of the punctures and of the waveform phase to model the angle \( \xi \) in our heuristic formula Eq. (11). In Sec. VI we analyze the generic configuration that led us to discover the large recoil velocities produced by the spin projection on the orbital plane of the binary. Here we use more refined tools to analyze the individual hole spins and momenta near merger time, when most of the recoil is generated. We end the paper with a Discussion section pointing out the need for further runs with higher accuracy to improve our first results, and an Appendix including the post-Newtonian analysis of the maximum recoil configuration.

II. TECHNIQUES

We use the puncture approach [40] along with the TwoPunctures [50] thorn to compute initial data. In this approach the 3-metric on the initial slice has the form \( \gamma_{ab} = (\psi_{BL} + u)^4 \delta_{ab} \), where \( \psi_{BL} \) is the Brill-Lindquist conformal factor, \( \delta_{ab} \) is the Euclidean metric, and \( u \) is (at least) \( C^2 \) on the punctures. The Brill-Lindquist conformal factor is given by \( \psi_{BL} = 1 + \sum_{i=1}^{n} m_i (\frac{1}{2\hat{r}^2 - \hat{r}_i^2}) \), where \( n \) is the total number of ‘punctures’, \( m_i \) is the mass parameter of puncture \( i \), \( \hat{r}_i \) is the coordinate location of puncture \( i \), and \( \hat{r} \) is the horizon mass associated with puncture \( i \). We evolve these black-hole-binary data-sets using the LAL2EV [51] implementation of the moving puncture approach [2]. In our version of the moving puncture approach [2, 49] we replace the BSSN [34, 53, 54] conformal exponent \( \phi \), which has logarithmic singularities at the punctures, with the initially \( C^4 \) field \( \chi = \exp(-4\phi) \). This new variable, along with the other BSSN variables, will remain finite provided that one uses a suitable choice for the gauge. An alternative approach uses standard finite differencing of \( \phi \) [3].

We use the Carpet [55, 56] mesh refinement driver to provide a ‘moving boxes’ style mesh refinement. In this approach refined grids of fixed size are arranged around the coordinate centers of both holes. The Carpet code then moves these fine grids about the computational domain by following the trajectories of the two black holes.

We obtain accurate, convergent waveforms and horizon parameters by evolving this system in conjunction with a modified 1+log lapse and a modified Gamma-driver
shift condition 2, 57, and an initial lapse \( \alpha(t = 0) = 2/(1 + \psi^3_{BL}) \). The lapse and shift are evolved with

\[
\begin{align*}
\partial_t \beta^a &= 0 \\
\partial_t \alpha &= -2\alpha K \\
\partial_t B^a &= 3/4\partial t \hat{\Gamma}^a - \eta B^a.
\end{align*}
\]

These gauge conditions require careful treatment of \( \chi \), the inverse of the three-metric conformal factor, near the puncture in order for the system to remain stable 2, 4, 12. In Ref. 58 it was shown that this choice of gauge leads to a strongly hyperbolic evolution system provided that the shift does not become too large.

We use AHFINDERDIRECT 59 to locate apparent horizons. We measure the magnitude of the horizon spin using the Isolated Horizon algorithm detailed in 60. This algorithm is based on finding an approximate rotational Killing vector (i.e. an approximate rotational symmetry) on the horizon, and given this approximate Killing vector \( \varphi^a \), the spin magnitude is

\[
S_{[\varphi]} = \frac{1}{8\pi} \int_{AH} (\varphi^a R^b K_{ab}) d^2 V
\]

where \( K_{ab} \) is the extrinsic curvature of the 3D-slice, \( d^2 V \) is the natural volume element intrinsic to the horizon, and \( R^a \) is the outward pointing unit vector normal to the horizon on the 3D-slice. We measure the direction of the spin by finding the coordinate line joining the poles of this Killing vector field using the technique introduced in 3. Our algorithm for finding the poles of the Killing vector field has an accuracy of \( 2° \) (see 8 for details).

We also use an alternative quasi-local measurement of the spin and linear momentum of the individual black holes in the binary that is based on the coordinate rotation and translation vectors 39. In this approach the spin components of the horizon are given by

\[
S_{[i]} = \frac{1}{8\pi} \int_{AH} \phi_{[i]}^a R^b \delta_{ab} d^2 V,
\]

where \( \phi_{[i]}^a = \delta_{ij} \delta_{mn} r^m \epsilon^{ij} \), and \( r^m = x^m - x^m_0 \) is the coordinate displacement from the centroid of the hole, while the linear momentum is given by

\[
P_{[i]} = \frac{1}{8\pi} \int_{AH} \xi_{ni}^a R^b (K_{ab} - K_{\gamma ab}) d^2 V,
\]

where \( \xi_{ni}^a = \delta_{i}^a \).

We measure radiated energy, linear momentum, and angular momentum, in terms of \( \psi_4 \), using the formulae provided in Refs. 61, 62. However, rather than using the full \( \psi_4 \) we decompose it into \( \ell \) and \( m \) modes and solve for the radiated linear momentum, dropping terms with \( \ell \geq 5 \). The formulae in Refs. 61, 62 are valid at \( r = \infty \). We obtain highly accurate values for these quantities by solving for them on spheres of finite radius (typically \( r/M = 25, 30, 35, 40 \)), fitting the results to a polynomial dependence in \( l = 1/r \), and extrapolating to \( l = 0 \). We perform fits based on a linear and quadratic dependence on \( l \), and take the final values to be the average of these two extrapolations with the differences being the extrapolation error.

We obtain a new determination of \( H \) in Eq. 29 using results from simulations performed by the NASA/GSFC 32, PSU 27, and AEI/LSU 41 groups. The simulations performed by these groups include runs with \( q = 1 \), and thus provide an accurate measurement of \( v_\perp \). We calculate \( H \) for each simulation (via \( H = v_\perp (\alpha_1^2 - \alpha_2^2)(1 + q)^{2}/q^2 \) and take the weighted average \( \langle H \rangle \pm \delta \langle H \rangle \), where

\[
\langle X^a \rangle = \sum_i X_i^a w_i,
\]

\[
w_i = (\delta X_i)^{-2}/\sum_j (\delta X_j)^{-2},
\]

\[
\delta \langle X \rangle = \sqrt{\langle X^2 \rangle - \langle X \rangle^2},
\]

\( X \) is the quantity to be averaged, \( n \) is some specified power, and \( \delta X_i \) is the uncertainty in a particular measurement of \( X \). Note that we weight \( H \) and \( H^2 \) by the same \( w_i \). We find \( \langle H \rangle = (6895 \pm 513) \) km s\(^{-1} \). Figure 1 shows the values of \( H \) obtained from each simulation as well as the average value of \( H \). We can see that based on the AEI/LSU data, which take into account the initial recoil at the beginning of the full numerical simulations, one could fit linear corrections to \( H \). However, the deviations from \( H = \text{const} \) are only significant near \( D = q^2/(1 + q)^2(\alpha_1^2 - \alpha_2^2) \) = 0, when the spin-induced recoil is small (and hence the relative error in the spin-induced recoil is large). The absolute differences between the predicted and measured recoil velocities for the AEI/LSU results are within 20 km s\(^{-1} \), when we take \( H = 6895 \) km s\(^{-1} \).

### III. POST-NEWTONIAN ANALYSIS

In order to compare results from the recoil due to unequal masses and those due to spin effects as well, we will study systems with similar orbital trajectories. Since the radiated momentum due to unequal masses is a function of the orbital acceleration, these systems will all exhibit very similar unequal-mass contributions to the net recoil, which allows us to isolate the spin-induced contributions to the recoil. To generate families of binaries with similar trajectories we use the formulae for the leading order post-Newtonian accelerations and choose configurations that minimize the effects of the spins on the trajectories, but have non-negligible spin contributions to the net recoil.

The relative one-body accelerations can be written as

\[
\ddot{a} = \ddot{a}_N + \ddot{a}_{PN} + \ddot{a}_{2PN} + \ddot{a}_{RR} + \ddot{a}_{SO} + \ddot{a}_{SS},
\]

with
FIG. 1: The value of $H$ calculated by inverting Eq. (2b) as determined from simulations by the AEI, PSU, and NASA/GSFC groups. The thick line is the weighted average and the thin lines are the expected uncertainty in the average.

\[ \vec{a}_N = -\frac{m}{r^2} \hat{n}, \]  
\[ \vec{a}_{PN} = -\frac{m}{r^2} \left\{ \hat{n} \left[ (1 + 3\nu)v^2 - 2(2 + \nu)\frac{m}{r} - \frac{3}{2}\nu \frac{v^2}{r^2} \right] - 2(2 - \nu)\hat{r}\hat{v} \right\}, \]  
\[ \vec{a}_{2PN} = -\frac{m}{r^2} \left\{ \hat{n} \left[ \frac{3}{4}(12 + 29\nu)\frac{m}{r}^2 + \nu(3 - 4\nu)v^4 + \frac{15}{8}\nu(1 - 3\nu)v^4 \right. \right. \]  
\[ \left. \left. - \frac{3}{2}\nu(3 - 4\nu)v^2\hat{r}^2 - \frac{1}{2}\nu(13 - 4\nu)\frac{m}{r}v^2 - (2 + 25\nu + 2\nu^2)\frac{m}{r}\frac{r^2}{v^2} \right] \right. \]  
\[ \left. - \frac{1}{2}\hat{r}\hat{v} \left[ \nu(15 + 4\nu)v^2 - (4 + 41\nu + 8\nu^2)\frac{m}{r} - 3\nu(3 + 2\nu)v^2 \right] \right\}, \]  
\[ \vec{a}_{RR} = \frac{8}{5} \frac{\nu m^2}{r^3} \left\{ \hat{r}\hat{n} \left[ 18v^2 + \frac{2m}{r} - 25\hat{r}^2 \right] - \hat{v} \left[ 6v^2 - \frac{2m}{r} - 15\hat{v}^2 \right] \right\}, \]  
\[ \vec{a}_{SO} = \frac{1}{r^3} \left\{ 6\hat{n}[(\hat{n} \times \hat{v}) \cdot (2\vec{S} + \frac{\delta m}{m}\Delta)] - [\hat{v} \times (7\vec{S} + 3\frac{\delta m}{m}\Delta)] + 3\hat{r}[(\hat{n} \times (3\vec{S} + \frac{\delta m}{m}\Delta))] \right\}, \]  
\[ \vec{a}_{SS} = -\frac{3}{\mu r^4} \left\{ \hat{n}(\vec{S}_1 \cdot \vec{S}_2) + \vec{S}_1(\hat{n} \cdot \vec{S}_2) + \vec{S}_2(\hat{n} \cdot \vec{S}_1) - 5\hat{n}(\hat{n} \cdot \vec{S}_1)(\hat{n} \cdot \vec{S}_2) \right\}, \]

where $\vec{x} \equiv \vec{x}_1 - \vec{x}_2$, $\vec{v} = d\vec{x}/dt$, $\hat{n} \equiv \vec{x}/r$, $m = m_1 + m_2$, $\mu \equiv m_1 m_2/m$, $\nu \equiv \mu/m$, $\delta m \equiv m_1 - m_2$, $\vec{S} \equiv \vec{S}_1 + \vec{S}_2$. 
and \( \Delta \equiv m(\vec{S}_2/m_2 - \vec{S}_1/m_1) \), and an overdot denotes \( d/dt \).

The first four terms in Eq. (8) correspond to the Newtonian, first-post-Newtonian (1PN), second-post-Newtonian, and radiation reaction contributions to the equations of motion. The last two terms in Eq. (8) are the spin-orbit (SO) and spin-spin (SS) contributions to the acceleration.

The radiated linear momentum due to the motion of the two holes has the form [17]

\[
\dot{P}_N = -\frac{8}{105} \frac{\delta m}{m^2} \left( \frac{m}{r} \right)^4 \left\{ \dot{\vec{r}} \hat{n} \left[ 55v^2 - 45r^2 + 12 \frac{m}{r} \right] \\
+ \vec{v} \left[ 38r^2 - 50v^2 - 8 \frac{m}{r} \right] \right\},
\]

(10)

plus higher post-Newtonian terms [63], while the radiated linear momentum due to spin-orbit effects has the form

\[
\dot{P}_{SO} = -8 \frac{\mu^2}{15} \left( \frac{m}{r} \right)^5 \left\{ 4\dot{r}(\vec{v} \times \hat{\Delta}) - 2v^2(\hat{n} \times \hat{\Delta}) \\
- (\hat{n} \times \vec{v}) \left[ 3\dot{r}(\hat{n} \cdot \hat{\Delta}) + 2(\vec{v} \cdot \hat{\Delta}) \right] \right\}.
\]

(11)

Note also that the spin-spin coupling does not contribute to the radiated linear momentum to this order.

In order to best study and model how the final remnant recoil velocity depends on the mass ratio and spins, we will chose configurations with black holes spinning along the orbital angular momentum. In this way the orbital plane will not precess and we can write [47]

\[
\vec{S} = \vec{S}_1 + \vec{S}_2 = S^z \hat{z},
\]

(12)

and

\[
\vec{v} = \dot{\vec{r}} \hat{n} + r \omega \hat{\lambda},
\]

(13)

where \( \vec{L}_N = \mu(\vec{x} \times \vec{v}) \) is the Newtonian orbital angular momentum, \( \hat{\lambda} = \vec{L}_N / |\vec{L}_N| \), and \( \omega = \delta \dot{\phi} / dt \) is defined as the orbital angular velocity.

Taking into account that the velocity remains in the orbital plane, i.e. Eq. (13), we find that the spin-orbit acceleration [96] is given by

\[
\vec{a}_{SO} = \frac{1}{r^3} \left\{ r \omega \left( 5S^z + 3 \frac{\delta m}{m} \Delta^z \right) \hat{n} - 2\dot{r}S^z \hat{\lambda} \right\},
\]

(14)

and the radiated linear momentum is given by

\[
\dot{P}_{SO} = \frac{16}{15} \mu^2 m \Delta^z \left( \frac{m}{r} \right)^5 \left\{ (\dot{r}^2 - r^2 \omega^2) \hat{\lambda} - 2\dot{r} \omega \hat{n} \right\},
\]

(15)

and

\[
\dot{P}_N = -\frac{8}{105} \frac{\delta m}{m^2} \left( \frac{m}{r} \right)^4 \left\{ \dot{\vec{r}} \hat{n} \left[ 5\dot{r}^2 - 2\dot{r}^2 + 4 \frac{m}{r} \right] \\
- r \omega \hat{\lambda} \left[ 50r^2 \omega^2 + 12\dot{r}^2 + 8 \frac{m}{r} \right] \right\}.
\]

(16)

Note that if we take the scalar product of these two instantaneous radiated momenta we obtain

\[
\dot{P}_N \cdot \dot{P}_{SO} / \sqrt{g(r, \dot{r}, \omega)} = \cos(\xi_{PN})
\]

(17)

where \( f = 4r^4 + (8m + 24r^3\omega^2)r^2 - r^2\omega^2(4m + 25r^4\omega^2) \) and \( g(r, \dot{r}, \omega) = 4r^8 - (16m - 124r^2\omega^2 - r^4) / r + (16m^2 + 232r^3\omega^2m + 1225r^6\omega^4 + 2r^4\omega^2)^2 / r^2 + \omega^2 (2\dot{m} + 800r^3\omega^2m + 2500r^6\omega^4 + r^4\omega^2) \). The fact that the factor of \( \Delta^z \) drops out of Eq. (17) suggests that \( \xi \) (which is the angle between the cumulative radiated linear momenta) will depend only weakly on the spins through their affects on the orbital motion. Binaries with similar orbital trajectories should therefore have similar values for \( \xi \). Note that \( \xi \) may still be a strong function of trajectory and \( q \).

We will now turn to the question of identifying a subset of physical parameters of the binary that produce similar trajectories for unequal-mass, non-spinning and unequal-mass, spinning binaries in order to compare their recoil velocities and extract the spin contribution to the total recoil.

### A. similar radial trajectories

An analysis of how \( \xi \) depends on configuration is greatly simplified if the trajectories of the spinning binaries are similar to the trajectory for a non-spinning binary with the same mass ratio. In order to accomplish this, we use the post-Newtonian expression for the spin-orbit induced acceleration Eq. (14), and choose configuration that minimize its effect.

The radial component of the spin-orbit induced acceleration will vanish if \( 5S^z + 3 \frac{\delta m}{m} \Delta^z = 0 \). This leads to the condition

\[
F = (3q + 2) + (3 + 2q)\hat{\alpha} = 0,
\]

(18)

where \( \hat{\alpha} = q_0 \alpha_1 / \alpha_2 \) can take any positive or negative value. However, if we consider the algebraic average over the range \( 0 \leq q \leq 1 \) at fixed \( F \) we find

\[
\langle \hat{\alpha} \rangle = \frac{1}{2} [\hat{\alpha}(q = 0) + \hat{\alpha}(q = 1)] = \frac{4}{15} F - \frac{5}{6},
\]

(19)

and that \( \hat{\alpha} = \langle \hat{\alpha} \rangle \) when \( q = 3/8 \) (independent of \( F \)).

We will thus study configurations with this mass ratio (which also produces a nearly maximum recoil velocity of \( \approx 175 \text{ km s}^{-1} \) for non-spinning unequal mass black hole binaries [22]).

Hence the first family of black-hole-binary configurations that we will study is given by the choice

\[
F = 0, \quad q = 3/8,
\]

(20)

thus

\[
\alpha_2 / \alpha_1 = -q(3 + 2q)/(2 + 3q) = -9/20.
\]

(21)

The total spin of the binary will in general be non-vanishing with

\[
S^z / m^2 = (\alpha_2 + q^2 \alpha_1)/(1 + q^2) = 4\alpha_2 / 11.
\]

(22)
TABLE I: Initial data parameters for quasi-circular orbits with orbital frequency \( \omega/M = 0.05 \). All sets have mass ratio \( q = m_1^H/m_2^H = 3/8 \). The ‘F’ series has \( \alpha = \alpha_2/\alpha_1 = -9/20 \) (hence \( F = q_0\alpha_1/2(2q + 3) + 3q + 2 = 0 \)), and the ‘S’ series has \( S = S_1^2 + S_2^2 = 0 \). The punctures are located along the \( x \)-axis with momenta \( \vec{P}_1 = (0, P, 0) \) and \( \vec{P}_2 = (0, -P, 0) \), and spins \( \vec{S}_i \) along the \( z \)-axis. \( m_i^p \) are the puncture masses, \( m_i^H \) are the horizon masses.

<table>
<thead>
<tr>
<th>Config</th>
<th>( Q_{38} )</th>
<th>( F_{+0.2} )</th>
<th>( F_{-0.2} )</th>
<th>( F_{+0.4} )</th>
<th>( F_{-0.4} )</th>
<th>( S_{+0.64} )</th>
<th>( S_{-0.64} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1/M )</td>
<td>-4.74555652</td>
<td>-4.6898329</td>
<td>-4.8008847</td>
<td>-4.6310312</td>
<td>-4.8548401</td>
<td>-1.5310235</td>
<td>-4.9561392</td>
</tr>
<tr>
<td>( x_2/M )</td>
<td>1.76045721</td>
<td>1.8161037</td>
<td>1.7042740</td>
<td>1.8711641</td>
<td>1.6475993</td>
<td>1.8975924</td>
<td>1.618224</td>
</tr>
<tr>
<td>( S_1/M^2 )</td>
<td>0.00000000</td>
<td>0.01521962</td>
<td>-0.01522140</td>
<td>0.030437161</td>
<td>-0.030447242</td>
<td>0.048726127</td>
<td>-0.04869700</td>
</tr>
<tr>
<td>( S_2/M^2 )</td>
<td>0.00000000</td>
<td>-0.0048702791</td>
<td>0.004870847</td>
<td>0.009738914</td>
<td>0.009743175</td>
<td>0.0048726127</td>
<td>0.004869700</td>
</tr>
<tr>
<td>( P/M )</td>
<td>0.10682112</td>
<td>0.10707929</td>
<td>0.10656747</td>
<td>0.10734244</td>
<td>0.10631792</td>
<td>0.10676439</td>
<td>0.10692958</td>
</tr>
<tr>
<td>( L^2/M^2 )</td>
<td>0.69498063</td>
<td>0.695546816</td>
<td>0.6932382744</td>
<td>0.697961678</td>
<td>0.6913258685</td>
<td>0.6863415524</td>
<td>0.702841096</td>
</tr>
<tr>
<td>( J/M^2 )</td>
<td>0.69498063</td>
<td>0.6630715129</td>
<td>0.72672691819</td>
<td>0.6309988643</td>
<td>0.7583097977</td>
<td>0.6863415524</td>
<td>0.702841096</td>
</tr>
<tr>
<td>( m_1^H/M )</td>
<td>0.257487827988</td>
<td>0.25319314</td>
<td>0.253279647</td>
<td>0.239665153</td>
<td>0.239816706</td>
<td>0.205915971</td>
<td>0.206131153</td>
</tr>
<tr>
<td>( m_2^H/M )</td>
<td>0.718534207968</td>
<td>0.71621170</td>
<td>0.716211394</td>
<td>0.70903409</td>
<td>0.70903488</td>
<td>0.715832591</td>
<td>0.715746409</td>
</tr>
<tr>
<td>( m_1^S/M )</td>
<td>0.27582974</td>
<td>0.27577886</td>
<td>0.27591869</td>
<td>0.27575065</td>
<td>0.27577578</td>
<td>0.2756959</td>
<td>0.2755812</td>
</tr>
<tr>
<td>( m_2^S/M )</td>
<td>0.73541100</td>
<td>0.73541402</td>
<td>0.735444371</td>
<td>0.735334919</td>
<td>0.735402505</td>
<td>0.7351861</td>
<td>0.734888095</td>
</tr>
<tr>
<td>( \alpha_1 )</td>
<td>0.000</td>
<td>0.20012582</td>
<td>-0.20013825</td>
<td>0.400282516</td>
<td>-0.400383662</td>
<td>0.6411766</td>
<td>-0.6411984</td>
</tr>
<tr>
<td>( \alpha_2 )</td>
<td>0.000</td>
<td>-0.00053523</td>
<td>0.090061919</td>
<td>-0.180130779</td>
<td>0.18015874</td>
<td>-0.00153</td>
<td>0.0015883</td>
</tr>
<tr>
<td>( M_{ADM}/M )</td>
<td>1.00001</td>
<td>1.00001</td>
<td>0.999997</td>
<td>1.00001</td>
<td>0.999997</td>
<td>1.00001</td>
<td>0.999991</td>
</tr>
</tbody>
</table>

**B. similar tangential trajectories**

We can also choose a configuration where the tangential component of the acceleration due to the spin-orbit coupling vanishes, i.e.

\[
S^2 = S_1^2 + S_2^2 = 0. \tag{23}
\]

This translates into the condition

\[
\alpha_2/\alpha_1 = -q^2 = -9/64 \tag{24}
\]

when \( q = 3/8 \). Note that now, the radial acceleration, as parametrized by \( F \), is non vanishing

\[
F = (3q + 2) + (3 + 2q)\alpha_1/\alpha_2 = -55/8. \tag{25}
\]

Thus, for \( q \neq 1 \), we cannot make both the radial and tangential components of the spin-orbit acceleration vanish at the same time by a simple choice of physical parameters of the binary.

**IV. INITIAL CONFIGURATIONS**

We choose quasi-circular initial configurations with mass ratio \( q = m_1/m_2 = 3/8 \) from four families of parameters that we will denote by Q, F, S, and A. The Q-series has initially non-spinning holes, the F-series has \( F = 0 \) (See Eq. \( \langle 19 \rangle \)); hence zero PN spin-orbit-induced radial acceleration, the S-series has total spin \( \vec{S} = 0 \); hence zero PN spin-orbit-induced tangential acceleration, and the A-series has neither \( F = 0 \) nor \( S = 0 \); hence both spin-orbit-induced accelerations are non-vanishing. The puncture masses were fixed by requiring that the total ADM mass of the system be 1 and that the mass ratio of the horizon masses be \( 3/8 \). The initial data parameters for these configurations are given in Tables I and II. We obtained initial data parameters by choosing spin and linear momenta consistent with 3PN quasi-circular orbits for binaries with mass ratio \( q = 3/8 \) and then solve for the Bowen-York ansatz for the initial 3-metric and extrinsic curvature. This method was pioneered by the Lazarus project \( \[64 \] \) (See Fig. 35 there), and then used in the rest of the breakthrough papers \( \[6, 7, 8, 39, 48, 62 \] \) by the authors (in Ref. \( \[28 \] \) we used the PN expressions for the radial component of the momentum as well).

**V. RESULTS**

We evolved all configurations given in Tables I and II using 10 levels of refinement with a finest resolution of \( h = M/80 \) and outer boundaries at \( 320M \) except configuration \( \Lambda_{+0.9} \) where we used an additional coarse level to push the outer boundaries to \( 640M \). In all cases, except where noted otherwise, we set the free Gamma-driver parameter in Eq. \( \langle 57 \rangle \) to \( \eta = 6/M \) \( \[2, 57 \] \).

In a generic simulation both the unequal mass and spin components of the recoil are functions of the trajectory. To single out each individual effect we perform runs cho-
TABLE II: Initial data parameters for quasi-circular orbits with orbital frequency $\omega/M = 0.05$. All sets have mass ratio $q = m_1^2/m_2^2 = 3/8$. The punctures are located along the $x$-axis with momenta $P_1 = (0, P, 0)$ and $P_2 = (0, -P, 0)$, and spins $\vec{S}_i$ along the $z$-axis. $m_i^r$ are the puncture masses, $m_i^H$ are the horizon masses. In this series neither $F$ nor $S$ vanishes.

<table>
<thead>
<tr>
<th>Config</th>
<th>$A_{+0.9}$</th>
<th>$A_{-0.9}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1/M$</td>
<td>-4.5443438</td>
<td>-4.8662563</td>
</tr>
<tr>
<td>$x_2/M$</td>
<td>1.573114</td>
<td>1.9275192</td>
</tr>
<tr>
<td>$S_1^2/M^2$</td>
<td>0.0000000</td>
<td>0.0000000</td>
</tr>
<tr>
<td>$S_2^2/M^2$</td>
<td>0.48873779</td>
<td>-0.48518609</td>
</tr>
<tr>
<td>$P/M$</td>
<td>0.10276456</td>
<td>0.11089099</td>
</tr>
<tr>
<td>$L^2/M^2$</td>
<td>0.6286584770</td>
<td>0.7533827395</td>
</tr>
<tr>
<td>$J/M^2$</td>
<td>1.171396265</td>
<td>0.267566537</td>
</tr>
<tr>
<td>$m_1^r/M$</td>
<td>0.2545666</td>
<td>0.2545806</td>
</tr>
<tr>
<td>$m_2^r/M$</td>
<td>0.2822299</td>
<td>0.284150275</td>
</tr>
<tr>
<td>$m_1^H/M$</td>
<td>0.2733564</td>
<td>0.2726296</td>
</tr>
<tr>
<td>$m_2^H/M$</td>
<td>0.728824</td>
<td>0.7270093</td>
</tr>
<tr>
<td>$\alpha_1^r$</td>
<td>0.0000000</td>
<td>0.0000000</td>
</tr>
<tr>
<td>$\alpha_2^r$</td>
<td>0.920196524</td>
<td>-0.9192121</td>
</tr>
<tr>
<td>$\mathcal{M}_{ADM}/M$</td>
<td>1.000000</td>
<td>0.999991</td>
</tr>
</tbody>
</table>

TABLE III: The rotation angle needed to align the trajectories of each simulation with the $Q_{38}$ configuration as measured directly from the orbital trajectories ($\Phi_{\text{track}}$) and using the phase of the waveform at the point of maximum amplitude ($\Phi_{\text{q}}$). Note that $\Phi_{\text{q}}$ provides the rotation angle modulo 180$^\circ$.

| Config | $\Phi_{\text{track}}$ | $\Phi_{\text{q}}$ | $|\Phi_{\text{track}} - \Phi_{\text{q}}|_0$ |
|--------|----------------------|-----------------|----------------------------------|
| $F_{+0.2}$ | 35.4$^\circ$ | 35.4$^\circ$ | 9.5$^\circ$ |
| $F_{-0.2}$ | -28.5$^\circ$ | -28.5$^\circ$ | 7.5$^\circ$ |
| $F_{+0.4}$ | 63.1$^\circ$ | 63.1$^\circ$ | 7.1$^\circ$ |
| $F_{-0.4}$ | -40.0$^\circ$ | -40.0$^\circ$ | 4.0$^\circ$ |
| $S_{+0.6}$ | 9.7$^\circ$ | 9.7$^\circ$ | 4.7$^\circ$ |
| $S_{-0.6}$ | 44.6$^\circ$ | 44.6$^\circ$ | 11.4$^\circ$ |
| $A_{+0.9}$ | *** | 12.3$^\circ$ | *** |
| $A_{-0.9}$ | *** | -15.9$^\circ$ | *** |

TABLE IV: Results of the recoil velocity for the $Q_{38}$ configuration with $\eta = 2/M$ at two different resolutions. After correcting for the phase error, equivalent to a rotation, the two recoils agree. Here $\text{R}_{\text{track}}$ denotes the velocity after rotating by the angle $\Phi_{\text{track}}$ and $\text{R}_{\text{q}}$ denotes the velocity after rotating by the angle $\Phi_{\text{q}}$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\Phi_{\text{track}}$</th>
<th>$\Phi_{\text{q}}$</th>
<th>$V_{x}$</th>
<th>$V_{y}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M/80$</td>
<td>34$^\circ$</td>
<td>36.5$^\circ$</td>
<td>-163$^{\pm}$12</td>
<td>-46$^{\pm}$11</td>
</tr>
<tr>
<td>$M/80$</td>
<td>$R_{\text{q}}$</td>
<td>***</td>
<td>-103$^{\pm}$12</td>
<td>-134$^{\pm}$11</td>
</tr>
<tr>
<td>$M/80$</td>
<td>$R_{\text{track}}$</td>
<td>***</td>
<td>-109$^{\pm}$12</td>
<td>-129$^{\pm}$11</td>
</tr>
<tr>
<td>$M/100$</td>
<td>0</td>
<td>0</td>
<td>-109$^{\pm}$14</td>
<td>-133$^{\pm}$12</td>
</tr>
</tbody>
</table>

The recoil velocity, $\vec{V}_{\text{recoil}}$, rotates by an angle $\Phi_{\text{track}}$ or $\Phi_{\text{q}}$ to obtain the recoil velocity in the $xy$ plane, $\vec{V}_{\text{Q38}}$. The rotation is given by Eq. (1), and the measured values of the recoil magnitude are given by Eq. (1) with $\xi$ calculated as in Eq. (27) for the rotation angle $\Phi_{\text{q}}$. Note that there is no rotation which will make the $A_{+0.9}$ or $A_{-0.9}$ trajectories line up with the $Q_{38}$ trajectory. In these cases the hangup-effect due to spin-orbit coupling significantly alters the trajectories (See Fig. 3).

Once we have found the correct rotation angle we obtain $\hat{\xi}$ via

$$\vec{V}_{\text{recoil}} = R(\Phi)\vec{V}_{\text{recoil}},$$

$$\vec{V}_{\text{spin}} = \hat{\vec{V}}_{\text{recoil}} - \vec{V}_{\text{Q38}},$$

$$\cos(\xi) = \hat{\vec{V}}_{\text{spin}} \cdot \vec{V}_{\text{Q38}}.$$  \hspace{1cm} (26)

where $\hat{\vec{V}}_{\text{recoil}}$ is the measured recoil velocity, $R(\Phi)$ rotates $\vec{V}_{\text{recoil}}$ by an angle $\Phi$ in the $xy$ plane, and $\vec{V}_{\text{Q38}}$ is the recoil of the $Q_{38}$ configuration. Note that when $\alpha_2^r - q_1 < 0$ we need to replace $\eta$ by $\pi - \xi$ in formula (27) since the coefficient $v_\perp$ in Eq. (1) is negative. We calculate two different values of $\xi$, $\xi_{\text{track}}$ and $\xi_{\text{q}}$, based on the rotation angles $\Phi_{\text{track}}$ and $\Phi_{\text{q}}$, respectively. We obtain an additional measurement of $\xi$ by solving for $\cos(\xi)$ using Eq. (1) and the measured values of the recoil magnitude. We denote this latter measurement of $\xi$, which is unaffected by rotations, by $\xi_{\text{Formula}}$, where

$$\xi_{\text{Formula}} = \cos^{-1}\left(\frac{v^2 - v_m(q)^2 - v_\perp(q, \alpha_1^r, \alpha_2^r)^2}{2 v_m(q) v_\perp(q, \alpha_1^r, \alpha_2^r)}\right),$$  \hspace{1cm} (27)

$v_m$ is given by Eq. (26), $v_\perp$ is given by Eq. (27), and $v$ is the measured magnitude of the recoil velocity.

We summarize the results of our simulations in Tables V and VI. All configuration, with the exception of the ‘A’ series, have radiated energies in the range $E_{\text{rad}}/M = 0.021 \pm 0.002$ and radiated angular momenta in the range $J_{\text{rad}}/M^2 = 0.15 \pm 0.01$, which is consistent
FIG. 2: The trajectory differences $\vec{r} = \vec{r}_1 - \vec{r}_2$ for the ‘F’ and ‘S’ series rotated so that the late-inspiral matches the $Q_{38}$ trajectory. The plots show the rotation angle $\Phi_{\text{track}}$.

with these trajectories being essentially the same for all configurations (See Fig. 2).

We obtain weighted averages for $\xi$ for the ‘F’ and ‘S’ series of $\langle \xi_{\text{track}} \rangle = (152 \pm 9)^\circ$, $\langle \xi_{\psi_4} \rangle = (143 \pm 14)^\circ$, and
TABLE VI: The recoil velocities (prior to any rotation), radiated energy and angular momentum, and $\xi$ for the ‘S’ and ‘A’ series. $\xi_{\text{track}}$ is calculated using $\Phi_{\text{track}}$ and Eq. (29). $\xi_{\psi4}$ is calculated using $\Phi_{\psi4}$ and Eq. (26). $\xi_{\text{Formula}}$ is calculated from the given recoil magnitude using Eq. (27). $|\vec{V}_{\text{track}}^{\text{pred}}|$, $|\vec{V}_{\psi4}^{\text{pred}}|$, and $|\vec{V}_{\text{avg}}^{\text{pred}}|$ are the recoil velocities as predicted by Eq. (1) with $\xi = \xi_{\text{track}}$, $\xi = \xi_{\psi4}$, and $\xi = \xi_{\text{Formula}}$ respectively.

<table>
<thead>
<tr>
<th>Config</th>
<th>$Q_{38}$</th>
<th>$A_{\text{9.0}}$</th>
<th>$A_{\text{9.0}}$</th>
<th>$A_{\text{9.0}}$</th>
<th>$A_{\text{9.0}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{\text{rad}}/M$</td>
<td>0.0210 ± 0.0003 &amp; 0.0202 ± 0.0003 &amp; 0.0219 ± 0.0004 &amp; 0.0193 ± 0.0002 &amp; 0.0228 ± 0.0004</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$J_{\text{rad}}/M^2$</td>
<td>0.1503 ± 0.0030 &amp; 0.1471 ± 0.0005 &amp; 0.1576 ± 0.0015 &amp; 0.1399 ± 0.0016 &amp; 0.1625 ± 0.0010</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V^x$ [km s$^{-1}$]</td>
<td>$-94 \pm 11$ &amp; $-177 \pm 10$ &amp; $-15 \pm 14$ &amp; $-223 \pm 12$ &amp; $15 \pm 14$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V^y$ [km s$^{-1}$]</td>
<td>$-141 \pm 5$ &amp; $-85 \pm 12$ &amp; $-155 \pm 5$ &amp; $33 \pm 18$ &amp; $-127 \pm 4$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$</td>
<td>\vec{V}</td>
<td>$ [km s$^{-1}$]</td>
<td>169.5 ± 7.4 &amp; 196.4 ± 10.4 &amp; 155.7 ± 7.2 &amp; 225.4 ± 12.2 &amp; 127.9 ± 4.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\xi_{\text{track}}$ [deg]</td>
<td>0° &amp; (143 ± 31)° &amp; (178 ± 73)° &amp; (147 ± 20)° &amp; (169 ± 21)°</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\xi_{\psi4}$ [deg]</td>
<td>0° &amp; (154 ± 43)° &amp; (127 ± 41)° &amp; (173 ± 25)° &amp; (179 ± 21)°</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\xi_{\text{Formula}}$ [deg]</td>
<td>0° &amp; (127 ± 26)° &amp; (131 ± 15)° &amp; (134 ± 20)° &amp; (144 ± 6)°</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$</td>
<td>\vec{V}_{\text{track}}^{\text{pred}}</td>
<td>$ [km s$^{-1}$]</td>
<td>175 &amp; 202 ± 9 &amp; 142 ± 3 &amp; 232 ± 10 &amp; 112 ± 9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$</td>
<td>\vec{V}_{\psi4}^{\text{pred}}</td>
<td>$ [km s$^{-1}$]</td>
<td>175 &amp; 205 ± 9 &amp; 158 ± 21 &amp; 240 ± 5 &amp; 110 ± 5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$</td>
<td>\vec{V}_{\text{avg}}^{\text{pred}}</td>
<td>$ [km s$^{-1}$]</td>
<td>175 &amp; 203 ± 3 &amp; 150 ± 4 &amp; 231 ± 5 &amp; 127 ± 8</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

TABLE VI: The recoil velocities (prior to any rotation), radiated energy and angular momentum, and $\xi$ for the ‘Q’ and ‘F’ series. $\xi_{\text{track}}$ is calculated using $\Phi_{\text{track}}$ and Eq. (29). $\xi_{\psi4}$ is calculated using $\Phi_{\psi4}$ and Eq. (26). $\xi_{\text{Formula}}$ is calculated from the given recoil magnitude using Eq. (27). $|\vec{V}_{\text{track}}^{\text{pred}}|$, $|\vec{V}_{\psi4}^{\text{pred}}|$, and $|\vec{V}_{\text{avg}}^{\text{pred}}|$ are the recoil velocities as predicted by Eq. (1) with $\xi = \xi_{\text{track}}$, $\xi = \xi_{\psi4}$, and $\xi = \xi_{\text{Formula}}$ respectively.

<table>
<thead>
<tr>
<th>Config</th>
<th>$S_{E_{\text{0.9}}}$</th>
<th>$S_{E_{\text{0.9}}}$</th>
<th>$A_{\text{0.9}}$</th>
<th>$A_{\text{0.9}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{\text{rad}}/M$</td>
<td>0.0209 ± 0.0003 &amp; 0.0203 ± 0.0002 &amp; 0.050668 ± 0.000574 &amp; 0.01274 ± 0.00003</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$J_{\text{rad}}/M^2$</td>
<td>0.152 ± 0.0007 &amp; 0.146 ± 0.001 &amp; 0.2999857 ± 0.00708 &amp; 0.092 ± 0.001</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V^x$ [km s$^{-1}$]</td>
<td>$-122 \pm 18$ &amp; $-119 \pm 5$ &amp; $13 \pm 30$ &amp; $48 \pm 24$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V^y$ [km s$^{-1}$]</td>
<td>$-181 \pm 15$ &amp; $31 \pm 4$ &amp; $-63 \pm 2$ &amp; $-340 \pm 8$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$</td>
<td>\vec{V}</td>
<td>$ [km s$^{-1}$]</td>
<td>218.3 ± 16.0 &amp; 123.0 ± 4.9 &amp; 64.1 ± 5.9 &amp; 343.4 ± 8.6</td>
<td></td>
</tr>
<tr>
<td>$\xi_{\text{track}}$ [deg]</td>
<td>(160 ± 31)° &amp; (148 ± 11)° &amp; *** &amp; ***</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\xi_{\psi4}$ [deg]</td>
<td>(142 ± 28)° &amp; (137 ± 7)° &amp; (158 ± 7)° &amp; (93 ± 7)°</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\xi_{\text{Formula}}$ [deg]</td>
<td>(124 ± 22)° &amp; (150 ± 7)° &amp; (159 ± 2)° &amp; (149 ± 19)°</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$</td>
<td>\vec{V}_{\text{track}}^{\text{pred}}</td>
<td>$ [km s$^{-1}$]</td>
<td>237 ± 10 &amp; 125 ± 10 &amp; *** &amp; ***</td>
<td></td>
</tr>
<tr>
<td>$</td>
<td>\vec{V}_{\psi4}^{\text{pred}}</td>
<td>$ [km s$^{-1}$]</td>
<td>230 ± 16 &amp; 135 ± 7 &amp; 68 ± 22 &amp; 259 ± 18</td>
<td></td>
</tr>
<tr>
<td>$</td>
<td>\vec{V}_{\text{avg}}^{\text{pred}}</td>
<td>$ [km s$^{-1}$]</td>
<td>231 ± 5 &amp; 127 ± 8 &amp; 108 ± 28 &amp; 340 ± 9</td>
<td></td>
</tr>
</tbody>
</table>
TABLE VII: The average value $|⟨\vec{r}⟩|$ of the trajectories for each configuration. The larger the value of $|⟨\vec{r}⟩|$ the slower the inspiral.

| Config $|⟨\vec{r}⟩|$ | Config $|⟨\vec{r}⟩|$ | Config $|⟨\vec{r}⟩|$ |
|----------------|----------------|----------------|
| $Q_{38}$ 0.858303 | $F_{+0.4}$ 0.902234 | $F_{-0.2}$ 0.7982157 |
| $F_{+0.4}$ 0.936172 | $F_{-0.4}$ 0.76745 | $S_{+0.64}$ 0.845197 |
| $S_{-0.64}$ 0.95333 | $A_{+0.9}$ 0.365654 | $A_{-0.9}$ 1.30869 |

$⟨\xi_{\text{Formula}}⟩ = (144 \pm 7)\degree$, where we use Eq. (12) to obtain the weighted average and uncertainty. These weighted averages are consistent with the measured values of $\xi$. The weighted average over all three measurements of $\xi$ is $⟨\xi⟩ = (145 \pm 10)\degree$. Interestingly, $⟨\xi⟩$ provides an accurate prediction for the recoil velocity of the $A_{-0.9}$ configuration. This result is unexpected because the recoil due to unequal masses is a function of the mass ratio and the trajectories (i.e. the accelerations of the masses over time). For the ‘F’ and ‘S’ configuration the trajectories are very similar to $Q_{38}$, and hence the unequal mass components of the recoil are expected to be very similar to $Q_{38}$. However, the spin-orbit coupling induced hangup effect in both $A_{+0.9}$ and $A_{-0.9}$ greatly affects the trajectories (See Fig. 3), as well as the radiated energy and angular momenta. If we consider the radiated linear momentum averaged over an orbit, then we see that the slower the inspiral (i.e. the closer to a closed orbit), the smaller the average recoil. Hence we expect that $A_{+0.9}$ will have a smaller unequal-mass recoil than $Q_{38}$, while $A_{-0.9}$ will have a larger one. To quantify how much the orbits close we take the average of $\vec{r} = \vec{r}_1 - \vec{r}_2$ over the trajectory from the beginning of each simulation until $|\vec{r}| \sim 0.1$. The resulting averages $|⟨\vec{r}⟩|$ for the ‘Q’, ‘F’, ‘S’, and ‘A’ families are given in Table VII. The mean and standard deviation of $|⟨\vec{r}⟩|$ for the ‘Q’, ‘F’, and ‘S’ configurations is $|⟨\vec{r}⟩| = 0.865 \pm 0.070$. The $A_{+0.9}$ and $A_{-0.9}$ configuration lie below the orbital plane, and the smaller hole having negligible spin. We also evolved a similar configuration, which we will denote by SP6R, that is identical to SP6, but with the spin rotated by $90\degree$ about the $z$-axis. We evolved both configurations using the same grid structure as in the previous section, but used $\eta = 2/M$ rather than $6/M$. This choice of smaller $\eta$ has the effect of reducing the effective resolution, but makes calculations of the quasi-local linear momentum and spin direction more accurate (See Ref. [39]) by reducing coordinate distortions. The initial data parameters for the two configurations are given in Table VIII. The drop in effective resolution when reducing $\eta$ from 6/M to 2/M is significant. In our simulations we found that a $Q_{38}$, $\eta = 2/M$ run with central resolution of $M/100$ had a slightly larger waveform phase error than an equivalent $M/80$ resolution run with $\eta = 6/M$, while an $M/80$ run with $\eta = 2/M$ displayed a significant phase error. We have found in general that, with our choice of gauge, the coordinate dependent measurements, such as spin and linear momentum direction, become more accurate as $\eta$ is reduced (and $h \to 0$). However, if $\eta$ is too small ($\eta \lesssim 1/M$), the runs may become unstable. Similarly, if $\eta$ is too large ($\eta \gtrsim 10/M$), then grid stretching effects can cause the remnant horizon to continuously grow, eventually leading to an unacceptable loss in accuracy at late-times. We have found that a value of $\eta = 6/M$ provides both very high accuracy in the com-

FIG. 4: $\xi$ versus $\Delta/m = S_2/m_2 - S_1/m_1$ as calculated in this work for a mass ratio $q = 3/8$ and from the data published by the NASA/GSFC group for a mass ratio $q = 2/3$ provided in Ref. [38]. We plot $\xi_{\text{track}}$, $\xi_{\text{eq}}$, and $\xi_{\text{Formula}}$ for the ‘F’ and ‘S’ configurations and $\xi_{\text{Formula}}$ for ‘A’ configurations. The thick horizontal line and the two thin horizontal line show the average value $⟨\xi⟩$ and its uncertainty (as calculated in this work from our simulations). The data displays significant scatter, but appears to be consistent with $\xi = \text{const.}$
TABLE VIII: Initial data parameters for the SP6 and SP6R configurations. $m_p$ is the puncture mass parameter of the two holes. SP6 has spins $\vec{S}_1 = (0, S, -S)$ and $\vec{S}_2 = (0, 0, 0)$, momenta $\vec{P} = \pm (P_t, P_r, 0)$, puncture positions $\vec{x}_i = (x_i, d, d)$ and $\vec{x}_2 = (x_-, d, d)$, masses $m_1$ and $m_2$, and $M_{ADM}/M = 1.00000 \pm 0.00001$. SP6R has the same parameters as SP6 with the exception that $\vec{S}_1 = (-S, 0, -S)$.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$m_p/M$</th>
<th>$d/M$</th>
<th>$m_1/M$</th>
<th>$m_2/M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_+ / M$</td>
<td>2.68773</td>
<td>$P_t / M$</td>
<td>$-0.0013947$</td>
<td>$m_2/M$</td>
</tr>
<tr>
<td>$x_- / M$</td>
<td>$-5.20295$</td>
<td>$P_+ / M$</td>
<td>0.10695</td>
<td>$S/M^2$</td>
</tr>
</tbody>
</table>

The computed waveform at modest resolutions, while keeping the remnant horizon size nearly fixed at late-times.

We measure a net recoil of $V_{\text{recoil}} = 375 \pm 18$ km s$^{-1}$ and $V_{\text{recoil}} = 848 \pm 20$ km s$^{-1}$ for SP6 and SP6R respectively.

The analysis of the recoil in SP6 and SP6R is complicated by the fact that the orbital plane precesses significantly during the merger. Thus, we cannot associate the $xy$ components of the precession of the orbital plane we need an accurate measurement of the orbital angular momentum. Here we derive a post-Newtonian model for the recoil produced to the recoil. Given this observation, it becomes very important to accurately model this recoil. In Appendix A we derive a post-Newtonian model for the recoil produced by this in-plane component and show that it predicts the $\cos \Theta$ dependence in our empirical formula.

VII. DISCUSSION

Interestingly, most of the recoil velocity imparted to the remnant is generated around 300 km s$^{-1}$ and $\alpha_\parallel = 0.62 \pm 0.03$ (which are within errors of the initial values). However, the SP6R configuration does show a definite change in $\alpha$ over time, with merger values of $\alpha_\parallel = -0.69 \pm 0.03$ and $\alpha_\perp = 0.54 \pm 0.03$. We can use Eq. (10) to give estimates for the predicted recoil velocity if we make the following assumptions: (1) $\xi = (\xi)$, (2) $\Theta$ for SP6R is rotated by $\pi/2$ radians with respect to SP6, and (3) $\Theta_0$ is the same for SP6 and SP6R. Given these assumptions and the above range of the values for $\alpha_\parallel$ and $\alpha_\perp$, we can perform a non-linear least-squares fit of the recoil magnitude for SP6 and SP6R to obtain $\Theta_0$. The resulting predictions for the recoil magnitude are $V_{\text{SP6}} = (500 \pm 60)$ km s$^{-1}$ and $V_{\text{SP6R}} = (1120 \pm 130)$ km s$^{-1}$. Both predictions are within $2\sigma$ of the actual measured values and have an absolute error of 32%. If we fix $\alpha_\parallel$ and $\alpha_\perp$ to their average values and vary our guess for $\xi$ over the range $(0, 360^\circ)$, we find that the predicted values for $V_{\text{SP6}}$ and $V_{\text{SP6R}}$ lie in the ranges $(402, 495)$ km s$^{-1}$ and $(1048, 1120)$ km s$^{-1}$ respectively.

The SP6 configuration demonstrated that the in-plane component of the spin can be the dominant contribution to the recoil. Given this observation, it becomes very important to accurately model this recoil. In Appendix A we derive a post-Newtonian model for the recoil produced by this in-plane component and show that it predicts the $\cos \Theta$ dependence in our empirical formula.

Note the rapid change in the direction at late times.
excluded from the calculation. Note that peak in \( \frac{dV}{dt} \) can be successfully applied \[19\].

not expected to work, but where the ‘Lazarus’ approach linear regime where post-Newtonian approximations are \( M \) after merger. See also Refs. \[21, 28, 37\], a non-linear regime where post-Newtonian approximations are not expected to work, but where the ‘Lazarus’ approach \[64, 65, 66, 67, 68, 69\] can be successfully applied \[19\].

Although an accurate modeling of \( \xi \) is challenging, starting from an ansatz that \( \xi = \xi(q, \Delta) \), we have found that, for quasi-circular orbits, \( \xi \) is qualitatively independent of either \( \Delta \) or \( q \) for \( q = 3/8 \), \( q = 2/3 \) (based on the results of Ref. \[32\]), and \( q = 1/2 \) (based on SP6). Note that the \( \xi \) that we measure is consistent with a similar parameter introduced in Ref. \[32\], where they found \( \xi = 147^\circ \) (in our notation), based on a least-squares fit of the magnitude of the recoil versus a simplified version of Eq. \[11\]. We know from the results for headon collision (where \( \xi = \pi/2 \) ), that \( \xi \) is a function of eccentricity. However, for quasi-circular orbits, it appears to vary only marginally with either \( q \) or \( \Delta \). Further long-term simulations with high-accuracy (including extrapolations to \( h \to 0 \) and \( \eta \to 0 \) ) and further separated binaries will be needed in order to obtain a highly accurate model for \( \xi \).

In particular, the \( \eta \to 0 \) limit will be important because the recoil depends sensitively on the linear momenta and spin directions of the individual black holes near merger (where gauge effects are most severe), and hence we need to take the \( \eta \to 0 \) limit in order to accurately measure \( \hat{n}, \hat{L} \), and \( \Theta \). Nevertheless, our simple formula holds with enough accuracy for astrophysical applications. In particular we have seen that the determination of an average value for the angle \( \xi \) of \( 145^\circ \) seems to work not only for the \( F \) and \( S \) sequences, but also when we move off of these sequences towards more generic binaries. However, the formula should definitely be used with caution in an untested regime, especially when the trajectories are significantly altered by spin-orbit effects.

Acknowledgments

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APPENDIX A: POST-NEWTONIAN MODELING

Here we provide a brief post-Newtonian analysis of the configurations that maximize the recoil velocity for spinning black holes. The spin-orbit-coupling (SO) contribution to the radiated linear momentum is given by Eq. \[11\].

We will restrict our analysis to planar orbits. Hence we have

\[
\vec{v} = \dot{r}\hat{n} + r\omega\hat{\lambda},
\]

where \( \hat{\lambda} = \hat{L}_N \times \hat{n}, \hat{L}_N = \frac{\hat{L}_N}{|\hat{L}_N|}, \) \( \omega \) is the orbital angular velocity, and \( \hat{L}_N \equiv \mu(\vec{x} \times \vec{v}) \) is the Newtonian orbital angular momentum. We shall take \( \hat{L}_N \equiv \hat{z} \). Hence

\[
\hat{\lambda} = \hat{z} \times \dot{\hat{n}} \quad \text{and} \quad \hat{n} \times \hat{\lambda} = \dot{\hat{n}} \times (\hat{z} \times \hat{n}) = \hat{z}
\]

We observe that the third and fourth terms in Eq. \[11\] only contribute to the recoil along the \( z \)-axis since

\[
\dot{\hat{n}} \times \vec{v} = r\omega\hat{z}
\]

This contribution to the recoil velocity might well be the leading one, hence, in order to maximize the total recoil we seek to align, as much as possible, the first two terms in Eq. \[11\] with the \( z \)-axis. This is achieved by having the spin of the black holes lie in the orbital plane, i.e.

\[
\hat{\Delta} = \Delta_n \hat{n} + \Delta_\lambda \hat{\lambda}.
\]

We then explicitly obtain the following products

\[
\vec{v} \times \Delta = (\dot{r}\Delta_n - r\omega\Delta_n)\hat{z},
\]

\[
\dot{\hat{n}} \times \Delta = \Delta_\lambda \hat{z},
\]

\[
\vec{v} \cdot \Delta = r\Delta_n + r\omega\Delta_\lambda.
\]

Plugging this into Eq. \[11\] we find

\[
\begin{align*}
\vec{P}_{\text{SO}} &= -\frac{8}{15} \frac{\mu^2 m}{r^5} \left\{ (2r^2 - 4r^2\omega^2)\Delta_\lambda - 9r\omega\Delta_n \right\} \hat{z}.
\end{align*}
\]
This clearly displays the fact that the recoil will be maximized when $\Delta$ takes the maximum magnitude (equal mass and opposite maximally rotating black holes) and varies sinusoidally with its projection along the line joining the holes. Note that if we define the angle between $\hat{n}$ and $\vec{\Delta}$ as $\theta$ we can write the above equation as

$$\vec{P}_{SO} = A(r)\Delta|\cos \theta + B(r)|\Delta|\sin \theta = C(r)|\Delta|\cos(\theta - \theta_0(r)).$$

(A9)

This $\cos \theta$ dependence in the recoil was the motivation for proposing the now-verified $\cos \Theta$ dependence in our empirical formula Eq. (24) for the recoil.

Note that this analysis applies to the radiated linear momentum flux. Hence we have assumed that the larger the radiated linear momentum flux, the larger the total radiated linear momentum.

It is also interesting to see if the unexpectedly large magnitude of the maximum out-of-plane recoil, compared to the in-plane recoil, can be understood using the post-Newtonian expression for the radiated linear momentum, i.e. Eqs. (15) and (A8) (See Ref. [38] for a similar analysis). To do this, we used the post-Newtonian formulae for the radiated linear momentum along with the numerical trajectories for runs with the spins in the plane and perpendicular to the plane. We found that the post-Newtonian formulae predicted that the maximum out-of-plane recoil will be approximately twice (almost 9/4) as large, rather than (the observed) $\approx 8$ times as large, as the maximum in-plane recoil. Thus we see that the magnitude of the out-of-plane recoil arises from nonlinear dynamics at merger not fully captured by the post-Newtonian formalism. One may then conclude that, while the post-Newtonian approximation gives the correct dependence of the recoil on the physical parameters, such as the scaling of the recoil velocities with the components of the spins parallel and perpendicular to the angular momentum, it is much less accurate when describing the amplitude of the recoils. Thus we find that post-Newtonian formalisms provides the correct form for our semi-empirical formula [10], but does not provide accurate measurements of the magnitudes of the constants in that formula.
