A new upper bound for the Ramsey number $R(5,5)$

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A NEW UPPER BOUND FOR THE RAMSEY NUMBER $R(5, 5)$

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Abstract.

We show that, in any colouring of the edges of $K_{53}$ with two colours, there exists a monochromatic $K_5$, and hence $R(5, 5) \leq 53$. This is accomplished in three stages: a full enumeration of $(4,4)$-good graphs, a derivation of some upper bounds for the maximum number of edges in $(4,5)$-good graphs, and a proof of the nonexistence of $(5,5)$-good graphs on 53 vertices. Only the first stage required extensive help from the computer.

1. Introduction.

The two-colour Ramsey number $R(k, l)$ is the smallest integer $n$ such that, for any graph $F$ on $n$ vertices, either $F$ contains $K_k$ or $\bar{F}$ contains $K_l$, where $\bar{F}$ denotes the complement of $F$. A graph $F$ is called $(k,l)$-good if $F$ does not contain a $K_k$ and $\bar{F}$ does not contain a $K_l$. The best upper bound known previously, $R(5, 5) \leq 55$, is due to Walker (1971 [7]). The best lower bound, $R(5, 5) \geq 43$, was obtained by Exoo (1989 [1]), who constructed a $(5,5)$-good graph on 42 vertices.

Throughout this paper we will also use the following notation:

$N_F(x)$ — the neighbourhood of vertex $x$ in graph $F$
$\deg_F(x)$ — the degree of vertex $x$ in graph $F$
$n(F), e(F)$ — the number of vertices and edges in graph $F$
$t(F)$ — the number of triangles in $F$
$l(F)$ — the number of independent 3-sets in graph $F$; i.e. $t(\bar{F})$
$V(F)$ — the vertex set of graph $F$
$(k,l,n)$-good graph — a $(k,l)$-good graph on $n$ vertices
$e(k,l,n)$ — the minimum number of edges in any $(k,l,n)$-good graph
$E(k,l,n)$ — the maximum number of edges in any $(k,l,n)$-good graph
$t(k,l,n)$ — the minimum number of triangles in any $(k,l,n)$-good graph

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Let \( n = |V(F)| \) and let \( n_i \) be the number of vertices of degree \( i \) in \( F \). The well-known theorem of Goodman [2] says that

\[
t(F) + \bar{t}(F) = \binom{n}{3} - \frac{1}{2} \sum_{i=0}^{n-1} i(n - i - 1)n_i. \tag{1}
\]

In his study of the Ramsey numbers \( R(k, l) \), Walker [6] observed that if \( F \) is a \( (k, l, n) \)-good graph then

\[
t(F) + \bar{t}(F) \leq \frac{1}{3} \sum_{i=0}^{n-1} \left( E(k-1, l, i) - e(k, l-1, n-i-1) + \binom{n-i-1}{2} \right)n_i.
\]

Let \( x \in V \) be a fixed vertex in a \( (k, l) \)-good graph \( F \) and consider the two induced subgraphs of \( F \), \( G_x \) and \( H_x \), where \( V(G_x) = N_F(x) \) and \( V(H_x) = V - \{x\} \cup V(G_x) \). Note that \( G_x \) and \( H_x \) are \((k-1, l)\)-good and \((k, l-1)\)-good graphs, respectively. We define the \textit{edge-deficiency} \( \delta(x) \) of vertex \( x \) to be

\[
\delta(x) = E(k-1, l, n(G_x)) - e(G_x) + e(H_x) - e(k, l-1, n(H_x)).
\]

The edge deficiency \( \delta(x) \) measures how close to extremal graphs the subgraphs \( G_x \) and \( H_x \) are. Clearly, \( \delta(x) \geq 0 \). One can also easily see that

\[
\delta(x) = E(k-1, l, n(G_x)) - e(G_x) + E(l-1, k, n(H_x)) - e(H_x).
\tag{2}
\]

It is convenient to define the \textit{edge deficiency} \( \Delta(F) \) of a \( (k, l) \)-good graph \( F \) by

\[
\Delta(F) = \sum_{x \in V(F)} \delta(x). \tag{3}
\]

The first lemma below, similar to (1) in [6], gives a strong condition which permits us to restrict the search space for \((k, l)\)-good graphs.

\textbf{Lemma 1.} If \( n_i \) is the number of vertices of degree \( i \) in a \( (k, l, n) \)-good graph \( F \) then

\[
0 \leq 2\Delta(F) = \sum_{i=0}^{n-1} \left( 2E(k-1, l, i) + 2E(l-1, k, n-i-1) + 3i(n-i-1) - (n-1)(n-2) \right)n_i. \tag{4}
\]

\textbf{Proof.} Observe that for all \( x \in V(F) \) the number of triangles containing \( x \) is equal to \( e(G_x) \) and the number of independent 3-sets containing \( x \) is equal to \( e(H_x) \). Hence by (2),

\[
3(t(F) + \bar{t}(F)) = \sum_{x \in V(F)} \left( e(G_x) + e(H_x) \right)
= \sum_{x \in V(F)} \left( E(k-1, l, n(G_x)) + E(l-1, k, n(H_x)) - \delta(x) \right),
\]

and so by (3) we have

\[
0 \leq \Delta(F) = \sum_{i=0}^{n-1} \left( E(k-1, l, i) + E(l-1, k, n-i-1) \right)n_i - 3(t(F) + \bar{t}(F)).
\]

Now using (1) and \( \sum_{i=0}^{n-1} n_i = n \), we obtain (4). \( \blacksquare \)
2. Generation of all $(4, 4)$-good graphs.

This section describes how we generated the set of all $(4,4)$-good graphs. Let us denote by $R(4, 4, n)$ the set of all $(4,4,n)$-good graphs and let $R'(4, 4, n)$ be the subset of those $F \in R(4, 4, n)$ with maximum degree $D$ at most $(n - 1)/2$. The result of applying the permutation $\alpha$ to the labels of any labelled object $X$ will be denoted by $X^\alpha$, and also $\text{Aut}(F)$ is the automorphism group of the graph $F$, as a group of permutations of $V(F)$.

Suppose that $\theta$ is a function defined on $\bigcup_{n \geq 2} R'(4, 4, n)$ which satisfies these properties:

(i) $\theta(F)$ is an orbit of $\text{Aut}(F)$,
(ii) the vertices in $\theta(F)$ have maximum degree in $F$, and
(iii) for any $F$, and any permutation $\alpha$ of $V(F)$, $\theta(F^\alpha) = \theta(F)^\alpha$.

It is easy to implement a function satisfying the requirements for $\theta$ by using the program nauty [3]. Given $\theta$, and $F \in R'(4, 4, n)$ for some $n \geq 2$, the parent of $F$ is the graph $\text{par}(F)$ formed from $F$ by removing the first vertex in $\theta(F)$ and its incident edges. The properties of $\theta$ imply that isomorphic graphs have isomorphic parents. It is also easily seen that $\text{par}(F) \in R'(4, 4, n-1)$. Since $R'(4, 4, 1) = \{K_1\}$, we find that the relationship “par” defines a rooted directed tree $T$ whose vertices are the isomorphism classes of $\bigcup_{n \geq 1} R'(4, 4, n)$, with the graph $K_1$ at the root. If $\nu$ is a node of $T$, then the children of $\nu$ are those nodes $\nu'$ of $T$ such that for some $F \in \nu'$ we have $\text{par}(F) \in \nu$. The set of children of $\nu$ can be found by the following algorithm, whose correctness follows easily from the definitions:

(a) Let $F$ be any representative of the isomorphism class $\nu$.

Suppose that $F$ has $n$ vertices and maximum degree $D$.

(b) Let $L = L(F)$ be a list of all subsets $X$ of $V(F)$ such that

(b.1) either $|X| > D$, or $|X| = D$ and $X$ does not include any vertex of degree $D$,
(b.2) $X$ intersects every independent set of size $3$ in $F$,
(b.3) $X$ does not include any triangle of $F$, and
(b.4) if $F(X)$ is the graph of order $n + 1$ formed by joining a new vertex $x$ to $X$,
    then $x \in \theta(F(X))$.

(c) Remove isomorphs from amongst the set $\{F(X) \mid X \in L\}$.

The remaining graphs form a set of distinct representatives for the children of $\nu$.

The primary advantage of this method is that isomorph rejection need only be performed within very restricted sets of graphs. For example, even though $|R'(4, 4, 12)| = 909767$, no isomorphism class of $R'(4, 4, 11)$ has more than 58 children.

The full set $\bigcup_{n \geq 1} R'(4, 4, n)$ was found by this method. Altogether, 5623547 sets $X$ passed conditions (b.1)-(b.3), and 2165034 passed condition (b.4) as well. The total size of $R'(4, 4, n)$ for all $n$ is 2065740, which is only slightly less because most $(4,4)$-good graphs have no nontrivial automorphisms. There are altogether 3432184 nonisomorphic $(4,4)$-good
graphs. The total execution time on a 12-mip computer was 9.4 hours, or 6 milliseconds per invocation of the program nauty. In particular, we obtained the information gathered in Table I.

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Table I. Some data on (4,4)-good graphs


Walker [7] established the best upper bound so far of 28 for $R(4,5)$, so we know that any (4,5)-good graph has at most 27 vertices. No (4,5,n)-good graph is known for $n \geq 25$. The goal of this section is to derive some upper bounds for $E(4,5,n)$ for $24 \leq n \leq 27$, provided such graphs exist.

Let $F$ be a (4,5,n)-good graph and let $a_i$ denote the number of edges in $F$ contained in $i$ triangles. Note that $a_i = 0$ for $i \geq 5$ since $F$ is (4,5)-good. For each $x \in V(F)$ consider induced subgraphs $G_x$ and $H_x$ as in Section 1, which in this case are (3,5)-good and (4,4)-good graphs, respectively.

Lemma 2.

$$\sum_{x \in V(F)} t(H_x) = 4a_4 - 2a_2 - 2a_1 + \sum_{x \in V(F)} (n/3 + 3 - \text{deg}_F(x))e(G_x). \quad (5)$$

Proof. For an arbitrary triangle $T = ABC$ in $F$ let $b_i(T)$ denote the number of vertices in $V(F) - T$ adjacent to exactly $i$ vertices in $T$, and let $\text{deg}_F(T) = \text{deg}_F(A) + \text{deg}_F(B) + \text{deg}_F(C)$. Note that $b_i(T) = 0$ for $i \geq 3$, since $F$ has no $K_4$. By counting the 4-sets of vertices formed by any triangle $T$ and any vertex $x$ not adjacent to $T$ in two different ways we have

$$\sum_{x \in V(F)} t(H_x) = \sum_{T \text{ - triangle}} b_0(T), \quad (6)$$
and one also easily notes that for each triangle $T$

$$b_0(T) = n - 3 - b_1(T) - b_2(T)$$  \hspace{1cm} (7)

and

$$b_1(T) + 2b_2(T) + 6 = \deg_F(T).$$ \hspace{1cm} (8)

Now (7) and (8) give

$$b_0(T) = n + 3 + b_2(T) - \deg_F(T).$$ \hspace{1cm} (9)

Using (9) in (6) we obtain

$$\sum_{x \in V(F)} t(H_x) = (n + 3)t(F) + \sum_{T \text{ - triangle}} (b_2(T) - \deg_F(T)).$$ \hspace{1cm} (10)

Counting edges adjacent to points in triangles by two methods gives

$$\sum_{T \text{ - triangle}} \deg_F(T) = \sum_{x \in V(F)} \deg_F(x)e(G_x),$$ \hspace{1cm} (11)

and one can also easily see that

$$3t(F) = \sum_{x \in V(F)} e(G_x) = \sum_{i=1}^{4} i a_i.$$ \hspace{1cm} (12)

By recalling the definitions of $b_2(T)$ and $a_i$ we conclude that

$$\sum_{T \text{ - triangle}} b_2(T) = \sum_{i=2}^{4} i(i-1)a_i = 4a_4 - 2a_2 - 2a_1 + 2 \sum_{i=1}^{4} i a_i.$$ \hspace{1cm} (13)

Now applying (11), (12) and (13) in (10) we obtain

$$\sum_{x \in V(F)} t(H_x) = \frac{1}{3}(n+3) \sum_{x \in V(F)} e(G_x) + 4a_4 - 2a_2 - 2a_1 + 2 \sum_{x \in V(F)} e(G_x) - \sum_{x \in V(F)} \deg_F(x)e(G_x),$$

which can be easily converted to (5).

We know that for each vertex $x$ the number of triangles in $H_x$ is at least $t(4, 4, n(H_x))$, where $n(H_x) = n - 1 - \deg_F(x)$. Define the triangle deficiencies $\gamma(x)$ of a vertex $x$ and $\Gamma(F)$ of a graph $F$ as

$$\gamma(x) = t(H_x) - t(4, 4, n(H_x)), \quad \Gamma(F) = \sum_{x \in V(F)} \gamma(x).$$ \hspace{1cm} (14)

For any vertex $x$ we obviously have $\gamma(x) \geq 0$. 

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**Lemma 3.** If $F$ is any $(4, 5, n)$-good graph on at least 24 vertices and $F$ has $n_i$ vertices of degree $i$ for each $i$, then

$$0 \leq 3\Gamma(F) \leq \sum_{i=6}^{13} ((n + 9 - 3i)E(3, 5, i) + 6i - 3i(4, 4, n-i-1))n_i. \quad (15)$$

**Proof.** Since $R(3, 5) = 14$ and $R(4, 4) = 18$, by (5) we have

$$3 \sum_{x \in V(F)} t(H_x) = 12a_4 - 6a_2 - 6a_1 + \sum_{i=6}^{13} \sum_{\deg_F(x) = i} (n + 9 - 3i)e(G_x).$$

Note that for $n \geq 24$ the coefficient $n + 9 - 3i$ is negative only for $i = 13$ or for $i = 12$ and $n = 24, 25, 26$, hence we can use $E(3, 5, i)$ in place of $e(G_x)$ in the following inequality except in those cases.

$$3 \sum_{x \in V(F)} t(H_x) \leq 12a_4 + \sum_{i=6}^{13} (n + 9 - 3i)E(3, 5, i)n_i$$

$$+ \sum_{\deg_F(x) \geq 12} (E(3, 5, \deg_F(x)) - e(G_x))(3 \deg_F(x) - n - 9). \quad (16)$$

All $(3,5)$-good graphs are known ([5] and independently [4]). In particular, there exists a unique $(3,5,13)$-good graph, which implies that the terms in the last summation for $\deg_F(x) \geq 13$ are equal to zero. It is also known that $E(3, 5, 12) = 24$ is achieved only by 4-regular graphs, and furthermore any $(3,5,12)$-good graph has only vertices of degree 3 and/or 4. Thus if for some vertex $x$ of degree 12 in $F$ the graph $G_x$ is not maximal, i.e. $e(G_x) < 24$, then for each vertex $y$ of degree 3 in $G_x$ the edge $\{x, y\}$ contributes to $a_3$, and each edge appearing in three triangles can be accounted at most twice this way. Thus the second summation in the right hand side of (16) is at most $3a_3$ for $n \geq 24$. Hence by $e(F) \geq a_4 + a_3$ and (16) we find

$$3 \sum_{x \in V(F)} t(H_x) \leq 12e(F) + \sum_{i=6}^{13} (n + 9 - 3i)E(3, 5, i)n_i. \quad (17)$$

Finally, we can easily obtain (15) by using (14), (17) and $12e(F) = \sum_{i=6}^{13} 6im_i$. 

**Theorem 1.** If we interpret $e(k,l,n)$ as $\infty$ and $E(k,l,n)$ as 0 for $n \geq R(k,l)$ then

- $153 \leq e(4,5,27)$ and $E(4,5,27) \leq 160$, $130 \leq e(4,5,26)$ and $E(4,5,26) \leq 154$, $116 \leq e(4,5,25)$ and $E(4,5,25) \leq 148$, $101 \leq e(4,5,24)$ and $E(4,5,24) \leq 139$.

**Proof.** Let $F$ be any $(4, 5, n)$-good graph for some $24 \leq n \leq 27$ with $e$ edges and $n_i$ vertices of degree $i$. Consider the set of constraints formed by $\sum_{i=6}^{13} n_i = n$ and the conditions for $\Delta(F)$ and $\Gamma(F)$ given by Lemmas 1 and 3, respectively. This gives a simple instance
(for a computer) of a non-negative integer linear programming optimization problem with variables $n_i$ and objective function $2e = \sum_{i=0}^{13} m_i$. For $n = 27$ we have to minimize or maximize

$$9n_9 + 10n_{10} + 11n_{11} + 12n_{12} + 13n_{13}$$

subject to

$$27 = n_9 + n_{10} + n_{11} + n_{12} + n_{13},$$
$$0 \leq -21n_9 - 10n_{10} - n_{11} + 2n_{12} - n_{13},$$

(18)

and

$$0 \leq n_9 + 4n_{10} + 6n_{11} - n_{12} - 17n_{13},$$

(19)

where constraint (18) is obtained from (4) and constraint (19) is obtained from (15), using the numerical data from Table I for $t(4,4,j)$, $E(4,4,i)$, and some of the results listed in [5], namely $E(3,5,i) = 2i$ for $10 \leq i \leq 13$ and $E(3,5,9) = 17$. Also in [5] we find the values $E(3,5,8) = 16$, $E(3,5,7) = 12$ and $E(3,5,6) = 9$, which are needed for the calculations in the cases of $24 \leq n \leq 26$. For $n = 27$ the maximal number of edges $e$ is 160 with the unique possible degree sequence $n_{12} = 23$ and $n_{11} = 4$. The other bounds are obtained similarly. We used a simple computer program to perform these calculations, and another to check them.

The numbers of edges in the known $(4,5,24)$-good graphs range from 118 to 132 (personal communication from G. Exoo). The lower bounds for $e(4,5,n)$ are not needed for the proof of $R(5,5) \leq 53$; they are included in Theorem 1 for completeness.

4. An upper bound for $R(5,5)$.

We are now in a position to prove our major result.

**Theorem 2.** $R(5,5) \leq 53$.

**Proof.** Assume that $F$ is a $(5,5)$-good graph on 53 vertices and let $n_i$ be the number of vertices of degree $i$ in $F$. Since $R(4,5) \leq 28$ we have in this case $n_{25} + n_{26} + n_{27} = 53$. The calculation of bounds for $2\Delta(F)$ from Lemma 1, using Theorem 1, gives

$$0 \leq (2.308 + 3.252.27 - 52.51)(n_{25} + n_{27}) + (2.308 + 3.26.26 - 52.51)n_{26}$$

$$= -11(n_{25} + n_{27}) - 8n_{26},$$

which is a contradiction.  

The same method does not disprove the existence of a $(5,5,52)$-good graph, but such a result would be possible if we could sufficiently improve the bounds of Theorem 1.
References.