Process Performance Measures When Process Distribution is Non-Normal

Donald Holmes
Stochos, Inc.

A. Erhan Mergen
Rochester Institute of Technology

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PROCESS PERFORMANCE MEASURES WHEN PROCESS DISTRIBUTION IS NON-NORMAL

Donald S. Holmes
Sirochrs Inc. 14 N. College Street Schenectady, N.Y. 12305 U.S.A.

A. Erhan Mercen
Rochester Institute of Technology College of Business,
Decision Sciences 107 Lomb Memorial Drive Rochester,
N.Y. 14623-5608 U.S.A.

Abstract

This paper addresses the problems associated with process performance measures, such as Cp, Cpk, when the frequency distribution of the variable being evaluated is not Normal. These measures, also known as capability indices, are commonly used in industry, yet they may not reflect the true process performance if the process distribution is not Normal. Gunter (1), in his four-part series articles, emphasized this point and other problems associated with measures like Cpk. In this paper, we discuss various scenarios with respect to process stability and frequency distribution, and provide an example using non-Normal process curve.

1. Introduction

There are some processes for which the data is not expected to be Normally distributed, for example, plating, drilling, etc. operations. When this situation occurs, some non-Normal frequency curve is used to fit the data. This activity is usually undertaken for one of two reasons:

1. Making $X$ control charts because of low rates of data accumulation or
2. Calculation of the process performance measures (Cp, Cpk, etc.)

The use of $X$ charts should not be an issue since there are several charts which allow the plotting of each data point that have much better average run lengths and are not nearly as sensitive to non-Normality of the data. These charts include the
Cumulative Sum (Cusum), the Exponentially Weighted Moving Average (EWMA) and the Dynamic Histogram charts (see, for example, Montgomery (2), and Holmes and Mergen (3) for these charts.)

This paper addresses non-Normality problems with respect to the second issue: process performance measures. For definitions of these measures, please consult Gitlow, Oppenheim and Oppenheim (4), for example.

DISCUSSION

Capability vs. Performance:

Let's concentrate on the performance measure, which deals with the width of a process relative to the allowed (tolerance) width. This measure is defined as:

$$\frac{USL - LSL}{Process\,\,\,Width}$$  \hspace{1cm} (1)

where USL and LSL stand for the upper and lower specification limit, respectively.

The process width used in the equation above can be one of two possibilities:

1. The width of the process as it exists - the performance width - or
2. The width of the process as it could be if the process were in control - the capability width.

The relative width measure is referred to as the Pp in the first case and the Cp in the second case. Note that the term Cp is reserved for processes, which are in control, i.e., statistical control (the process is said to be in statistical control when it is influenced only by common causes of variation).

If the process is not in control, the capability width may be significantly smaller in magnitude than the performance width. This causes differences, of course, in the reported process performance measures.

Normal Process:

If the frequency curve is approximately Normal, the width is usually taken to be ±3 standard deviations around the average. The performance standard deviation is one calculated for the entire data set without regard as to whether or not the data gave evidence of lack of statistical control. The capability standard deviation, on the other hand, is one, which is independent of chances in average values. It can be obtained from control chart calculations such as $$\frac{\overline{R}}{d_2}$$ or $$\frac{\overline{S}}{c_4}$$, where $$\overline{R}$$ and $$\overline{S}$$ are the average of the subgroup ranges and the average of the subgroup standard deviation, respectively. The values of $$d_2$$ and $$c_4$$ can be read from tables for control chart constants for a given subgroup size. Another method for calculating the capability standard deviation is the mean square successive difference (MSSD) (see Holmes and Mergen (5) and Hild(6), for example). Using MSSD, the capability standard deviation would be estimated as follows:

$$X = \bar{X} \pm T \times MSSD$$

where

$$X$$ are the inc

Roes. Does SD using MSSD as

which will converge

Non-Normal

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width may be det

99.7% of the data,

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process width is to

Problem#1:

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standardized curve

be very difficult.

Problem#2:

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(see Appendix I fo

$$(\bar{X})$$) and target ($$T$$)

$$\bar{X} = 5$$, and $$T$$

standard deviation

As you see in th
with the width of a is defined as:

\[ w = \frac{X_{\text{USL}} - X_{\text{LSL}}}{6} \tag{1} \]

n limit, respectively, two possibilities: 4th or were in control - the 1st case and the Cp processes, which are in statistical control when it significantly smaller es, of course, in the is usually taken to be standard deviation is whether or not the standard deviation, on age values. It can be where \( \overline{R} \) and \( s \) are the standard deviation, limits for control chart 2) calculating the capability SD (see Holmes and y standard deviation using MSSD as

\[ \sigma_{MSSD} = \sqrt{\frac{MSSD}{2}} \tag{2} \]

where

\[ MSSD = \frac{1}{n-1} \sum_{i=1}^{n-1} (X_{i+1} - X_i)^2 \tag{3} \]

and \( X_i \) are the individual observations, \( n \) is the number of observations.

Roes, Does Schuring (7), gave the unbiased estimate of the standard deviation using MSSD as

\[ \sigma_{MSSD} = \sqrt{\frac{\sigma_{MSSD}}{1 - \frac{1}{2n}}} \tag{4} \]

which will converge rapidly to the expression given in equation (2) as \( n \) gets large.

**Non-Normal and Stable (i.e., In-Control) Process:**

If the process follow a non-Normal distribution, then the width of the curve is determined by fitting a non-Normal curve to the data. Once the curve is fitted, the width may be determined by calculating the values of the variable, which include 99.7% of the data. The minimum \( X \) is usually taken to be the one which has 0.15% below it; the maximum \( X \) is taken to be the one which has 0.15% above it. The process width is taken to be \( X_{\text{max}} - X_{\text{min}} \).

**Problem #1:** The curve type used in fitting the data impacts on the estimate of the process width.

The curve type selected is subjective and one can only reject unfit curves, not guarantee the selection of the "right" curve, so consistent results must be based on a standardized curve fitting approach. We think to get a consensus on this matter will be very difficult.

**Problem #2:** For every curve expect the Normal the average and standard deviation are not independent.

Thus striving for centering the process on nominal value will also change the width of the process, which is used in the denominator of \( Cp \) and \( Cpk \) type measures. For example, assume we have fitted Rayleigh distribution to a non-Normal process data (see Appendix I for a description of the Rayleigh distribution). The process average (\( \overline{X} \)) and target (\( T \)) (i.e., nominal) are given as

\[ \overline{X} = 5 \text{, and } T = 10 \text{, respectively.} \]

From equation (4) and (8) in Appendix I, the standard deviation can be estimated as

\[ s = (1.1284)(0.4633)\overline{x} \tag{5} \]

As you see in the above equation, if the average is moved to be closer to the target, the standard deviation will get larger and have a direct impact on the \( Cp \). Thus improving one quality index automatically worsens another.
Non-Normal and Unstable (i.e., not In-Control) Process:

Fitting a curve to an unstable process is inherently dangerous. Should the issue of stability (control) be ignored when fitting a non-Normal curve to the data, then incorrect conclusions relative to the quality level may occur. The example below demonstrates this point in the context of the Rayleigh distribution.

**Problem #3:** If the process is not in control (i.e., not stable), then estimation of curve parameters that depend on the average will not be reasonable since the average does not reflect the changes in process.

In other words, using the average ($\bar{X}$) to estimate the value for curve parameter does not allow one to distinguish between the capability and the performance of the process. However, there are ways to calculate the standard deviation that will enable the distinction between capability and performance to be made. The standard deviation estimated through MSSID provides a method to distinguish between capability and performance (see, for example, Holmes and Mergen (5)).

**EXAMPLE**

Consider the histogram and the descriptive statistics summary obtained recently from a process (see Figure 1 and Table 1) which is known to generate an approximate Rayleigh distribution. The complete data set is listed in Appendix II.

(Approximate location for Figure 1 and Table 1)

It is clear from the histogram that the data is not Normally distributed (a chi-square test also rejected Normality). The specification limits for the variable in question (i.e., quality characteristic) are:

\[ USL = 5.00 \text{ Nominal} = 3.00 \text{ LSL} = 1.00 \]

During the time period in which the data is collected, the regular standard deviation ($\sigma_R$) is 1.102 and the mean square successive difference (MSSID) standard deviation ($\sigma_{MSSID}$) is 0.595. The two variance are significantly different as per the $Z$ test described in Dixon and Massey (8). This means, in turn, that the process is not in control. This is also evident from the X-bar chart shown in Figure 2 (for subgroup size five).

(Approximate location for Figure 2)

Thus, in turn, there will be a significant difference between the $Cp$ and the $Pp$ for this process. The process performance calculations based on process width estimated with the Rayleigh distribution using the mean, the regular, and also the MSSID standard deviations are shown in the table below (Table 2).

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The $Cp$ of 1.2 were to be brought estimate of $q$ is a difference between mentioned above hi unstable (i.e., r)

This paper de Normal data is sub to a specific fam careful analysis bel result may be to percent defective c center is from the

**KEY WORD:**

[1] Gunter, B.F.
The table obtained recently for variable in Table 1 shows an approximate distribution (a chi-square for the variable in regular standard deviation (MSSD) standard deviation as per the Z at the process is not sure 2 (for subgroup process:

- A. Should the issue to the data, then
  - The example below

- The standard deviation 

for curve parameter estimates performance of the equation that will enable

- The standard deviation between capability

- The CP of 1.24 is an indication of what the process is capable of doing if it were to be brought into control, whereas the Pp (0.428 or 0.667 depending on which estimate of q is used) indicates how well the process is currently performing. The difference between these estimates is significant and points out the potential problem mentioned above in estimating performance measures from a curve fitted to data from an unstable (i.e., not in control) process.

CONCLUSION

This paper demonstrates that the use of process performance indices for non-Normal data is subject to many problems. Problem 1 is solvable by everyone agreeing to a specific family of curves. Problem 2 cannot be resolved. Problem 3 calls for careful analysis before fitting the curve. But, problem 2 indicates the only meaningful result may be to give up Cp, Cpk, etc. and go back to the more universally accepted percent defective or parts per million (ppm) plus a measure of how far the process center is from the nominal.

KEY WORDS: Non-Normal process distribution, process performance measures.

REFERENCES

APPENDIX I

Rayleigh Distribution:

The Rayleigh distribution is a special case of Weibull where the shape parameter, $\beta$, is equal to 2 (Nelson (9)). The probability density function for the Rayleigh curve is:

$$f(x) = \frac{x}{q^2} e^{-\frac{x^2}{2q^2}}$$

The curve is usually fitted using the average of the data to find the value for $q$ (the "scale" parameter) as shown below.

$$E(X) = q \frac{3}{2} = q \frac{\sqrt{\pi}}{2}$$

where $\Gamma$ represents gamma function and $\pi \approx 3.141593$.

Hence the value of the scale parameter is usually estimated using:

$$q = \frac{2E}{\sqrt{\pi}} = 1.12842$$

This value for $q$ is then used to determine the value of $x$ which has 0.15% of the curve below it ($x_{\text{min}}$) and the value of $x$ which has 0.15% of the data above it ($x_{\text{max}}$). The values for $x_{\text{max}}$ and $x_{\text{min}}$ may be calculated as:

$$x_{\text{max}} = q \sqrt{-\ln 0.9995} = 2.54996q$$

$$x_{\text{min}} = \sqrt{-\ln 0.0005} = 0.03874q$$

The variance for the Rayleigh distribution, on the other hand, is:

$$V(X) = q^2 [\Gamma(2) - \Gamma(2)^2]$$

and the standard deviation, $s$, would be

$$s = 0.4633q$$

Hence the value of the scale parameter may also be obtained using:

$$q = 2.1584s$$

Once expressed in this fashion, one can distinguish between the capability and performance distribution using the appropriate standard deviation mentioned above. The capability estimate of $s$ may be obtained using $\bar{R}/d_2$ or MSSD as mentioned earlier.
the shape parameter, or the Rayleigh curve

\[ \text{(6)} \]
d the value for \( q \) (the

\[ \text{(7)} \]
\]

\[ \begin{align*}
1 & : 2.17 & 51 & 2.62 & 101 & 3.60 & 151 & 4.44 \\
2 & : 3.53 & 52 & 2.35 & 102 & 2.57 & 152 & 4.52 \\
3 & : 2.62 & 53 & 2.07 & 103 & 3.07 & 153 & 4.35 \\
4 & : 2.55 & 54 & 3.08 & 104 & 3.49 & 154 & 4.33 \\
5 & : 2.18 & 55 & 3.49 & 105 & 2.43 & 155 & 4.87 \\
6 & : 3.13 & 56 & 2.88 & 106 & 4.47 & 156 & 4.92 \\
7 & : 3.10 & 57 & 3.13 & 107 & 2.15 & 157 & 4.09 \\
8 & : 2.60 & 58 & 2.97 & 108 & 3.06 & 158 & 5.55 \\
9 & : 2.09 & 59 & 2.48 & 109 & 2.68 & 159 & 4.11 \\
10 & : 2.08 & 60 & 2.47 & 110 & 3.02 & 160 & 5.63 \\
11 & : 2.62 & 61 & 3.84 & 111 & 3.33 & 161 & 4.97 \\
12 & : 2.45 & 62 & 2.42 & 112 & 2.51 & 162 & 5.21 \\
13 & : 3.49 & 63 & 2.76 & 113 & 3.06 & 163 & 5.35 \\
14 & : 3.35 & 64 & 2.40 & 114 & 2.10 & 164 & 4.97 \\
15 & : 3.70 & 65 & 3.21 & 115 & 2.15 & 165 & 4.06 \\
16 & : 2.92 & 66 & 2.16 & 116 & 4.42 & 166 & 4.69 \\
17 & : 2.59 & 67 & 2.77 & 117 & 4.03 & 167 & 5.13 \\
18 & : 2.48 & 68 & 2.23 & 118 & 3.47 & 168 & 5.38 \\
19 & : 2.69 & 69 & 3.95 & 119 & 2.81 & 169 & 5.50 \\
20 & : 3.01 & 70 & 2.07 & 120 & 2.67 & 170 & 5.44 \\
21 & : 2.74 & 71 & 2.30 & 121 & 2.09 & 171 & 4.46 \\
22 & : 2.01 & 72 & 2.89 & 122 & 2.20 & 172 & 5.66 \\
23 & : 2.81 & 73 & 2.58 & 123 & 2.37 & 173 & 5.83 \\
24 & : 2.22 & 74 & 2.34 & 124 & 4.80 & 174 & 5.06 \\
25 & : 2.09 & 75 & 2.98 & 125 & 2.64 & 175 & 5.46 \\
26 & : 3.19 & 76 & 2.22 & 126 & 2.53 & 176 & 5.31 \\
27 & : 2.29 & 77 & 2.60 & 127 & 2.79 & 177 & 5.28 \\
28 & : 4.06 & 78 & 2.50 & 128 & 2.65 & 178 & 5.07 \\
29 & : 2.88 & 79 & 2.58 & 129 & 2.20 & 179 & 4.05 \\
30 & : 3.40 & 80 & 2.14 & 130 & 3.80 & 180 & 5.21 \\
31 & : 2.94 & 81 & 2.65 & 131 & 2.39 & 181 & 6.80 \\
32 & : 2.30 & 82 & 2.76 & 132 & 3.40 & 182 & 5.36 \\
33 & : 2.98 & 83 & 2.33 & 133 & 2.11 & 183 & 6.21 \\
34 & : 2.07 & 84 & 2.57 & 134 & 2.43 & 184 & 4.05 \\
35 & : 2.32 & 85 & 2.70 & 135 & 2.67 & 185 & 5.30 \\
36 & : 2.84 & 86 & 2.43 & 136 & 2.22 & 186 & 4.39 \\
37 & : 2.22 & 87 & 2.53 & 137 & 2.15 & 187 & 4.77 \\
38 & : 2.52 & 88 & 2.03 & 138 & 3.06 & 188 & 5.96 \\
39 & : 2.66 & 89 & 3.53 & 139 & 2.08 & 189 & 6.05 \\
40 & : 2.88 & 90 & 2.30 & 140 & 2.88 & 190 & 4.41 \\
41 & : 3.39 & 91 & 3.43 & 141 & 2.97 & 191 & 5.07 \\
42 & : 2.34 & 92 & 2.07 & 142 & 2.08 & 192 & 4.07 \\
43 & : 2.34 & 93 & 2.23 & 143 & 2.24 & 193 & 4.19 \\
44 & : 2.34 & 94 & 2.58 & 144 & 2.59 & 194 & 5.16 \\
45 & : 2.51 & 95 & 2.25 & 145 & 2.86 & 195 & 4.78 \\
46 & : 2.57 & 96 & 3.16 & 146 & 2.70 & 196 & 4.06 \\
47 & : 3.19 & 97 & 2.79 & 147 & 2.30 & 197 & 5.17 \\
48 & : 4.20 & 98 & 2.30 & 148 & 2.62 & 198 & 5.14 \\
49 & : 2.62 & 99 & 2.33 & 149 & 2.39 & 199 & 4.04 \\
50 & : 2.88 & 100 & 3.18 & 150 & 2.89 & 200 & 4.09 \\
\end{align*} \]

**APPENDIX II**
Table and Figure Captions
(Holmes and Mergen)

Figure 1. Histogram of the data.

Table 1. Summary of descriptive statistics of the data.

Figure 2. $\bar{x}$ and Range charts of the data.

Table 2. Process performance calculations using Rayleigh distribution.

<table>
<thead>
<tr>
<th>Performance</th>
<th>Basis for $q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(using mean)</td>
<td></td>
</tr>
<tr>
<td>(using regular std. dev.)</td>
<td></td>
</tr>
<tr>
<td>(using MSSD std. dev.)</td>
<td></td>
</tr>
</tbody>
</table>

Table 1. Summary...
<table>
<thead>
<tr>
<th>Mean</th>
<th>3.298</th>
<th>Median</th>
<th>2.880</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reg. std. dev. ($\sigma_r$)</td>
<td>1.102</td>
<td>SE Mean</td>
<td>0.078</td>
</tr>
<tr>
<td>Range</td>
<td>4.790</td>
<td># Observ</td>
<td>200</td>
</tr>
<tr>
<td>Minimum</td>
<td>2.010</td>
<td>Maximum</td>
<td>6.800</td>
</tr>
<tr>
<td>Skewness</td>
<td>0.952</td>
<td>Kurtosis</td>
<td>2.855</td>
</tr>
<tr>
<td>Cap. SD</td>
<td>0.595</td>
<td>Cap. Ratio</td>
<td>0.540</td>
</tr>
</tbody>
</table>

Mean Square Successive Difference (MSSD) Tests:

| Normal Z   | 10.069 | MSSD(SD) | 0.395 |

Table 1. Summary of descriptive statistics.

<table>
<thead>
<tr>
<th>Basis for q</th>
<th>$q$</th>
<th>Std. Dev.</th>
<th>$X_{max}$</th>
<th>$X_{min}$</th>
<th>Width ($X_{max} - X_{min}$)</th>
<th>Relative Width (Pp or Cp)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Performance</td>
<td>3.72</td>
<td>**</td>
<td>9.486</td>
<td>0.144</td>
<td>9.342</td>
<td>0.428 (Pp)</td>
</tr>
<tr>
<td>(using mean)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Performance</td>
<td>2.38</td>
<td>1.102</td>
<td>6.069</td>
<td>0.092</td>
<td>5.977</td>
<td>0.667 (Pp)</td>
</tr>
<tr>
<td>(using regular std. dev.)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Capability</td>
<td>1.28</td>
<td>0.595</td>
<td>3.264</td>
<td>0.049</td>
<td>3.215</td>
<td>1.244 (Cp)</td>
</tr>
<tr>
<td>(using MSSD std. dev.)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

Table 2. Process performance calculations using Rayleigh distribution.
A large class of hydrodynamics, ge reduced to the find
In spite of actual difficulties. There is
In present workation of domain an
This method of effective algorithm

Let $K$ be any $G \in \mathcal{R}$, is such a
$S_D = \delta D$.
We consider th