Theories and experiments for determination of nonsteady loads on turbomachine blades

James F. Crofoot
THEORIES AND EXPERIMENTS FOR DETERMINATION OF NONSTEADY LOADS ON TURBOMACHINE BLADES

by

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A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of MASTER OF SCIENCE in Mechanical Engineering

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This thesis collates the state of the art for flow induced excitation in a turbomachine stage under subsonic incompressible flow conditions. Theoretical developments are considered first, beginning with the vortex theories of Kemp and Sears, Henderson, Horlock and Holmes. Actuator disk analyses of Whitehead, Henderson and Horlock, and Horlock, Greitzer and Henderson are then considered followed by field theories of Osborne, Warner and Steele. Computer programs based on these theories are described and detailed input/output instructions are given. Results of these theories are compared and the state of the art for theoretical prediction of nonsteady blade loading is discussed. It is concluded that the selection of an appropriate analysis is dependent on the stage geometry and on the type of excitation, i.e. low per revolution or nozzle passing excitation.

The theoretical development of the hydraulic analogy is discussed. Experimental studies of turbomachine stage flow by Harleman and Ippen, Heen and Mann, Johnson, Bryant, Owczarek, and Rhomberg are described. The development of the RIT rotating water table is also described. Water table tests, designed to determine the agreement between theoretically and experimentally predicted nonsteady loading of turbomachine blades are described and sample results are given. It is concluded that the nonsteady blade loading results obtained from water table tests compare favorably with those obtained from state of the art theoretical analyses for nozzle exit Mach numbers above 0.5. For Mach numbers below this value viscous effects become significant, and further studies to evaluate the accuracy of the hydraulic analogy under such conditions are recommended.
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1. INTRODUCTION

Turbomachine blades operate in a nonuniform flow field created by the interaction of other machine components with the steady flow field. These nonuniformities can cause significant non-steady excitation of the blades. For example, early steam turbines were massive low power machines so the unsteady forces on the blading were low. As the demand for higher power output from units of similar overall size increased, turbines with higher inlet pressures and temperatures and lighter blading were developed. The increased unsteady forces associated with high inlet pressures coupled with the increased flexibility of the blading caused many fatigue blade failures.


"Vibration cracked buckets and wheels and wrecked turbines, sometimes within a few hours and sometimes after years of operation."

If the frequency of the excitation is close to a natural frequency of the blades, dynamic amplification of the blade response can occur causing high stress levels and possibly fatigue failure of the blades. It is common practice to "tune" the blades so that the resonant frequencies are not integer multiples of the machine speed or of the excitation frequency (typically nozzle passing frequency in turbines).

Tuning the blades in this manner has reduced the number of fatigue failures of turbomachinery blading. In some cases, however, the non-steady excitation is so large that fatigue failures can occur without dynamic amplification of the blade response. A partial solution to this problem was the use of high strength stainless steel for the internal components. However, the problem of blade failures due to high unsteady loading still exists. Such failures reportedly have occurred in gas turbines, compressors and marine steam turbines, i.e. in small high power turbomachines.

Due to the hostile environment within a typical turbomachine of this type (high temperatures, high pressures, high rotational speeds, etc) it is difficult with existing strain gage technology, to directly measure in-situ the magnitude of the non-steady forces on the blades, thus creating a need for effective procedures for the prediction of these forces.
Two effective experimental procedures which have been developed to meet the need are the air test rig model which reduces many of the environmental problems involved in direct measurement of non-steady blade loading by employing air as the working fluid instead of the prototype fluid and the hydraulic analogy water table which substitutes low speed water flow for the high temperature, high speed prototype fluid flow.

A significant time lag existed between the development of large modern turbomachines and the development of theories for the prediction of the unsteady loading on the blades of such machines, due in part to the complexity of the aeroelastic interaction problem. Modern analytical methods for calculating unsteady forces on turbomachine blades began in the early 1950's. The problems associated with the calculation of the magnitude of the non-steady loading on the blades are twofold: namely (i) the quantification of the complex nonuniformities in the flow field and (ii) the determination of the aerodynamic excitation of the blades due to these nonuniformities.

State of the art vortex theories typically address the latter problem above, and the analytical model is generally limited to flat plate blades. A recent analysis has extended the range of the analytical models to include thin cambered airfoils: See Mukhopadhyay [18]. Reports on experimental verification of vortex theory analyses in the open literature are rare and the reported correlation varies from good to poor: See Holmes [14].

Actuator disk theories are limited to the case of low per revolution excitation of cascades. Such analyses generally are restricted to flat plate blading although a recent "semi-actuator disk" analysis has relaxed this restriction to include thin cambered blades: See Horlock, Greitzer & Henderson [35]. Experimental verification of actuator disk analyses has not been reported in the literature.

Field theories typically are employed to define the flow field nonuniformities in a turbomachine stage. Some analyses of this type have been applied to stage excitation studies: See Osborne [4].
The purpose of this thesis is to review in detail the significant theories for the prediction of non-steady loading on turbomachine blades and to compare and contrast these theories with each other, and where possible, with experimental data.

In Section 2 of this thesis the significant contributions to nonsteady excitation of turbomachine blades are discussed in general terms. Major analyses in each of the three categories (vortex theories, actuator disk theories and field theories) are discussed in more detail in Sections 3, 4, and 5 and the state of the art is summarized in Section 6. Considerable confusion exists in the open literature regarding notation and definitions. Rather than add to the confusion the notation used in Sections 3, 4, and 5 is identical wherever possible to the authors. As a result, a separate notation listing is given at the end of each section.

Computer programs, based on these state of the art analyses, which were developed and written by the author are described in Sections 7 and 8. Included in these sections are detailed input and output instructions. The limitations and future developmental requirements of the computer program library are discussed in Section 9.

The hydraulic analogy between compressible gas flow and water flow and its application to the study of turbomachine stage flows on the RIT water table apparatus is discussed in detail in Chapter 10. A full description of the modeling techniques necessary for such studies is also given.

In Chapter 11 a two phase water table test program designed to provide quantitative data for comparison to theoretical blade excitation results is described. Qualitative trends in the water table data are discussed to fully demonstrate the data reduction procedure used in water table testing.

The results of this test program are compared to results from the applicable theories and from an air test rig in Section 12. Included is a discussion of the quantitative accuracy of the water table data for the subsonic incompressible flow case.


2. REVIEW OF LITERATURE SOURCES

2.1 General

Several recent reviews of excitation literature have been published. Sisto [5] has summarized the present status of non-steady flow analyses with emphasis on vortex theories. This is a brief introduction to the existing excitation literature applied to turbine L.P. stage flows. Rao [6] has discussed vortex analyses with reference to turbine blade vibration in a general survey paper. Samoylovich [7] has published (in Russian) a monograph concerned with turbine blade excitation and vibration, which reviews excitation sources and describes blade measurement and test procedures for dynamic pressure distributions and for blade response. Osborne [4] has given a comparative review of developments in non-steady subsonic compressible flow theories. Existing procedures for calculating unsteady interactions between blade rows are reviewed and compared with results from Osborne's matched asymptotic expansion approach for subsonic flows ($0 < M < 0.9$). Gostelow [8] has reviewed steady-state compressible theories for potential flow through cascades with reference to the transonic flow problem, and has indicated some numerical techniques for the solution of problems of this type.

In the remainder of this section the major milestones in the development of vortex theories, actuator disk theories and field theories listed in table 1 will be discussed in approximately chronological order.

2.2 Development of Vortex Theories

Vortex theories involve the mathematical modeling of airfoils as plane vortex sheets. Early development of vortex theories for isolated airfoils was done by Theodorsen [9], von Karman and Sears [10], Sears [11] and Kemp [12] over a seventeen-year period (1935-1952). The extension of vortex theories to turbomachine stages was first reported by Kemp and Sears [2] in 1953. The...
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<td>Henderson and Horlock [33]</td>
<td>1972</td>
<td>Actuator disk analysis for nonsteady lift ratio on thin, cambered, high solidity cascade blades in subsonic incompressible flow.</td>
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<td>Warner [38]</td>
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<td>Henderson [15]</td>
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<td>Mukhopadhyay [18]</td>
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assumptions made in these analyses are common to all vortex analyses. The flow through the stage is assumed to be inviscid, subsonic and incompressible. The airfoils are assumed to be isolated, i.e. the non-steady circulation of neighboring blades is neglected in calculating the non-steady effects on a particular airfoil. For rotor blades however, the effects of the vortex wakes shed by the stators are included. The rotor and stator blades are assumed to be flat plates though they may be thin, slightly cambered, lightly loaded airfoils. The elementary stage on which the analysis is based consists of adjacent cascades of staggered flat plate airfoils, as shown in Figure 1. It is evident that this stage more closely models a low-pressure turbine stage or a compressor stage than a high-pressure turbine stage. Closed form expressions for unsteady forces on rotor and stator blades due to aerodynamic interference (potential flow effects) between the blade rows for sonic incompressible flow conditions were given. The sample results given by Kemp and Sears, shown in Figure 2 are for the first two harmonics of the absolute value of the ratio of the non-steady lift to the steady lift plotted against the axial spacing ratio and the pitch ratio. The dashed curves correspond to an elliptical steady load distribution on both the rotor and stator blades, while the solid curves are for an elliptical distribution on the stator blades, and a flat-plate distribution on the rotor blade. The authors concluded that "the non-steady part of the lift may be as large as 18 percent of the steady lift and therefore may be of practical importance." This low value is a feature of the simple model chosen. It is by no means an upper limit for practical blade conditions. In a subsequent paper [3] 1955, Kemp and Sears gave closed form solutions for the unsteady forces on the rotor blades due to the viscous wakes of the stators for the same elementary cascade. The stator wakes are modeled as inviscid, symmetrical shear perturbations of the undisturbed free stream. The strength of the is assumed to be the same as that behind an isolated airfoil: See Figure 2. Comparing the results of this analysis with those obtained previously suggests that the viscous-wake effects on a rotor blade may be of the same magnitude as the circulation-induced non-steady lift. The authors suggest addition of these two results in the complex plane to eliminate phasing e
Figure 1. Kemp-Sears Elementary Turbomachine Stage.
Figure 2. Sample Results of Komp-Sears Analysis. From [2].

(- - - -) Elliptical steady load distribution on rotor and stator blades.

(-----) Elliptical steady load distribution on stator blades and flat plate distribution on rotor blades.

m = 1: first harmonic
m = 2: second harmonic
stagger angle: 45°
Figure 3. Viscous Wake Model Used in Kemp-Sears Analysis [3].
Both of the above analyses consider only those velocity perturbations which are perpendicular to the chord, i.e. transverse gusts in the calculation of unsteady forces.

Horlock [13] in 1968 gave a closed form solution for the unsteady force on isolated airfoils due to velocity perturbations parallel to the blade chord for subsonic incompressible flow conditions. The Kemp-Sears assumptions for thin airfoil theory were again imposed. Since the flow inlet velocity is not necessarily parallel to the chord, the author has included small angles of attack in this analysis. It was assumed that the gusts do not decay over the length of the airfoil, i.e. convecting streamwise gusts. Horlock suggested the incorporation of this solution into the Kemp-Sears analysis.

Holmes [14], in a 1973 paper, described a closed form solution for the non-convecting streamwise gust in subsonic incompressible flow. An experimental investigation of the accuracy of an analysis which combined Kemp's theory [12], Horlock's theory [13] and Holmes' theory was made. A wind tunnel apparatus was constructed which produced either transverse or streamwise gusts relative to an isolated airfoil. The form of the gusts is shown in figure 4. Since the time-variation of velocity was known during the testing, comparisons between measured and theoretical values of the lift response were obtained, together with pressure distributions around the airfoil. Figure 5 shows the predicted and measured amplitudes and the phase of the lift fluctuation due to a transverse gust. The theoretical curve represents the results of the Kemp-Sears theory for an airfoil subjected to transverse gusts. The curves generally agree to within 10 percent. Holmes found that the Kemp-Sears theory "Accurately predicts the variation in the lift response from one flow condition to another, but not the absolute magnitude of the response." Figure 6 shows the predicted and measured amplitude and phase of pressure fluctuations versus percent chord length for frequency parameter, \( \omega = 0.016 \). Smoke tests indicated the existence of a separation point with a strong associated vortex. Holmes also states that "any periodic variation in the strength of this vortex must create a wake and so will invalidate the simple mathematical model used... by the author."
1.3 TRANSVERSE GUST FLOW

Velocity in x-direction: constant
Velocity in y-direction: sinusoidal in time and x-direction

1.4 STREAMWISE GUST FLOW

Velocity in x-direction: sinusoidal in time and x-direction
Velocity in y-direction: constant

Figure 4. Gust Notation.
Figure 5. Amplitude and Phase of Nonsteady Lift Due to a Transverse Gust. From Holmes [14].

Figure 6. Amplitude and Phase of Nonsteady Pressure Fluctuation Due to a Transverse Gust. From Holmes [14].
Figure 7 shows the predicted and measured amplitude and phase of the lift fluctuation due to a streamwise gust. The magnitude of the response agrees only at low frequencies while the phase is never in agreement. The author concludes that this airfoil theory for streamwise gusts was invalidated where viscous effects led to flow separation.

Henderson [15] conducted an experiment to determine the unsteady response of thin airfoils in a rotating air cascade. Using an expression given by Henderson and Daneshyar [16] the author calculated the unsteady circulation of a rotating blade row in terms of the circumferential distribution of the time mean total pressure. Kemp and Sears [2] have shown that the fluctuating lift is a direct function of the fluctuating circulation. The unsteady response of a rotating blade row to spatial variations in inlet velocity can therefore be obtained from the time mean total pressure. Figure 8 shows the cascade configuration used in the study. The stagger angle was held constant at 45 degrees and tests were run for pitch/chord ratios of 0.676 and 1.353. Results obtained by Henderson for unsteady circulation are shown plotted against reduced frequency in figures 9(a) and 9(b). Also shown are the components of the unsteady circulation for infinite pitch to chord ratio, corresponding to the Kemp-Sears results for an isolated airfoil. Both curves exhibit the same trends with reduced frequency and with pitch/chord ratio. Discrepancies are attributed to airfoil thickness (assumed to be zero in the theory) and to viscous (boundary layer) effects. Henderson concludes that, "the representation of the unsteady response of a turbomachine blade row as an isolated airfoil is not valid for values of reduced frequency less than 1.2."

Horlock [13] described an analysis by Holmes in which a closed form solution for the unsteady lift on an airfoil of parabolic camber subjected to a streamwise convecting gust is derived. This analysis was performed in 1968, making it one of the first deviations from flat plate airfoil theory. The analysis was independently verified in 1973 by Naumann and Yeh [17].
Figure 7. Amplitude and Phase of Nonsteady Lift Due to a Streamwise Gust. From Holmes [14].

Figure 8. Cascade Configuration Used by Henderson [15].
a) Real component of unsteady circulation.

b) Imaginary component of unsteady circulation.

Figure 9. Real and Imaginary Components of Unsteady Circulation Obtained by Henderson [15]. From Reference [15].
Mukhopadhyay [18] has derived a closed form solution for the unsteady forces on an airfoil of general camber which is subjected to simultaneous transverse and streamwise non-convecting gusts in a subsonic compressible flow field. No experimental verification of vortex theories for cambered airfoils is reported in the open literature.

2.3 Development of Actuator Disk Theories

Actuator disk analyses were first developed for studies of propeller performance and are therefore limited to cascade studies. The wavelength of the excitation is assumed to be very large compared to the blade chord. If the flow is viewed from a very great distance, the excitation wavelength is finite while the blade row appears to be a thin 'actuator' disk.

The early development of actuator disk analyses was directed toward the study of flutter in axial flow compressors: See, for example, Lilley [19], Lane and Wang [20], Lane and Friedman [21] and Yeh [22]. Actuator disk theory has been applied by Hawthorne and Horlock [23] to study flow in axial compressors, and by Hawthorne and Ringrose [24] to the study of free-vortex turbomachines. Horlock [25] has discussed the actuator disk method in his book. These references pertain only to steady-state turbo-machine flows.

One of the earliest applications of the actuator disk method to the non-steady excitation of blade cascades was given by Whitehead [26]. As stated above, in this type of analysis the blade row is replaced by a narrow actuator disk as shown in figure 10. The blades in the immediate vicinity of the origin are shown in figure 11. The following assumptions are applied to the model:

1) The time required for the fluid to pass through the actuator disk is small compared to the period of vibration of the blades. In order for this requirement to be fulfilled, the reduced frequency parameter based on the blade chord (\( \omega \)) must
Figure 10. Actuator Disk and Velocity Notation.
Figure 11. Actuator Disk Cascade in Immediate Vicinity of Origin. From Whitehead [26].
be small, and is, in fact, limited to a value of zero for the true actuator disk analysis.

ii) Adjacent blades vibrate with a small, constant phase angle between them. This angle is the interblade phase angle ($\delta$). It will be shown in section 4.6 that this assumption is necessary in order to satisfy item (i) above.

iii) The fluid is incompressible and inviscid.

iv) The fluid flow and the blade vibration are two-dimensional.

v) Only vibrations in a single degree of freedom are considered. Whitehead first considered vibration in a direction perpendicular to the chord and subsequently torsional vibration about the leading edge of the blade.

vi) Only small perturbations are considered. The resulting solution is then linear so that any number of solutions may be superposed to obtain results for any required excitation spectrum.

vii) All blades are assumed to be flat plates which vibrate with a constant maximum velocity $q$ and amplitude $h_0$.

Sample results given by Whitehead for the magnitude of the non-steady lift coefficient as a function of the ratio of the interblade phase angle to the reduced frequency for a cascade with a pitch to chord ratio of 2.0 and a stagger angle of 75° are shown in figure 12. This analysis is limited to the case of zero reduced frequency so that only those values of non-steady lift coefficient at the zero value of frequency ratio are valid.
Figure 12. Sample Results from Whitehead's Actuator Disk Analysis. From Reference [26].
n order to relax this restriction and extend the range of allowable
educed frequency from zero to unity, Whitehead developed semi-actuator
disk theories for bending and torsional vibration of unstalled blades
ith zero mean deflection, [27], [28] and [29]. The cascade geometry is
he same as that shown in figures 10 and 11. The following assumptions
re then applied to the cascade.

i) All blades vibrate with the same amplitude and velocity
although a constant phase angle $\beta$ exists between adjacent
blades. Note that the interblade phase angle is not required
to be small as was the case in the actuator disk analysis [26].

ii) The fluid is incompressible and inviscid.

iii) The fluid flow and the blade vibration are two-dimensional.

iv) Only single degree of freedom vibrations are considered.
Translational motion in a direction perpendicular to the
chord is considered first while motion parallel to the chord
is neglected as a second order effect. Torsional vibration
about an axis located at the lead edge of the blade is also
considered.

v) Only small perturbations are allowed. The resulting linear
theory allows the principle of superposition to be employed.

vi) All the blades are assumed to be flat plates operating at
zero mean flow incidence. This means that the steady force and
therefore the mean deflection are zero and that the blades and
their wakes can be replaced by vortex sheets.

vii) The blades do not stall so that the flow always follows the
surface of the blade.
The above analysis was subsequently extended to the case of finite mean deflection by Whitehead [30] thus allowing non-zero flow incidence angles. The basic procedure in this analysis is identical to the analysis above except that velocity perturbations perpendicular to the chord are included. Whitehead [31] and Smith [32] gave analyses which include the effect of generated acoustic waves for cascade with zero mean deflection.

Henderson and Horlock [33] have approximated the unsteady lift on airfoils for cascades having low pitch-to-chord ratios and finite chord length. The pitch is assumed to be small with respect to the wavelength of the inlet disturbance, as shown in figure 13. The rotor blades are thin and may have considerable camber but the lift coefficient is assumed to be small due to the low pitch to chord ratio. The flow is assumed to be two-dimensional, inviscid and incompressible. Sinusoidal axial disturbances are considered, of finite frequency parameter based on the blade chord but low frequency parameter based on blade pitch.

The authors applied the pitch-averaging technique of Horlock and Marsh [34] to the unsteady equations of motion for the flow through the moving blade passage. The total force in the tangential and axial directions was obtained by integrating the pressure difference between the pressure and suction face of the blade along the chord. Sample calculations of the unsteady lift were made for a cascade of flat plate airfoils at a stagger angle of 45° and a pitch to chord ratio of unity. Results of these calculations are compared with results obtained by Whitehead [26] and [27] for various values of reduced frequency parameter in figure 14. The authors attribute the poor agreement with Whitehead's results to the assumption of small blade chord made by Whitehead.

Actuator disk procedures have been reviewed by Horlock, Greitzer and Henderson [35] who presented a further alternative semi-actuator disk analysis. The stated purpose of this analysis is to resolve the discrepancy of results obtained by Whitehead [26][31], Smith [32] and Henderson and Daneshyar [16] for low values of reduced frequency as shown in figure 15. The results of
Figure 13. Cascade Notation of Henderson and Horlock. From [33].

Figure 14. Comparison of Results Obtained by Henderson-Horlock [33] and Whitehead [26]. From Henderson-Horlock [33].
Figure 15. Comparison of Unsteady Lift Coefficient Predicted by Sears [11], Henderson-Daneshyar [16], Whitehead [26][31], and Smith [32]. From Horlock-Greitzer-Henderson [35].
this analysis agree with those obtained by Whitehead [31] and Smith [32] for low values of pitch to chord ratio and reduced frequency. The authors therefore conclude that "in the limited range of very small reduced frequency, it...seems appropriate to use the present analysis or that of Smith..."

2.4 Development of Field Theories

Gostelow [8] has recently surveyed the state-of-the-art for steady-state subsonic potential flow analysis. Miles [36] has presented a comprehensive review of subsonic and supersonic non-steady potential flow theory. A review of computational techniques for isolated airfoils in steady subsonic-supersonic flow (elliptic-hyperbolic problem) has been given by Choroszylow [37].

Miles [36] has described solutions for several special forms of the potential flow equation for steady and non-steady flows, and has given certain airfoil solutions. Of particular interest for turbine and compressor blades studies are his results for supersonic gust loading on a two-dimensional airfoil, and on a supersonic rectangular wing.

Warner [28] used a finite element approach to analyze the two-dimensional, inviscid, incompressible, steady-state potential flow around an iterative cascade of cylinders. The quadrilateral fluid element was formulated in terms of the potential function \( \phi \). To test the functioning of this procedure Warner calculated the surface velocity on an infinite cascade of right circular cylinders. A typical mesh used in the calculation is shown in figure 16, and figure 17 is a comparison of calculated results for various numbers of surface/fluid boundary nodes with theoretical values for the same cascade. The element mesh at a distance from the cylinder was maintained as the interface mesh was refined. The author suggests that the accuracy of the solution would probably improve if the outer mesh were refined in the same manner as the interface mesh.
Figure 16. Finite Element Mesh for the Calculation of Potential Flow Around a Cylinder. From Warner [28].
Figure 17. Surface Velocity vs Angular Position on a Cylinder in an Infinite Cascade.

From Warner [38].
Steele [39] has extended Warner's analysis to a higher order element based on the Navier-Stokes equations. Sample calculations were made for Stokes flow (creeping flow) through a cascade of airfoils as shown in figure 18. The author notes that creeping flow is not a practical model of flow through a turbomachine stage and proposes the development of a viscous flow element with inertial effects included.

2.5 Comments

At present the accurate calculation of non-steady forces on turbomachine blades is restricted to certain special configurations and flow conditions. In general, the blade profiles must be thin and of small camber, though specific airfoil shapes can be considered by using certain specialized analyses, transform procedures or numerical methods. The frequency and velocity of the flow harmonics must be known in advance. The inclusion of compressibility effects in the analysis is possible, but not usual. So, the accuracy of results diminishes in the vicinity of sonic conditions. Non-steady supersonic cascade solutions do not appear to have been published, though certain supersonic single airfoil results are available, and trans-sonic results have been described: See Fleeter [40].

Two-dimensional field theories are in an early developmental stage and even the steady-state incompressible case remains as a computation problem for practical airfoil geometries. Finite element procedures are presently being developed for this case. While the finite element procedure overcomes the boundary-node problem and simplifies problem formulation, large amounts of computer time are still required. Time-dependent cascade flow does not appear to have yet been examined by this procedure.

It appears that non-steady forces on low-pressure, low-camber, thin turbine blades may now be calculated for sensibly incompressible flows with reasonable accuracy, provided that suitable input data is available. The same cannot be said for high-pressure stage blading because of the high turning angles and the highly compressible flow conditions under which these stages frequently operate.
Figure 18. Finite Element Model of Turbine Cascade. From Steele [39].
3. VORTEX THEORIES

3.1 General

Vortex theories involve the mathematical modeling of airfoils as plane vortex sheets. Kemp [12] discussed the results of earlier analyses by von Karman and Sears [10][11] and gave an expression for the unsteady lift on an isolated airfoil due to convecting and non-convecting sinusoidal transverse gusts (perpendicular to the chord). Kemp and Sears [2] extended this theory to study the unsteady forces on blades in an elementary turbine stage in which the blades are idealized as flat plates. The authors [3] subsequently included the effects of viscous stator wakes in calculating the unsteady blade forces. Horlock [13] performed an analysis similar to that in reference [2] using a convecting streamwise gust pattern (parallel to the chord). Holmes [14] extended Horlock's analysis to the case of non-convecting streamwise gusts. Inclusion of blade camber effects in such analyses were first reported by Horlock [41] for an airfoil of circular arc camber subjected to convecting streamwise gusts. Identical results were independently obtained by Naumann and Yeh [17]. Mukhopadhyay [18] performed an analysis to determine the unsteady forces on an airfoil of general camber which is subjected to simultaneous transverse and streamwise non-convecting gusts. These theories will be discussed in detail in the remainder of this section.

3.2 Kemp-Sears Theory for Non-Steady Lift on Airfoils Due to Transverse Gusts

Kemp and Sears [2][3] applied line-vortex theory to calculate the non-steady lift forces acting on both the moving blades and the stationary blades of the typical elementary turbine cascade shown in figure 11. The flow through the cascade is assumed to be inviscid and incompressible. The blades are assumed to be isolated airfoils, which means that the non-steady circulation of neighboring blades is neglected in the calculation of the non-steady effects on a particular airfoil. However, for the rotor blades, effects arising from the vortex wakes and from the viscous wakes shed by the sta
are included in the non-steady lift calculation. The rotor and stator blades are assumed to be flat plates in the analysis, though they may be thin, slightly cambered, lightly loaded airfoils in practice. This analysis is therefore more suited to turbine LP blade airfoil sections than to IP or HP blades with highly curved impulse sections. This section contains an integrated version of the two Kemp-Sears theories, in expanded detail.

In vortex theories each airfoil is replaced by a plain vortex sheet, as shown in figure 19. The vortex strength $\gamma(x,t)$ is such that at every instant in time the relative velocity component normal to the airfoil surface is zero, thus satisfying the boundary condition at the surface. Because the total circulation about the whole system is invariant, any changes in the airfoil circulation $\gamma(t)$ must be accompanied by the shedding of free vortices at the trailing edge, which are then carried downstream by the flow. The transient velocity field of the trailing wake of free vortices induces velocity components normal to the airfoil. These components are superimposed on other relative normal velocity components caused by the airfoil incidence, motion, and by non-uniformities in the stream in order to determine $\gamma(x,t)$. Kemp and Sears have shown that the expression for lift can be calculated by separating it into three distinct terms:

$$\tau_l(t) = q_L(t) + \gamma_L(t) + \eta_L(t)$$  \hspace{1cm} (3.1)

where

$q_L(t) = \text{Quasi-steady lift, which is calculated for instantaneous motion and flow conditions, neglecting entirely the wake effects.}$

$\gamma_L(t) = \text{Lift corresponding to variations in the apparent mass of the airfoil, which is not considered here.}$

$\eta_L(t) = \text{Lift due to wake effects.}$
Figure 19. Representation of an Airfoil by a Vortex Sheet.

Figure 20. Representation of Rotor Blade Circulation for Determination of Induced Velocity at a Stator Blade. From Rao [44].
Considerable simplification can be achieved by assuming sinusoidal variations in time. When the quasi-steady circulation of the airfoil is given by

\[ q^* \equiv q(t) = F e^{i \nu t} \]  

(3.2)

it can be shown that

\[ qL(t) + 2L(t) = qL(t) C(\omega) \]  

(3.3)

where \( \omega \) is the reduced frequency parameter \( \nu c/V \), \( c \) is the semichord and \( C(\omega) \) is the Theodorsen function given by

\[ C(\omega) = \frac{K_1(i\omega)}{K_0(i\omega) + K_1(i\omega)} \]  

(3.4)

In the above expression \( K_0 \) and \( K_1 \) are modified Bessel functions of the second kind.

If the airfoil operates in a sinusoidal gust pattern such that the relative upwash velocity at the airfoil is given by

\[ v(x,t) = v_0 e^{i \nu(t-x/W)} \]  

(3.5)

i.e. a convecting gust, then Kemp [12] has shown that the total non-steady lift is

\[ L(t) = 2\pi cWv_0 S(\omega) e^{i \nu t} \]  

(3.6)

with the corresponding quasi-steady circulation given by
where

\[ S(\omega) = \frac{1}{i\omega(K_0(i\omega) + K_1(i\omega))} \] (3.8)

and

\[ J(\omega) = J_0(\omega) - iJ_1(\omega) \] (3.9)

where \( J_0 \) and \( J_1 \) are Bessel functions of the first kind. From equation (3.7), the strength of the vortex distribution in the wake, \( \varepsilon(x,t) \) can be obtained

\[ \varepsilon(x,t) = 2\pi v_0 J(\omega) i\omega S(\omega) e^{i\nu(t-x/W)} \] (3.10)

Kemp [12] modified the equation for upwash velocity given in equation (3.10) to consider the effect of stator wakes on rotor blades. The gust is then a non-convecting gust of the following form:

\[ v(x,t) = v_0 e^{i\nu t} e^{-i\mu x/W} \] (3.11)

where \( \mu \) may be any arbitrary complex number.

For this case, it can be shown that the unsteady lift is

\[ tL(t) = 2\pi cW_0 S(\omega, \lambda) e^{i\nu t} \] (3.12)
with the quasi-steady circulation given by
\[ q^n(t) = 2\pi cv_0 J(\lambda)e^{i\nu t} \] (3.13)

The wake vortex strength can be shown to be
\[ c(x,t) = -2\pi v_0 J(\lambda)i\omega S(\omega)e^{i\nu(t-x/W)} \] (3.14)

where \( S(\omega,\lambda) \) is the modified Sears function.
\[ S(\omega,\lambda) = J(\lambda)c(\omega) + \frac{i\omega}{\lambda} J_1(\lambda) \] (3.15)

Equations (3.6), (3.7), (3.10), (3.12), (3.13) and (3.14) will be used in the application to stator-rotor problem.

Interference Effects Between Stator and Rotor Blades

In order to apply the above expressions to a turbomachine stage it is necessary to model the effects exerted on a rotor or stator blade by the other components of the stage as a series of upwash gusts of the form of equation (3.5) or (3.11). For example, consider the elementary turbomachine stage shown in figure 11. In general, each airfoil must be considered to be acting in a velocity field induced by (a) its own wake, (b) the variable bound vortices of the other blades in its own blade row, (c) their wakes, (d) the variable bound vortices of members of the other blade row, and (e) their wakes. However, this leads to a very complex problem of solving two simultaneous integral equations. Kemp and Sears [2] used the successive approximation approach outlined below:

1. The entire unsteady effect on the circulation of any blade is assumed to be small compared with the steady circulation carried by the blade.
2. In the calculation of nonsteady effects on a typical stator blade, it is assumed that the only significant contribution arises from the steady circulation of the rotor blades. The effects of the unsteady terms of rotor circulation, the unsteady parts of the circulation of all other blades and all vortex wakes except those shed by the stator itself are neglected.

3. In a similar way, to calculate the unsteady effects on a typical rotor blade, it is assumed that the significant contribution arises from the steady stator circulation. However, the effect of the vortex wakes shed by the stator blades is considered.

4. The unsteady forces on the rotor blades due to viscous wakes shed by the stator blades are also accounted for in the analysis.

The upwash velocity normal to the x-axis, produced by equal vortices of strength \( \Gamma' \) located at any \( N \) complex points \( \zeta_n \), is given by:

\[
V = -\frac{\Gamma'}{2\pi} \sum_{n=1}^{N} \frac{1}{(x-\zeta_n)} 
\]

(3.16)

With the rotor circulation represented by single point vortices as shown in figure 20, the induced velocity at the stator blade may be written as

\[
v^S = -\frac{\Gamma'}{2\pi} \sum_{n=-\infty}^{\infty} \frac{1}{x^S-(\xi+i\nu+ind_r)d^i\alpha_s} 
\]

(3.17)

Equation (3.17) can be transformed to a more useful form, given by:

\[
v^S = \frac{\Gamma'e^{-i\alpha}}{2d_r} [1+2 \sum_m \exp\{-\frac{2\pi m}{d_r} e^{-i\alpha}(\zeta+i\eta-x_S)\}] 
\]

(3.18)
Replacing \( \Gamma' \) by a distribution of vortices, \( \psi_{\Gamma'}(x_r) \) along a line representing a typical rotor blade, as shown in figure 20 and noting that

\[
\xi + i\eta = (b+ih_s)e^{i\alpha_s} + x_r e^{i(\alpha_s + \alpha_r)}
\]
equation (3.18) becomes

\[
\psi = \frac{r e^{-i\alpha_s}}{2d_r} \left[1 + 2 \sum_m \exp\left(-\frac{2\pi m}{d_r} (b+ih_s - x_r e^{-i\alpha_s})\right) H^r_m\right] (3.19)
\]

where

\[
H^r_m = \sum_{c_r} \frac{\psi_{\Gamma'}(x_r)}{\psi_{\Gamma'}} \exp\left(-\frac{2\pi m}{d_r} x_r e^{i\alpha_r}\right) dx_r
\] (3.20)

and

\[
\psi_{\Gamma'} = \sum_{c_r} \psi_{\Gamma'}(x_r) dx_r
\] (3.21)

To simplify equation (3.20), \( \psi_{\Gamma'} \) is assumed to be of a form similar to that given by Glauert [47].

\[
\psi_{\Gamma'} = 2W_r \left[A_0^\Gamma \frac{(1 - \cos \theta)}{\sin \theta} + 2 \sum_m A_m^\Gamma \sin m\theta\right] (3.22)
\]

with

\[
x_r = c_r \cos \theta
\]

The constants \( A_0, A_1, A_2 \ldots \) can be determined for the blade under consideration by conformal transformation of the airfoil to a circle.
Using equation (3.22) in equation (3.21)

\[
o^r_r = 2\pi c r W_r (A^r_0 + A^r_1)
\]  

(3.23)

Using equation (3.22) in equation (3.20) and using the properties of Bessel functions, the following equation for \( H^r_m \) can be obtained:

\[
H^r_m = J_0 (\pi m \alpha_r e^{-i(\pi/2-\alpha_r)}) + \sum_n (-i)^n \frac{A^r_{n+1} - A^r_{n-1}}{A^r_0 + A^r_1} J_n (\pi m \alpha_r e^{-i(\pi/2-\alpha_r)})
\]  

(3.24)

To this point the rotor was assumed to be stationary, and it is convenient to take the positions of the reference rotor and stator blades (\( n=0 \)) to be as shown in figure 1 at \( t=0 \). Measured from this position,

\[
b + h_s = b[1+i(\tan \alpha_r - \frac{V}{W_s \cos \alpha_s})] - \frac{iVc_r}{W_r} - iVt
\]  

(3.25)

using equation (3.25) in equation (3.19), the following expression for the induced velocity of the stator blade can be obtained:

\[
v^s(x_s,t) = \frac{o^r e^{i\alpha_s}}{2d_r} + \frac{o^r}{2\pi c_s} \sum_m G^s_m \exp(\frac{2\pi m}{d_r} e^{-i\alpha_s x_s}) e^{i\nu s m t}
\]  

(3.2)

where

\[
G^s = \frac{\pi c_s d_s}{d_r} e^{-i\alpha_s} \cdot H^r_m \cdot x
\]
\[ x \exp[-\pi \alpha_r r_c r_{rb} (1+i \tan \alpha_r - \frac{iV}{W_s \cos \alpha_s} - \frac{iV}{W_r})] \] (3.27)

and \( v_s = 2\pi V / d_r \)

In a similar way, the velocity \( V_r^l \) induced at a rotor blade by the motion of the steady stator circulation \( O_t^S \), can be determined using the notation shown in figure 21. The following equations are used to obtain the final expressions for \( V_r^l \).

\[ O_t^S = 2W_s \left[ A_0^S (1-\cos \theta) \frac{\sin \theta}{\sin \theta} + 2 \sum_n A_n^S \sin \theta \right] \] (3.28)

\[ x_s = c_s \cos \theta \] (3.29)

\[ O_t^R = 2\pi c_s W_s (A_0^S + A_1^S) \] (3.30)

The final expression for \( V_r^l \) is given by:

\[ V_r^l(x_r, t) = \frac{O_t^S e^{i\alpha_r}}{2d_s} + \frac{O_t^S}{2\pi c_r} \sum_m G_m^r \exp\left(-\frac{2\pi m}{d_s} e^{i\alpha_r} x_r \right) \]

\[ x \exp \left( -\frac{\pi \sigma_r d_r}{d_s} \right) \exp \left( \frac{b}{c_r} (1+i \tan \alpha_r - \frac{iV}{W_s \cos \alpha_s} - \frac{iV}{W_r}) \right) \] (3.31)

where

\[ G_m^r = \frac{-\pi \sigma_r d_r}{d_s} e^{i\alpha_r} H_m^S x \]

\[ \times \exp \left( -\frac{\pi \sigma_r d_r}{d_s} \right) \exp \left( \frac{b}{c_r} (1+i \tan \alpha_r - \frac{iV}{W_s \cos \alpha_s} - \frac{iV}{W_r}) \right) \] (3.32)
Figure 21. Representation of Stator Blade Circulation for Determination of Induced Velocity at a Rotor Blade. From Rao [44].

Figure 22. Velocity at a Point P Due to Vorticity $\varepsilon(x)$. 
Unsteady Lift of the Stator Blade:

Equation (3.26) is similar in form to that given in equation (3.11) so that equation (3.12) can be used to determine the lift expression for stator blades. Letting

\[ \omega_s = \frac{2\pi W_s}{d_s} e^{i[\pi/2-\alpha_s]} \]  

and

\[ \nu_s = \frac{2\pi c_s}{d_s} e^{i[\pi/2-\alpha_s]} \]

The unsteady lift on the stator blade can be obtained as

\[ \frac{t^L_S(t)}{o^L_s} = \frac{0^{n_r}}{o^{r_s}} \sum_m G^L_m S(\omega_s m, s_m) e^{i\nu_s m t} \]

where

\[ o^L_s = o^{r_s} \rho W_s \]
Application of equation (3.13) gives

$$q^{R,S}(t) = \rho \sum_{m} F_{m} e^{i \nu_{m} t}$$

(3.37)

$$F_{m} = G_{m} J(\lambda_{r}, m)$$

Unsteady Lift of Rotor Blade Due to Steady-State Circulation

The unsteady lift of the rotor blade due to steady-stator circulation can be obtained from the following:

$$\mu_{r} = \frac{2\pi \psi_{r}}{d_{r}} e^{-i[\pi/2 - \alpha_{r}]}$$

(3.38)

$$\lambda_{r} = \frac{2\pi c_{r}}{d_{r}} e^{-i[\pi/2 - \alpha_{r}]}$$

(3.39)

$$t^{1r}_{l}(t) = t^{1s}_{l} \sum_{m} G_{m} S(\omega_{r}, \lambda_{r}, m) e^{i \alpha_{r} t}$$

(3.40)

$$l^{1r}_{l}(t) = l^{1s}_{l} \sum_{m} G_{m} J(\lambda_{r}, m) e^{i \nu_{r} t} \{ F_{m} = G_{m} J(\lambda_{r}, m) \}$$
Unsteady Lift of Rotor Blade Due to Stator Vortex Wakes

The strength of the stator vortex wakes is determined by means of equations (3.14), (3.26), and (3.37), considering the appropriate phase relationships between the stator wakes. If \( f_n^s(t) \) is any function of time associated with the \( n \)th stator blade, (such as lift or circulation), then

\[
f_{n+k}^s(t) = f_n^s \left[ t + \frac{kd}{V} \right]
\]

(3.41)

The strength of the wake produced by the reference stator blade, due to periodic variations of its circulation may be found by applying equation (3.10) to equation (3.37), as follows:

\[
\varepsilon_0^s(x_s^0, t) = \frac{-i}{c_s} \sum_{m} f_m^s i \omega_s m s(\omega_s m) e^{i \omega_s m (t-x_s^0/W_s)}
\]

(3.42)

If the strength of vortex distribution is

\[
\varepsilon(x) = R(-B e^{-i\nu x})
\]

(3.43)

then the component of induced velocity \( w \) at point \( P \) in figure 22 taken normal to an arbitrary direction \( \beta \), is given by

\[
w = \pm \left[ \begin{array}{cc} \frac{1}{2} i B e^{+i\beta e^{-i\nu x} - |\nu L|} & \text{L > 0} \\ \frac{1}{2} i B e^{-i\beta e^{-i\nu x} - |\nu L|} & \text{L < 0} \\ \frac{1}{2} i B \cos \beta e^{-i\nu x} & \text{L = 0} \end{array} \right] \]

(3.44)

\( \nu > 0 \) upper sign

\( \nu < 0 \) lower sign
For the \( n^{th} \) stator wake, equation (3.42) may be modified by replacing \( t \) by \( t + nd_s/V \) and \( s_s^n = x_s^0 + nd_s \sin \alpha_s \), as

\[
e_{n}^{0}(x_s^0, t) = - \sum_k B_k^s e^{i \omega_s k[t + nd_s/V - nd_s + nd_s \sin \alpha_s/W_s]} \quad (3.45)
\]

where

\[
B_k^s = \frac{r^r}{c_s} p_k^s i \omega_s k S(\omega_s k) \quad (3.46)
\]

The upwash velocity \( v^r_2 \) due to stator wakes on the reference rotor blade can be obtained as

\[
v^r_2(x_r^0, t) = \frac{q^r}{2 \pi c_r} \sum_m g_m^r e^{i \omega_r m[t - x_r^0/W_r]} \quad (3.47)
\]

where

\[
g_m^r = - \frac{\pi \sigma r \frac{d^r}{d_x} \cos \alpha_s}{\cos \alpha_s} \sum_k p_k^s q_{mk}^r e^{i \omega_r m} \quad (3.48)
\]

\[
p_k^s = \frac{\pi \alpha_s \frac{d_s^r}{d_r}}{\frac{b}{c_r}} e^{-i \alpha_s} H_r^s S(\omega_s k) J(\lambda_s k) \times \exp(-\pi \alpha_r \frac{b}{c_r} (1 + i \tan \alpha_r))
\]

and

\[
q_{mk}^r = \frac{md_s^r W_s}{kd_s^r W_r} \{1 + \cot^2 \beta (1 - \frac{md_s^r W_s}{kd_s^r W_r} \sec \beta)^2 \}^{-1} \quad (3.49)
\]
Using equations (3.5), (3.6) and (3.47), the rotor blade lift due to stator wakes can be obtained as

$$ L_r^2(t) = \sum_{m} G_m^2 S(\omega_r, m) e^{i\nu_r mt} $$

(3.50)

When expanded, the above expression becomes

$$ L_r^2(t) = -\sum_{m} \frac{\sigma_r d_r}{d_s \cos \alpha_s} \times \exp\left\{-im\pi r \left( \frac{bV}{r_s c_r \cos \alpha_s} + \frac{V_r}{W_r} \right) + i\omega_r k \right\} \times
\frac{f_m^s S(\omega_r m) S(\omega_r k)(W_s d_r / r_s d_s) (k/m)}{1 + \cot^2 \beta [1 - (W_s d_r / W_r d_s) (k/m) \sec \beta]^2} e^{i\nu_r kt} $$

(3.51)

Equations (3.36), (3.40), and (3.51) were programmed by Rao and Rieger [55] to determine the unsteady stator and rotor blade forces without considering the viscous effects.

3.3 Kemp-Sears Analysis of Unsteady Lift of Rotor Blade Due to Viscous Wakes

In order to study viscous-wake effects it must be assumed that the flow downstream of a cascade of blades with viscous wakes can be represented by an inviscid shear flow. The velocity profile of the wake behind each blade is made to resemble the wake behind an isolated airfoil. The following three relations from the experimental results of Silverstein, Katzoff and Bullivant [43] are used to completely define the wake as shown in figure 23.
Figure 23. Representation of the Viscous Wake Behind an Isolated Airfoil.

Figure 24. Elementary Turbomachine Stage Including Viscous Stator Wakes.
\[ Y = 0.68 \, c \quad \text{and} \quad \frac{w_c}{c} = \frac{c_D (x/c) - 0.7}{x/c} \]

\[ 1 - \left( \frac{q_c}{q_{\infty}} \right) = 4.84 \, c^1/2 \left( \frac{x}{c} - 0.4 \right) \quad (3.52) \]

\[ \frac{1 - q/q_{\infty}}{1 - q_c/q_{\infty}} = \cos^2 \left( \frac{\pi \, y}{2 \, Y} \right) \quad |y| \leq Y \]

For a small perturbation velocity \( u \) in the x-direction \( (u \ll W) \),

\[ 1 - \left( \frac{q}{q_{\infty}} \right) = 1 - \left[ \frac{(W+u)^2}{W^2} \right] = - \frac{2u}{W} \quad (3.53) \]

Using the above expression, equations (3.52) can be converted from dynamic pressure to velocity, as

\[ \frac{u_c}{W} = - 2.42 \, c^1/2 \left( \frac{x}{c} - 0.4 \right) \]

\[ \frac{u}{u_c} = \cos^2 \left( \frac{\pi \, y}{2 \, Y} \right) \quad (3.54) \]

The mathematical analysis can be simplified by using Gaussian error curve in the above expressions instead of a cosine wake profile. By using the momentum flow conditions, equations (3.54) can be approximated by

\[ \frac{u}{u_c} = \exp \left[ -\pi \left( \frac{Y}{y} \right)^2 \right] \quad Y \ll y \ll Y \quad (3.55) \]
If equations (3.59) represent the profile of the wake shed by the reference stator blade, the profile of the wake shed by the \( n \)th stator blade can be obtained by replacing \( y \) by \( y - nd_s \).

The total velocity field \( u_T \) of all the wakes can then be obtained by the summation

\[
\frac{u_T}{u_c} = \sum_{n=-\infty}^{+\infty} \exp[-\pi\left(\frac{(y - nd_s)\cos\alpha_s}{y}\right)\frac{2}{\alpha}]
\]

(3.60)

It should be noted that the above equation does not satisfy the principle of continuity of the axial flow through the stage. To satisfy this condition a constant equal to the average value of \( u_T \) should be subtracted from the right-hand side. Expanding the equation in complex Fourier series by using Poisson summation formula and subtracting the average value of \( u_T \), one obtains

\[
\frac{u_T}{u_c} = \frac{2\sqrt{\pi}}{K} \sum_{m=1}^{\infty} \exp\left\{-2\pi \text{Im} \left\{ \frac{y}{d_s} - \frac{\pi m^2}{K^2} \right\}\right\}
\]

(3.61)

where

\[
K = \sqrt{\pi} \frac{d_s \cos\alpha_s}{y}
\]

If \( t=0 \) is taken in the position shown in figure 24, the time dependence of \( y' \) coordinate can be obtained as

\[
y' = V\left(\frac{x_r}{W_r} - t\right)
\]

(3.62)
Equations (3.52) and (3.54) are rewritten in terms of

\[
x^* = x - 0.7c
\]
\[
Y = 0.68 \left( 2C_D x^* c \right)^{1/2}
\]  
(3.56)
\[
\frac{u_c}{W} = \frac{2.42 C_D^{1/2}}{x^* + 0.3}
\]

From equations (3.56) it can be noted that the wake is relatively narrow compared to the airfoil chord, (e.g. at \( x = 3c \), \( Y/c \approx 0.2 \)).

Equations (3.55) and (3.56) are now used to determine the velocity inc at the rotor. Consider the simple stage shown in figure 24. It is convenient to use oblique coordinates with the relations

\[
x^*_s = s' - y' \sin \alpha_s x'
\]  
(3.5)
and
\[
y = y' \cos \alpha_s
\]  
(3.5)

Equations (3.55) and (3.56) now become

\[
\frac{u}{u_c} = \exp \left\{ -\pi \left[ \frac{y' \cos \alpha_s}{y} \right]^2 \right\}
\]
\[
Y = 0.68 \left( 2C_D x^* c \right)^{1/2}
\]  
(3.5)
\[
\frac{u_c}{W} = \frac{2.42 C_D^{1/2}}{x'^* / c + 0.3}
\]
Noting that the upwash velocity at the blade is \(-u_\theta \sin \beta\), \(v^{rv}\) can be obtained from equation (3.61)

\[
v^{rv} = -u_c \sin \beta \frac{2\sqrt{\pi}}{k} \sum_{m=1}^{\infty} \exp\left(-\frac{m^2}{k^2}\right) e^{i\nu_r m(t-x_r/W)}
\]  

(3.63)

Both \(u_c\) and \(K\) are functions of \(x'/c_s\). From the relationship between \(x'\) and \(x_r\)

\[
\frac{x'}{c_s} = \frac{c_r}{c_s} \left\{ \frac{b}{c_r} \sec \alpha_s + \frac{x_r}{c_r} \frac{W_s}{W_r} \right\} - 0.07
\]  

(3.64)

which is then substituted in equations (3.59) and (3.61). Equation (3.61) can then be used to determine the lift forces using the non-steady thin airfoil theory.

It is noted that \(v^{rv}\) is still dependent on \(x_r\) so a further approximation is used in assuming a value for \(x_r/c_r\). This value may be anywhere between -1 and 1. Values of \(x_r/c_r\) between -1/2 and 1/2 seem to be more appropriate at this stage, and the differences in the magnitudes of forces obtained for this range have been shown by Rao [44] to be marginal. Denoting \(x_r\) in equation (3.64) as \(x_{r0}\), the following expression for \(v^{rv}\) can be obtained

\[
v^{rv}(x_{r0}, t) = \frac{W_r}{2\pi} \sum_{m=1}^{\infty} G_m^{rv} e^{i\nu_r m(t-x_{r0}/W)}
\]  

(3.65)

where

\[
G_m^{rv} = 4.65446 \frac{\pi W_s C_s^S \sin \beta}{W_r \cos \alpha_s \left( \frac{x_{r0}}{c_s} + 0.3 \right)} \cdot \frac{x'}{c_s} \cdot \exp\left[ -\frac{m^2}{2} \left( \frac{0.68 \sigma}{c_s} \right)^2 \right] \cdot \frac{C_D}{c_s} \frac{x'}{c_s}
\]  

(3.66)
using equations (3.5) and (3.6), the unsteady lift due to viscous effects can be obtained.

\[
t_{L_r}^{rv}(t) = \frac{t_{L_r}^{rv}(t)}{\frac{1}{2} \rho W_r^2 c_r}
\]

\[
= \sum_{m=1}^{\infty} G_m^{rv} S(m\omega_r) e^{im\omega_r t}
\]

Writing

\[
o_{L_r}^r = c_L^r \cdot \frac{1}{2} \rho W_r^2 c_r
\]

the ratio of nonsteady lift to steady lift can be shown to be

\[
\frac{t_{L_r}^{rv}(t)}{o_{L_r}^r} = \frac{1}{D_L^r} \sum_{m=1}^{\infty} G_m^{rv} S(m\omega_r) e^{im\omega_r t}
\]

Sample calculations and a computer program using the above analyses are given in Section 7.1.

3.4 Horlock Analysis for Non-Steady Lift on Airfoils Due to Convecting Streamwise Gusts

The Kemp-Sears analysis given in section 3.2 is based on the assumption that velocity perturbations parallel to the chord are negligible compared to those perpendicular to the chord. This assumption is justified if interference effects are considered when the angle of attack is small. If the gusts parallel to the chord are not negligible an analysis developed
by Horlock [13] may be used.

Consider a flat plate airfoil which is subjected to gusts parallel to the chord as shown in figure 25. The lift can be expressed as the integral over the chord of the pressure difference across the blade.

\[
L = \sum_{-1}^{1} (p_B - p_A)dx
\]

(3.69)

Bernoulli's equation for the unsteady flow is

\[
\frac{p_B}{\rho} + \frac{q_B^2}{2} + \sum_{-1}^{B} \frac{\partial q}{\partial t} dx = \frac{p_A}{\rho} + \frac{q_A^2}{2} + \sum_{-1}^{A} \frac{\partial q}{\partial t} dx
\]

(3.70)

where \( q_A \) and \( q_B \), the local velocities at points A and B, are given by the expressions

\[
q_A = [(U + u_A)^2 + v_A^2]^{1/2}
\]

(3.71)

\[
q_B = [(U + u_B)^2 + v_B^2]^{1/2}
\]

By substituting equation (3.70) into (3.69), and noting that \( \frac{\partial q}{\partial t} \) is the same on both sides of the airfoil at a point just upstream of the leading edge, the lift can be expressed in terms of the velocities.

\[
L = \rho \sum_{-1}^{1} \frac{q_B^2 - q_A^2}{2} dx + \rho \sum_{-1}^{1} \left[ \sum_{-1}^{x} \frac{\partial q_A}{\partial t} - \frac{\partial q_B}{\partial t} \right] dx
\]

(3.72)
Figure 25. Streamwise Gust Loading of a Flat Plate.

Figure 26. General Convecting Gust Loading of a Flat Plate.
Noting that
\[ q_A - q_B = \gamma \]
and
\[ \sum_{-1}^{1} (q_A - q_B)dx = \Gamma \]
and neglecting second order terms allows equation (3.72) to be rewritten as
\[
L = \rho \sum_{-1}^{1} (U + u)\gamma dx + \rho \frac{\partial \Gamma}{\partial t} - \rho \frac{\partial}{\partial t} \sum_{-1}^{1} \gamma y dx
\]

The steady lift may be expressed in terms of the steady bound vorticity \( \gamma_s(x) \)
\[
L_s = \rho \sum_{-1}^{1} U\gamma_s(x)dx
\]

The fluctuating lift (\( \Delta L \)) is defined as the difference between the total lift and the steady lift.
\[ \Delta L = L - L_s \]

If the instantaneous circulation \( \Gamma \) is written as
\[ \Gamma = \sum_{-1}^{1} \gamma dx \]
the fluctuating lift can be expressed as the sum of four integrals.

\[
\Delta L = \rho U \sum_{-1}^{1} (\gamma_0 + \gamma_1)dx + \rho \sum_{-1}^{1} u\gamma_s dx + \rho \frac{\partial}{\partial t} \sum_{-1}^{1} (\gamma_0 + \gamma_1)dx
\]

\[
- \rho \frac{\partial}{\partial t} \sum_{-1}^{1} (\gamma_0 + \gamma_1)dx
\]

This expression may be simplified using procedures similar to those of von Karman and Sears [10], Glauert [42] and Grobner and Hofreiter [45] to obtain a simple expression for fluctuating lift

\[
\Delta L = [2\pi c U u_0 \delta \omega T(\omega)] e^{i\nu t}
\]

where \( \delta \) is the angle of attack and \( T(\omega) \) is the Horlock function

\[
T(\omega) = X(\omega) + iY(\omega)
\]

and

\[
X(\omega) = (2-a)J_0(\omega) - bJ_1(\omega)
\]

\[
Y(\omega) = (a+1)J_1(\omega) - bJ_0(\omega)
\]

The coefficients \( a \) and \( b \) are given by

\[
\frac{K_0(i\omega)}{K_0(i\omega) + K_1(i\omega)} = a + ib
\]

where \( K_0(i\omega) \) and \( K_1(i\omega) \) are modified Bessel functions of the second kind.
Thus for an isolated flat plate airfoil at an angle of attack $\delta$ to the steady relative flow $W$, and subjected to a general convecting gust of the form

$$\omega = u_0 e^{i\nu(t-x/u)} \quad (3.77)$$

as shown in figure 26, the fluctuating lift is given by the sum of equations (3.6) and (3.76)

$$\Delta L = \omega_0 c2\pi U [\delta\cos\beta T(\omega) + \sin\beta S(\omega)] e^{i\nu t} \quad (3.78)$$

where $\beta$ is the flow angle relative to the axial direction (i.e. the sum of the stagger angle and the angle of attack).

### 3.5 Holmes Analysis for Non-Steady Lift on Airfoils Due to Non-Convecting Streamwise Gusts

Holmes [14] reported the extension of Horlock's theory to the case of non-convecting streamwise gusts of the form

$$u = u_0 e^{i\nu(t-\lambda x)} \quad (3.79)$$

The fluctuating lift on a flat plate subjected to the gust $u$ is given by

$$\Delta L = \rho c U u_0 \delta H(\omega,\lambda) e^{i\nu t} \quad (3.80)$$

where $H(\omega,\lambda)$ is the Holmes function

$$H(\omega,\lambda) = S(\omega,\lambda) + J_0(\lambda) + iJ_1(\lambda)$$

where $S(\omega,\lambda)$ is the modified Sears function defined by equation (3.15). Following the same logic as previously, the response to a general non-convecting gust of the form

$$\omega = u_0 e^{i\nu t} e^{-i\lambda x} \quad (3.81)$$
is simply the sum of equations (3.12) and (3180)

\[ \Delta L = \omega_o c 2\pi \rho U [\delta \cos \delta H(\omega, \lambda) + \sin \delta S(\omega, \lambda)] e^{i\nu t} \]  

(3.82)

3.6 Holmes Theory for Parabolic Cambered Airfoils

Horlock [13] reported an extension of the analysis by Holmes to airfoils with camber specified by

\[ y_B = y_{B_{\text{max}}} \left( \frac{1 - \cos \theta}{2} \right) \]  

(3.83)

and subjected to a streamwise convecting gust. The expression for unsteady lift obtained by Holmes is

\[ \Delta L = 2\pi \rho U \omega_o y_{B_{\text{max}}} T'(\omega) e^{i\nu t} \]  

(3.84)

where \( T'(\omega) \) is the Holmes camber function\(^1\) given by

\[
T'(\omega) = 2[J_0(\omega) + J_2(\omega)] + (b-a)[J_0(\omega) - J_2(\omega)] \\
- 2bJ_1(\omega) - 2i[J_1(\omega) - aJ_1(\omega)]
\]

and \( a \) and \( b \) are defined previously.

For an airfoil of the camber described in equation (3.83) and subjected to a general convecting gust of the form of equation (3.77) the total fluctuating lift is given by

\[ \Delta L = 2\pi c \omega_o U [\sin \delta S(\omega) + \cos \delta [\delta T(\omega) + \frac{y_{B_{\text{max}}}}{c} \cdot T'(\omega)]] e^{i\nu t} \]  

(3.86)

Naumann and Yeh[17] independently obtained identical results for the response of the camber described by equation (3.83) to a gust of the form of equation

\(^1\)This function is also referred to as the Holmes function in the literature. The author will refer to this function as the Holmes camber function, to avoid confusion with the Holmes function \( H(\omega, \lambda) \).
It should be noted that the Holmes camber function relates a specified camber function to the fluctuating lift due to that camber. In general, a different Holmes camber function is required for every different blade camber function.

3.7 Mukhopadhyay's Analysis for General Camber Airfoils

Mukhopadhyay [18] has derived an expression for the fluctuating lift on an airfoil with a general camber function expressed in terms of a polynomial

$$y' = \sum_{m} a_m x^m$$ \hspace{1cm} (3.87)

where \(x\) is an axial coordinate. The slope of the camber line is then given by

$$\frac{dy'}{dx} = \sum_{m} m a_m x^{m-1}$$ \hspace{1cm} (3.88)

If a change of variables is made such that

$$x = -\cos \theta$$ \hspace{1cm} (3.89)

the slope of the camber line is given by

$$\frac{dy'}{dx} = \sum_{m=0}^{n} A_m \cos \theta$$ \hspace{1cm} (3.90)

where \(n\) is one less than the highest power of the polynomial in \(x\).

For a general non-convecting gust of the form

$$\omega = \nu_0 e^{iut} e^{-lut}$$ \hspace{1cm} (3.91)
the fluctuating lift is given by

\[
\Delta L = \pi \rho c U \sum_{m=0}^{n} \Delta m [F_m(\omega, \lambda) + i m+1 (1 - \frac{\omega}{\lambda}) [J_{m+1}(\lambda) - J_{m-1}(\lambda)]] \\
- i m+1 2 m (1 + \frac{m \omega}{\lambda}) \frac{J_m(\lambda)}{\lambda} + Z \nu \sigma S(\omega, \lambda) \\
+ 2 u_o (\alpha - A_o) [J_0(\lambda) + i J_1(\lambda)] + \frac{4 u_o}{\pi \lambda} (1 - \frac{\omega}{\lambda}) \sin \lambda \tag{3.92}
\]

where \( J_n(\lambda) \) are Bessel functions of the first kind of order \( n \), \( S(\omega, \lambda) \) is the modified Sears function and \( F_m(\omega, \lambda) \) is a function defined by equation (3.93)

\[
F_m(\omega, \lambda) = 2 i m [i J_{m-1}(\lambda)] - C(\omega) - J_m(\lambda) C(\omega) (1 - \frac{im}{\lambda}) + \frac{im}{\lambda}] \tag{3.93}
\]

where \( C(\omega) \) is the Theodorsen function.

3.8 Comments

Vortex theories for the calculation of unsteady forces on turbomachine blading in the subsonic, incompressible flow regime are all based on isolated airfoil theory. Horlock Greitzer and Henderson (35) have shown that this is an acceptable approximation for pitch to chord ratios greater than 1.5. It would appear that the analysis given by Mukhopadhyay is the most general of the vortex theories in that the blades may have any camber. Inclusion of blade thickness effects is the next logical extension of vortex theories.
3.9 Notation

- $b$ - axial spacing
- $c$ - blade semichord
- $C_L$ - lift coefficient
- $C(\omega)$ - Theodorsen function
- $d$ - blade pitch
- $e$ - exponential function
- $F_m(\omega, \lambda)$ - Mukhopadhyay function
- $H(\omega, \lambda)$ - Holmes function
- $L$ - lift
- $p$ - pressure
- $S(\omega)$ - Sears function
- $S(\omega, \lambda)$ - modified Sears function
- $T(\omega)$ - Horlock function
- $T'(\omega)$ - Holmes camber function
- $u$ - flow velocity parallel to chord
- $U$ - flow velocity in axial direction
- $v$ - flow velocity perpendicular to chord
- $V$ - flow velocity in tangential direction
- $w$ - general flow velocity
- $W$ - freestream flow velocity
- $\alpha$ - stagger angle
- $\beta$ - flow angle
- $\gamma$ - vortex strength
- $\Gamma$ - circulation
- $\delta$ - angle of attack
- $\mu$ - complex constant
- $\nu$ - excitation frequency
- $\rho$ - density
- $\omega$ - reduced frequency

subscripts

- $o$ - free stream condition
- $r$ - referred to rotor
- $s$ - referred to stator
4. ACTUATOR DISK THEORIES

4.1 General

One of the earliest developments of the basic theory for the analysis of cascade blading vibration by actuator disk methods was given by Lilley [19]. Subsequent refinements to the basic theory were made by Lane and Wang [20] and by Lane and Friedman [21]. These investigations were primarily directed towards the determination of the critical values of flow parameters resulting in bending and torsional flutter. Whitehead [26], extending the work of these authors, developed an actuator disk analysis to determine the unsteady lift and moment coefficients for blades vibrating in cascade. In this and all true actuator disk analyses, the blade chord is assumed to be very small compared to the wavelength of the upstream disturbance. The flow field may therefore be considered as being observed from a very great distance while a disturbance of finite wavelength, generated far upstream of the blade row propagates downstream. Since the blade chord is extremely small in this reference frame, the blade row appears to be a thin "actuator" disk. Actuator disk analyses similar to Whitehead's have been given by Ehrich [46], Yeh [22], Plourde and Stenning [47], Hawthorne and Horlock [48].

Analyses which are similar to the actuator disk analysis, but which relax one or more of the assumptions of the actuator disk analysis, are called semiactuator disk methods. Whitehead [27] developed a semiactuator disk analysis relaxing the restriction of small pitch to chord ratios. A similar analysis by Smith [32] twelve years later gave results identical to those of Whitehead. Henderson and Horlock [33] gave an analysis for closely spaced blades with finite turning angle. Henderson and Daneshyar [16] developed an analysis which showed good agreement with that of Whitehead [27] and Smith [32] for moderate values of frequency parameter \( \omega \). For low frequency parameters, agreement is poor. Horlock, Greitzer, and Henderson [35] employed two separate techniques to derive a method which resolved...
this discrepancy and showed good agreement with the analysis of Whitehead [27] and Smith [32] for low values of the frequency parameter. In the remainder of this section the analyses of Whitehead [26] and [27], Smith [32], Henderson and Horlock [33], and Horlock, Greitzer and Henderson will be developed.

4.2 Whitehead's Actuator Disk Analysis [26]

As stated above, in this type of analysis the blade row is replaced by narrow actuator disk as shown in figure 10. The blades in the immediate vicinity of the origin are shown in figure 11. The following assumptions are applied to the model:

i) The time required for the fluid to pass through the actuator disk is small compared to the period of vibration of the blade. In order for this requirement to be fulfilled, the reduced frequency parameter based on the blade chord ($\omega$) must be small and is in fact limited to a value of zero for the true actuator disk analysis.

ii) Adjacent blades vibrate with a small, constant phase angle between them. This angle is the interblade phase angle ($\beta$). It will be shown in section 4.6 that this assumption is necessary in order to satisfy item (i) above.

iii) The fluid is incompressible and inviscid.

iv) The fluid flow and the blade vibration are two dimensional.

v) Only vibrations in a single degree of freedom are considered. Whitehead first considered vibration in a direction perpendicular to the chord and subsequently torsional vibration about the leading edge of the blade.
vi) Only small perturbations are considered. The resulting solution is then linear so that any number of solutions may be superposed to obtain results for any required excitation spectrum.

vii) All blades are assumed to be flat plates which vibrate with a constant maximum velocity $q$ and amplitude $h_0$.

The reduced frequency parameter based on the blade chord is given by:

$$\omega = \frac{\nu c}{U \sec \alpha} \quad (4.1)$$

where $\nu$ is the disturbance frequency, $c$ is the blade chord, $U$ is the mean axial gas velocity, and $\alpha$ is the angle of the relative gas velocity $W$ measured from the axial direction as shown in figure 10. The denominator of equation (4.1) is equal to the magnitude of the relative gas velocity:

$$W = U \sec \alpha \quad (4.2)$$

so that:

$$\omega = \frac{\nu c}{W} \quad (4.3)$$

The exciting disturbance is assumed to propagate in the positive $y$ direction with a velocity $V_s$. The phase angle $\beta$ is then given by:

$$\beta = -\frac{\nu s}{V_s} \quad (4.4)$$

where $\nu$ is the frequency of blade vibration, and $s$ is the blade pitch. The displacement of the blade at the origin is given by:

$$H_0 = h_0 e^{i\nu t} \quad (4.5)$$
while the displacements of two reference blades spaced \( N \) blade pitches away from the origin in the positive and negative \( y \) directions are respectively:

\[
H_+N = h_0 e^{i(\nu t + NB)} \\
H_-N = h_0 e^{i(\nu t - NB)}
\]  

(4.6)

where \( N \) is large enough that effects of any single blade are negligible, but small enough so that \( NB << 1 \). The small angle \( \phi \) through which the cascade turns due to its vibration is, to the first order:

\[
\phi = \frac{ih_0 \bar{q} \sin \phi e^{i\nu t}}{s}
\]  

(4.7)

but

\[
q \bar{e}^{i\nu t} = i \nu h_0 e^{i\nu t}
\]  

(4.8)

so that

\[
\phi = \frac{q \bar{e} \sin \phi e^{i\nu t}}{\nu sr}
\]  

(4.9)

The relative velocity components just upstream of the blades are given by

\[
u_{r1} = U + (u_{01} + q \sin \phi) e^{i\nu t}
\]

(4.10)

\[
u_{r1} = V_1 + (v_{01} - q \cos \phi) e^{i\nu t}
\]

and the downstream relative velocity components can be shown to be
\[
\begin{align*}
    u_{r2} &= U + (u_{02} + q\sin\phi)e^{ivt} \\
    v_{r2} &= V_2 + (v_{02} - q\cos\phi)e^{ivt} 
\end{align*}
\]

for the case when \(\phi = 0\). For \(\phi \neq 0\) the upstream relative velocity components normal and parallel to the inclined cascade are, respectively:

\[
\begin{align*}
    u'_{r1} &= u_{r1}\cos\phi + v_{r1}\sin\phi \\
    v'_{r1} &= -u_{r1}\cos\phi + v_{r1}\sin\phi 
\end{align*}
\]

and the downstream velocities are

\[
\begin{align*}
    u'_{r2} &= u_{r2}\cos\phi + v_{r2}\sin\phi \\
    v'_{r2} &= -u_{r2}\cos\phi + v_{r2}\sin\phi 
\end{align*}
\]

The continuity condition requires the relative velocity perpendicular to the cascade to be constant, i.e.

\[
U'_{r1} = U'_{r2}
\]

or

\[
u_{r1}\cos\phi + v_{r1}\sin\phi = u_{r2}\cos\phi + v_{r2}\sin\phi
\]

substituting for the relative velocities and the inclination angle \(\phi\), and retaining only first order terms gives

\[
(u_{01} - u_{02}) + \frac{(V_1 - V_2)\sin\phi\theta}{\nu s} = 0
\]

It is assumed that the relative exit velocity of the stage is constant and that the flow exhausts at the same flow angle as the blade stagger angle. The downstream relative velocity components are therefore related by the expression
\[
\frac{v_{r2}}{u_{r2}} = \tan \theta
\]

or

\[
\frac{v_2 + (v_{02} - q \cos \theta)e^{i \omega t}}{U + (u_{02} + q \sin \theta)e^{i \omega t}} = \tan \theta
\]  \hspace{1cm} (4.16)

Simplifying the above yields

\[
v_2 + (v_{02} - q \cos \theta)e^{i \omega t} = [U + (u_{02} + q \sin \theta)e^{i \omega t}] \tan \theta
\]  \hspace{1cm} (4.17)

The steady velocity \( v_2 \) is related to the steady velocity \( U \) by the expression

\[
v_2 = U \tan \theta
\]

so that

\[
v_{02} - q \cos \theta = (u_{02} - q \sin \theta) \tan \theta
\]  \hspace{1cm} (4.18)

Let the force per unit length acting on each blade in the inclined cascade be \( x' \) and \( y' \), in the perpendicular and parallel directions, respectively. The total force per unit blade length between the two reference blades in the perpendicular direction is

\[
2Nx' = (p_1 - p_2)D
\]  \hspace{1cm} (4.19)

where \( 2N \) is the number of blades between the reference blades, \( (p_1 - p_2) \) is the pressure drop across the cascade, and \( D \) is the pitch-wise distance between the reference blades. For quasi-steady flow with no losses the pressure drop is given by
The distance between the reference blades in the stationary cascade is 2Ns where \( s \) is the blade pitch. The reference blades are deflected from their rest positions as indicated in equations (4.6) so that the total distance \( D \) is given by

\[
D = 2Ns + h_0 e^{i(\nu t - N_\beta)} \cos \theta - h_0 e^{i(\nu t - N_\beta)} \cos \theta
\]

Substituting equations (4.20) and (4.21) into equation (4.19) gives

\[
X' = \frac{1}{2} \rho (v'_{r2}^2 - v'_{r1}^2)(s + i h_0 \beta \cos \theta) e^{i \nu t}
\]

Similarly, the force parallel to the inclined cascade may be shown to be

\[
Y' = \rho u'_{r1} (v'_{r1} - v'_{r2})(s - i h_0 \beta \cos \theta) e^{i \nu t}
\]

Neglecting higher order terms, the force in the direction of vibration is

\[
F e^{i \nu t} = [-X' \sin(\vartheta - \phi) + Y' \cos(\vartheta - \phi)] e^{i \nu t}
\]

After substituting from equations (4.19) to (4.15), and (4.19) to (4.23) and after considerable simplification the ratio \( F/\rho s \) is found to be

\[
\frac{F}{\rho s} = (V_1 v_{o1} - V_2 v_{o2}) \sin \theta + (V_1 u_{o2} - V_2 u_{o1}) \cos \theta
\]

\[
+ U(v_{o1} - v_{o2}) \cos \theta - U(u_{o1} - u_{o2}) \cos \theta \cot \theta
\]
Considering the vorticity of the inlet flow to be zero and assuming no stagnation pressure losses, the following velocity equations may be derived from the stream function for the cascade.

\[ u_{02} + i v_{02} = \frac{i[(V_1 - V_2)\cos \theta \delta \alpha + (V_{01} - V_{02})]}{(v_2/V_s) - 1 + i(U/V_s)} \]  
\[ (4.26) \]

\[ u_{01} + i v_{02} = 0 \]  
\[ (4.27) \]

Using equations (4.15), (4.18), (4.26) and (4.27) equation (4.25) can be written in the form below to define the unsteady lift coefficient

\[ C_F = \frac{s \sec \theta}{\pi c} \left\{ \frac{C_{FR} + C_{FI}}{4 + x^2 \delta^2} \right\} \]  
\[ (4.28) \]

where

\[ C_{FR} = [\tau x^2 \delta (\tau \delta + x \delta + 2t) + 2x(2\tau \varepsilon + x \delta - 2t) + 4] \]

\[ C_{FI} = i[2x^2 \delta \tau (\tau \tau + 1) + 2x\varepsilon (1+\tau^2) - x^3 \delta^2 \tau (1+\tau^2)] \]

\[ x = \frac{U/V_s} = - \frac{U \beta \varepsilon}{V_s} = - \frac{c_\infty \cos \theta}{2 \delta \omega} \]

\[ \omega = \frac{V_c}{2U \sec \alpha_1} \]

\[ t = \tan \theta \]

\[ \delta = \sec^2 \theta \]

\[ \varepsilon = 1 - \tan^2 \theta \]

\[ \tau = (\tan \alpha_1 - \tan \theta) \cos^2 \theta \]
Note again that the above analysis is based on zero reduced frequency and is therefore only applicable when \( \omega = 0 \). Also note that this analysis is for transverse vibration of unstalled blades with finite mean deflection. Whitehead also developed an expression for the unsteady torque coefficient.

\[
C_M = \frac{\xi_p - \xi_T}{c} \cdot C_F 
\]

where \( \xi_p \) is the chordwise coordinate of the center of pressure on the blade and \( \xi_T \) is the torsional axis of the blade. For the case of excitation due to wakes from upstream obstructions, Whitehead gives the following expression for the unsteady force coefficient.

\[
C_{FW} = (q - w)C_F 
\]

where \( w \) is the velocity which would be induced in the vibration direction at the origin of the actuator disk if the disk were removed from the flow. Also given in the paper are expressions for the unsteady force coefficients for cascades of stalled and unstalled blades with zero mean deflection (implying no steady blade loading) and for fixed, stalled blades. Sample calculations and results of this analysis are given in section 8.4.

### 4.3 Whitehead's Modified Actuator Disk Analysis

The above analysis is limited to the case where the reduced frequency parameter \( \omega \) is zero. To relax this restriction and extend the range of reduced frequency to \( 0 < \omega < 1.0 \), Whitehead developed semiactuator disk theories in a series of papers dealing with the following conditions:
a) bending vibration of unstalled blades with zero mean deflection \[27][28]\n
b) bending vibration of unstalled blades with finite mean deflection \[29]\n
c) torsional vibration of unstalled blades with zero mean deflection \[27][28][20]\.

The analysis of reference \[27\] will be reviewed here since it is representative of the other analyses.

The cascade notation for this analysis is identical to that shown in figures 10 and 11. The following assumptions are then applied to the cascade.

i) All bladed vibrate with the same amplitude and velocity although a constant phase angle $\beta$ exists between adjacent blades. Note that the interblade phase angle is not required to be small as was the case in the actuator disk analysis \[26\].

ii) The fluid is incompressible and inviscid.

iii) The fluid flow and the blade vibration are two dimensional.

iv) Only single degree of freedom vibrations are considered. Translational motion in a direction perpendicular to the chord is considered first while motion parallel to the chord is neglected a second order effect. Torsional vibration about an axis located at the lead edge of the blade is also considered.

v) Only small perturbations are allowed. The resulting linear theory allows the principle of superposition to be employed.

vi) All the blades are assumed to be flat plates operating at zero mean flow incidence. This means that the steady force and the fore the mean deflection are zero. Also the blades and their wakes can be replaced by vortex sheets.

vii) The blades do not stall so that the flow always follows the surface of the blade.
The general method for calculating the vorticity is to determine the chordwise distribution of vorticity which induces the required upwash velocity normal to the blade surface. The vorticity associated with the blade will be termed the 'bound' vorticity $\gamma$. An element of bound vorticity at a distance $x_1$ from the origin (lead edge of the reference blade) will be defined as

$$\gamma dx \, e^{i\nu t} \quad (4.31)$$

The corresponding element on the blade immediately above the reference blade will lead the element on the reference blade by the interblade phase angle $\beta$ and will be advanced in location due to the stagger of the cascade as shown in figure 27. The element of vorticity on this blade is then given by

$$\gamma dx \, e^{i(\nu t + \beta)} \quad (4.32)$$

and the coordinates of its location are

$$x_1 + s \sin \xi, \quad s \cos \xi \quad (4.33)$$

Generalizing this concept to the $j^{th}$ blade above the reference blade the element of vorticity and its coordinates are given by

$$\gamma dx \, e^{i(\nu t + j\beta)}$$

$$x_1 + js \sin \xi, \quad js \cos \xi \quad (4.34)$$
Figure 27. Relative Positions of Bound Vorticity Elements Due to Interblade Phase Angle and Stagger Angle Effects. From Whitehead [26].
A wake consisting of a vortex sheet of strength $\varepsilon e^{i\nu t}$ is associated with each element of vorticity on the blade. The vorticity $\varepsilon$ is termed the free vorticity and is being carried downstream from the blades at the relative velocity of the mainstream $W$. The free vorticity due to the element of bound vorticity at $x_1$ on the reference blade is given by

$$\varepsilon e^{i\nu t} = D e^{i\nu(t - x_1/u)}$$  \hspace{1cm} (4.35)

where $D$ is a constant which is determined by the time rate of change of the vorticity at a fixed point on the blade. Solving for the constant and substituting into equation (4.35) gives an expression for the free vorticity.

$$\varepsilon = -\gamma dx \frac{i\nu}{W} e^{i\nu(x - x_1)/W}$$  \hspace{1cm} (4.36)

The total free vorticity at any point on the blade is made up of contributions from all the bound vorticity elements between the lead edge and the point so that an expression for the total free vorticity at a point $(x_1,0)$ can be obtained by integrating equation (4.36) from $x = 0$ to $x = x_1$.

$$\varepsilon = -\frac{i\nu}{W} \int_0^{x_1} \gamma e^{i\nu(x - x_1)/W} dx$$  \hspace{1cm} (4.37)

Removing constant terms from the integral and rearranging gives a relationship between the bound and free vorticity.

$$\varepsilon e^{i\nu x_1/W} = -\frac{i\nu}{W} \int_0^{x_1} \gamma e^{i\nu x/W} dx$$  \hspace{1cm} (4.38)
Differentiating this equation with respect to $x_1$ and rewriting the resulting equation with $x$ as the independent variable gives the following differential equation.

$$
\frac{ds}{dx} + \frac{i\nu}{W} (\gamma + \varepsilon) = 0 \tag{4.39}
$$

where $(\gamma + \varepsilon)$ is the total vorticity on the blade. It should be noted that the bound vorticity is zero in the wake.

The strength of the vorticity is determined by matching the vorticity induced in the flow by the vorticity (the upwash velocity) to the velocity of the vibrating blade. If the blade is only vibrating in a direction perpendicular to the chord with velocity $q$ the flow velocity at the blade surface must be equal to the blade velocity.

$$
\nu_q = q \tag{4.40}
$$

If the blade vibrates in torsion about the lead edge so that the angular displacement is $ae^{iwt}$ the velocity of the blade at the distance $n$ from the lead edge is given by

$$
n \frac{d}{dt}(ae^{iwt}) = n \alpha e^{iwt} \tag{4.41}
$$

The fluid velocity normal to the chord must be

$$
(\nu_\alpha - \alpha W)e^{iwt} \tag{4.42}
$$

Equating (4.42) and (4.41) gives the flow velocity at the blade surface due to torsional motion of the blade.

$$
U_\alpha = \alpha W(1 + i\omega n) \tag{4.43}
$$
This expression is only valid if $\eta$ is a chordal coordinate i.e. $\eta = 0$ at the lead edge and $\eta = 1$ at the trail edge. If the blade is vibrating due to interaction with the wake from an upstream obstruction, it can be shown that the fluid velocity normal to the blade surface is given by the expression

$$v_w = -we^{-i\omega \eta}$$  \hspace{1cm} (4.44)

If all three types of disturbances are considered, the total fluid velocity normal to the chord is simply the sum of these components.

$$v = v_q + v_\alpha + v_w$$

$$v = q + \alpha W(l + i\omega \eta) - we^{-i\omega \eta}$$  \hspace{1cm} (4.45)

The strength of the bound vorticity which would be required to induce this velocity is given by

$$\gamma = q\gamma_q + \alpha W\gamma_\alpha - W\gamma_w$$

$$\gamma = q\gamma_q + \alpha W\gamma_\alpha - W\gamma_w$$  \hspace{1cm} (4.46)

The vorticities $\gamma_q$, $\gamma_\alpha$ and $\gamma_w$ are given by the solution to the following integral equations

$$\int_0^1 \gamma_q K(x-\eta)dx = 1$$

$$\int_0^1 \gamma_\alpha K(x-\eta)dx = 1 + i\omega \eta$$

$$\int_0^1 \gamma_w K(x-\eta)dx = e^{-i\omega \eta}$$  \hspace{1cm} (4.47)
where $K(x-n)$ is a kernel function defined by the expression

$$K(z) = V(-z) - V(-\infty) - i\omega e^{i\omega z} \int_{z}^{\infty} e^{-i\omega z} L(V(z) - V(-\infty)) dz$$

(4.48)

and

$$V(z) = \frac{1}{4}(a+ib) \frac{\exp{(-\pi\beta)(a+ib)}}{\sinh(\pi(a+ib)z)} + \frac{1}{4}(a-ib) \frac{\exp{(\pi\beta)(a-ib)z}}{\sinh(\pi(a-ib)z)}$$

(4.49)

and

$$a = \frac{C_s}{5}\cos\zeta$$

$$b = \frac{C_s}{5}\sin\zeta$$

It is now necessary to derive an expression relating the vorticity to the aerodynamic force and the moment acting on the blade. If second order terms are neglected, the equation of motion in the $x$ direction is

$$\frac{\partial}{\partial t}(wei^{\lambda t}) + W \frac{\partial}{\partial x} (wei^{\lambda t}) = -\frac{1}{\rho} \frac{\partial \rho}{\partial x}$$

(4.50)

For a point slightly below the blade this becomes

$$\frac{\partial}{\partial t}(w_1 e^{\lambda t}) + W \frac{\partial}{\partial x} (w_1 e^{\lambda t}) = -\frac{1}{\rho} \frac{\partial \rho_1}{\partial x}$$

(4.51)

and for a point slightly above the blade

$$\frac{\partial}{\partial t}(w_2 e^{\lambda t}) + W \frac{\partial}{\partial x} (w_2 e^{\lambda t}) = -\frac{1}{\rho} \frac{\partial \rho_2}{\partial x}$$

(4.52)
Subtracting equation (4.52) from (4.51) gives

\[
\rho \frac{\partial}{\partial t} (w_1-w_2) + \rho \frac{\partial}{\partial x} (w_1-w_2)e^{ivt} = -\frac{1}{\rho} \frac{\partial}{\partial x} (p_1 - p_2) \tag{4.53}
\]

The total vorticity on the blade is equal to the difference in the velocities \( u \) so that

\[
(w_1 - w_2) - (\gamma + \varepsilon) \tag{4.54}
\]

and

\[
\left[ \frac{\partial}{\partial t} (\gamma + \varepsilon) + \rho \frac{\partial}{\partial x} (\gamma + \varepsilon) \right] e^{ivt} = -\frac{1}{\rho} \frac{\partial}{\partial x} (p_1 - p_2) \tag{4.55}
\]

simplifying gives

\[
(iu(\gamma + \varepsilon) + \frac{d\gamma}{dx} + U \frac{d\varepsilon}{dx})e^{ivt} = -\frac{1}{\rho} \frac{\partial}{\partial x} (p_1 - p_2) \tag{4.56}
\]

Substituting equation (4.139) into the above gives

\[
\rho \frac{d\gamma}{dx} e^{ivt} = -\frac{1}{\rho} \frac{\partial}{\partial x} (p_1 - p_2) \tag{4.57}
\]

Integrating the above yields

\[
p_1 - p_2 = -\rho \gamma e^{ivt} \tag{4.58}
\]

The lift force on the blade \( Fe^{ivt} \) can be obtained by integrating the pressure difference over the chord length,
The moment acting on the blade is given by

\[ M = -cW \int_0^c y \, dx \] (4.61)

Substituting equation (4.46) into equation (4.60) expressions gives

\[ F = -cW \int_0^c \left( q_Y q + \alpha W Y_{\alpha} - W_Y \right) \, dx \] (4.62)

\[ M = -cW \int_0^c \left( q_Y q + \alpha W Y_{\alpha} - W_Y \right) \, dx \]

The equation for the lift force may be written in terms of lift coeffi

\[ F = \pi b W C (q C_{Fq} + \alpha W C_{F\alpha} - W C_{Fw}) \] (4.63)

Making the substitution

\[ x = \frac{1}{2}(1 - \cos \theta) \] (4.64)

the coefficients can be written as
Similar expressions can be derived for the moment coefficients. It is obvious that these coefficients are best determined by a matrix procedure. First it is necessary to derive an expression for the vorticities of equations (4.47). These equations may be written in matrix form as

\[ [A][\alpha] = [B] \]  

(4.66)

where the A matrix is a nxn 'Kernel function' matrix whose \( \lambda \)th column and \( m \)th row is given by

\[ K(\frac{\pi}{2} \cos \frac{\pi(2m+1)}{2n} - \frac{1}{2} \cos \frac{\pi \lambda}{n}) \]  

(4.67)

\([\alpha] \) is a nx3 matrix in which the \( \lambda \)th row is

\[ \gamma_q \cdot \frac{\pi}{2n} \sin \frac{\pi \lambda}{n}, \quad \gamma_\alpha \cdot \frac{\pi}{2n} \sin \frac{\pi \lambda}{n}, \quad \gamma_w \cdot \frac{\pi}{2n} \sin \frac{\pi \lambda}{n} \]  

(4.68)

except for the first row which is multiplied by a factor of one half, and

\([B] \) is a nx3 matrix in which the \( m \)th row is given by

\[ 1, \quad \{1 + \frac{i\omega}{2} (1-\cos \frac{\pi(2m+1)}{2n})\}, \quad \exp\{-i\omega[1-\cos \pi(2m+1)/2n]/2\} \]
In the above, \( n \) is an integer number which determines the order of approximation of the solution. Whitehead achieved sufficient accuracy for \( 3 \leq n \leq 5 \). Equation (4.67) may be solved by standard matrix procedure

\[
[\Gamma] = [A]^{-1}[B]
\] (4.68)

The equations for the force and moment coefficients can be written in form as

\[
[C] = \frac{1}{\pi} [X][\Gamma]
\] (4.70)

where

\[
[C] = \begin{bmatrix}
  c_{Fq} & c_{Fo} & c_{Fw} \\
  c_{Mq} & c_{Mo} & c_{Mw}
\end{bmatrix}
\] (4.71)

and \([X]\) is a 2x\( n \) matrix in which the \( i^{th} \) column is given by

\[
\begin{bmatrix}
  1 \\
  \frac{1}{2}(1 - \cos \frac{\pi \phi}{n})
\end{bmatrix}
\] (4.72)

Substituting equation (4.69) into (4.70) yields

\[
[C] = \frac{1}{\pi} [X][A]^{-1}[B]
\] (4.73)

This matrix equation may be used to obtain the six force and moment coefficients for the cascade. Whitehead [28] and Smith [32] developed an
which extend the above analysis to include the effects of generated acoustic waves for cascades of zero near deflection. If the elements in the matrices which deal with acoustic effects are neglected, both analyses reduce to the analysis above.

Whitehead [29] also developed an analysis for the case of finite angle of incidence of the mean flow relative to the blade, i.e. for the case of finite mean deflection of the blades. The basic steps in the analyses are the same as for the analyses above, except that velocity perturbations perpendicular to the chord (assumed to be zero previously) are considered. The result of this analysis is a matrix equation of the form:

\[ \eta[C_F] = [\bar{A}_1] + \tau[\bar{A}_2] + \tau^2[\bar{A}_3] \]  \hspace{1cm} (4.74)

where

\[ [\bar{A}_1] = -(D)^T[A]^{-1}(D) \]

\[ [\bar{A}_2] = ((D)^T[A_0]^{-1}[B_0]^T - (D)^T[I]\tan\xi)[H] + [I]A^{-1}(D) \]

\[ + ((D)^T[A]^{-1}[B]^T - (D)^T[A]^{-1}[Q])[A_0]^{-1}(D) \]

\[ [\bar{A}_3] = ((D)^T[A_0]^{-1}[B_0]^T - (D)^T[I]\tan\xi)[H] + [I]A^{-1}[Q][A_0]^{-1}(D) \]

\[ + (D)^T[A_0]^{-1}([B][A]^{-1}[Q] - [P])[A_0]^{-1}(D) \]

The matrices \([A], [A_0], [B], [B_0], [D], [H], [I], [P] and [Q] are defined in table 2. The variable \(\tau\) specifies the incidence of the flow and is given by

\[ \tau = \frac{\tan\alpha_1 - \tan\xi}{\sec^2\xi} \]  \hspace{1cm} (4.75)

and the superscript \(T\) denotes the matrix transpose.
TABLE 2. Matrices Used in Actuator Disk Analysis for Finite Mean Blade Deflection. [29].

<table>
<thead>
<tr>
<th>Matrix</th>
<th>Order</th>
<th>Element in ((m + 1))th row and ((\ell + 1))th column</th>
</tr>
</thead>
<tbody>
<tr>
<td>([A]_2)</td>
<td>(n \times n)</td>
<td>(K(z) + i\omega e^{-i\omega z} \left[ \frac{1}{2 \pi} \ln 2 + \frac{1}{\pi} \sum_{r=1}^{n} \frac{1}{r} \cos \frac{\pi r (2m + 1)}{2n} \cos \frac{\pi \ell}{n} - \frac{1}{2\pi} \log</td>
</tr>
<tr>
<td>([A_0]_2)</td>
<td>(n \times n)</td>
<td>(V_0(z) - V_0(-\omega))</td>
</tr>
<tr>
<td>([B])</td>
<td>(n \times n)</td>
<td>(J \left( \frac{1}{2} \cos \frac{\pi \ell}{n} - \frac{1}{2} \cos \frac{\pi m}{n} \right))</td>
</tr>
<tr>
<td>([B_0]_2)</td>
<td>(n \times n)</td>
<td>(U_0 \left( \frac{1}{2} \cos \frac{\pi \ell}{n} - \frac{1}{2} \cos \frac{\pi m}{n} \right) - U_0(-\omega))</td>
</tr>
<tr>
<td>([D])</td>
<td>(n \times 1)</td>
<td>(1.0)</td>
</tr>
<tr>
<td>([H])</td>
<td>(n \times n)</td>
<td>(- \frac{i\omega}{2n} \left[ \frac{\pi m}{n} + 2 \sum_{r=1}^{n-1} \frac{1}{r} \sin \frac{\pi rm}{n} \cos \frac{\pi \ell}{n} \right] \times \left[ \exp \left{ \frac{\omega}{2} \left( \cos \frac{\pi m}{n} - \cos \frac{\pi \ell}{n} \right) \right} \right] \times \sin \frac{\pi m}{n} )</td>
</tr>
<tr>
<td>([I])</td>
<td>(n \times n)</td>
<td>(1.0) (Identity Matrix)</td>
</tr>
<tr>
<td>([P]_2)</td>
<td>(n \times n)</td>
<td>(M \left( \frac{1}{2} \cos \frac{\pi \ell}{n} - \frac{1}{2} \cos \frac{\ell m}{n} \right))</td>
</tr>
<tr>
<td>([Q]_2)</td>
<td>(n \times n)</td>
<td>(N(z))</td>
</tr>
</tbody>
</table>

\(^1\) \(0 \leq \ell \leq n - 1; \ 0 \leq m \leq n - 1; \) and \(z = \frac{1}{2} \left\{ \cos \frac{\pi \ell}{n} - \cos \frac{\pi (2m + 1)}{2n} \right\}\)

\(^2\) The functions \(K(z), J(z), M(z), N(z), U_0(z), \) and \(V_0(z)\) are defined in the text.
Six special functions must be defined in order to obtain the terms in these matrices. The functions $K(z)$ and $J(z)$ are used to obtain the induced velocities parallel and perpendicular to the chord and are very similar to the kernel function in reference [27]. These functions are given by the expressions

$$K(z) = V(z) + i\omega e^{-i\omega z} \left[ \int \frac{1}{z} (e^{i\omega z} V(z_1) - \frac{1}{2\pi z_1}) dz_1 \right] - \frac{1}{2\pi} \log|z| + \frac{1}{2} (a+ib) \left\{ \sum_{r=0}^{\infty} \frac{\exp[-(2\pi r + \beta)(a+ib) - i\omega]}{(2\pi r + \beta)(a+ib) + i\omega} \right\} - \frac{1}{2} (a-ib) \left\{ \sum_{r=1}^{\infty} \frac{\exp[-(2\pi r - \beta)(a-ib) - i\omega]}{(2\pi r - \beta)(a-ib) + i\omega} \right\}$$

$$J(z) = U(z) + i\omega e^{-i\omega z} \left[ \int \frac{1}{z} (e^{i\omega z} U(z_1) dz_1 \right] - \frac{1}{2} (a-ib) \left\{ \sum_{r=0}^{\infty} \frac{\exp[-(2\pi r + \beta)(a+ib) - i\omega]}{(2\pi r + \beta)(a+ib) + i\omega} \right\} + \frac{1}{2} (a-ib) \left\{ \sum_{r=1}^{\infty} \frac{\exp[-(2\pi r - \beta)(a-ib) - i\omega]}{(2\pi r - \beta)(a-ib) + i\omega} \right\}$$

where

$$a = \frac{c}{s} \cdot \cos \xi \quad \quad b = \frac{c}{s} \cdot \sin \xi$$

The integrals in these expressions can be evaluated using numerical integration methods. The infinite series converge rapidly to very small values and can be truncated after four or five terms. The functions $U(z)$ and $V(z)$ which determine the induced velocities due to the unsteady vorticity are also required.
\[ V(z) = \frac{1}{4\pi i} \cdot \left( \frac{c}{s} \right) \left( e^{-2i\xi f(x)} - e^{i f(x)} - e^{i \xi f(\bar{x})} \right) \]  
(4.78)

\[ U(z) = \frac{1}{4\pi} \cdot \left( \frac{c}{s} \right) \left( e^{-2i\xi f(x)} + e^{i \xi f(\bar{x})} \right) \]

where

\[ x = -\frac{c}{s} \cdot \text{ize}^{-i\xi} \]  
(4.79)

\[ \bar{x} = \frac{c}{s} \cdot \text{ize}^{i\xi} \]

and

\[ f(x) = \begin{cases} 
\pi e^{i(\pi-\beta)x} \csc \pi x, & 0 < \beta < 2\pi \\
\pi \cot \pi x, & \beta = 0 
\end{cases} \]

The functions \( U_0(z) \) and \( V_0(z) \) are the functions given in equations (4) when \( \beta = 0 \). The functions \( M(z) \) and \( N(z) \) give the velocities induced by mean displacement of the blade and its associated steady vorticity.

\[ M(z) = \frac{i}{4\pi \omega} \cdot \left( \frac{c}{s} \right)^2 \left( e^{-2i\xi h(x)} + e^{2i\xi h(\bar{x})} \right) \]  
(4.80)

\[ N(z) = \frac{1}{4\pi \omega} \cdot \left( \frac{c}{s} \right)^2 \left( e^{-2i\xi h(x)} + e^{2i\xi h(\bar{x})} \right) \]

where \( x \) and \( \bar{x} \) are given in equations (4.79) and

\[ h(x) = \frac{\pi}{2} \csc^2 \pi x \left[ (1 - \frac{3}{2\pi}) e^{-i\beta x} + \frac{3}{2\pi} e^{i(2\pi - \beta x)} - 1 \right] \]  
(4.81)
4.4 Henderson-Horlock Analysis [33]

All of the analyses discussed above consider the blades as flat plates. The present analysis applies an averaging technique with respect to the pitch in order to obtain the unsteady lift on highly cambered airfoils. Henderson and Horlock apply the following assumptions.

i) The pitch (spacing between adjacent blades) is small compared to the disturbance wavelength. It will be shown in section 4.6 that this is equivalent to requiring the interblade phase angle $\beta$ to be small.

ii) The blades are thin and highly cambered but the lift coefficient is small. This is required because of the low pitch.

iii) The flow is two dimensional, incompressible and inviscid.

iv) The blades are not stalled so that at every point on the blade the direction of the fluid flow is controlled by the blade surface.

The general procedure is to apply the pitch averaging technique of Horlock and Marsh [34] to the equations of momentum and continuity for the flow in the blade channel. The resulting expressions will then be written in terms of a pressure difference across the blade which is integrated over the length of the chord to give the lift on the blade. For the blade coordinate system shown in figure 13 the equations of motion with respect to the $x,y$ coordinate system moving with the blades may be written as

\[
\begin{align*}
- \frac{1}{\rho} \frac{\partial p}{\partial x} &= \frac{\partial u}{\partial t} + \frac{\partial (u^2)}{\partial x} + \frac{\partial (uv)}{\partial y} \\
- \frac{1}{\rho} \frac{\partial p}{\partial y} &= \frac{\partial v}{\partial t} + \frac{\partial (uv)}{\partial x} + \frac{\partial (v^2)}{\partial y} \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0
\end{align*}
\]
where \( u \) is the axial velocity in the blade channel and \( v \) is the tangential velocity. These velocities are related at the blade surface by the condition that the flow always follows the blade surface

\[
v = u \tan \theta \tag{4.8}
\]

where \( \theta \) is the angle of the blade surface relative to the axial direction as shown in figure 13. The cross products of the velocity components can be obtained from equation (4.83).

\[
u v = u^2 \tan \theta
\]

\[
u v \tan \theta = v^2
\]

Using these equations and the pitch averaging technique of [34] all equations (4.82) to be rewritten for the averaged flow in the blade

\[
\frac{1}{\rho g} \frac{\partial (\rho \bar{v})}{\partial x} + \frac{(p_s \tan \theta_s - p_p \tan \theta_p)}{\rho g} = \frac{\partial \bar{u}}{\partial t} + \frac{1}{g} \frac{\partial (g \bar{u}^2)}{\partial x}
\]

\[
\frac{(p_s - p_p)}{\rho g} = \frac{\partial \bar{v}}{\partial t} + \frac{1}{g} \frac{\partial (g \bar{u} v)}{\partial x}
\]

\[
\frac{\partial (g \bar{u} v)}{\partial x} = 0
\]

where the subscripts \( s \) and \( p \) refer to the suction and pressure faces of the blades respectively, \( g \) is the width of the blade passage, and the quantities are average quantities across the passage. These expressions are completely general and, as yet, no assumptions have been included regarding the blade geometry. Considerable simplification of the equations results when it is assumed that the blades are thin so that
\[ \theta_s = \theta_p = \theta \]

and

\[ g = s \]

Since \( g \) is no longer a function of axial location \( (x) \) equations (4.85) become

\[
- \frac{l}{p} \frac{\partial p}{\partial x} + \frac{(p_s - p_p) \tan \theta}{\rho s} = \frac{\partial u}{\partial t} + \frac{\partial (\bar{u}^2)}{\partial x}
\]

(4.87)

\[
- \frac{(p_s - p_p)}{\rho s} = \frac{\partial u}{\partial t} + \frac{\partial (uv)}{\partial x}
\]

\[
\frac{\partial u}{\partial x} = 0
\]

Following the same logic as that used to arrive at equation (4.83) an expression relating the pitch averaged velocity components can be deduced.

\[
\bar{v} = \bar{u} \tan \bar{\alpha}
\]

\[
\bar{uv} = \bar{u}^2 \tan \bar{\alpha}
\]

(4.88)

where \( \bar{\alpha} \) is the mean flow angle in the channel as shown in figure 13. Using equations (4.88) and neglecting cross products of pitch averaged terms (valid if \( s/2 \ll 1 \)) equations (4.87) become

\[
- \frac{l}{p} \frac{\partial p}{\partial x} + \frac{(p_s - p_p) \tan \theta}{\rho s} = \frac{\partial u}{\partial t}
\]

(4.89)

\[
- \frac{(p_s - p_p)}{\rho s} = \tan \bar{\alpha} \frac{\partial u}{\partial t} + \bar{u}^2 \frac{\partial}{\partial x} (\tan \bar{\alpha})
\]
where the continuity equation has also been used.

The lift forces acting on the blade in the axial and tangential directions may be written in terms of the integral of the pressure difference across the blade.

\[ L_x = \int_0^c (p_p - p_s) \tan \theta \, dx \]

\[ L_y = \int_0^c (p_p - p_s) \, dx \]  \hspace{1cm} (4.90)

This pressure difference will be calculated using equations (4.89). Assuming that a linear variation of pressure exists across the blade channels\(^1\) on either side of the reference blade the pitch averaged pressures can be written as

\[ \bar{p}_1 = \frac{p_{s1} + p_{p1}}{2} \hspace{1cm} \bar{p}_2 = \frac{p_{s2} + p_{p2}}{2} \]  \hspace{1cm} (4.91)

where the subscripts 1 and 2 denote the channel across which the average is taken. The pressure difference across the reference blade is there given by

\[ p_{p1} - p_{s2} = \bar{p}_1 - \bar{p}_2 - (\frac{p_{s1} - p_{p1}}{2}) - (\frac{p_{s2} - p_{p2}}{2}) \]  \hspace{1cm} (4.92)

\(^1\) Henderson and Horlock also derived lift expressions assuming linearwise velocity variation. The results showed good agreement with the pressure case. It should be noted that any variation of pressure or city may be assumed although no studies have been made regarding the effect of non-linear variations.
Using equations (4.89) and (4.92) the time dependant pressure difference across the blade at an axial location $x_1$ is found to be

$$
\Delta p(x_1, t) = \rho \left[ \left( \frac{\partial \bar{u}}{\partial t} \right)_{x=0} (x_1 + \int_0^{x_1} \tan \theta \tan \bar{\alpha} dx) \right] \\
+ \rho \left[ \left( \bar{u}^2 \right)_{x=0} \int_0^{x_1} \tan \theta \frac{d}{dx} (\tan \bar{\alpha}) dx \right] \\
+ \frac{\rho s}{2} \left[ \left( \frac{\partial \bar{u}}{\partial t} \right)_{x=0} \tan \bar{\alpha}(x_1) \right] + \left( \bar{u}^2 \right)_{x=0} \frac{d}{dx} (\tan \bar{\alpha}) \right] 
$$

(4.93)

Assuming the inlet velocity consists of a steady component $U$ and an unsteady component $u_0 \sin \nu(t - y/V)$ the pitch averaged axial velocity in the two channels can be expressed as

$$
\bar{u}_1 = U + \frac{u_0}{\lambda} \left[ \sin \lambda \sin \nu t - (\cos \lambda - 1) \cos \nu t \right]
$$

(4.94)

$$
\bar{u}_2 = U + \frac{u_0}{\lambda} \left[ \sin \lambda \sin \nu t + (\cos \lambda - 1) \cos \nu t \right]
$$

(4.95)

where $\lambda$ is the reduced frequency parameter based on pitch

$$
\lambda = \frac{2 \pi s}{1}
$$

The mean flow angle $\bar{\alpha}$ will be assumed to vary in the same manner as the blade surface angle $\theta$ so that
\[
\tan \alpha = \tan \theta = \tan \theta_{LE} + \frac{x}{c'} (\tan \theta_{TE} - \tan \theta_{LE}) \tag{4.96}
\]

where \( c' \) is the width of the cascade in the axial direction and \( \theta_{LE} \) and \( \theta_{TE} \) are the angles of the blade surface relative to the axial direction at the lead edge and trail edge of the blade. The steady and tangential forces respectively are obtained by evaluating the independant terms of equation (4.93) to yield

\[
L_{xs} = \frac{\rho U^2}{2} (\tan^2 \theta_{TE} - \tan^2 \theta_{LE}) \tag{4.97}
\]

\[
L_{ys} = - \rho s U^2 (\tan \theta_{TE} - \tan \theta_{LE})
\]

The unsteady axial and tangential forces are obtained by integrating every term in equation (4.93) and subtracting the steady forces of equations (4.97) from the resulting total force equations to get

\[
\frac{\bar{L}_x}{\rho c V u_o} = - \left[ \frac{2 c h_5}{s \sec^2 \xi} (1-\cos \lambda) - \frac{s U h_8}{c V \lambda} \sin \lambda - \frac{s}{c \lambda} \tan \theta_{LE} \sin \lambda \right] \sin
\]

\[
- \left[ \frac{4 U h_6}{\lambda V \sec^2 \xi} (1-\cos \lambda) + \frac{h_7}{\sec \xi} \sin \lambda \right] \cos \omega t \tag{4.98}
\]

\[
\frac{\bar{L}_y}{\rho c V u_o} = \left[ \frac{2 c h_2}{s \sec^2 \xi} (1-\cos \lambda) - \frac{s}{\lambda c} \sin \lambda - \frac{2 U s h_1}{\lambda V c} \sin \lambda \right] \sin \omega t
\]

\[
+ \left[ \frac{4 U h_3}{\lambda V \sec^2 \xi} (1-\cos \lambda) + \frac{h_4}{\sec \xi} \sin \lambda \right] \cos \omega t
\]
where $\xi$ is the stagger angle of the blades, $V$ is the blade tangential velocity and the functions $h_1$ through $h_8$ are given by

\begin{align*}
h_1 &= \tan \theta_{TE} - \tan \theta_{LE} \\
h_2 &= \frac{\sec^2 \theta_{LE}}{2} + \frac{h_1 \tan \theta_{LE}}{3} + \frac{(h_1)^2}{12} \\
h_3 &= \frac{h_1 \tan \theta_{LE}}{2} + \frac{(h_1)^2}{6} \\
h_4 &= \tan \theta_{LE} + \frac{h_1}{2} \\
h_5 &= \frac{\tan^3 \theta_{LE}}{2} + \frac{h_1}{3} + \frac{\tan \theta_{TE}}{2} + \frac{2h_1 \tan^2 \theta_{LE}}{3} + \frac{\tan \theta_{LE}(h_1)^2}{3} + \frac{(h_1)^3}{15} \\
h_6 &= \frac{h_1 \tan \theta_{LE}}{2} + \frac{h_1^2 \tan \theta_{LE}}{2} + \frac{h_1^3}{8} \\
h_7 &= \tan^2 \theta_{LE} + h_1 \tan \theta_{LE} + \frac{(h_1)^2}{3} \\
h_8 &= \tan^2 \theta_{TE} - \tan^2 \theta_{LE}
\end{align*}

If the total unsteady lift force on the blade is desired, the equations (4.98) may be combined using the stagger angle as follows

\[ \frac{\Gamma}{\rho c V u_0} = \frac{L_x \sin \xi - L_y \cos \xi}{\rho c V u_0} \]  

(4.100)

for a turbine cascade or

\[ \frac{\Gamma}{\rho c V u_0} = \frac{L_y \cos \xi - L_x \sin \xi}{\rho c V u_0} \]  

(4.101)
for a compressor cascade. The authors also give an expression for the unsteady lift for the special case of a compressor cascade of flat plate airfoils at a stagger angle of 45 degrees and at zero mean incidence to a convecting gust of the form:

\[ u = U + u_0 e^{i \nu (t-y/V)} \]  (4.102)

An extension of this analysis to the generalized case of turbine compressor cascades of cambered airfoils at any stagger angle and mean incidence made by the author is reported in reference [94]. For a turbine cascade, the unsteady lift is

\[
\frac{\overline{L}}{\rho c V U_0} = \left\{ -\frac{2c}{s} \cdot \frac{(1-\cos \lambda)}{\sec^2 \xi} \left[ h_2 \cos \xi + h_5 \sin \xi \right] \\
+ \frac{s}{c} \cdot \frac{U}{V} \cdot \frac{\sin \lambda}{\lambda} \left[ 2h_1 \cos \xi + h_8 \sin \xi \right] \\
+ \frac{s}{c} \cdot \frac{\sin \lambda}{\lambda} \left[ \cos \xi + \tan \xi \tan \lambda \sin \xi \right] \\
- i \left\{ \frac{4U(1-\cos \lambda)}{\sec \xi} \left[ h_3 \cos \xi + h_6 \sin \xi \right] \\
+ \frac{\sin \lambda}{\sec \xi} \left[ h_4 \cos \xi + h_7 \sin \xi \right] \right\} 
\]  (4.103)

and for a compressor cascade

\[
\frac{\overline{L}}{\rho c V U_0} = \left\{ \frac{2c}{s} \cdot \frac{(1-\cos \lambda)}{\sec^2 \xi} \left[ h_2 \cos \xi + h_5 \sin \xi \right] \\
- \frac{s}{c} \cdot \frac{U}{V} \cdot \frac{\sin \lambda}{\lambda} \left[ 2h_1 \cos \xi + h_8 \sin \xi \right] \\
- \frac{s}{c} \cdot \frac{\sin \lambda}{\lambda} \left[ \cos \xi + \tan \xi \tan \lambda \sin \xi \right] \\
+ i \left\{ \frac{4U(1-\cos \lambda)}{\sec \xi} \left[ h_3 \cos \xi + h_6 \sin \xi \right] \\
+ \frac{\sin \lambda}{\sec \xi} \left[ h_4 \cos \xi + h_7 \sin \xi \right] \right\} 
\]  (4.104)
It is evident, however, that it is also possible to determine the ratio of unsteady forces to steady forces in both the axial and tangential directions using equations (4.97) and (4.98) or (4.103) and (4.104). Sample calculations and results obtained using this method are discussed in section 8.3.

4.5 Horlock-Grietzer-Henderson Analysis[35]

The stated purpose of this analysis is the resolution of the inconsistent results obtained by various authors for low values of reduced frequency parameter as discussed in section 4.1. To accomplish this, the authors employ two methods. The first method may be described as a combination of the analyses of Whitehead [26] and Henderson-Horlock [33] in that it shares major assumptions with both analyses and, in fact, follows the analytical procedure of Henderson-Horlock quite closely. It is this method which will be presented here. The second method used was developed by Rannie and Marble [49]. The results obtained by the two methods are identical. The following assumptions are applied to the cascade shown in figure 28.

i) The flow is inviscid, incompressible and two dimensional

ii) The cascade consists of a set of flat plates at stagger angle \( \xi \) with finite chord and low pitch.

iii) The velocity within the blade passage is constant along the passage.

iv) The flow enters the cascade at zero mean incidence, i.e. the mean blade deflection is zero.

v) The streamlines within the blade passage are straight and parallel to the blade.

vi) There exists three distinct regions in the flow field: a) upstream of the blade row; b) within the blade row, and c) downstream of the blade row. At the interfaces of these regions (the lead and trail edges of the blade row) certain
Figure 28. Cascade and Flow Notation for Semi-Actuator Disk Analysis. From Horlock, Greitzer and Henderson [33].
boundary conditions must exist which match the potential flow in one region with that in the adjacent region.

The boundary conditions to be applied are as follows:

- far upstream \((x = -\infty)\): a disturbance in the axial velocity which varies harmonically in the \(y\) direction exists far upstream of the blade row.
- blade lead edge \((x = 0)\): the stream function is continuous but its slope is discontinuous due to the discontinuity in the tangential velocity. The relative stagnation pressure must also be continuous.
- blade trail edge \((x = x_c = c \cos \xi)\): the stream function and stagnation pressure are continuous.
- far downstream \((x = \infty)\): there are flow variations in the \(y\) direction.

For the far upstream region, the stream function is assumed to be of the following form

\[
\psi_1(x) = [Ee^{-i\kappa y} + Fe^{\kappa(x-i\nu y)}]e^{i\nu t}
\]

\((4.105)\)

where \(E\) and \(F\) are complex coefficients which will be determined later, \(\nu\) is the excitation frequency and \(\kappa\) is a dimensionless disturbance wavelength parameter given by

\[
\kappa = \frac{2\pi}{L}
\]

\((4.106)\)
where \( \lambda \) is the disturbance wavelength. This function matches the upstream boundary condition since at \( x = -\infty \)

\[
\psi_1(-\infty) = E e^{i(\nu t - \kappa y)} \quad (4.107)
\]

At the blade leading edge \( (x=0) \)

\[
\psi_1(0) = (E+F) e^{i(\nu t - \kappa y)} \quad (4.108)
\]

the stream function within the blade row is assumed to be

\[
\psi_2(x) = P e^{i\nu t} e^{-i\eta \sec \xi} \quad (4.109)
\]

where \( P \) is a complex coefficient and \( \eta \) is the coordinate direction perpendicular to the blade as shown in figure 28. The reference blade is represented by a continuous body force distribution \( f \) which matches that of the fluid at the lead edge. In the downstream region, the stream function is

\[
\psi_3(x) = \left[ Ge^{-i\kappa y} + He^{-\kappa(x+iy)} \right] e^{i\nu t} \quad (4.111)
\]
By matching the upstream stream function with the stream function in the blade row at the leading edge of the reference blade the following expression relating the flow coefficients results

\[ E + F = P \] \hspace{1cm} (4.112)

Similarly, by matching the stagnation pressure gradient at the lead edge to the gradient at a point just downstream of the lead edge \((x=s)\) and taking the limit at \(s \to 0\) yields

\[-UF^{2} \cdot \sin F = [PK^{2} \tan^{2} \xi + PK \cot \xi + B \cos \xi] \] \hspace{1cm} (4.113)

Matching the stream function in the blade row to that downstream of the blade row at the trailing edge of the reference blade yields

\[ Pe^{2i\sigma} = G + He^{-2\sigma \cot \xi} \] \hspace{1cm} (4.114)

where \(\sigma\) is the reduced frequency parameter based on the blade semi-chord

\[ \sigma = \frac{\omega c}{2W} \] \hspace{1cm} (4.115)

and \(W\) is the mean velocity relative to the blade.

Matching the tangential velocity at the trailing edge yeilds

\[ iPe^{2i\sigma} = -He^{-2\sigma \cot \xi} \] \hspace{1cm} (4.116)
The stagnation pressure must be matched at the trail edge so that

\[ U_k^2 p e^{2i\sigma \cot \xi} \tan \xi + P \omega e^{2i\sigma} + B e^{2i\sigma \cos \xi} - \kappa^2 \sec^2 \xi \sin Pe^{2i\sigma \cot \xi} \cos \xi = i \kappa \nu H e^{-2i\sigma \cot \xi} - U_k^2 H e^{-2i\sigma \cot \xi} \]  

(4.117)

Equations (4.112) through (4.117) can be used to solve for the coefficients \(B, F\) and \(P\) with respect to the amplitude of the upstream disturbance \(E\).

\[
\begin{align*}
\frac{P}{E} &= \frac{1}{\beta} \{t[3t^2 + 2\sigma t + 1] - i[2\sigma - 2t^3 + t + 1]\} \\
\frac{B}{E} &= -\frac{\nu}{s \cos \phi} \{-2[3t^2 + \sigma t(2\sigma t + 1) + 1 - \beta/2] + i[\beta(t+1) - 3t^3 + \delta(1+2\sigma)]\} \\
\frac{F}{E} &= \frac{1}{\beta} \{[3t^2 + t(2\sigma t + 1) - \beta] + i[2t^3 - \alpha]\} \\
\end{align*}
\]

(4.118)

where

\[ t = \tan \theta \]
\[ \delta = \sec^2 \theta \]
\[ \beta = -\frac{2\pi s}{\lambda} \]
\[ \alpha = 2\sigma t + t + 1 \]
The loading on the lead edge \( (L_a) \) can be expressed in terms of the \( P \) and \( F \) coefficients

\[
L_a = \frac{2\pi U_s}{\lambda \cos \theta} (F - iP) e^{i(vt - 2\pi y/\lambda)} \quad (4.119)
\]

By integrating the body force over the pitch \( s \) and over the chord \( c \), the lift on the remainder of the blade \( L_b \) is obtained:

\[
L_b = \rho s \cos \theta [Bc - \frac{i4Uc^2\pi^3}{\lambda^3 \cos^2 \theta} P] e^{i\nu t} \quad (4.120)
\]

The total unsteady lift acting on any blade can be given in terms of the sum of equations (4.116) and (4.117)

\[
C_L = \frac{L_a + L_b}{\rho c \pi \nu u e^{i\nu t}}
\]

\[
C_L = -\frac{1}{\pi c} \left[ \frac{P}{E} \left[ 1 - \frac{2\sigma^2}{\sin \theta \cos \theta} \right] - i \left[ \frac{B}{E} \left( \frac{\sigma^2 \cos^2 \theta}{2U^2 \sin \theta} - F \right) \right] \right] \quad (4.121)
\]

Sample calculations and results obtained using this method are discussed in section 8.2

4.6 Reduced Frequency Based on Blade Semi-Chord

Horlock, Greitzer and Henderson [35] gave a semi-actuator disk analysis for cascades of low pitch to chord ratios. The reduced frequency parameter \( \sigma \) is defined as:

\[
\sigma = \frac{c}{2} \cdot \frac{\nu}{\bar{U}} \quad (4.122)
\]
where $c$ is the blade semi-chord, $v$ is the excitation frequency and $W$ is the flow velocity relative to the leading edge of the blade. The disturbance frequency is defined as

$$\nu = \frac{2\pi v}{\lambda}$$  \hspace{1cm} (4.123)

where $V$ is the blade speed and $\lambda$ is the circumferential wavelength of the disturbance. Substituting equation (4.23) into equation (4.122):

$$\sigma = \frac{\pi c V}{\lambda W}$$  \hspace{1cm} (4.124)

Rearranging terms allows equation (4.124) to be written as:

$$\sigma = \frac{\pi s}{\lambda} \cdot \frac{V}{W} \cdot \frac{c}{s}$$  \hspace{1cm} (4.125)

where $s$ is the blade pitch. The authors define the interblade phase angle $\beta$ (the phase angle between the lift fluctuations of adjacent blades) as

$$\beta = -\frac{2\pi s}{\lambda}$$  \hspace{1cm} (4.126)

so that

$$\sigma = \frac{|\beta|}{2} \cdot \frac{V}{W} \cdot \frac{c}{s}$$  \hspace{1cm} (4.127)

The reduced frequency parameter based on the semi-chord is given by the product of the absolute value of half the interblade phase angle, the relative velocity ratio and the inverse of the pitch to chord ratio.
4.6.1 Reduced Frequency Based on Blade Chord

Whitehead [26][27][28][29][30][31] has developed actuator disk theories using a reduced frequency parameter based on the blade chord:

\[ \omega = \frac{\nu C}{W} \quad (4.128) \]

This reduced frequency is related to the reduced frequency of Horlock, Greitzer and Henderson [35] by the expression

\[ \omega = 2\sigma \quad (4.129) \]

so that

\[ \omega = |\beta| \cdot \frac{V}{W} \cdot \frac{C_s}{s} \quad (4.130) \]

4.6.2 Reduced Frequency Based on Blade Pitch

Henderson and Horlock [33] presented an analysis for the case when the blade pitch was much smaller than the wavelength of the disturbance. The reduced frequency parameter based on pitch was defined as:

\[ \lambda = \frac{2\pi s}{\lambda_s} \quad (4.131) \]

Comparing this equation to equation (4.123) it is obvious that the reduced frequency parameter based on pitch and the interblade phase angle are related by the following expression.

\[ \lambda = -\beta \quad (4.132) \]
The authors also used the reduced frequency based on the blade semi-chord as defined in equation (4.122).

4.6.3 **Comments on the Interblade Phase Angle**

In all of the above analyses the interblade phase angle $\beta$ was assumed to be small. The implication of this assumption is that the circumferential wavelength of the disturbance is large compared to the blade pitch. It should be noted that the interblade phase angle is a flow related quantity giving the phase relationship of the excitation of adjacent blades. The interblade phase angle represents a blade response related quantity if and only if every blade responds with an equal, constant phase relative to the excitation.

4.7 **Notation**

- $c$ - blade chord
- $C_F$ - lift coefficient
- $p$ - pressure
- $s$ - blade pitch
- $u$ - nonsteady axial velocity
- $U$ - steady axial velocity
- $v$ - nonsteady tangential velocity
- $V$ - steady tangential velocity
- $V_s$ - disturbance propagation velocity
- $w$ - nonsteady relative velocity
- $W$ - steady relative velocity
- $x,y$ - coordinate directions
- $\beta$ - interblade phase angle
- $\gamma$ - vorticity
- $\Gamma$ - circulation
- $\theta$ - blade surface angle
- $\lambda$ - reduced frequency based on pitch
- $\nu$ - disturbance frequency
- $\xi$ - stagger angle
- $\rho$ - density
- $\sigma$ - reduced frequency based on semichord
- $\omega$ - reduced frequency based on chord
5. Field Theories

5.1 General

Field theories are based on the definition of the flow field in terms of velocity (or pressure) potential functions. The solution of the potential equations may take several forms. Osborne [4][50] and Mani [51] closely follow the Kemp-Sears analysis for subsonic flow, with an extension to include compressibility effects. Pigott and Yeh [52] applied the time marching technique and the finite area approximation method to numerically solve the equations of motion of the flow through a blade channel for the unsteady aerodynamic pressure and forces on a blade. The forces obtained are then used to determine the flutter characteristics of the blade. Ni and Sisto [53] applied the time marching technique to compressible subsonic and supersonic flow through a cascade of flat plate airfoils. Good agreement with results obtained from Smith's [32] analysis was obtained. Warner [38] used a finite element approach to analyze the two-dimensional, inviscid, incompressible, steady potential flow around an interactive cascade of cylinders. Steele [39] extended Warner's analyses to a higher order element for Stokes flow around a cascade of cylinders and performed a preliminary analysis for flow through a cascade of airfoils. Gostelow [8] reviewed existing field theories for subsonic compressible flow through cascades.

5.2 Subsonic Compressible Interactions Between Blade Rows

Osborne [4] developed expressions for the unsteady force on an isolated two-dimensional airfoil subjected to convecting and non-convecting transverse gusts in a subsonic compressible flow field. The basic approach used by Osborne was to apply a Prandtl-Glauert transformation to convert the cascade parameters to an equivalent incompressible (E.I.) plane. The upwash velocity components at a point in the E.I. plane are found using the Kemp-Sears [2] approach. The results are then transformed

\[\text{See section 4.3}\]
back into the compressible plane and the unsteady lift is determined. Osborne's assumptions are identical to those of Kemp and Sears with three major exceptions: i) the flow through the cascade is compressible; ii) the airfoils are two dimensional, i.e. thin, cambered airfoils and; iii) the vortex wakes of the stators are neglected as their contribution to the total rotor unsteady lift is a second order effect.

For a convecting transverse gust of the form

$$v = v_0 e^{i\nu(t-x/V)}$$  \hspace{1cm} (5.1)

the unsteady blade force is given by

$$\frac{L(t)}{2\pi \rho c V v_0} = \frac{J(\tau) S(\alpha) e^{i\nu t}}{(1-M^2)^{1/2}}$$  \hspace{1cm} (5.2)

For a non-convecting transverse gust of the form

$$v = v_0 e^{i(\nu t - \mu x/V)}$$  \hspace{1cm} (5.3)

the unsteady blade force is given by

$$\frac{L(t)}{2\pi \rho c V v_0} = (1-M^2)^{-1/2} \left[ J(\delta)[C(\omega) J(\tau) + i J_1(\tau)]ight]$$

\[ - 2i \delta^{-1} \left[ 1 - \frac{\Omega}{\delta} \right] \sum_{n=1}^{\infty} n J_n(\delta) J_n(\tau) \]

\[ + i \delta^{-1} \left[ J_0(\tau) J_1(\delta) - J_1(\tau) J_0(\delta) \right] e^{i\nu t} \hspace{1cm} (5.4) \]

where
\[ \Omega = \frac{\nu c}{V(1-M^2)^{1/2}}; \quad \tau = M^2 \Omega; \quad \delta = \tau + \frac{u c}{V} \]

\[ J(x) = J_0(x) - iJ_1(x) \]

and \( S(\Omega) \) and \( C(\Omega) \) are the Sears and Theodorsen functions defined in section 2.2.

The reader is encouraged to compare these expressions with those obtained by Kemp and Sears \([2]\) for unsteady lift. The cascade configuration used by Osborne is shown in figure 29 and is similar to that of Kemp and Sears. The formulation of the rotor lift force is derived from two types of interaction; namely, rotor upwash due to the relative motion of the rotor and stator and rotor upwash due to the viscous wakes of the stators. The vortex wakes of the stator are not considered since their contribution to the total rotor lift is negligible.

5.2.1 Upwash Due to the Relative Motion of the Rotor and Stator Cascades

The compressible flow field in the stator cascade can be expressed in terms of the linearized steady potential flow equation.

\[ \beta_s^2 \psi_{xx} + \psi_{yy} = 0 \quad (5.5) \]

where \( \beta_s^2 \) is the stator Mach number parameter

\[ \beta_s^2 = 1 - M_s^2 = 1 - (V_s/a_0)^2 \quad (5.6) \]

and \( a_0 \) is the sonic velocity of the undisturbed free stream. Since the flow velocity at the blade surface must be zero, the no penetration boundary condition is
where \( Y_s(x) \) is the stator blade shape, i.e. the camber function. To apply the Prandtl-Glauert transformation along the freestream direction of the stator flow field requires the following equalities.

\[
x = \beta_s x; \quad y = Y; \quad \psi = \beta_s^{-1} \phi
\]

With these substitutions equations (5.5) and (5.6) become

\[
\phi_{xx} + \phi_{yy} = 0 \tag{5.8}
\]

\[
\phi_y = V_s Y_s'(x) \tag{5.9}
\]

The result of this transformation is to convert the stage (both flow field and profile) to an equivalent incompressible E.I. plane. Thus \( Y_s'(x) \) is the E.I. camber function. The pitch, stagger angle and chord are also converted to E.I. values as shown in figure 30.

\[
d_s' = \beta_s^{-1} (1 - M_s^2 \cos^2 \alpha_s)^{1/2} ds
\]

\[
tan \alpha_s' = \beta_s^{-1} tan \alpha_s \tag{5.10}
\]

\[
c_s' = \beta_s^{-1} c_s
\]

By mapping velocity in the E.I. plane, the transverse unsteady velocity at a point in the flow field is given by
Figure 29. Cascade Configuration for Osborne Analysis. From Osborne [4].
a) Stator in Compressible flow plane

b) Stator in equivalent incompressible flow plane

Figure 30. Conversion of Stage Parameters from the Compressible Plane to the Equivalent Incompressible Plane. From Osborne [4].
\[ v = [1 - (1 - \beta_s) e^{-i\chi \cos \chi}] e^{-2\omega_i t} \]  
where \( \chi = \alpha_e + \alpha_s \) and \( \omega_i \) is the complex velocity in the incompressible plane, i.e.

\[ \omega_i = v_i + iu_i \]

The transverse upwash on the rotor blades is given by

\[ v = [1 - (1 - \beta_s) e^{-i\chi \cos \chi}] \sum_{m=1}^{\infty} G_m e^{[2\pi m/\ell_r - i\omega_i t]} \]

where

\[ G_m = -\pi r (dr) e^{i\alpha_r} D H_m \exp(-\pi m_\ell_r (dr) [\frac{\chi}{\ell_r} r + \tan \alpha_r + i \frac{U_s}{\ell_r} \cos \alpha_s] e^{-im\alpha_s} A \]

\[ H_m = J_0(\pi m \ell_r e^{-i\alpha_s} A) + \sum_{n=1}^{\infty} (-i)^n A_{n+1} A_{n-1} \]

\[ A = [-1 - (1 - \beta_s) e^{i\alpha_s \cos \alpha_s}] / (1 - M_s^2 \cos^2 \alpha_s) \]

\[ B = 1 - i(1 - \beta_s) e^{-i\chi \sin \chi} \]

\[ C = AB \quad D = \beta_s^{-1} A \]

and

\[ \Gamma_s = 2\pi c_s V_s (A_0 + A_1) \]
The coefficients $A_n$ are determined by the blade profile as given by

$$
Y_s(\xi) = -A_0 - 2 \sum_{n=1}^{\infty} A_n \cos n \theta
$$

(5.16)

where

$$
\xi = c_s \cos \theta
$$

The steady rotor blade circulation is given by

$$
\Gamma_r = 2\pi c_r V_r (A_0 + A_1)
$$

(5.17)

where the coefficients $A_n$ are given by

$$
Y_r(\xi) = -A_0 - 2 \sum_{n=1}^{\infty} A_n \cos n \theta
$$

(5.18)

and

$$
\xi = c_r \cos \theta
$$

The ratio of the unsteady rotor blade lift to the steady blade lift is given by

$$
\frac{L(t)}{L_r} = \frac{S}{r_r} \beta_r^{-1} [1 - (1 - \beta_s) e^{-u^r \cos \xi}] \sum_{m=1}^{\infty} G_m K_{L_m} (\mu_{m_r}, m_{\delta_r}, M_r) e^{i \nu_r t}
$$

(5.19)

where the coefficient $G_m$ is given by equation (5.14)

$$
\Omega_r = \frac{\omega_r}{r_r} \beta_r^{-2} ; \quad \tau_r = M_{r_r}^2 \Omega_r ; \quad \delta_r = \tau_r + u_r c_r / V_r
$$

$$
\omega_r = \frac{v_r c_r}{V_r} ; \quad \mu_r = -2m r e^{i \alpha_r} C / ds
$$

and $K_L$ is given by the expression in the {} brackets in equation (5.4).
5.2.2 Upwash Due to Viscous Stator Wakes

The expression for the upwash velocity at a point in the flow field as a function of time due to stator viscous wakes is

\[ v(x_r, t) = \frac{V_r}{2\pi} \sum_{m=1}^{\infty} G_m \exp[im\tau_r(t-x_r/V_r)] \]  

(5.20)

where

\[ G_m = 4\pi \frac{V_s}{V_r} c C \left( \frac{1.6456 \sin \alpha_s}{(z/c_s) + 0.3} \cdot 1.4142 \cos \alpha_s \right)^{1/2} e^{im\omega_r} \]  

(5.21)

\[ \cdot \exp\left[-\left(\frac{0.68 \sigma_s}{1.4142 \cos \alpha_s}\right)^2 \frac{\pi c_D z}{c_s} \right] \]  

(5.21)

and

\[ \frac{z}{c_s} = \left( \frac{s}{c_s} \sec \alpha_s + \frac{z_r}{c_s} \frac{V_s}{V_r} \right) - 0.7 \]  

(5.22)

As was the case with the Kemp-Sears analysis, \( z_r \), may range from +1 to -1 and \( \sigma_s = 2D_s/d_s \) is the solidity of the stator cascade. The ratio of the unsteady lift to the steady lift due to viscous stator wakes is given by

\[ \frac{L(t)}{L_0} = \beta_r^{-1} \sum_{m=1}^{\infty} G_m j(m\tau_r) S(m\Omega_r) e^{im\mu_r t} \]  

(5.23)

where the \( G_m \) coefficients are given in equation (5.21) and

\[ \tau_r = M_r^2 \Omega_r; \quad \Omega_r = \omega_r \beta_r^{-2} \]
Sample calculations were made by Osborne for the following stage parameters:

\[
\frac{s}{c_r} = 0.4 \\
\frac{d_r}{d_s} = 1.0 \\
\sigma_s = \sigma_r = 1.0 \\
\alpha_s = \alpha_r = 45^\circ \\
C_D = 0.01 \\
z_r = -0.25 \\
\Gamma_r/\Gamma_s = -1.0
\]

In the calculations the stator and rotor blades were considered as flat plates. Figure 31(a) is a plot of the magnitude of the unsteady lift ratio vs rotor Mach number for potential flow interaction of the rotor and stator rows. As the rotor Mach number approaches the sonic condition, i.e. \(M_r > 0.8\), the resulting lift ratio abruptly increases. These values are a result of the factor \(B_s\) approaching zero in the denominator as the flow becomes transonic thereby establishing an upper limit of \(M_r = 0.8\) for the theory to be valid. Figure 31(b) is a similar plot for the viscous stator wake/rotor interaction. It is noted that the values from these plots may not be added directly but must be summed in the complex plane to avoid phasing errors.

5.3 Finite Element Procedures

Warner [38] used a finite element approach to analyze the two-dimensional, inviscid, incompressible, steady potential flow around an iterative cascade
Figure 31. Unsteady Lift Ratio vs. Mach Number for Potential Flow and Viscous Wake Interactions. From Osborne [4].
of cylinders. The quadrilateral fluid element was formulated in terms of the potential function $\phi$. For irrotational flow the velocity field is governed by the Laplace equation:

$$\nabla^2 \phi = 0$$

(5.22)

Using a variational approach, solutions to this expression require the functional $X$ to be stationary within the region enclosed by the element boundary $S$. $q(s)$ is the outflow normal to the boundary curve. For linear potential variation along the sides of the elements, $X$ may be expressed as

$$X = \sum_{ \text{ELEM} } \Phi^T K_e \phi - \sum_{ \text{NODES} } R_i \phi_i$$

(5.23)

where $\Phi$ is the potential vector associated with an element, $K_e$ is the influence matrix associated with an element, $\phi_i$ is the potential associated with node $i$, and $R_i$ is the 'nodal loading' associated with node $i$. To make $X$ stationary, the total differential of $\phi_i$ must vanish, i.e.

$$K \phi = R$$

(5.24)

where $K$ is the global influence matrix.

The unknown potential values are then found from the appropriate boundary conditions.

To test the functioning of this procedure Warner has calculated the surface velocity on an infinite cascade of right circular cylinders. A typical mesh used in the calculation is shown in figure 16 and figure 17 is a comparison of calculated results for various numbers of surface/liquid boundary nodes with theoretical values for the same cascade. The element mesh at a distance
from the cylinder was maintained as the interface mesh was refined. The accuracy of the solution would probably improve if this outer mesh were refined in the same manner as the interface mesh (Warner).

Steele [39] developed a two-dimensional eight-node isoparametric finite element procedure based on Stokes flow. The major difference between Steele's Stokes flow formulation and Warner's potential flow formulation is that the potential flow formulation neglects viscosity and compressibility while the Stokes formulation includes these effects but neglects inertial effects. Since inertial effects are important in turbomachine stage flow, Steele's analysis must be regarded as an intermediate step in the development of a suitable element. Figure 32 is a comparison of the results obtained by Steele with those of Warner for flow through a cascade of cylinders.

Finite element procedures for the calculation of unsteady blade forces appear to have great promise. The aerelastic interaction between the blading and the working fluid could be modeled with relative ease if suitable fluid flow elements are generated.

5.7 Notation

- a - sonic velocity
- c - blade semichord
- \(C_L\) - lift coefficient
- d - blade pitch
- L - lift
- M - Mach number
- s - axial spacing
- \(u\) - nonsteady velocity perpendicular to chord
- \(U\) - steady velocity perpendicular to chord
- \(v\) - nonsteady velocity parallel to chord
- \(V\) - steady velocity parallel to chord
- \(\alpha\) - stagger angle
- \(\beta\) - Mach number parameter
- \(\Gamma\) - circulation
- \(\mu\) - complex constant
- \(\nu\) - excitation frequency
- \(\rho\) - density
a) Stokes flow finite element solution for flow through a cascade of cylinders. Steele [39].

b) Potential flow finite element solution for flow through a cascade of cylinders. Warner [38].

Figure 32. Comparison of Finite Element Results Obtained by Steele [39] and Warner [38] for Flow Through a Cascade of Cylinders.
6. COMPARISON OF THEORIES FOR THE CALCULATION OF UNSTEADY FORCES

6.1 General

A comparison of the analytical models used by the authors referenced in Sections 3, 4 and 5 indicates two general categories into which all of the above analyses fall regardless of the analytical method used. The first category is typified by the use of isolated airfoil methods to determine the unsteady forces on the blades due to excitations at stator passing frequency. These analyses are applicable to low solidity blade rows such as those found in LP turbine stages and compressor stages. Included in this category are all of the vortex theory analyses and some of the field theory analyses. The analyses in the second category do not employ isolated airfoil theory. These theories are based on the assumption that the blade pitch is small compared to the excitation wavelength, i.e., the blade row is subjected to low per-rev excitation. This category includes all of the actuator disk analyses. Since a majority of these theories model the blades as flat plates, the analyses are more applicable to LP turbine stages and compressor stages than to HP and IP turbine stages. In the remainder of this section the theories in each category will be compared.

6.2 Isolated Airfoil Theories for Nozzle Passing Frequency Excitation

The Kemp and Sears analyses [2] [3] are based on an elementary turbine stage consisting of a cascade of stators and an adjacent rotor cascade. The blades in each cascade are staggered flat plates. The velocity field downstream of the stator cascade due to potential effects and to the viscous stator wakes are calculated. The calculated velocity field is then into convecting sinusoidal gusts. Only those gusts perpendicular to the
rotor blade chord are used to calculate the unsteady forces on the rotor blades, using isolated airfoil theory. Gusts parallel to the blade chord were neglected as being second order effects. Horlock [13] and Holmes [14] derived expressions relating convecting and non-convecting streamwise gusts respectively to the unsteady force on isolated flat plate airfoils. The Horlock and Holmes functions in these expressions are similar in form to the Sears function. Holmes [14] derived an expression relating a non-convecting streamwise gust to the unsteady force on an isolated airfoil of circular arc camber. This expression is highly specialized since no corresponding expressions relating convecting and non-convecting transverse gusts or convecting streamwise gusts to the unsteady force on this type of airfoil were presented. The total unsteady force is therefore impossible to calculate. Mukhopadhyay [18] subsequently derived an expression which relates a general non-convecting gust, i.e., a gust with both transverse and streamwise gust, to the unsteady force on an isolated airfoil of general camber. This analysis is also specialized because no relationship between a general convecting gust and the unsteady force was presented.

All of the above analyses are based on subsonic incompressible flow theory. Osborne [4] extended the Kemp-Sears analyses to the subsonic compressible flow case. As in the Kemp-Sears analyses, the blades are represented as flat plates and streamwise gusts are neglected. There is no reference in the open literature which relates streamwise gusts, either convecting or non-convecting to the unsteady forces on airfoils in cascade. A comparison of the theories mentioned above is given in table 3.
<table>
<thead>
<tr>
<th>Author(s)</th>
<th>Type of Flow</th>
<th>Type of Blades</th>
<th>Type of Gusts</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>incompressible</td>
<td></td>
<td>nonconvecting transverse</td>
<td></td>
</tr>
<tr>
<td>Kemp-Sears [3]</td>
<td>subsonic</td>
<td>isolated flat plates</td>
<td>convecting transverse</td>
<td>Viscous wake effects.</td>
</tr>
<tr>
<td></td>
<td>incompressible</td>
<td></td>
<td>nonconvecting transverse</td>
<td></td>
</tr>
<tr>
<td></td>
<td>incompressible</td>
<td></td>
<td>nonconvecting streamwise</td>
<td></td>
</tr>
<tr>
<td>Holmes [14]</td>
<td>subsonic</td>
<td>isolated parabolic cambered</td>
<td>convecting streamwise</td>
<td>No stage formulation.</td>
</tr>
<tr>
<td></td>
<td>incompressible</td>
<td></td>
<td>nonconvecting streamwise</td>
<td></td>
</tr>
<tr>
<td>Mukhopadhyay [18]</td>
<td>subsonic</td>
<td>isolated general cambered</td>
<td>general nonconvecting</td>
<td>No stage formulation.</td>
</tr>
<tr>
<td></td>
<td>incompressible</td>
<td></td>
<td>nonconvecting</td>
<td></td>
</tr>
<tr>
<td>Kaji-Okazaki [54]</td>
<td>subsonic</td>
<td>isolated flat plate</td>
<td>convecting transverse</td>
<td>Interaction effects between blade rows.</td>
</tr>
<tr>
<td></td>
<td>compressible</td>
<td></td>
<td>nonconvecting transverse</td>
<td></td>
</tr>
<tr>
<td></td>
<td>compressible</td>
<td></td>
<td>nonconvecting transverse</td>
<td></td>
</tr>
</tbody>
</table>
6.3 Actuator Disk Theories for Per-Rev Excitation

All actuator disk theories have a common assumption, namely that the pitch of the rotor blades must be small compared to the wavelength of the excitation. The most prolific contributor to the actuator disk literature is Whitehead.

Whitehead's original actuator disk analysis [26] assumes that both the blade pitch and the blade chord are much smaller than the excitation wavelength. The analysis is based on the assumption that the reduced frequency is zero, implying an infinite excitation wavelength and/or zero blade chord. It is further assumed that the mean deflection of the blades is zero, i.e., that the steady load on the blade is very small. In a subsequent analysis [27] the author considered reduced frequencies greater than zero but less than unity for the case of zero mean deflection. In yet another analysis [28] the requirement that the mean deflection be zero was relaxed to allow finite mean deflections, i.e., significant steady loading, and sound generation by the blade row was considered, implying compressible flow. Henderson, Greitzer and Horlock [35] reported an analysis based on the assumption that the blade pitch is small compared to the excitation wavelength. The analysis is restricted to cases in which the reduced frequency is less than 0.4. Although a different analytical method was used, excellent agreement was obtained with the Whitehead analyses for the zero mean deflection case. All the above analyses have been based on the assumption that the blades are flat plates. Henderson and Horlock [33] presented an analysis based on the assumption that the pitch is small compared to the excitation wavelength. The blades are assumed to be thin cambered airfoils. This paper greatly extends the state-of-the-art of actuator disk analyses and brings the capabilities of these analyses closer to those of isolated airfoil analyses.
Most actuator disk analyses are based on subsonic incompressible flow theory though the analyses by Whitehead [31], Smith [32] and Kaji and Okazaki [54] are for subsonic compressible flow. A comparison of actuator disk theories is given in table 4.

6.4 State-of-the-Art for the Calculation of Unsteady Forces on Turbomachine Blading

Substantial progress has been made in the development of subsonic incompressible flow theories for unsteady force prediction. A complete set of theories for calculating the unsteady forces on a low solidity turbomachine stage consisting of flat plate airfoils has been given [2] [3] [13] [14]. Experimental verification of the theories has been attempted and the degree of correlation is strongly dependent on the extent to which the analytical model agrees with the experimental model. Similar theories for stages with airfoils of general camber are being developed [18], though no experimental verification has yet been attempted.

The extension of such stage flow analyses to the subsonic compressible flow case has been done [4] though streamwise gust effects and experimental verification have yet to be done.

Whitehead's actuator disk analysis [26] appears to work well for cascades in which both the pitch and the chord are small compared to the excitation wavelength and for low reduced frequency values \((\omega < 1.0)\). The upstream velocity profile of the flow entering the actuator disk is required, or must be assumed. Actuator disk analyses thus appear to be suited to analysis of low-per-rev harmonic effects on blading for subsonic incompressible flow. No comparison study of actuator disk vs experiment appears to have been made in the open literature.
### TABLE 4. Comparison of Major Actuator Disk Theories.

<table>
<thead>
<tr>
<th>Author(s)</th>
<th>Type of Flow</th>
<th>Type of Blades</th>
<th>Reduced Frequency Range</th>
<th>Pitch to Chord Ratio Restrictions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Whitehead [26]</td>
<td>subsonic incompressible</td>
<td>flat plate</td>
<td>$\omega = 0.0$</td>
<td>s/c : finite</td>
</tr>
<tr>
<td>Whitehead [27]</td>
<td>subsonic incompressible</td>
<td>flat plate</td>
<td>$0 \leq \omega \leq 1.0$</td>
<td>s/c : finite</td>
</tr>
<tr>
<td>Whitehead [29]</td>
<td>subsonic incompressible</td>
<td>flat plate</td>
<td>$0 \leq \omega \leq 1.0$</td>
<td>s/c : finite</td>
</tr>
<tr>
<td>Whitehead [31]</td>
<td>subsonic compressible</td>
<td>flat plate</td>
<td>$0 \leq \omega \leq 1.5$</td>
<td>s/c : finite</td>
</tr>
<tr>
<td>Smith [32]</td>
<td>subsonic compressible</td>
<td>flat plate</td>
<td>$0 \leq \omega \leq 1.5$</td>
<td>s/c : finite</td>
</tr>
<tr>
<td>Henderson and Daneshyar [16]</td>
<td>subsonic incompressible</td>
<td>flat plate</td>
<td>$0 \leq \omega \leq 1.5$</td>
<td>s/c : finite</td>
</tr>
<tr>
<td>Henderson and Horlock [33]</td>
<td>subsonic incompressible</td>
<td>thin cambered</td>
<td>$0 \leq \omega \leq 1.0$</td>
<td>s/c $\ll 1.0$</td>
</tr>
<tr>
<td>Horlock, Greitzer and Henderson [35]</td>
<td>subsonic incompressible</td>
<td>flat plate</td>
<td>$0 \leq \omega \leq 0.4$</td>
<td>s/c : finite</td>
</tr>
</tbody>
</table>
for steam turbine blading. The prediction of non-steady lift for higher values of reduced frequency parameter \(0<\omega<0.4\) also appears to be possible if the semi-actuator disk analysis of Horlock, Greitzer, and Henderson [35] is used. Most actuator disk methods do not account for significant flow turning angles. Cascades with thin cambered blades can be analyzed using the approximate mass-flow method of Henderson and Horlock [33] to give values for non-steady lift. This also is an incompressible subsonic flow procedure.

Actuator disk methods for the subsonic compressible flow case have been developed [31] [32] [54] though minimal experimental verification has been attempted.

The actuator disk analyses are linearized so that results obtained for any number of given excitation frequencies can be superposed to give the total response to all the frequencies.
7. **COMPUTER PROGRAM FOR NOZZLE PASSING EXCITATION**

7.1 Program Development and Verification

A computer program based on the Kemp-Sears analysis including viscous effects was developed by Rao and Rieger [55] and by Rao [44]. A listing of the program is given in section 7.3. The computer program was written and verified in two steps. First, the section of the program which calculates lift fluctuations due to interference effects (no viscous effects) was developed. Figures 33 and 34 are plots of unsteady lift vs. pitch ratio for flat plate and elliptical steady load distribution respectively. In most cases there is good agreement between the computed results and those of Kemp and Sears [2]. Rao and Rieger [55] in discussing these results mention the approximate formulae used by Kemp and Sears for the calculation of Bessel functions with negative and/or imaginary arguments as a likely source for the differences in figure 33. Figures 35 and 36 are plots of unsteady lift vs. axial spacing ratio for flat plate and elliptical distributions respectively. These plots show good to excellent agreement with the Kemp-Sears theory for both the first and the second harmonic, considering the computation differences.

The second developmental step reported by Rao [44] was to extend the program to include the effect of viscous wakes on the rotor lift. The extended program was verified by comparing the results with those of Kemp-Sears [3]. In general good to excellent agreement existed between the computed results. To demonstrate this correlation results from four test runs using the following input from an example calculated by Kemp and Sears [3] are compared to the Kemp-Sears results.
Figure 33. Comparison of Unsteady Lift Ratio Obtained by Rao [44] and Kemp-Sears [2] for Flat Plate Steady Load Distribution. From Rao [44].

Figure 34. Comparison of Unsteady Lift Ratio Obtained by Rao [44] and Kemp-Sears [2] for Elliptical Steady Load Distribution. From Rao [44].
Figure 35. Comparison of Unsteady Lift Ratio Obtained by Rao [44] and Kemp-Sears [2] for Flat Plate Steady Load Distribution. From Rao [44].
\[ \sigma_r = 1.0 \]
\[ \sigma_s = 1.0 \]
\[ \frac{d_r}{d_s} = 1.0 \]
\[ \alpha_r = 0.7854 \text{ rad (45°)} \]
\[ \alpha_s = 0.7854 \text{ rad} \]
\[ V_s = 1.0 \]
\[ V_r = 1.0 \]
\[ U = 1.4142 (2) \]
\[ b'/c_r = 0.20 \]
\[ C_L = 1.00 \]
\[ C_D = 0.02 \]
\[ x_{ro}/c_r = -1/2; +1/2 \]

The results of these test runs are given in table 5. It can be seen that:

1. The viscous effects determined by the computer program agree with the Kemp-Sears results.
2. An increase in the drag coefficient \( C_D \) increases the viscous effects (in this case).
3. A large change in \( x_{ro}/c_r \) from -1/2 to +1/2 produces changes of order of 25% in the viscous effects. More conservative results are obtained when the lower value is used, i.e. \( U_c \) and \( K \) to be evaluated at the quarter chord point.
4. The second harmonic produced by viscous wakes is generally larger than that produced by the steady circulation and stator vortex wakes.

7.2 Input and Output Instructions

The input data required by the program are given below. The column numbers in which the data must appear are listed in the right-hand column. Excep
TABLE 5  Comparison of Computed Results With Kemp-Sears Analyses [2] and [3].

<table>
<thead>
<tr>
<th>$c_D^S$</th>
<th>$\frac{x_{ro}}{c_r}$</th>
<th>1st Harmonic Nonsteady Lift Ratio (ΔL/L) Due to Viscous Wakes Only</th>
<th>2nd Harmonic Nonsteady Lift Ratio Due to Viscous Wakes Only</th>
<th>Total Nonsteady Lift Ratio</th>
</tr>
</thead>
<tbody>
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<td>0.01</td>
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<td>0.0289</td>
<td>0.020 0.020</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>0.0195 0.0195</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.0779 (0.0705)*</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.0245 (0.0081)*</td>
</tr>
<tr>
<td>0.01</td>
<td>$\frac{1}{2}$</td>
<td>0.023</td>
<td>0.023</td>
<td>0.014 0.0148</td>
</tr>
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<td>0.0446</td>
<td>0.026 0.0263</td>
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<td></td>
<td>0.0310 0.0310</td>
</tr>
</tbody>
</table>

Note: Asterisks (*) indicate values of nonsteady lift ratio obtained without viscous effects included.
for the variable $M$, all variables are floating point numbers. The variable $M$ must be right justified to avoid input errors.

<table>
<thead>
<tr>
<th>Notation Used in Text</th>
<th>Notation Used In Program</th>
<th>Data Card Number</th>
<th>Location on Data Card</th>
</tr>
</thead>
<tbody>
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<td>1-10</td>
</tr>
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<td>RAS1</td>
<td>1</td>
<td>11-20</td>
</tr>
<tr>
<td>$A^S_2$</td>
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<td>21-30</td>
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<tr>
<td>$m$</td>
<td>$M$(integer value)</td>
<td>3</td>
<td>1-2</td>
</tr>
<tr>
<td>$V_s$</td>
<td>RVS</td>
<td>3</td>
<td>3-12</td>
</tr>
<tr>
<td>$V_r$</td>
<td>RVR</td>
<td>3</td>
<td>12-22</td>
</tr>
<tr>
<td>$U$</td>
<td>RU</td>
<td>3</td>
<td>23-32</td>
</tr>
</tbody>
</table>
The output of the program consists of the input data in the same order as it is read and the following information:

(1) Stator lift first harmonic \[ \frac{t_{L_s}^{L}(t)}{o_{L_s}^{L}} \] \[ m=1 \]

(2) Stator lift second harmonic \[ \frac{t_{L_s}^{L}(t)}{o_{L_s}^{L}} \] \[ m=2 \]

(3) Rotor lift first harmonic \[ \frac{t_{L_r}^{L1}(t) + t_{L_r}^{L2}(t)}{o_{L_r}^{L}} \] \[ m=1 \]

(4) Rotor lift second harmonic \[ \frac{t_{L_r}^{L1}(t) + t_{L_r}^{L2}(t)}{o_{L_r}^{L}} \] \[ m=2 \]

(5) Rotor lift first parts \[ \frac{t_{L_r}^{L1}(t)}{o_{L_r}^{L}} \] \[ m=1 \] \[ \frac{t_{L_r}^{L1}(t)}{o_{L_r}^{L}} \] \[ m=2 \]

(6) Rotor lift second parts \[ \frac{t_{L_r}^{L2}(t)}{o_{L_r}^{L}} \] \[ m=1 \] \[ \frac{t_{L_r}^{L2}(t)}{o_{L_r}^{L}} \] \[ m=2 \]
(7) Rotor lift parts due to viscous wakes

\[ \sum_{m=1}^{m=2} \frac{t^{LrV}(t)}{0^{Lr}} \]

(8) Total rotor lift first harmonic

\[ \sum_{m=1}^{m=2} t^{Lr1}(t) + t^{Lr2}(t) + t^{Lrv}(t) \]

(9) Total rotor lift second harmonic

\[ \sum_{m=2}^{m=2} t^{Lr1}(t) + t^{Lr2}(t) + t^{Lrv}(t) \]

NOTE: If there are any errors in computing the Bessel functions, the corresponding error code is printed. If there are no errors, no error code is printed.

7.3 Program Listing

Included in the following listing are the master program; subroutine CBESJ, for calculating Bessel functions of the first kind of order \( n \) with complex arguments of the form \( J_n(\rho e^{i\phi}) \); subroutine SUMK, for calculating factorials; and subroutine BEJIM, for calculating Bessel functions of the second kind of order \( n \) with negative real arguments. In addition to the above subroutines, two standard subroutines included in the IBM scientific subroutine library are required to calculate Bessel functions of the first and second kind of order \( n \) with real positive arguments.
\begin{verbatim}
! = J - 1
VHILL = ARS (\texttt{HILLS})
CALL RESH \texttt{(HILL, J, J1, 0, 0, 1, L) ;}
IF \texttt{(P \leq 4)} \texttt{WRITE (10, 10) LEX}
\texttt{IF (P \leq 4)} \texttt{WRITE (10, 10) LEX}  
\texttt{OUT}
\end{verbatim}
SUBROUTINE Fholm(n, X, W, NRES)  
C THIS SUBROUTINE EVALUATES HOML FUNCTION J WITH COMPLEX ARGUMENTS  
C N IS THE ORDER OF THE HOML FUNCTION J  
C X IS THE COMPLEX ARGUMENT  
C W IS THE COMPLEX RESULT  
COMMON X, W  
C RESULT stored in X, W  
C NRES = 1 REQUIRES ACCURACY NOT UNATTAINED WITH 100 ITERATIONS  
C USES SUBROUTINE SUMA  
C COMPLEX X, W, NRES  
XX=0D0  
WW=0D0  
DO 5 XX=X, 1,100  
DO 5 WW=W, 1,100  
5 CONTINUE  
C CALL SUMA(X, NRES)  
CALL SUMA(W, NRES)  
XX=X**2(1**2+1)**2*X**2/NRES/NRES  
WW=W**2(1**2+1)**2*W**2/NRES/NRES  
IF (XX(W)**2(1**2+1)**2*X**2/NRES/NRES  
CONTINUE  
C RETURN  
END  

SUBROUTINE SUMA(RH, NRES)  
C THIS SUBROUTINE EVALUATES FACTORIAL OF RH  
C RH = RESULT  
C RH MUST BE GREATER THAN OR EQUAL TO ZERO  
C IF (RH**2=15,15,20  
15 RH=1,0  
20 CONTINUE  
25 RH=1,0  
35 RH=1,0  
45 CONTINUE  
RETURN  
END
8. COMPUTER PROGRAMS FOR PER-REV EXCITATION

8.1 General

As stated previously, the Kemp-Sears analysis is based on isolated airfoil theory and upon induced circulation interference effects between blade rows. These assumptions limit the Kemp-Sears analysis to the calculation of excitation at nozzle passing frequency for a state of low solidity. Computer programs have been written to calculate per-rev excitation of high solidity blade rows using the semi-actuator disk analysis of Horlock, Grietzer and Henderson (HGH)[35](see section 4.5), by the analysis of Henderson and Horlock [33](see section 4.4) and by Whitehead's [26] actuator disk analysis (see section 4.2). Listings and input instructions for these programs are given in the following sections.

8.2 Development and Verification of the Semi-Actuator Disk Analysis Program [35]

The semi-actuator disk analysis is applicable to high solidity cascades (s/c<<1) when the reduced frequency is small (ω<<1). High pressure turbine stages meet these conditions approximately when the stage is operating near the design condition, i.e. when pressure gradients across the passage width are small. The program described below has been verified by comparing the output with results given in reference [35] for the magnitude of the unsteady lift coefficient as a function of pitch to chord ratio for various values of reduced frequency: See table 6. To facilitate this comparison, the unsteady lift coefficient is normalized on the blade pitch s instead of the blade chord c, as is usually the case. At present there is no explanation for the lack of agreement between the computed values and those of HGH for a reduced frequency of ω = 0.01. Excellent agreement is obtained for all other reduced frequencies.

8.2.1 Input and Output Instructions

The program is intended to be run from a remote terminal using an interactive input and control method. After the initial run command, two requests for data
<table>
<thead>
<tr>
<th>Reduced Frequency Parameter</th>
<th>Pitch to Chord Ratio (s/c)</th>
<th>Magnitude of Nonsteady Lift Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>1.0</td>
<td>0.7145</td>
</tr>
<tr>
<td>0.01</td>
<td>0.6</td>
<td>0.7145</td>
</tr>
<tr>
<td>0.10</td>
<td>0.6</td>
<td>0.7937</td>
</tr>
<tr>
<td>0.10</td>
<td>0.2</td>
<td>0.7937</td>
</tr>
<tr>
<td>0.30</td>
<td>1.0</td>
<td>1.0300</td>
</tr>
<tr>
<td>0.30</td>
<td>0.6</td>
<td>1.0300</td>
</tr>
<tr>
<td>0.30</td>
<td>0.2</td>
<td>1.0300</td>
</tr>
<tr>
<td>0.40</td>
<td>1.0</td>
<td>1.1650</td>
</tr>
<tr>
<td>0.40</td>
<td>0.6</td>
<td>1.1650</td>
</tr>
<tr>
<td>0.40</td>
<td>0.2</td>
<td>1.1650</td>
</tr>
</tbody>
</table>

TABLE 6. Comparison of Computed Results With Semi-Actuator Disk Analysis of Reference [35].
are issued by the computer. The prompt character for this data input is a question mark. The first question mark is a request for an identifying title of up to 80 alphanumeric characters. The second question mark is a request for the blade chord in inches, blade pitch in inches, mean axial gas velocity in inches per second, blade circumferential velocity in inches per second, blade stagger angle in degrees and disturbance frequency in radians per second.

These variables are entered in the order given above and must be separated by commas. Note that all the variables are real numbers and, as such, require a decimal point. The output consists of the data in the order in which it was read, the calculated reduced frequency, the real and imaginary components of the unsteady lift coefficient and the magnitude of the unsteady lift coefficient. Following the results, the program offers the option of changing any or all of the input data.

To demonstrate this procedure, a sample run using the following data is given.

Blade chord (in.) = 2.0
Blade pitch (in.) = 2.0, 1.5
Axial gas velocity (ips) = 2400.0
Blade speed (ips) = 2400.0
Stagger angle (deg.) = 45.0
Disturbance frequency (rad/sec) = 339.4

!RUN
?DEMONSTRATION RUN FOR PER-REV EXCITATION PROGRAM
?2.0, 2.0, 2400.0, 2400.0, 45.0, 339.4

DEMONSTRATION RUN FOR PER-REV EXCITATION PROGRAM

MODE: TURBINE
DATA GIVEN
BLD.CHORD, IN. = 2.00  BL.SPACING, IN. = 2.00
MN.AX.GAS VELOCITY, IPS = 2400.0  BLD.SPEED, IPS = 2400.0
STAGGER ANGLE, DEGREES = 45.00  DISTRUB.FREQ., RAD/SEC = 339.400

REDUCED FREQ:  \Omega = 1.0000 00

\begin{tabular}{ccc}
\textbf{REAL} & \textbf{IMAG} & \textbf{MAGNITUDE} \\
\hline
\textbf{NON-STEADY} & -6083E 00 & 5098E 00 & 7937E 00 \\
\hline
\textbf{LIFT COEFF} & & & \\
\end{tabular}

DO YOU WISH TO CHANGE ANY OF INPUT DATA? (TYPE 'TRUE' OR 'FALSE')?

?T

?S = 1.5*

REDUCED FREQ:  \Omega = 1.0000 00

\begin{tabular}{ccc}
\textbf{REAL} & \textbf{IMAG} & \textbf{MAGNITUDE} \\
\hline
\textbf{NON-STEADY} & -6083E 00 & 5098E 00 & 7937E 00 \\
\hline
\textbf{LIFT COEFF} & & & \\
\end{tabular}

DO YOU WISH TO CHANGE ANY OF INPUT DATA? (TYPE 'TRUE' OR 'FALSE')?

?F

*EXIT*

8.2.2 Program Listing

C PROGRAM NAME: HGH
C THIS PROGRAM CALCULATES THE NON-STEADY FORCES ON A
C CASCADE OF AIRFOIL BLADES ACCORDING TO THE THEORIES
C OF J.H. HORLOCK, E.M. GREENBERG & R.E. HERBERSON.
C ANALYSIS BY DR. NEVILLE F. RIEGER, R.I.T. APRIL 1977
C
C INPUT INFORMATION AND OTHER RELEVANT VARIABLES:
C C = BLADE CHORD(PITCH), INCHES
C S = BLADE SPACING(PITCH), IN,
C U = MEAN AXIAL GAS VELOCITY, IN. PER SEC. (IPS)
C V = BLADE SPEED, BLADE TANGL. VELOCITY, IPS
C TH = STAGGER ANGLE, DEGREES
C DU = DISTRUBANCE FREQUENCY, RAD/SEC
C TITLE = IDENTIFYING TEXT FOR THE RUN NAME, DATE, ETC.
C AS = REDUCED FREQUENCY PARAMETER
C
REAL L, LR, LI, LX, LY, LO, N
REAL L, TURB, LCOMP, L190, LGO, LGO6, LGO8, LEM, LAM70A
LOGICAL FIRST, TEST
INTEGER TITLE(20)

NAMELIST

FIRST=.TRUE.,
TEST=.FALSE.,
PI=3.141592653589

INPUT DATA

READ (105, 500) TITLE
READ (105, 510) C, S, U, V, TH, GU

WRITE(108, 600)
WRITE(108, 602)
WRITE(105, 500), TITLE
WRITE(108, 601)
WRITE(108, 611)
WRITE(108, 615)
WRITE(108, 620) C, S
WRITE(108, 621) U, V
WRITE(108, 622) TH, GU

CONTINUE

TA = TH / 57, 3
STA = SIN(TA)
CTA = COS(TA)
TTA = STA / CTA

U = SORT(U**2 + V**2)
C1 = TAN(TA)
C2 = C1**2
C3 = C2*C1
C4 = C3*C1

SS = GU**C / 2.0/U
FI = 2.0*SS*(1.0 + C2) + (1.0 + C1)
CM = 4.0*C4 + FI**2
DI = 2.0*C2 + FI*C1
D2 = 2.0*C3 - FI
D3 = D1 - CM
D4 = 2.0*C2 - DI
D5 = 2.0*C2 + FI*C3
FR = D1
FY = D2
FR = D3
FY = D4
BM = 2.0*(C4 - 2.0*C2 - 1.0) + CM - 2.0*FI*(1.0 + C2)*C1
BY = CM*C1 + (1.0 + C2)**(FI - 4.0*C3)
01 = 2. * C1 * (8 * S) / SSTAR**2 * 1.0
02 = 2. * S / C1
04 = 1. / 6H
CLR = 0.3 * (D1 * 01 - 62 * 8Y / 02)
CLY = 0.3 * (02 * 01 + 02 * D0 - 03)
06 = SORT (CLR**2 + CLY**2)

C PRINT OUT RESULTS
C
WRITE (108, 630) SS
WRITE (108, 632)
WRITE (108, 633) CLR, CLY, 06
C
C CHANGE ANY VARIABLES?
C
WRITE (108, 501)
WRITE (108, 700)
READ (105, 501) TEST
IF (.NOT. TEST) GO TO 999
WRITE (108, 501)
INPUT (105) .
FIRST = 'FALSE
GO TO 400
C
C FORMAT STATEMENTS
C
500 FORMAT (20A4)
501 FORMAT (L5)
502 FORMAT (I0610.0)
503 FORMAT (1H1)
504 FORMAT (1H0)
505 FORMAT (/)
600 FORMAT (6X, 'MODE', 2X, 'TURBINE', '/)
615 FORMAT (1X, 'DATA GIVEN', '/)
620 FORMAT (1X, 'BLDerek 1n=. ', F7.3, 'BLD.SPARCING 1n. = ', F7.2
1 )
621 FORMAT (1X, 'AN. AX. GASP VEL., IPS= ', F7.1, 3X,
1 'OLD SPEED, IPS= ', F7.1)
622 FORMAT (1X, 'STAGGER ANGL.DEGR= ', F7.2, 3X,
1 'DISTRUB. FREQ., RAD/SEC= ', F8.3, '/)
630 FORMAT (1X, 'REDUCED FREQ; OMEGA= ', E10.4, '/)
632 FORMAT (1X, 'REAL', 8X, 'IMAG', 8X, 'MAGITUDE',
1 / ', 15X, 10(1H_), 2X, 10(1H_), 5X, 10(1H_))
633 FORMAT (1X, 'NON-STEADY', '/ ', 1X, 'LIFT COEFF', 4X, 3(E10.4, 2X,
1 E10.4, 5X))
700 FORMAT ('DO YOU WISH TO CHANGE ANY OF INPUT DATA?',
1 ' (TYPE "TRUE" OR "FALSE")' )
C
99 CALL EXIT
END
8.3 Development and Verification of the Program for the Henderson-Horlock Analysis [33]

This analysis is applicable to stages where the rotor blade pitch is small compared to the stator pitch and where the rotor blade lift coefficient is small. High pressure stages with impulsive blading may be modeled by this procedure. The program described below has been verified by comparing the output to results given in reference [33] for the real and imaginary parts of the unsteady lift coefficient for various values of reduced frequency parameter: See table 7. Acceptable agreement is obtained for all reduced frequency parameters below 0.20.

8.3.1 Input and Output Instructions

The program is intended to be run from a remote terminal using an interactive input and control method. After the initial run command, the computer issues two requests for data. The prompt character for this data input is a question mark. The first question mark is a request for the blade chord in inches, the blade pitch in inches, the stagger angle in radians, the blade surface angle at the lead edge in radians and the blade surface angle at the trail edge in radians. The second question mark is a request for the mean axial gas velocity in inches per second, the blade speed in inches per second, the excitation frequency in radians per second and the excitation wavelength in inches. These variables are real numbers and must be entered in the order given, separated by commas. The program then outputs the data and issues a third request-for-data prompt character. This input, 1 for a turbine analysis and 2 for a compressor analysis, directs the computer to the correct program segment. The output consists of the magnitude of the unsteady lift coefficient for comparison to the theory and the following results which are normalized on the density:

1) unsteady axial force \((\Delta A)\)
2) unsteady tangential force \((\Delta F)\)
3) unsteady lift force \((\Delta L)\)
TABLE 7. Comparison of Computed Results With Analysis of Reference [33].

<table>
<thead>
<tr>
<th>Reduced Frequency Parameter</th>
<th>Pitch to Chord Ratio</th>
<th>Stagger Angle</th>
<th>Magnitude of Nonsteady Lift Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Program</td>
</tr>
<tr>
<td>0.10</td>
<td>1.0</td>
<td>45°</td>
<td>0.429</td>
</tr>
<tr>
<td>0.20</td>
<td>1.0</td>
<td>45°</td>
<td>0.335</td>
</tr>
<tr>
<td>0.30</td>
<td>1.0</td>
<td>45°</td>
<td>0.390</td>
</tr>
</tbody>
</table>
Following the results, the program offers the option of changing any or all of the input data. To demonstrate this procedure, a sample run using the following data is given.

Blade chord (in): 1.873
Blade pitch (in): 1.315
Stagger angle (rad): 1.174
Lead edge blade surface angle (rad): 0.6417
Trail edge blade surface angle (rad): 1.2217
Mean axial gas velocity (ips): 5.8095
Blade speed (ips): 23.124
Excitation frequency (rad/sec): 69.385
Excitation wavelength (in): 2.094

Data Given

Chord: 1.87 Pitch: 1.31 Stagger: 1.12
Le angle: 0.64 Te angle: 1.22 Ax. Gas Vel: 5.81
Blade speed: 23.12 Exc. Freq: 69.38 Exc. Length: 2.09
Vel. Defect: .0400

Enter 1 if this is a turbine, 2 if compressor
MAGNITUDE OF UNSTEADY LIFT FOR COMPARISON
TO HENDERSON-HORLOCK PLOT: 1.56641

ALL STEADY & UNSTEADY FORCES BELOW HAVE
BEEN DIVIDED BY THE DENSITY

UNSTEADY AXIAL FORCE: 168.638
UNSTEADY TANGENTIAL FORCE: -31.074
UNSTEADY LIFT FORCE: 171.477
STEADY AXIAL FORCE: 117.939
STEADY TANGENTIAL FORCE: 88.784
STEADY LIFT FORCE: 147.610
AXIAL FORCE RATIO: 1.430
TANGENTIAL FORCE RATIO: -.350
LIFT RATIO: 1.162

8.3.2 Program Listing

C PROGRAM NAME: MHHUK
C THIS PROGRAM IS BASED ON THE HENDERSON-
C HORLOCK ANALYSIS.
C
C INPUT VARIABLE DEFINITION:
C THL=ANGLE OF BLADE SURFACE RELATIVE TO
C THE AXIAL DIRECTION AT THE LEAD
C EDGE (RAD)
C THTE=SAME AS ABOVE BUT AT TRAIL EDGE (RAD)
C EX=EXCITATION FREQ (RAD/SEC)
C EXL=EXCITATION CIRCUMFERENTIAL WAVELENGTH
C (IN)
C SA=BLADE STAGGER ANGLE (RAD)
C B=BLADE PITCH (IN)
C SC=BLADE CHORD (IN)
C U=MEAN AXIAL GAS VELOCITY (FPS)
C V=BLADE SPEED (IPS)
C AMP=VELOCITY DEFFECT IN WAKE (PERCENT)
30 WRITE(108,600)
WRITE(108,621)
READ(105,500) BC,BP,SA,THLE,THTE
WRITE(108,601)
WRITE(103,622)
READ(105,501) U,V,EXF,EXL
WRITE(108,630)
READ(105,510) AMP
WRITE(108,602)
WRITE(108,603)
WRITE(108,604) BC,BP,SA
WRITE(108,605) THLE,THTE,U
WRITE(108,606) V,EXF,EXL
WRITE(108,640) AMP
T1=TAN(THLE)
T2=TAN(THTE)
T3=(1.0*COS(THLE))**2
H1=T2-T1
H2=(T3/2.0)+((H1+T1)/3.0)+((H1**2)/12.0)
H3=((H1**2)/2.0)+(-1.0**2)/6.0
H4=T1**2
H5=(T1/2.0)+((H1**2)+((T2/2.0)+((2.0**2)+(T1**2))/3.0)
1+(T1**2)/12.0)+((H1**2)/2.0)+((H1**3)/6.0)
H6=(T1**2)+((H1**2)/2.0)+((H1**3)/6.0)
H7=(T2**2)+((H1**2)/2.0)+((H1**3)/6.0)
H8=(T2**2)-((T1**2)/2.0)
PI=3.14159265
XL=(2.0*PI*BP)/EXL
SXL=SIN(XL)
CXL=COS(XL)
SSA=1.0*COS(SA)
SSA2=SSA**2
SXLIFT=(U**2)*H8/2.0
SYLIFT=(-1.0)*BP+(U**2)*H1
A1=((1.0-CXL)*H2+BC+2.0)/(BP*SSA2)
A2=(-1.0)*(BP*SXL)/(XL*BC)
A3=(-1.0)*(2.0*U*BP+H1*SXL)/(XL*BC*V)
A=A1+A2+A3
B1=(4.0*U*H3*(1.-CXL))/(XL*V*SSA)
B2=H3*SXL/SSA
B3=B1+B2
C1=(2.0*BC+H5*(1.-CXL))/(BP*SSA2)
C2=(BP+U*H8+SXL)/(BC*V*XL)
C3=(BP*T1+SXL)/(XL*BC)
C=C2+C3-C1
D1=(4.0*U*H8*(1.-CXL))/(XL*V*SSA)
D2=(H7+SXL)/SSA
D=(-1.0)*(D1+D2)
WRITE(108,607)
READ(105,502) M
IF(M,NE.1) GO TO 10
G1=((C*X*SIN(SA))-(A*COS(SA)))**2
G2=((D*X*SIN(SA))-(B*COS(SA)))**2
TEST1=ABS(G1+G2))/PI
YLIFT=(-1.0)*((SORT((A**2)+(B**2)))**2*COS(SA)+B*C*Y*AMP*U)
XLIFT=(SORT((C**2)+(D**2)))*SIN(SA)*BC*V*AMP**2
ULIFT=SORT((YLIFT**2)+(XLIFT**2))
GO TO 20
10 G1=(A*COS(SA))-(C*SIN(SA))***2
G2=(B*COS(SA))-(D*SIN(SA))***2
TEST1=(SORT(G1+G2))/'PI
YLIFT=(SORT((A**2)+(B**2)))*COS(SA)*BC*V*AMP**2
XLIFT=(-1.)*(SORT((C**2)+(D**2)))*SIN(SA)*BC*V*AMP**2
ULIFT=SORT((YLIFT**2)+(XLIFT**2))
20 SLIFT=SORT((SYLIFT**2)+(SXLIFT**2))
WRITE(108+602)
ULIFT=ABS(ULIFT)
SXLIFT=ABS(SXLIFT)
SYLIFT=ABS(SYLIFT)
AXRAT=XLIFT/SXLIFT
TANRAT=YLIFT/SYLIFT
SIFRAT=ULIFT/SLIFT
WRITE(108+650)
WRITE(108+680)TEST1
WRITE(108+602)
WRITE(108+670)
WRITE(108+680)
WRITE(108+602)
WRITE(108+610)XLIFT
WRITE(108+611)YLIFT
WRITE(108+612)ULIFT
WRITE(108+613)SXFLFT
WRITE(108+614)SYLIFT
WRITE(108+615)SLIFT
WRITE(108+616)AXRAT
WRITE(108+617)TANRAT
WRITE(108+618)SIFRAT
WRITE(108+620)
READ(105+502)
IF(R.20.1) GO TO 2:
600 FORMAT(1X,'ENTER CHORD,PITCH,STAGGER ANGLE')
621 FORMAT(1X,'ENT GRACE ANGLES AT LE & TE')
601 FORMAT(1X,'ENTER AX, GAS VEL, BLADE SPEED')
622 FORMAT(1X,'EXCITATION FREQ, & WAVELENGTH')
602 FORMAT(1X)
603 FORMAT(5X,'DATA GIVEN')
604 FORMAT(1X,'CHORD ',F4.2,3X,'PITCH ',F6.2,3X,'STAGGER ',F6.2,3X)
605 FORMAT(1X,'LE ANGLE ',F6.2,3X,'TE ANGLE ',F6.2,3X)
613X,'AX, GAS VEL ',F6.2,3X)
606 FORMAT(1X,'BLADE SPEED ',F6.2,3X,'EXC. FREQ ',F6.2,3X,'LENGTH ',F6.2)
607 FORMAT(1X,'TURBINE = 1 IF THIS IS A TURBINE = 2 IF COMPRESSOR')
608 FORMAT(1X,'TURBINE MODE')
609 FORMAT(9X,'COMPRESSOR MODE')
610 FORMAT(2X,'UNSTEADY AXIAL FORCE ',F6.3,F6.3)
611 FORMAT(2X,'UNSTEADY TANGENTIAL FORCE ',F6.3,F6.3)
612 FORMAT(2X,'UNSTEADY LIFT FORCE ',F6.3,F6.3)
613 FORMAT(2X,'STEADY AXIAL FORCE ',F6.3,F6.3)
614 FORMAT(2X,'STEADY TANGENTIAL FORCE ',F6.3,F6.3)
8.4 Development and Verification of the Actuator Disk Analysis Program

The actuator disk analysis is applicable to cascades in which the disturb wavelength is much larger than both the pitch ($\lambda >> s$) and the chord ($\lambda >> c$). The only restriction placed on the solidity is that it must be finite. A result of the above assumptions, the reduced frequency must be very small ($\omega < 1$). The program described below has been verified by comparing the output with sample results given by Whitehead [26]: See table 8. Good agreement is obtained for zero and thirty degree stagger angles at all incidence angles for the pitch to chord ratio 0.25. At present there is no explanation for the lack of agreement at the pitch to chord ratio 0.50.

8.4.1 Input and Output Instructions

The program is intended to be run from a terminal using an interactive input and control method. After the run command, the computer prints a request for data, followed by a question mark. The data required are the blade stagger angle in radians, the blade pitch in inches, the blade chord
<table>
<thead>
<tr>
<th>Pitch to Chord Ratio</th>
<th>Stagger Angle</th>
<th>Inlet Flow Angle</th>
<th>Program Real</th>
<th>Program Imaginary</th>
<th>Whitehead Real</th>
<th>Whitehead Imaginary</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>0°</td>
<td>0°</td>
<td>-0.1572</td>
<td>0.0124</td>
<td>-0.1572</td>
<td>0.0124</td>
</tr>
<tr>
<td>0.25</td>
<td>0°</td>
<td>45°</td>
<td>0.7652</td>
<td>0.0247</td>
<td>0.7652</td>
<td>0.0247</td>
</tr>
<tr>
<td>0.25</td>
<td>0°</td>
<td>-45°</td>
<td>-1.2348</td>
<td>0.0247</td>
<td>-1.2348</td>
<td>0.0247</td>
</tr>
<tr>
<td>0.25</td>
<td>30°</td>
<td>0°</td>
<td>-0.5611</td>
<td>0.2296</td>
<td>-0.5612</td>
<td>0.1328</td>
</tr>
<tr>
<td>0.25</td>
<td>30°</td>
<td>45°</td>
<td>0.1335</td>
<td>-0.2462</td>
<td>0.1335</td>
<td>-0.2980</td>
</tr>
<tr>
<td>0.25</td>
<td>30°</td>
<td>-45°</td>
<td>-1.3572</td>
<td>0.9247</td>
<td>-1.3572</td>
<td>0.2026</td>
</tr>
<tr>
<td>0.50</td>
<td>0°</td>
<td>0°</td>
<td>-0.3173</td>
<td>0.0126</td>
<td>-0.3144</td>
<td>0.0247</td>
</tr>
<tr>
<td>0.50</td>
<td>0°</td>
<td>45°</td>
<td>3.5246</td>
<td>0.0252</td>
<td>1.5304</td>
<td>0.0494</td>
</tr>
<tr>
<td>0.50</td>
<td>0°</td>
<td>-45°</td>
<td>-4.4755</td>
<td>0.0252</td>
<td>-2.4696</td>
<td>0.0494</td>
</tr>
<tr>
<td>0.50</td>
<td>30°</td>
<td>0°</td>
<td>1.9812</td>
<td>1.0276</td>
<td>-1.1223</td>
<td>0.2656</td>
</tr>
<tr>
<td>0.50</td>
<td>30°</td>
<td>45°</td>
<td>0.9112</td>
<td>-0.8321</td>
<td>0.2669</td>
<td>-0.5960</td>
</tr>
<tr>
<td>0.50</td>
<td>30°</td>
<td>-45°</td>
<td>-5.0793</td>
<td>3.7563</td>
<td>-2.7144</td>
<td>0.4052</td>
</tr>
</tbody>
</table>
in inches and the relative flow velocity in inches per second. These variables are real numbers and must be separated by commas. Upon receiving the above data, the computer prints another request for data, followed by a question mark. The data required are the excitation frequency in radians per second, the excitation circumferential wavelength in inches, the angle between the relative velocity and the axial direction in radians, and the blade speed in inches per second. Again the variables must be real numbers and must be separated by commas. The data is reprinted by the computer for a check. Since the analysis takes different forms for turbine and compressor stages, the next request for data determines which of the two forms the program will follow. Following the prompt character, the user enters "1" for a turbine stage analysis and "2" for a compressor stage analysis. The output consists of the real and imaginary components of the unsteady lift coefficient and the magnitude of the unsteady lift coefficient. Following the results, the program offers the option of changing the data. To demonstrate this procedure, a sample run using the following data is given.

<table>
<thead>
<tr>
<th>Stagger angle (rad.)</th>
<th>= 0.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blade pitch (in.)</td>
<td>= 1.0</td>
</tr>
<tr>
<td>Blade chord (in.)</td>
<td>= 4.0</td>
</tr>
<tr>
<td>Relative velocity (ips)</td>
<td>= 10,000.0</td>
</tr>
<tr>
<td>Excitation wavelength (rad/sec)</td>
<td>= 0.05</td>
</tr>
<tr>
<td>Excitation wavelength (in.)</td>
<td>= 15916.0</td>
</tr>
<tr>
<td>Relative flow angle (rad.)</td>
<td>= 0.0, -0.7854</td>
</tr>
<tr>
<td>Blade speed (ips)</td>
<td>= 795.80</td>
</tr>
</tbody>
</table>

ENTER STAGGER ANGLE, PITCH, CHORD, AND RELATIVE VELOCITY
?0.0, 1.0, 4.0, 10000.

ENTER EXCITATION FREQUENCY, EXCITATION WAVELENGTH,
RELATIVE FLOW ANGLE, AND BLADE SPEED
?0.05, 15916., 0.0, 795.80

DATA GIVEN:

STAGGER = 0.000  PITCH = 1.000  CHORD = 4.000

EXCITATION FREQUENCY = 0.050  EXCITATION WAVELENGTH = 15916.000

RELATIVE VELOCITY = 10000.000  RELATIVE FLOW ANGLE = 1.000
BLADE SPEED = 795.800

ENTER 1 IF TURBINE, 2 IF COMPRESSOR
?1

REAL COMPONENT OF LIFT COEFF = -1.15718

IMAG. COMPONENT OF LIFT COEFF = .123529E-01

UNSTEADY LIFT COEFFICIENT = .157673

ENTER 1 TO RUN NEW DATA, 2 TO QUIT
?1

ENTER STAGGER ANGLE, PITCH, CHORD, AND RELATIVE VELOCITY
?0.5236, 1.0, 4.0, 10000.

ENTER EXCITATION FREQUENCY, EXCITATION WAVELENGTH,
RELATIVE FLOW ANGLE, AND BLADE SPLT
?0.05, 15916, -.7854, 795.80

DATA GIVEN:

STAGGER = .524 PITCH = 1.000 CHORD = 4.000

EXCITATION FREQUENCY = .050 EXCITATION WAVELENGTH = 15916.000

RELATIVE VELOCITY = 10000.000 RELATIVE FLOW ANGLE = -.785

BLADE SPEED = 795.800

ENTER 1 IF TURBINE, 2 IF COMPRESSOR
?1

REAL COMPONENT OF LIFT COEFF = -1.15718

IMAG. COMPONENT OF LIFT COEFF = .123529E-01

UNSTEADY LIFT COEFFICIENT = 1.64227

ENTER 1 TO RUN NEW DATA, 2 TO QUIT
?2

*STOP* 0
8.4.2 Program Listing

C PROGRAM NAME: WHIT
C
C THIS PROGRAM IS BASED ON WHITEHEAD'S
C ACTUATOR DISK ANALYSIS (1959),
C (REFERENCE EQUATION 4.28) IN THESIS.
C
C INPUT VARIABLE DEFINITION
C
C SA=STAGGER ANGLE (RAD)
C BP=BLADE PITCH (IN)
C BC=BLADE CHORD (IN)
C W=RELATIVE VELOCITY OF GAS (IN/SEC)
C EXF=EXCITATION FREQUENCY (RAD/SEC)
C XL=EXCITATION CIRCUMFERENTIAL
C WAVELENGTH (IN)
C ALP=ANGLE OF \( \gamma \) WITH RESPECT TO THE
C AXIAL DIRECTION. FOR ZERO ANGLE
C OF ATTACK, ALP=SA.
C
C PROGRAM
C
10 WRITE(108,602)
   WRITE(108,600)
   READ(105,500)SA,BP,BC,W
   WRITE(108,601)
   WRITE(108,651)
   READ(105,501)EXF,XL,ALP,VS
   WRITE(108,602)
   WRITE(108,603)
   WRITE(108,602)
   WRITE(108,604)SA,BP,BC
   WRITE(108,602)
   WRITE(108,605)EXF,XL
   WRITE(108,602)
   WRITE(108,606)W,ALP
   WRITE(108,602)
   WRITE(108,660)VS
   WRITE(108,602)
   WRITE(108,650)
   READ(105,502)H
   BETA=EXF*BP/VS
   REOF=(EXF*BC)/(2.*W)
   X=(BC*BETA*COS(SA))/(2.*BP*REOF)
   T=TAN(SA)
   CSA=COS(SA)
   DEL=1./(CSA**2)
   EPS=1.-EPS**2
   TA=TAN(ALP)
   IF(N.EQ.1) GO TO 50
   T=-T
   ALP=-ALP
TA=TANH(ALP)
50 TAU=(TA-T)*((CSA**2)
TAU=TAU*TAU
X2=X**X
DEL2=DEL*DEL
X3=X*X
T2=T*T
T3=T2*T
FR1=TAU2*X2*DEL2
FR2=TAU2*(X3*DEL2)+(2.*X2*T*DEL)+(4.*X*EPS))
FR3=(2.*X2*DEL)-(4.*T*EPS)+4.
CFR=-1.4:(FR1+FR2+FR3)
FII=TAU2塞尔X3*DEL2)*2.**EPS)
FII=TAU2塞尔X2*DEL2)(2.*X**EPS)
FII=(2.*X2*DEL)*(2.*EPS)
CFII=-1.%((FII+FI2+FI3)
20 CONS=3.14129631.*CSA*(4.*((X2*DEL2)))
XLIIT=CONS*SQR((CFR*CFR)+(CFII*CFII))
CFR=CFI*CONS
CFII=CFI*CONS
WRITE(108,602)
WRITE(108,611)
WRITE(108,602)
WRITE(108,612)
WRITE(108,602)
WRITE(108,607)
WRITE(108,603)
WRITE(108,608)
READ(105,502)
IF(N.EQ.1) GO TO 10
600 FORMAT(IX, 'ENTER STAGGER ANGLE; PITCH=; IX, 1' CHORD AND RELATIVE VELOCITY')
601 FORMAT(IX, 'ENTER EXCITATION FREQUENCY; EXCITATION=; IX, 1' WAVELENGTH')
602 FORMAT(IX, 'RELATIVE FLOW ANGLE; AND BLADE SPEED')
603 FORMAT(IX, 'DATA GIVEN')
604 FORMAT(IX, 'STAGGER=; FS.3; IX, 5X; PITCH=; FS.3; L')
1., 'CHORD=; FS.3;')
605 FORMAT(IX, 'EXCITATION FREQUENCY=; FS.3; IX, 5X
1 EXCITATION WAVELENGTH=; FS.3;')
606 FORMAT(IX, 'RELATIVE VELOCITY=; FS.3; IX, 5X, 1 RELATIVE FLOW ANGLE=; FS.3;)
607 FORMAT(IX, 'UNSTEADY LIFT COEFFICIENT=; FS.3;')
608 FORMAT(IX, 'ENTER 1 TO RUN NEW DATA; 2 TO QUIT')
609 FORMAT(IX, 'REAL COMPONENT OF LIFT COEFFICIENT; FS.3;')
610 FORMAT(IX, 'IMAG. COMPONENT OF LIFT COEFFICIENT; FS.3;')
650 FORMAT(IX, 'ENTER 1 IF TURBINE; 2 IF COMPRESSOR')
660 FORMAT(IX, 'BLADE SPEED=; FS.3;')
500 FORMAT(610.0)
501 FORMAT(IX, 4010.0)
502 FORMAT(1)
503 STOP
END
9. **COMMENTS ON THE COMPUTER PROGRAMS FOR NONSTEADY FORCE CALCULATIONS**

9.1 General

The computer programs currently in the program library are listed in Table 9. The existing programs have been verified by comparing the results with sample results given by the authors. The verification is limited by the fact that, with the exception of the Kemp-Sears program, no comparison of theoretical results with experimental data was made. This is due in part to the difficulties involved in acquiring sufficient experimental data on turbine stage excitation in the open literature. Results obtained from the computer programs described in Sections 7 and 8 should therefore be used with great caution until correlation with experimental data is achieved.

In the remainder of this section the limitations of the programs currently in the library are discussed and suggestions are given for further development and expansion of the program library. Finally, a procedure is given for selecting a suitable program based on the physical parameters of the system to be modeled.

9.2 Compressibility Effects

All of the programs described in sections 7 and 8 are based on incompressible flow theory. Goslow [81] has reviewed existing theories based on incompressible flow. Of these, the analysis by Osborne [4] appears to be the most comprehensive for subsonic flow: See Section 5.2. Miles [36] has given an analysis for compressible supersonic flow. Results obtained by both authors show significant variations in the unsteady loading on the blades with varying Mach numbers. Since the Osborne analysis is similar to the Kemp-Sears analysis, many segments of the Kemp-Sears program can be directly incorporated into a program based on the Osborne analysis. This would result in a significant decrease in the time required to debug the program. A particular example is the Bessel function subroutine package associated with the Kemp-Sears program. Approximately 25 percent of the
TABLE 9. List of Programs in Current Program Library.

<table>
<thead>
<tr>
<th>Analysis</th>
<th>Reference Number</th>
<th>Program Identification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kemp - Sears</td>
<td>[2],[3]</td>
<td>LERAO</td>
</tr>
<tr>
<td>Henderson - Horlock</td>
<td>[33]</td>
<td>HENHOR</td>
</tr>
<tr>
<td>Horlock - Greitzer - Henderson</td>
<td>[35]</td>
<td>HGH</td>
</tr>
<tr>
<td>Whitehead</td>
<td>[26]</td>
<td>WHIT</td>
</tr>
</tbody>
</table>
development and debug time associated with the Kemp-Sears program was spent in formulating and debugging subroutine CBESJ for calculating Bessel functions of the first kind of order $n$ with complex arguments and subroutine BEJIM for calculating Bessel functions of the second kind of order $n$ with negative real arguments. These subroutines are now fully operational and may be directly incorporated into future programs.

9.3 Interaction Effects Between Adjacent Blades

Each of the programs in the library are based on the assumption that adjacent blades vibrate nearly in phase. This is a simplifying assumption which allows the unsteady motion of adjacent blades and the related induced unsteady flow to be neglected in the calculation of the unsteady forces on a blade. No studies published in the open literature have been made to determine the magnitude of the unsteady loads on blades due to the unsteady motion of adjacent blades.

Another implication of this assumption is that the blades are free standing with no mechanical coupling such as tie wires or shrouds between blades. Since many LP turbine blade rows consist of groups of blades connected by tie wires and shrouds, the programmed analyses are perhaps over simplified for these applications. Reformulation of the analyses to include possible mechanical coupling between blades may be required.

9.4 Velocity Defect Definition

The Kemp-Sears program is the only program which calculates the velocity field downstream of the stator cascade. All other programs require this velocity field to be input as data. The unsteady force values obtained by the Kemp-Sears analyses are therefore limited in accuracy by the accuracy of the calculated velocity field parameters. As shown in Section 12.5, the magnitude of the velocity defect in the viscous wakes calculated by the
Kemp-Sears program are significantly greater than those measured in test cascades. The reformulation of the Kemp-Sears analysis using the analytical model of the wake downstream of a stator blade given by Raj and Lakshminarayana [56] would be straightforward and could improve the accuracy of the results given by the Kemp-Sears analysis. Comments similar to those above also apply to the Osborne analysis. The actuator disk analyses all require the velocity field to be defined as input data. Since these analyses are valid only for low per-rev excitation this is not a severe difficulty.

9.5 Further Development of Program Library

Several analyses described in the previous sections would be valuable additions to the existing library of computer programs. As mentioned previously, a program based on the analysis by Osborne [4] would allow calculation of the unsteady forces on blades for Mach numbers up to approximately 0.8, with compressibility effects included. Such a program would be ideally suited for studies of low solidity. LP turbine stages if the modifications in the viscous wake calculation procedure discussed above are made. A program based on the modified Whitehead analysis [27], described in Section 4.3, would be much more flexible than the existing actuator disk programs in that a wider range of per-rev excitations could be studied. A program of this type, together with the modified version of the Kemp-Sears program, could determine the unsteady forces on a blade due to excitations varying from one-per-rev up to several times nozzle passing frequency for subsonic, incompressible flow. The programs based on the analyses of Henderson, Greitzer and Horlock [35], and the original Whitehead actuator disk analysis [26] would be replaced by the modified actuator disk program.

Instead the forces on thin blades of any camber in subsonic incompressible flow could be calculated using a program based on the analysis of Mukhopadhyay [18]: See Section 3.7.

The addition of these programs to the existing library of programs would
result in a comprehensive library based on state-of-the-art theories for the calculation of unsteady forces on turbomachine blades under subsonic flow conditions.

9.6 Selection of a Suitable Analysis

The primary consideration in selecting the appropriate analysis is the type of excitation to be modeled. At present excitation due to an upstream row of stators can only be modeled by the Kemp-Sears program (LERAO).

Per-rev excitation due to relatively few upstream obstructions can be modeled by either of three actuator disk analyses. In general, for closely spaced blades and typically five or less obstructions upstream, either the Herlock, Greitzer and Henderson program (HGH) or the Whitehead program (WHIT) may be applied. The Henderson-Horlock (HENHOR) program should be applied to the intermediate cases, i.e. more than five upstream obstructions typically. Note that this analysis is the only one in the library which allows cambered blades to be modeled.
10. **THE HYDRAULIC ANALOGY**

10.1 **General**

The hydraulic analogy as applied in this thesis is based on the mathematical similarity between the equations governing the two-dimensional, isentropic flow of a compressible, perfect gas with a specific heat ratio equal to two, and the equations governing the two-dimensional, isentropic, free surface flow of water over a horizontal surface. The analogous relationships between the two flows are defined in section 10.2.

10.1.1 **Development of the Hydraulic Analogy**

The qualitative similarity between water waves in a river near bridge piers and gas waves near similar blunt bodies was noted and published by Mach [57] in 1887. Jouget [58] derived the fundamental mathematical relationships of the analogy in 1920. Riabouchinsky [59] further developed the mathematical theory of the analogy in 1932 to supplement an experimental investigation of choking in supersonic diffusers. Several years later Binnie and Hooker [60] applied the analogy to study supersonic nozzle flow. Ippen and Knapp [61], crediting von Karman, applied the analogy "in reverse" by using aerodynamic methods to describe supercritical water flow. Von Karman [62] subsequently developed the mathematics of the analogy in great detail to highlight its application to studies of both aerodynamic and hydraulic phenomena. Further development of the theory of the hydraulic analogy with considerations of the primary assumptions have been given by Loh [63], Bryant [64], Thompson [65], Harleman and Ippen [66], Breugelmans [67], Johnson [68], Rieger [69] and Laitone [70]. Bibliographies giving 250 individual references have been prepared by Hoyt [71].
and Bryant [72]. A bibliography of papers reviewed by the author is given in Appendix II of this thesis.

10.1.2 Studies of Flow in Turbomachines by the Hydraulic Analogy.

One of the earliest investigations of flow in a turbine stage was published by Preiswerk [73] in 1942. Included were still photographs of subsonic, low supersonic, and high supersonic flows through a stationary cascade. For twenty years, the stationary cascade represented the state-of-the-art for hydraulic analogy studies of flow in turbines and compressors. Results from several investigations of this type have been compared to cascade wind tunnel test results with good correlation being observed: see Hoyt [71]. Harleman and Ippen [66] conducted an investigation to determine the quantitative accuracy of the hydraulic analogy for transonic flow conditions. Several wedge profiles were towed through a one inch deep water tank at speeds which modeled Mach numbers from 0.8 to 1.09. The ratio of the local pressure to the critical pressure (at $M=1.0$) was determined at several locations along the chord. Figure 37 shows results obtained for a cusped wedge profile and a $20^\circ$ wedge profile. Good agreement with the theory is obtained over most of the cusped wedge (maximum error 11%), though agreement is not so good for the $20^\circ$ wedge (maximum error 27%). The authors attribute this large error to the large vertical accelerations produced by the lead edge of the profile, which (locally) violate a primary assumption of the analogy.

More recent studies have used both stationary blades and moving blades to more accurately model the fluid-structure interaction in a turbine stage. Heen and Mann [74] studied flow in a partial admission turbine stage using a track and carriage apparatus: figure 38. The water depth was measured along a blade passage.
Figure 37. Ratio of Local Pressure to Critical Pressure vs. Percent of Chord for Several Free Stream Mach Numbers. From Harleman and Ippen [66].

a) 20° wedge profile

b) Cusped wedge profile
Figure 38. Partial Arc Admission Test Rig. From Heen and Mann [74].
A theoretical depth at each position was obtained from one-dimensional water flow theory. Typical results plotted against theoretical values are shown in figure 39. Johnson [68] has described a two stage axial flow compressor study made at the General Electric Research Laboratory. The model employed two rows of rotor blades which moved relative to adjacent stator rows. The apparatus was used to study off-design conditions as well as the effects of varying rotor and stator angles and spacings. Slow-motion moving pictures were taken during testing. Johnson has commented:

"Although much of the information obtained was qualitative, it has contributed to the understanding of many flow details that make up the gross performance of machines, and, would not be possible to obtain in prototype testing."

Similar tests have been performed on turbine stages using a similar apparatus. Rhomberg [75] has presented a paper which demonstrates the compatability of qualitative results obtained from a water table model and those obtained from a rotating transonic air cascade. Shadowgraphs of the flow downstream of a transonic air cascade are shown in figure 40(a). The dark lines are shock waves. Figure 40(b) is a photograph of the water flow pattern in the same region of the analogous hydraulic cascade. Good correlation exists in the location of the shock waves obtained by each method.

Owczarek [76] investigated a periodic wave phenomenon occurring from stage interactions between turbine blades and stators. This phenomenon occurs when a pressure wave is generated on the leading edge of the rotor blades. From a theoretical analysis Owczarek concluded that a particular speed range may exist in which this phenomenon could occur depending on:

(i) the ratio of the number of nozzles to the number of blades.
(ii) the axial distance between the trail edge of the nozzles and the lead edge of the blades.
Figure 39. Water Depth vs. Distance for Several Locations in Moving Blade Passages. From Heen and Mann [74].
(A) Air flow downstream of transonic cascade.

(B) Water flow downstream of transonic cascade.

Figure 40. Comparison of air and water flow patterns downstream of a transonic cascade. [67]
(iii) the angle and shape of the nozzle.
(iv) the Mach number of the nozzle exit flow.

Owczarek constructed a rotating radial inflow water table model of a suitable configuration for this phenomenon, according to theory. Photographs were taken of the flow between the stator and rotor which showed waves (analogous to gas shocks) which propagated as predicted by theory.

10.2 Theory of the Hydraulic Analogy.

As stated previously, the hydraulic analogy as used in this thesis is based on the mathematical similarity between the equations of motion for the two-dimensional, isentropic flow of a compressible, perfect gas and the equations of motion for the two-dimensional, isentropic, free surface flow of water over a horizontal surface. Loh has summarized the analogous expression for one-dimensional steady flow [77], one-dimensional unsteady flow [63] and two-dimensional steady flow [63]. The expression for two-dimensional unsteady flow in cylindrical coordinates has been given by Rieger [69]. Since the analogous relationships obtained for the unsteady flow case are identical to those obtained for the steady flow case, the two-dimensional steady flow equations given by Loh [63] will be discussed here.

10.2.1 Equations of Motion

Consider the two-dimensional gas flow field pictured in figure 41(a). At some initial reference position \((x_0, y_0)\) the pressure \(p_0\), density \(\rho_0\) and temperature \(T_0\) are given. At a second point \((x, y)\) in the flow field the velocity in the \(x\)-direction \(u\), the velocity in the \(y\)-direction \(v\), the pressure \(p\), the density \(\rho\) and the temperature
Figure 41. Analogous Two-Dimensional Gas and Water Flow Fields.
T are known. The continuity equation for these two positions may then be written.

\[ \bar{u} \left( \frac{\partial \bar{h}}{\partial x} \right) + \bar{v} \left( \frac{\partial \bar{h}}{\partial y} \right) + \bar{u} \left( \frac{\partial \bar{y}}{\partial y} \right) + \bar{v} \left( \frac{\partial \bar{y}}{\partial x} \right) = 0 \quad \text{(10.1)} \]

where

\[ \bar{u} = \frac{u}{a_0} ; \quad \bar{v} = \frac{v}{a_0} ; \quad \bar{p} = \frac{p}{p_0} ; \quad a_0 = \sqrt{g \gamma RT_0} \]

and

\[ \bar{x} = \frac{x}{l} ; \quad \bar{y} = \frac{y}{l} \]

In the above \( l \) is an arbitrary characteristic length and \( a_0 \) is the local sonic velocity.

A similar expression is obtained for the two-dimensional water flow field shown in figure 41(b).

\[ \bar{u} \left( \frac{\partial \bar{h}}{\partial x} \right) + \bar{v} \left( \frac{\partial \bar{h}}{\partial y} \right) + \bar{u} \left( \frac{\partial \bar{y}}{\partial y} \right) + \bar{v} \left( \frac{\partial \bar{y}}{\partial x} \right) = 0 \quad \text{(10.2)} \]

where

\[ \bar{u} = \frac{u}{c_0} ; \quad \bar{v} = \frac{v}{c_0} ; \quad \bar{h} = \frac{h}{h_0} ; \quad c_0 = \sqrt{g h_0} \]

and

\[ \bar{x} = \frac{x}{l} ; \quad \bar{y} = \frac{y}{l} \]

In the above \( h \) denotes the local water depth and \( c_0 \) is the local gravity wave propagation velocity.

The momentum equations in the \( x \) and \( y \) directions for the gas flow are

**x-direction**

\[ \bar{u} \left( \frac{\partial \bar{u}}{\partial x} \right) + \bar{v} \left( \frac{\partial \bar{u}}{\partial y} \right) = -\frac{1}{\gamma-1} \frac{3}{\bar{x}} \left[ \frac{\gamma-1}{\gamma} \frac{\partial \bar{c}}{\partial x} \right] \quad \text{(10.3a)} \]
y-direction \( \tau (\frac{\partial u}{\partial y}) + u (\frac{\partial \tau}{\partial y}) = - \frac{1}{\gamma - 1} \frac{\partial}{\partial x} \left[ \rho \left( \frac{y-1}{\gamma} \right) \right] \) \( (10.3) \)

where \( \gamma \) is the specific heat ratio of the gas and \( \rho = \frac{\rho}{\rho_0} \).

Similar expressions are obtained for the water flow field.

x-direction \( u (\frac{\partial u}{\partial x}) + \tau (\frac{\partial \tau}{\partial x}) = - \frac{\partial h}{\partial x} \) \( (10.4) \)

y-direction \( \tau (\frac{\partial u}{\partial y}) + u (\frac{\partial \tau}{\partial y}) = - \frac{\partial h}{\partial y} \)

The energy equation for the gas flow is \( \frac{u^2}{u_{\infty}^2} = 1 - \frac{T}{T_0} \) \( (10.5) \)

and for the water flow is \( \frac{u^2}{u_{\infty}^2} = 1 - \frac{h}{h_0} \) \( (10.6) \)

10.2.2 Analogous Relationships.

Comparing the continuity expressions, \((10.1)\) and \((10.2)\) and assuming that the sonic velocity in the gas is proportional to the wave velocity in the water i.e.

\[ \sqrt{\gamma \rho RT} \propto \sqrt{\gamma h} \] \( (10.7) \)

yields the following analogous relationship between the gas density ratio and the water depth ratio.
\[ \overline{p} = \overline{\rho} \quad (10.8) \]

or

\[ \frac{\overline{p}}{p_0} = \frac{h}{h_0} \quad (10.9) \]

Comparing the energy equations (10.5) and (10.6) yields the following relationship between the gas temperature ratio and the water depth ratio.

\[ \overline{T} = \frac{T}{T_0} \quad (10.10) \]

or

\[ \frac{T}{T_0} = \frac{h}{h_0} \]

From a comparison of the momentum equations (10.3) and (10.4), the following relationship between the gas pressure ratio and the water depth ratio is obtained.

\[ \frac{1}{\gamma - 1} \left( \frac{\overline{p}}{p_0} \right)^{\gamma - 1} = \frac{h}{h_0} \quad (10.10) \]

In order to satisfy this condition, the specific heat ratio of the gas \( \gamma \) must be equal to 2.0 so that

\[ \frac{\overline{p}}{p_0} = \left( \frac{\gamma}{\gamma - 1} \right)^2 \quad (10.11) \]
Since no naturally occurring gas has a specific heat ratio of two, it is necessary to apply a correction factor when modeling gas flow by the hydraulic analogy. Several correction factors are discussed in section 10.4. The above analogous relationships are summarized in table 10.

10.3 Assumptions of the Hydraulic Analogy.

The major and implied assumptions of the hydraulic analogy are given in table 11. Comments by major authors on the assumptions are given in table 12. Of the nine major assumptions the four discussed below appear to be of primary importance in determining the quantitative accuracy of the hydraulic analogy.

Isentropic flow of a gas implies shock-free flow. In water flow, the phenomena which is analogous to a shock is the hydraulic jump. Shocks frequently occur in many turbomachine stages i.e. in the nozzle throats of axial flow steam turbines. Shocks occur with equal frequency in water table models of turbomachine stages. Many authors [74] [66] [78] [79] [80] have concluded that for weak shocks, the entropy increase is small enough that it need not be considered a violation of the analogy. Gilmore, et al [79] have established a depth ratio across the shock of two as the maximum allowed for this assumption to remain valid.

Another implication of isentropic flow is that no viscous effects may occur. Thus no boundary layers in either the gas or water flow are included in the analogy. The flow of any real liquid must violate the assumption of zero boundary layer thickness. In the case of flow past a stationary model, there are two boundary layers which must be considered: (i) the bottom boundary layer, and (ii) the boundary
TABLE 10. Analogous Quantities Between Gas and Water Flow.

<table>
<thead>
<tr>
<th>Gas Flow</th>
<th>Water Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density Ratio ( (p_0/p) )</td>
<td>Depth Ratio ( (h_0/h) )</td>
</tr>
<tr>
<td>Temperature Ratio ( (T_0/T) )</td>
<td>Depth Ratio ( (h_0/h) )</td>
</tr>
<tr>
<td>Pressure Ratio ( (p_0/p) )</td>
<td>Depth Ratio Squared ( (h_0/h)^2 )</td>
</tr>
<tr>
<td>Sonic Velocity ( a = \sqrt{gRT} )</td>
<td>Gravity Wave Propagation Velocity ( c = \sqrt{gh} )</td>
</tr>
<tr>
<td>Mach Number ( (M) )</td>
<td>Froude Number ( (Fr) )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>MAJOR ASSUMPTIONS</th>
<th>IMPLIED ASSUMPTIONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Ideal liquid</td>
<td>a) Incompressible fluid</td>
</tr>
<tr>
<td>2. Perfect gas</td>
<td>a) Density is a function of pressure and temperature only.</td>
</tr>
<tr>
<td>3. Isentropic gas and water flows</td>
<td>b) Speed of sound: ( a = \sqrt{\frac{g}{g} Y R T} )</td>
</tr>
<tr>
<td>4. Two dimensional gas and water flows</td>
<td>a) Shock free flows</td>
</tr>
<tr>
<td>5. Small unsteady velocity perturbations</td>
<td>b) Inviscid flows (no viscous wakes or boundary layer)</td>
</tr>
<tr>
<td>6. Stationary coordinates</td>
<td>c) Irrotational flows</td>
</tr>
<tr>
<td>7. Kinematic similarity between gas and water flows</td>
<td>d) No flow separation</td>
</tr>
<tr>
<td>8. Negligible surface tension</td>
<td>a) Uniform flow through water depth</td>
</tr>
<tr>
<td>9. Specific heat ratio of gas: ( Y \approx 2.0 )</td>
<td>a) Wave propagation velocity: ( c = \sqrt{gh} )</td>
</tr>
<tr>
<td></td>
<td>b) No meniscus</td>
</tr>
</tbody>
</table>
Table 12. Review of Comments Concerning Assumptions and Conditions of Hydraulic Analogy.

<table>
<thead>
<tr>
<th>Assumption</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inviscid Water Flow</td>
</tr>
<tr>
<td>Reference</td>
</tr>
<tr>
<td>Adams, D. M.</td>
</tr>
<tr>
<td>Gilmore, F. R.</td>
</tr>
<tr>
<td>Harlmann, D. B. P. and Spinn, A. T.</td>
</tr>
<tr>
<td>Herr, B. X. and Mann, A. V.</td>
</tr>
<tr>
<td>Klein, E. J.</td>
</tr>
<tr>
<td>Laitone, E. V. and Nielsen, R.</td>
</tr>
<tr>
<td>Szzechely, V. G. and Whicker, L. F.</td>
</tr>
<tr>
<td>Bryant, R. A. A.</td>
</tr>
<tr>
<td>Bryant, R. A. A.</td>
</tr>
<tr>
<td>Laitone, E. V.</td>
</tr>
<tr>
<td>Loh, W. H. T.</td>
</tr>
<tr>
<td>Stahler, A. F.</td>
</tr>
</tbody>
</table>

Conclusions

No Reynolds number modeling is possible. Boundary layer effects are significant. Vorticity may be analogous. Weak shocks with negligible entropy increase can be analogous. h/l < 1.0. For a range of shallow shocks centered around 1.0, the assumption is approximately valid.
| Potential Flow | No Flow Separation Comment | Two-dimensional Flow Vertical Acceleration Negligible Wave Speed  
\(c = \sqrt{\rho g h} \) |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>No comment</td>
<td>No comment</td>
<td>Vertical accelerations are small for small slopes of table surface, and ( h &lt; 1.0 ) in. Non-sustained waves can be reduced by reduction of surface tension.</td>
</tr>
<tr>
<td>No comment</td>
<td>No comment</td>
<td>No comment</td>
</tr>
<tr>
<td>No comment</td>
<td>h = 1.0 in. resulted in small vertical velocity for flowing water.</td>
<td>h = 1.0 in. resulted in small vertical acceleration.</td>
</tr>
<tr>
<td>No comment</td>
<td>Assumed true, ( h &lt; 1.0 ) in.</td>
<td>Assumed true, ( h &lt; 1.0 ) in.</td>
</tr>
<tr>
<td>No comment</td>
<td>Assumed true</td>
<td>Assumed true</td>
</tr>
<tr>
<td>No comment</td>
<td>Towed model</td>
<td>Assumed true</td>
</tr>
<tr>
<td>No comment</td>
<td>Assumed true</td>
<td>Assumed true</td>
</tr>
<tr>
<td>No comment</td>
<td>Assumed true</td>
<td>Assumed true</td>
</tr>
<tr>
<td>No comment</td>
<td>Assumed true</td>
<td>Assumed true</td>
</tr>
<tr>
<td>No comment</td>
<td>Assumed true</td>
<td>Assumed true</td>
</tr>
<tr>
<td>No comment</td>
<td>One-dimensional analysis.</td>
<td>Assumed true</td>
</tr>
<tr>
<td>No comment</td>
<td>Assumed true</td>
<td>No comment</td>
</tr>
<tr>
<td>Assumed true</td>
<td>No comment</td>
<td>Radial outflow over horizontal surface.</td>
</tr>
</tbody>
</table>
| Assumed true  | Must be assumed.             | True, if depth is large compared to boundary layer thickness. | Reductio

of surface tension proper selection of depth can minimize error.
**Table 12. (con't)**

<table>
<thead>
<tr>
<th>No Meniscus</th>
<th>Small Unsteady Velocity Perturbations</th>
<th>Incompressible Water Flow</th>
<th>Kinematic Similarity of Gas and Water Flow</th>
<th>Stationary Coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>No comment</td>
<td>No comment</td>
<td>a priori assumption</td>
<td>a priori assumption</td>
<td>No comment</td>
</tr>
<tr>
<td>No comment</td>
<td>No comment</td>
<td>a priori assumption</td>
<td>a priori assumption</td>
<td>No comment</td>
</tr>
<tr>
<td>No comment</td>
<td>No comment</td>
<td>a priori assumption</td>
<td>a priori assumption</td>
<td>No comment</td>
</tr>
<tr>
<td>No comment</td>
<td>No comment</td>
<td>a priori assumption</td>
<td>a priori assumption</td>
<td>No comment</td>
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<tr>
<td>No comment</td>
<td>No comment</td>
<td>a priori assumption</td>
<td>a priori assumption</td>
<td>No comment</td>
</tr>
<tr>
<td>No comment</td>
<td>No comment</td>
<td>a priori assumption</td>
<td>a priori assumption</td>
<td>No comment</td>
</tr>
<tr>
<td>No comment</td>
<td>No comment</td>
<td>a priori assumption</td>
<td>a priori assumption</td>
<td>No comment</td>
</tr>
<tr>
<td>No comment</td>
<td>No comment</td>
<td>a priori assumption</td>
<td>a priori assumption</td>
<td>No comment</td>
</tr>
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<td>No comment</td>
<td>No comment</td>
<td>a priori assumption</td>
<td>a priori assumption</td>
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<td>No comment</td>
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<td>a priori assumption</td>
<td>a priori assumption</td>
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<td>a priori assumption</td>
<td>a priori assumption</td>
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<td>No comment</td>
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<td>a priori assumption</td>
<td>a priori assumption</td>
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<tr>
<td>No comment</td>
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<td>a priori assumption</td>
<td>a priori assumption</td>
<td>No comment</td>
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<tr>
<td>No comment</td>
<td>No comment</td>
<td>a priori assumption</td>
<td>a priori assumption</td>
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<tr>
<td>No comment</td>
<td>No comment</td>
<td>a priori assumption</td>
<td>a priori assumption</td>
<td>No comment</td>
</tr>
<tr>
<td>No comment</td>
<td>No comment</td>
<td>a priori assumption</td>
<td>a priori assumption</td>
<td>Must be assumed</td>
</tr>
</tbody>
</table>

Model surface can be treated Assumed in analysis, to reduce meniscus.
Table 12. (con't)

<table>
<thead>
<tr>
<th>Perfect Gas Assumed true for air.</th>
<th>Two-Dimensional Gas Flow Assumed true for air.</th>
<th>Irrotational Gas Flow (Inviscid Flow) For large Reynolds numbers in gas and water flow, similarity is not important, viscous effects are approximately analogous.</th>
<th>Shock Free Gas Flow Negligible effects for weak shocks.</th>
<th>Specific Heat Factor $\gamma = 1.4$ Can be adjusted using correction factors.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assumed true</td>
<td>Assumed true</td>
<td>Assumed true Isentropic flow assumed.</td>
<td>Negligible effects for weak shocks.</td>
<td>Small errors will reappear.</td>
</tr>
<tr>
<td>Assumed true</td>
<td>Assumed true</td>
<td>Assumed true No comment Analogy breaks down when shocks occur.</td>
<td>No comment</td>
<td>Channel shape varied to account for $\gamma \neq 1.4$.</td>
</tr>
<tr>
<td>Assumed true</td>
<td>Assumed true Vorticity downstream of bow wave has negligible effect.</td>
<td>No comment Can be adjusted using correction factors.</td>
<td>No comment No correction present.</td>
<td></td>
</tr>
<tr>
<td>Assumed true</td>
<td>Assumed true Isentropic flow is not vital for analogy to hold.</td>
<td>No comment No correction present.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Assumed true</td>
<td>Assumed true Isentropic flow implied. Assume weak shocks with negligible entropy increase.</td>
<td>No comment No correction present.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Assumed true</td>
<td>One-dimensional analysis. Assumed true Isentropic flow assumed.</td>
<td>Assumed true Isentropic flow assumed.</td>
<td>Channel shape varied to account for $\gamma \neq 1.4$.</td>
<td></td>
</tr>
<tr>
<td>Assumed true</td>
<td>Assumed true Isentropic flow assumed.</td>
<td>Assumed true Isentropic flow assumed.</td>
<td>Channel shape varied to account for $\gamma \neq 1.4$.</td>
<td></td>
</tr>
<tr>
<td>Assumed true</td>
<td>Assumed true Isentropic flow assumed.</td>
<td>Assumed true Isentropic flow assumed.</td>
<td>Assumed $\gamma = 1.4$. No correction attempted.</td>
<td></td>
</tr>
<tr>
<td>Assumed true in analysis. Can be assumed if radial velocity and radial velocity gradient are small.</td>
<td>Can be assumed if radial velocity and radial velocity gradient are small.</td>
<td>Weak shocks can be analyzed.</td>
<td>Can be accounted for use of correction factors.</td>
<td></td>
</tr>
</tbody>
</table>

Table 12. (con't)
layer around the model. The low speed boundary layer around the model will not behave as the high speed boundary layer of the gas flow. The bottom boundary layer has no analogous counterpart in the gas flow. It must be concluded that boundary layer effects are sources of inaccuracy and must be considered in modeling. Adams [78] has suggested that the growth of the bottom boundary layer be measured and an opposite slope be incorporated on the table surface. The result would be an apparent horizontal surface. Bryant [81] has presented an analytical method for incorporating the boundary layer thickness along the model into the scaling factor. The resulting water table model is scaled accordingly. Bryant also suggests the use of towed models through stationary water to eliminate the bottom boundary layer. Another effect of viscosity is viscous wakes. As shown in section 6, these wakes can be a major contribution to the total unsteady loading on a blade. Adams [78] states that viscous effects in the water flow field can be approximately analogous to those in the gas flow field, only if the following conditions exist: (i) geometric similarity of the gas and water table models; (ii) surface roughness similarity; (iii) dynamic similarity of the gas and water flows. It is noted that Reynolds number simulation is difficult to achieve without extreme measures such as heating the water or injecting additional water at some point in the flow field. If, however, the Reynolds numbers of the gas and water flow are sufficiently large, frictional effects are small and the need for similarity is reduced.

The assumption of two-dimensional water flow requires that the velocity be constant throughout the water depth. Adams [78] has concluded that for depths of one inch or less the flow through the depth is uniform. This is coupled with the small table surface slope, which is equal and opposite to the rate of boundary layer growth. Harleman and Ippen [66] and Laitone and Nielsen [82] have concluded that a towed model is desirable since this requirement is exactly satisfied. Harleman and Ippen also concluded that for flow past a stationary model this assumption is valid if the water depth is one
inch or less. Wicks [83] has measured the velocity at several vertical positions in water 0.55 inches deep. The difference between the maximum velocity (0.1 inches from the surface of the water) and the minimum velocity (0.1 inches from the water table surface) was less than 5%.

Surface tension effects may create severe inaccuracies in analogous flow conditions. The hydraulic analogy requires that the propagation speed of water waves be a function of depth only so that this speed is proportional to the local speed of sound in the gas flow. The surface tension of a liquid is responsible for the formation of capillary waves which propagate at a different speed changing the wave propagation velocity to a group velocity which is a combination of the two wave forms. This group velocity is given by the following expression.

\[ c^2 = \left( \frac{a^2}{2 \pi^2 \lambda^2} + \frac{2 \pi S}{\rho \lambda} \right) + \frac{2 \pi h}{\lambda} \tan h \frac{2 \pi h}{\lambda} \]  

(10.12)

where \( S \) is the surface tension and \( \lambda \) is the wavelength of the surface tension waves.

Adams [78] and Gilmore, et al [79] have concluded that surface tension reduction by the addition of a detergent improves the accuracy of the hydraulic analogy by decreasing the capillary wave amplitude. Heen and Mann [74] and Laitone and Nielsen [82] have concluded that pure water satisfies this requirement if the depth is sufficiently small (e.g., \( h \approx 0.25 \) inches). Szebehely and Whicker [84] and Laitone [70] have further clarified this by showing that the assumption is satisfied if the depth is small compared to the wavelength. Bryant [81] has stated that, for transonic flow, differences in wave propagation speed are negligible if the transonic similarity laws are applied to modify the water flow parameters.
Another surface tension effect is the meniscus, which has no analogous phenomenon in the gas flow. The meniscus makes depth measurements difficult and causes nonuniform wetting of the models. Surface tension reduction serves to decrease the meniscus as well as minimizing capillary wave amplitudes.

The requirement that the specific heat ratio of the gas must be equal to two is of primary concern to many authors. Heen and Mann [74] however, concluded that the errors resulting from the violation of this assumption are small, with little effect on the quantitative accuracy of the analogy. It should be noted that this is a minority opinion. It is common practice to model the flow of a certain gas by another gas with a different specific heat ratio. For example, many experimental investigations of flow in a steam turbine stage are made using air as the operating fluid. Similarity laws are used in such studies to adjust the model parameters to maintain similarity with the prototype conditions. The same procedures can be used to adjust the water table parameters. A further discussion of the correction factors used in water table studies is given in section 10.5.

10.4 The R.I.T. Rotating Water Table

A device which employs the hydraulic analogy to study gas flow phenomena is called a water table. Of the many water table studies made during the ninety-year history of the hydraulic analogy, only two references in the open literature describe rotating water tables. Owczarek [76] designed and built a water table with a blade row rotating concentrically around a stationary nozzle row. Although this was a pioneering effort, the small scale (17 inch diameter) precluded the study of many practical problems such as multi-stage operation. Meier [85] also studied stage flows using a rotating water table though this table also was of a small diameter.

A third rotating water table of a much larger size has been developed at R.I.T. by Rieger and Wicks [86]. The R.I.T. water table is shown
in figure 42. This apparatus is capable of modeling gas flows in either a turbine or a compressor stage. The range of tests which can be performed on the water table is shown in table 13. A brief description of the water table and typical tests performed to date is given below. A more complete description is given in reference [87].

10.4.1 Turbine Water Table.

Figure 43 is a schematic of the water table arranged to operate in the turbine mode. Water flows radially outward from a central plenum-diffuser on to the horizontal table surface and through a stationary nozzle row. The nozzles accelerate the flow and direct it into a variable speed rotating blade row. After passing through the moving blade row, the water flows out across the horizontal table surface to a fine mesh wire screen which simulates the back pressure of the stage. The water passes through the screen and returns to the sump via gutters.

Unsteady forces on either a rotor blade or a stator blade may be monitored using the semiconductor strain gage load cells shown in figure 44. The strain gage output is amplified on board the rotor before passing through a multi-channel gold slip ring assembly to be recorded.

10.4.2 Turbine Stage Tests.

Non-Steady Forces from Nozzle Wakes. A single stage test was made to determine the excitation levels associated with a certain nozzle design which caused inservice failures. The stage geometry is shown in figure 45. A typical force response amplitude vs. frequency spectrum from the moving blade strain gage output is shown in figure 46. A large amplitude spike exists at nozzle passing frequency (NPF). Such spikes represent significant forces applied harmonically to each moving
TABLE 13. Range of Tests Possible on the RIT Water Table. From Rieger and Crofoot [95].

<table>
<thead>
<tr>
<th>COMPRESSOR MODE</th>
<th>TURBINE MODE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radial Inflow:</td>
<td>Radial Inflow:</td>
</tr>
<tr>
<td>Axial flow fans</td>
<td>Radial flow steam turbines*</td>
</tr>
<tr>
<td>Axial flow blowers</td>
<td>Radial flow gas turbines</td>
</tr>
<tr>
<td>Axial flow compressors*</td>
<td></td>
</tr>
<tr>
<td>Radial Outflow:</td>
<td>Radial Outflow:</td>
</tr>
<tr>
<td>Centrifugal fans</td>
<td>Axial flow steam turbines (partial admission)*</td>
</tr>
<tr>
<td>Centrifugal blowers</td>
<td>Axial flow steam turbines (full admission) [HP*, IP*, and LP* stages]</td>
</tr>
<tr>
<td>Centrifugal compressors</td>
<td>Axial flow gas turbines</td>
</tr>
</tbody>
</table>
Figure 43. Flow Circuit Schematic for Water Table in Turbine Mode.
Figure 44. Strain Gaged Monitor Posts Showing Semi-Conductor Strain Gages (right) and Aluminum Shield (left).
NUMBER OF:  
   Blades   120  
   Nozzles   38  

NOZZLE TO BLADE RATIO: $N_r = 0.3167$

CHORD LENGTHS:  
   Blades   2.629 in  
   Nozzles   5.468 in.

PITCH LENGTH:  
   Blades   1.7528 in  
   Nozzles   1.584 in

AXIAL SPACING:   0.25 in.

Figure 45. Single Stage Turbine Test Configuration. From Rieger and Crofoot [95].
Figure 46. Tangential Load
Magnitude vs. Frequency.
From Rieger and Crofoot [95].

Velocity ratio: 0.5
Pressure ratio: 1.35

38 Nozzles 120 Blades
blades at N.P.F. Harmonics of the NPF are also clearly visible. Results for dimensionless non-steady force amplitude vs velocity ratio are shown in figure 47. In a subsequent test, similar data was obtained with certain design changes made in the prototype machine. A significant reduction in the magnitude of the normalized force ratios was observed. The water table is thus able to reproduce trends in non-steady load variations between specific turbine stage geometries.

Transient Forces From a Partial Admission Stage. This test was conducted to measure the transient load variation on a moving blade as it passed through the flow from a nozzle arc in a partial admission stage. The general form of the load transient on a moving blade in an actual partial admission stage is known from prototype strain gage tests. The purpose of this test was to determine the degree to which the form of the water table load transient obtained resembled that obtained from actual blading tests. The stage parameters and typical results are listed in figure 48. Similarity of the following features is observed on comparing the two charts:

(a) relative slope of the inlet response.
(b) inlet response spike magnitude.
(c) outlet response spike magnitude.
(d) relative magnitude of main curve.

Subsequent partial admission tests have demonstrated that this correlation is typical, and that quantitative comparison (e.g. peak/average) is representative of that observed in practice.

10.4.3 The Compressor Mode Water Table Apparatus.

In the compressor mode the water table is arranged for the water to flow in the radially inward direction as shown schematically in figure 49. The incoming flow enters a circular trough located around the table outer circumference and passes through a series of flow-smoothing screens. An adjustable overflow weir controls the inlet
1ST NORM. NON-STeadY FORCE RATIO
VS.
VEL0CITY RATIO

DIRECTION: Tangential
PRESSURE RATIOS:
NORM. FORCE: $+1.447 \times 2.097$
NORM. VEL. RATIO: .3000

Figure 47. Normalized, tangential force ratio vs. velocity ratio from Rieger and Crofoot [95].
<table>
<thead>
<tr>
<th>Type of stage</th>
<th>Impulse</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of moving blades</td>
<td>110</td>
</tr>
<tr>
<td>Number of partial admission arcs</td>
<td>3</td>
</tr>
<tr>
<td>Number of nozzles in arcs</td>
<td>10, 10, 12 (corresponds to 76 nozzles in complete circle)</td>
</tr>
<tr>
<td>Stage pressure ratio</td>
<td>1.7</td>
</tr>
<tr>
<td>Stage velocity ratio</td>
<td>0.45</td>
</tr>
<tr>
<td>Number of instrumented blades</td>
<td>1</td>
</tr>
<tr>
<td>Direction of force measurement</td>
<td>Tangential</td>
</tr>
</tbody>
</table>

Figure 48. Comparison of Water Table Partial Arc Result with Typical Steam Turbine Result. From Rieger and Crofoot [95].
Figure 49. Flow Circuit Schematic for Water Table in Compressor Mode.

Rotor with Blades

Water Table Surface

Overflow Gutter

Feed Trough

Valve

Plenum

Diffuser

Flow Return

Centrifugal Pump

Sump
flow depth. A set head is established by adjusting the height of the overflow weir with respect to an inner dam. This models the ambient pressure. During operation, the depth increases at the inner dam and flow discharges due to pumping.

The compressor rotor is mechanically driven at the required modeling velocity. Strain gage signals are obtained from both rotor blades and stator blades in the same manner as described for the turbine mode.

10.4.4 Tests Performed in the Compressor Mode.

To date, one test has been conducted on a compressor stage. The model consisted of a row of inlet guide vanes, a rotating blade row, and a stator row. Strain gage data was acquired for both the rotor and the stator blading. In addition to the usual non-steady force signals, an interesting non-synchronous rotating flow phenomenon was observed under certain flow conditions. This phenomenon appeared as several rotating 'cells' of sharply increased average water depth containing waves at blade passing frequency. The 'cells' were spaced at approximately equal intervals around the water table circumference. A strip chart recording of flow depth variation obtained from a stationary depth probe is shown in figure 50(a). The abrupt fluctuations represent water depth, i.e. pressure, variations as the 'cells' pass the probe. To study velocity variations within these regions, the strip chart recording shown in figure 50(b) was made. This clearly shows that sudden fluctuations in tangential velocity are associated with the passage of the flow cells. It is felt that this phenomenon may have been rotating stall, a phenomenon common to axial compressors. Unfortunately, the compressor stage tested was a prototype stage and no aerodynamic data was available for comparison.
Figure 50a. Strip chart recording of tangential velocity variation at a point as 'cells' pass. From Rieger and Crofoot [95].

Figure 50b. Strip chart recording of depth variation at a point as 'cells' pass. From Rieger and Crofoot [95].
The hydraulic analogy states that the two-dimensional inviscid flow of an ideal liquid (water) is analogous to the two-dimensional inviscid flow of a perfect gas, provided that the gas has a specific heat ratio of \( \gamma = 2.0 \). It is known, however, that no naturally occurring gas has a specific heat ratio of \( \gamma = 2.0 \). In order to model a prototype using a real gas, it is first necessary to model the prototype with the imaginary analog gas with \( \gamma = 2.0 \). The hydraulic analogy then models the analog gas. The primary step toward accurate modeling is therefore the conversion of the prototype gas parameters to the analog gas parameters.

To apply the water table hydraulic analogy to unsteady flow in turbomachines, the important modeling parameters are flow geometry and flow velocity. For maximum quantitative accuracy, similarity must be maintained between the prototype gas flow and the analogous water flow. Stage geometry for the water table model must therefore be scaled in proportion to that of the actual turbomachine. More importantly, the Froude number of the water table flow should be equal to the Mach number of the prototype gas flow at every point to maintain dynamic similarity between the prototype gas and water flow fields.

The geometric scaling of the water table model is a straightforward process as the size of the model is determined by the water table size. The Mach/Froude number similarity is achieved by using one of the correction factors described below. All of the correction factors are based on steady, one-dimensional isentropic flow theory. It is assumed, subject to experimental verification, that the results are applicable to the two-dimensional flow conditions on the water table: See section 11.3.

10.5.1 Modeling Parameters.

The exit Mach number of the converging-diverging nozzle, under steady, one-dimensional, isentropic flow conditions is given by:

\[
M = \left( \frac{2}{\gamma - 1} \left[ \left( \frac{P}{P_0} \right)^{\frac{\gamma - 1}{2}} - 1 \right] \right)^{1/2}
\]

(10.13)
It is obvious that two gases having different specific heat ratios will have different nozzle exit Mach numbers for identical pressure ratios. For the analog gas ($\gamma = 2.0$) the nozzle exit Mach number is given by equation (10.14)

\[
M = \left\{ 2 \left[ \left( \frac{p_{\infty}}{p} \right)^{\gamma/2} - 1 \right] \right\}^{1/2}
\]

(10.14)

The nozzle exit Froude number of water flow is

\[
Fr = \left\{ 2 \left[ \frac{h_{\infty}}{h} - 1 \right] \right\}^{1/2}
\]

(10.15)

The analog gas Mach number is exactly modeled by the water Froude number since

\[
\frac{h_{\infty}}{h} = \left( \frac{Fr}{Fr} \right)^{1/2}
\]

(10.16)

The critical pressure ratio ($P_{r*}$) in a turbomachine stage is defined as that pressure ratio at which the nozzle exit Mach number is unity. Substituting $M=1$ into equation (10.13) yields

\[
P_{r*} = \left( \frac{\gamma + 1}{2\gamma} \right)^{1/(\gamma-1)}
\]

(10.17)

The critical pressure ratio of the analog gas ($P_{r*}$) is obtained by
substituting $\gamma = 2.0$ into the above equation

$$P_{r_{m}}^* = 2.25$$ (10.18)

This value corresponds to the square of the critical depth ratio of $H_{r}^* = 1.25$.

10.5.2 The W-2 Correction Factor.

The W-2 correction factor was devised independently by P.C. Warner and A. L. Wicks in 1975. The correction factor is based on the common practice of expressing the stage pressure ratio as a given percent of the critical pressure ratio. The procedure is to establish the same percentage for the analog gas and hence the depth ratio on the water table i.e.

$$\frac{P_{r_{m}}}{P_{r_{m}}^*} = \frac{P_{re}}{P_{re}^*}$$ (10.19)

This method insures that the nozzle exit Mach number of the prototype flow is equal to the nozzle exit Mach number of the analog gas. Substituting equations (10.16) through (10.18) into equation (10.19) and rearranging gives

$$\frac{h_a}{h} = \left[2.25 \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{\gamma-1}} \cdot P_{rp} \right]^{\frac{1}{2}}$$ (10.20)

Figure 51 is a comparison of the W-2 correction factor with the
Figure 51. Comparison of the W-2 Specific Heat Ratio Correction Factor with those Given by Harleman and Ippen [66] and Schorr [88].
correction factors proposed by Harleman and Ippen [66] (derived
using the von Karman transonic similarity parameter) and by Schorr
[88] for a gas of $\gamma = 1.4$. Good agreement with the Schorr correction
factor is observed for all pressure ratios above critical. Good
agreement with the Harleman and Ippen correction factor is observed
in the above critical transonic range (1.893 $< P_{r} < 4.00$). Harleman and
Ippen suggest the use of a correction factor identical to that of
Schorr above the transonic range so the $W-2$ correction factor is valid
for all pressure ratios above critical. It is obvious from equation
(10.20) that the $W-2$ correction is in error for subcritical pressure
ratios. A prototype pressure ratio of unity (implying no flow) yields
a depth ratio greater than unity for all $\gamma < 2.0$. It is proposed that
the $W-2$ correction factor be modified. The modified $W-2$ ($MW-2$) correction
factor is derived by assuming a linear relationship between the square
of the depth ratio and the prototype pressure ratio in the range
$1.0 < P_{rp} < P_{r}^*$. For above critical pressure ratios the $MW-2$ correction
factor is identical to the $W-2$ correction factor. The form of the
$MW-2$ correction factor is

$$
H_r^2 = \begin{cases} 
\frac{1.25 P_{rp} + P_{r}^* - 2.25}{P_{r}^* - 1} & \text{for } P_{rp} < P_{r}^* \\
2.25 \frac{P_{rp}}{P_{r}^*} & \text{for } P_{rp} \geq P_{r}^* 
\end{cases} 
$$

Figure 52 is a comparison of the $MW-2$ correction factor with the
Schorr correction factor for a prototype gas of $\gamma = 1.4$. Good
agreement is obtained for all pressure ratios.

10.6 Advantages and Disadvantages of the Hydraulic Analogy.

Klein [80] concluded that "the hydraulic analogy has never failed to
reproduce qualitatively the gas dynamic features." Most authors agree
that this reliability of flow modeling is the greatest advantage of the
Figure 52. Comparison of the MW-2 and Schorr [88] Specific Heat Ratio Correction Factors.
analogy. The range of studies undertaken to date on the RIT water table seeking quantitative data indicates that the early agreement obtained by Harleman and Ippen [66] and others on simple models may also be obtained with complex models provided that the fundamental requirements of the analogy are carefully maintained eg. horizontal surface, smooth undisturbed inflow, adequate specific heat correction, etc. Johnson [68] indicates that the favorable (1000:1) time dilation feature, and the ease with which models can be manipulated are also major advantages of hydraulic analogy studies. Rieger and Nowak [89] have added the relatively low cost of the models vs. prototypes and the ease of data acquisition as other advantages.

Bomelburg [90] has stated that "It is by no means an easy experiment to run a shallow water channel in order to obtain reliable results as there are so many possibilities which can cause errors in the results of the water analogy." Bomelburg was particularly concerned with the problems of surface tension and surface contamination and his experiments were performed using kerosene as the working fluid.

The degree of quantitative accuracy obtained in a test must be questioned in any simulation, especially where complexity exceeds previously established test results. Several factors may reduce the water table quantitative accuracy stemming from possible violations of analogy assumptions. It must be concluded though, that with adequate surface tension reduction, adequate geometric modeling and with the use of pressure ratio correction factors, the quantitative accuracy of measurements of unsteady loads on model turbomachine stages can be very good.
10.7 Notation

a - sonic velocity in gas
b - wave propagation velocity in water
Fr - Froude number
g - gravitational acceleration
h - local water depth
Hr - depth ratio
M - Mach number
p - local pressure in gas
Pr - pressure ratio
R - gas contact
S - surface tension
T - temperature
u - velocity in x direction
v - velocity in y direction
x,y - coordinate direction
γ - specific heat ratio
λ - wavelength of capillary surface waves in water
ρ - gas density
11. WATER TABLE TESTS

11.1 Set Up Procedure for Water Table Tests of Turbine Stages

The blading used in water table stage tests is made from clear acrylic plexiglas to exact dimensions scaled from manufacturing drawings of the prototype blading. The nozzle blades are aligned on the water table at the prescribed radius, pitch and stagger using a special alignment fixture so that the circle defined by the nozzle blade centroids is centered precisely on the axis of rotation of the rotor. The rotor blades are aligned using the same fixture so that the rotor blade row and nozzle row are concentric. The clearance between the rotor blades and the table surface is set at 0.30 inches to allow free rotation of the blade row. One rotor blade is supported by an instrumented monitor post. The monitor post, shown in figure 44, has strain gages mounted in such a way as to measure displacements in two perpendicular directions as well as torsional displacement. Typically the monitor post is aligned so as to measure displacements in the radial and tangential directions. The radial direction on the water table corresponds to the axial direction on an axial flow turbine. The required stage depth ratio is determined from the prototype pressure ratio by using the MW-2 correction factor (see section 10.6.2). The stage depth ratio is defined as the ratio of the depth of the inlet to the stator row ($h_1$) and at the rotor exit ($h_3$); see figure 53. The depth at the inlet to the stator row is arbitrarily set at $h_1 = 1.125$ in. typically. The required depth ratio is set by adjusting the back pressure screen to allow more or less water to pass through. A difficulty associated with this procedure is that a uniform depth is required around the circumference downstream of the rotor row. Slight variations in the screens caused by, for instance, water contaminants can create a non-uniform depth distribution. An alternative method is to clamp a thin weir, machined to a uniform height, to the table surface downstream of the rotor row. The stator row inlet depth is then varied to achieve the correct depth ratio. Whichever method is used to set the pressure ratio, the depths are checked frequently during testing to insure that a constant pressure ratio is maintained throughout the test program. The nozzle exit velocity is
Figure 53. Definition of Stage Pressure Ratio.
measured with the rotor stationary using a hot film anemometer. Values are typically taken at four measurement stations spaced approximately 90° apart around the stator ring to insure that the flow is uniform around the test configuration. The exit velocity of several adjacent nozzles is also measured at each of the measurement stations, to insure that the effect of any assembly errors is minimized in the exit velocities measured. The four values of exit velocity are then averaged to give the nozzle exit velocity \( V_e \) used in calculating the stage velocity ratio. The rotor velocity required to calculate the velocity ratio for a given test point is measured by timing one complete revolution of the rotor on three separate occasions, and averaging the results. The leading edge radius, \( r \), of the rotor profile is then used to calculate the rotor speed, using the following equation.

\[
\frac{2\pi r}{T} = U
\]  

(11.1)

where \( T \) is the average time per revolution and \( U \) is the rotor speed. The velocity ratio is the ratio between the average nozzle exit velocity and the rotor speed. Typically 10 to 20 velocity ratio runs are made at each pressure ratio tested. During each velocity ratio run, strain gage data is acquired as follows. The output of the three sets of strain gages is amplified onboard the rotor and passed through a gold slip ring assembly. The amplified output of each gage is then recorded on a separate channel of magnetic tape by an FM tape recorder. A fourth channel is used to record the pulsed output of a shaft encoder mounted on the rotor shaft.

11.2 Data Reduction Procedure

The recorded strain gage data is reduced into three forms: (1) steady force, (2) non-steady force, and (3) force ratio.

11.2.1 Steady Force

The steady forces and torques are represented by the calibrated DC voltage
levels of the strain gage output. These results are obtained directly from the taped data using a high-pass filter with a frequency cut-off at 0.1 Hz. The high-pass filter uncouples the AC voltage component from the DC component. The DC voltages from each of the strain gages are then read out on a voltmeter and plotted against the corresponding velocity ratio. The steady forces in the tangential and radial directions are denoted by F and A respectively, and the steady torque is denoted by T.

11.2.2 Unsteady Forces

The unsteady forces and torques are obtained by processing the taped data through a real-time Fourier spectrum analyzer. The analyzer digitally samples the recorded signal and produces a spectral plot of the amplitude versus frequency. Amplitudes of the non-steady forces and torques for any given frequency of interest can then be obtained from the spectral plot. The frequencies of interest in turbine studies are typically the nozzle passing frequency (NPF) and its harmonics. The magnitude of the unsteady forces and torques is obtained from the spectral plot and is plotted against the corresponding velocity ratio. The unsteady forces in the tangential and radial directions are denoted by \( \Delta F \) and \( \Delta A \) respectively, and the unsteady torque is denoted by \( \Delta T \).

11.2.3 Force Ratios

Two types of non-steady force and non-steady torque ratio plots, obtained by two different methods, are typically prepared. One method is to divide the magnitude of the non-steady force or non-steady torque at a given velocity ratio by the corresponding steady force or steady torque magnitude at the same velocity ratio. Charts obtained by this method are titled, "Non-Steady Force Ratio vs. Velocity Ratio". One difficulty is encountered using this procedure. As the velocity ratio increases, the steady force and steady torque generally approach zero, resulting in artificially high values of the force and torque ratio. The difficulty can be avoided by dividing the unsteady force or unsteady torque at all velocity ratios by the steady force or steady torque at a single specified velocity ratio. This procedure allows the non-steady force and non-steady torque values to be related to a single
operating condition. Charts obtained by this second method are titled, "Normalized Non-Steady Force Ratio vs. Velocity Ratio". The normalizing steady forces in the tangential and axial directions are denoted by $F_n$ and $A_n$ respectively, and the normalizing steady torque is denoted by $T_n$.

11.3 Water Table Test Program

The water table test program consists of two phases as follows. Phase one of the test program was a test on a typical LP turbine stage. Since LP blading is more nearly flat plates than any other type of blading, i.e., HP or LP, this test data was chosen for comparison with results obtained from the Kemp-Sears analysis. Phase two of the test program was designed to study the velocity field downstream of a row of turbine nozzles. A detailed description of the magnitude and frequency of the spectral components of the velocity field may increase the reliability of unsteady excitation results obtained from the computer programs described in sections 7 and 8.

11.4 Phase One Water Table Test

The LP turbine stage geometry used in this phase of the water table test program is shown in figure 54. This stage was selected to provide a comparison of water table data with the Kemp-Sears gas dynamic excitation theories. The geometry of the stator row chosen was such that the turning angle was large at the lead edge and small through the throat section. This allowed the stator profile to simulate a flat plate, on which the Kemp-Sears analysis is based. The rotor blade geometry was selected from a low pressure stage because of the low camber and thickness to chord ratio associated with profiles of this type. Numerical values for the stage parameters are as follows:

- Rotor Solidity Ratio $s_r/c_r = 0.702$ (rotor pitch/rotor chord)
- Stator Solidity Ratio $s_s/c_s = 0.536$ (stator pitch/stator chord)
- Pitch Ratio $s_p/s_c = 0.641$ (rotor pitch/stator pitch)
- Spacing Ratio $b'/c_r = 0.5, 1.0, 1.5$ (axial clearance/rotor semi-chord)
- Number of stator blades 72
- Number of rotor blades 100
- Stage Pressure Ratio 2.79
Figure 54. Low Pressure Stage Geometry for Phase One Water Table Test.
Using the above stage geometry, a test program was conducted to evaluate the effect on the blade excitation ratios of varying a major state parameter while maintaining a constant pressure ratio. Other analog parametric effects such as absolute water depths and volumetric flow rates were held constant while the spacing ratio \( b'/c_r \) was varied. The spacing ratios tested were \( b'/c_r = 0.5, 1.0, \) and \( 1.5 \). At each of these settings, excitation data was taken for at least 15 velocity ratio data points ranging in value from 0.30 through 1.3. Data was recorded at each point for subsequent reduction and comparison with excitation theory. Results are presented in figures 55 through 85.

The pressure ratio of the prototype stage was 2.79. The stator inlet water depth was arbitrarily established at 1.125 inches and the MW-2 correction factor was used to calculate the required rotor exit depth of 0.600 inches. This pressure ratio was maintained throughout all tests in phase one by using the same height ratio \( (h_1/h_3) \). The reduced steady and non-steady data from the water table tests showed a number of distinct characteristics related to axial spacing and velocity ratio. The major trends in the data are described below.

11.4.1 Discussion of Spectral Data

**Tangential Spectra: Figures 55 to 59**

1. The tangential spectra show significant high amplitude, low frequency excitation up to 50 percent of nozzle passing frequency: see figure 58. As the velocity ratio increases the low frequency components tend to reduce in number, but to increase in amplitude. At the high velocity ratios, significant components do not exist above 25 percent of NPF, but the amplitudes are approximately 300 percent of the amplitude at nozzle passing frequency.

2. The effect of increased axial spacing on the tangential spectra is to broaden the bandwidth of the spectral component of nozzle passing frequency. At an axial spacing ratio of 0.5 a narrow band of excitation exists at nozzle passing frequency. At spacing ratios of 1.0 and 1.5 the bandwidth broadens so that significant
Figure 55. Unsteady Tangential Force Amplitude vs. Frequency.
Figure 56. Unsteady Tangential Force Amplitude vs. Frequency.
Figure 57. Unsteady Tangential Force Amplitude vs. Frequency.

Tangential
Spacing Ratio 1.5
Velocity Ratio 0.83

Broad Band Nozzle
Passing Excitation
Figure 58. Unsteady Tangential Force Amplitude vs. Frequency.
Figure 59. Unsteady Tangential Force Amplitude vs. Frequency.
amplitudes are observed at frequencies up to 3 Hz above and below nozzle passing frequency. It is felt that this is due to the increased width of the wake as the axial spacing increases.

Axial Spectra: Figures 60 to 64

1. The axial spectra exhibit the same low frequency characteristics as the tangential spectra, with increased amplitudes occurring at the higher velocity ratios, and high excitation components occurring in the low frequency range.

2. Distinct one hertz side-banding of the nozzle passing frequency component is shown in figure 60. These spectra did not seem to be sensitive to wake expansion as the tangential spectra, although the plots for the 1.5 axial spacing ratio show a somewhat broader nozzle passing frequency band than those of either the 0.5 or 1.0 axial spacing ratio.

3. The axial data spectral plots show a strong 1/8th harmonic of NPF consistently throughout the test range. Significant amplitudes of this harmonic do not appear in the tangential direction, but do appear in the torsional spectra. This fact is important because the strain gage monitor post is sensitive to water-table-generated noise and to background excitation in both the tangential and axial directions, but is insensitive to extraneous signals in the torsional mode. The lack of a strong 1/8th harmonic in the tangential spectra, and the presence of this harmonic in the torsional spectra tends to indicate that this harmonic is a flow-excited phenomenon.

Torsional Spectra: Figures 65 to 67

1. The torsional spectra support the idea that the broadening spectral band-width around NPF (and the associated reduction in amplitude) is due to the wake expansion as the axial spacing increases.
Figure 60. Unsteady Axial Force Amplitude vs. Frequency.
Figure 61. Unsteady Axial Force Amplitude vs. Frequency.

Axial
Spacing Ratio 1.0
Velocity Ratio 0.77

Broad Band Axial Nozzle Passing Excitation
Figure 62. Unsteady Axial Force Amplitude vs. Frequency.

Broad Band Axial Nozzle Passing Excitation

Axial Spacing Ratio 1.5
Velocity Ratio 0.83
Figure 63. Unsteady Axial Force Amplitude vs. Frequency.
Figure 64. Unsteady Axial Force Amplitude vs. Frequency.
Figure 65. Unsteady Torque Amplitude vs. Frequency.

Narrow band Torsional nozzle
Passing Excitation

Torsion
Spacing Ratio 0.5
Velocity Ratio 1.03
Figure 66. Unsteady Torque vs. Frequency
0.2

Figure 67. Unsteady Torque Amplitude vs. Frequency.
2. The torsional spectra contain more discrete low frequency components than the tangential and axial spectra, indicating that some of the low frequency components in the tangential and axial directions are due to background noise or random water table noise. The trends toward increasing NPF amplitudes with increasing axial spacing, and toward decreasing spectral band-widths around NPF at high velocity ratios are again evident.

11.4.2 Steady Force Data

Steady Tangential Force vs. Velocity Ratio: Figure 68

1. The curve of the steady tangential load decreases linearly as the velocity ratio increases. The slope of this curve shows a maximum variation of three percent between the three axial spacing ratio settings. This demonstrates the repeatability of the steady load versus speed relationship for the flow.

2. The form of the curve appears to be consistent with theory, i.e., the lift decreases due to the relative velocity change. For increasing velocity ratios, the lift coefficient also decreases thus reducing the effective tangential loading on the monitored blade, due to decreased angle of attack.

Steady Axial Force vs. Velocity Ratio: Figure 69

1. The general trend of the curves is a linear decrease in steady axial load with increasing velocity ratio. An inflection point occurs in all three curves at a velocity ratio in the range of 0.5 to 0.6. The slopes of all curves at velocity ratios below the inflection range agree to within ten percent, independent of the axial spacing. Similar comments apply for the slopes at velocity ratios greater than 0.6.

2. The general curve shape again seems compatible with airfoil theory for varying incidence angles. At higher velocity ratios both the drag and the lift functions decrease due to the decrease in angle of attack. The axial and tangential load is related
Figure 68. Variation of Steady Tangential Force vs. Velocity Ratio for Various Axial Spacing Ratios. From Rieger et. al. [94]
Figure 69. Variation of Steady Axial Force vs. Velocity Ratio for Various Axial Spacing Ratios.
From Rieger et al. [94]
230

vectorially to the lift and drag components and a direct correlation is not possible without the resolution of these forces, as discussed in Section 12.

3. Magnitudes for the steady axial load at a specific velocity ratio are approximately the same for each axial spacing ratio. This seems to indicate little change in the expansion of the nozzle exit flow due to changes in the axial spacing between blade rows. The small changes in the magnitude of the steady loads may be due to small errors in velocity ratio measurements.

Steady Torque vs. Velocity Ratio: Figure 70

1. The curves of steady torque versus velocity ratio appear to be slightly domed, with the peak value at a velocity ratio of about $V_r = 0.8$. Large scatter and drift in the results make meaningful analysis very difficult.

2. The form of these curves is difficult to define because of the complex flow pattern. It is believed that isolated airfoil theory does not give a reliable guide in the case of torsion. Subsequent tests at the same velocity ratios tend to suggest that the observed scatter is valid. It also appears that the data obtained for an axial spacing ratio of 1.0 is in error due to significant amplifier drift. The scattering observed in the data for axial spacing ratios of 0.5 and 1.5 may well be valid due to the complex flow regime surrounding the profile. Similar scatter does not appear in the axial and tangential data, since the flow effects tend to be averaged in the axial and tangential loading.

11.4.3 Non-Steady Force Ratio Data

Non-Steady Tangential Force Ratio vs. Velocity Ratio: Figure 71

1. The non-steady tangential force ratio curves for the three axial spacing ratios all follow the similar trends. The curves consistently peak at a velocity ratio of about 0.6, then
Figure 70. Variation of Steady Torque vs. Velocity Ratio for Various Axial Spacing Ratios.
From Rieger et al. [94]
Figure 71. Variation of Nonsteady Tangential Force Ratio vs. Velocity Ratio for Various Axial Spacing Ratios.  
From Rieger et.al. [94]

---

1ST NON-STEEP FORCE RATIO

VS.

VELOCITY RATIO

DIRECTION: Tangential

PRESSURE RATIOS: 2.72

Spacing Ratio: 4, 1.0, 1.5

---

232
decrease to a minimum at a velocity ratio of 1.0, and finally
increase as the velocity ratio approaches 1.3. The reason
for this curve shape is not known, but it is thought to be
due to the interaction of the impulsive and reaction loading
at the high attack angles.

2. The magnitude of the non-steady tangential force ratio seems
to be dependent on the axial spacing. For the lower axial
spacing ratio the peak occurring at $V_r = 0.6$ reaches a value
of 0.05. For the middle and upper axial spacing ratios the
corresponding peak reaches values of 0.032 and 0.035 respec-
tively. The minimum value occurring at a velocity ratio of
unity shows the same trend.

3. A plot of the normalized non-steady force ratio (dividing
by a constant steady force rather than by the instantaneous
steady load) in figure 72, shows the same trends as previously
mentioned. It is felt that this approach is more representa-
tive of the excitation and it can be seen that the peaking at
high velocity ratios becomes less significant than in the
non-normalized plots.

4. A possible explanation for the peak values which occur at a
velocity ratio of 0.6 is that separation of flow at the lower
velocity ratio causes high excitation. As the incidence angle
decreases, a more suitable flow regime is established reducing
the blade excitation. This theory is supported by the spectra
which show broad band excitation at the low velocity ratios.

Non-Steady Axial Force Ratio vs. Velocity Ratio: Figure 73

1. The non-steady axial force ratio curves follow much the same
trend as the tangential non-steady force curves, i.e., a peak
value occurs at a velocity ratio of about 0.6. The non-steady
axial force ratio decreases to a minimum at a velocity ratio of
1.0 but, unlike the non-steady tangential force ratio, remains
relatively constant for velocity ratios above 1.0.
Figure 72. Variation of Normalized Nonsteady Tangential Force Ratio vs. Velocity Ratio for Various Axial Spacing Ratios.
From Rieger et al. [94]
Figure 7. Variation of Nonsteady Axial Force Ratio vs. Velocity Ratio for Various Axial Spacing Ratios. From Rieger et al. [94]
2. The magnitude of the normalized non-steady axial force ratio decreases with increasing axial spacing ratio: see figure 74. This phenomenon is attributed to the decrease in wake strength associated with increased axial spacing.

3. The peaking effect observed in the velocity ratio range 0.6 to 0.7 may again be caused by flow separation which disappears as the incident flow angle approaches the design condition.

Non-Steady Torque Ratio vs. Velocity Ratio: Figure 75

1. The curve shapes for the normalized non-steady torsional loading appear to be consistent for all three tests, i.e., domed curves with peaks in the velocity ratio range of 0.8 to 0.9 as shown in figure 76. There is an indication of a minor peak around $V_r = 1.1$, which appears in all plots.

2. The peak value of the excitation is nearly constant for each of the axial spacing ratios. This peak value is approximately 0.04 although this value may be exceeded in certain instances due to the variation in the steady loading explained previously.

Additional plots of the steady tangential and axial forces and the steady torque at each of the axial spacing ratios tested are given in figures 77 through 79. Plots of the non-steady tangential force ratio, non-steady axial force ratio and non-steady torque ratio at each axial spacing ratio are given in figures 80 through 82. Plots of the normalized non-steady tangential force ratio, normalized non-steady axial force ratio and normalized non-steady torque ratio at each axial spacing ratio are given in figures 83 through 85.

11.5 Phase Two Water Table Test

This phase of the water table test program was intended to examine the extent to which the flow field downstream of a cascade of stator blades on the water table models that downstream of the same cascade in the prototype gas. No strain gage data was required to make this comparison.
Figure 74. Variation of Normalized Nonsteady Axial Force Ratio vs. Velocity Ratio for Various Axial Spacing Ratios.
From Rieger et al. [94]
Figure 75. Variation of Nonsteady Torque Ratio vs. Velocity Ratio for Various Axial Spacing Ratios.
From Rieger et al. [94]
Figure 76. Variation of Normalized Nonsteady Torque Ratio vs. Velocity Ratio for Various Axial Spacing Ratios.
From Rieger et al. [94]
STEADY FORCE VS. VELOCITY RATIO

DIRECTION: + Axial △ Tangential × Torsional

PRESSURE RATIOS: 2.79

Spacing Ratio: 0.50

Figure 77. Variation of Steady Forces and Torque vs. Velocity Ratio.
Axial Spacing Ratio 0.50.
From Rieger et.al. [94]
Figure 78. Variation of Steady Forces and Torque vs. Velocity Ratio. Axial Spacing Ratio 1.00. From Rieger et al. [94]
Figure 79. Variation of Steady Forces and Torque vs. Velocity Ratio. Axial Spacing Ratio 1.50. From Rieger et. al. [94]
Figure 80. Variation of Nonsteady Force and Torque Ratios vs. Velocity Ratio. Axial Spacing Ratio 0.50.
From Rieger et al. [94]
Figure 81. Variation of Nonsteady Force and Torque Ratios vs. Velocity Ratio. Axial Spacing Ratio 1.00.
From Rieger et al. [92]
Figure 82. Variation of Nonsteady Force and Torque Ratios vs. Velocity Ratio. Axial Spacing Ratio 1.50.
From Rieger et. al. [94]
Figure 83. Variation of Normalized Nonsteady Force and Torque Ratios vs. Velocity Ratio. Axial Spacing Ratio 0.50. From Rieger et al. [9]
Figure 84. Variation of Normalized Nonsteady Force and Torque Ratios vs. Velocity Ratio. Axial Spacing Ratio 1.00.
From Rieger et al. [94]
Figure 35. Variation of Normalized Nonsteady Force and Torque Ratios vs. Velocity Ratio. Axial Spacing Ratio 1.50. From Rieger et al. [94]
11.5.1 Experimental Procedure

A single row of stators, shown in figure 86, was mounted on the water table using the procedure described in section 11.1. No rotor blades were installed downstream of the stators. Numerical values for the stator cascade are as follows:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pitch</td>
<td>$s = 2.314$</td>
</tr>
<tr>
<td>Chord</td>
<td>$c = 4.092$ inches</td>
</tr>
<tr>
<td>Number of stator blades</td>
<td>82</td>
</tr>
<tr>
<td>Pressure ratios</td>
<td>1.05, 1.10, 1.20</td>
</tr>
<tr>
<td>Nozzle exit Mach numbers</td>
<td>0.28, 0.34, 0.46</td>
</tr>
</tbody>
</table>

The velocity field downstream of the stator cascade was measured at each of the three pressure ratios as follows. The hot film anemometer probe was mounted on a special carriage attached to the rotor. The carriage was designed to allow the probe to be positioned at the required distance from the trail edge of the stator cascade. With the probe locked in the carriage, the rotor was slowly rotated through four to six complete revolutions to obtain the circumferential variation of the stator exit velocity at the three values of axial spacing shown in figure 86. Plots of the anemometer output versus time for the three pressure ratios are given in figures 87 through 95. Spectral plots of the data at the nominal axial spacing (0.90) are given in figures 116 through 118.

11.5.2 Discussion of Data

Anemometer Output vs. Time Plots: Figures 87 through 95

The general form of the velocity field downstream of the stator cascade is shown in these plots. The ordinate of these plots (anemometer output) is a voltage. The anemometer output voltage must be converted to velocity using the calibration curve shown in figure 96 in order to obtain the true velocity profile. The relationship between the output voltage and the velocity is nearly linear in the range of voltage between 7.86 and 8.32
Figure 86. Stator Row Geometry for Phase Two Water Table Test.
Figure 87. Anemometer Output vs. Time. Pressure Ratio 1.20. Axial Spacing 0.45.
Figure 88. Anemometer Output vs. Time. Pressure Ratio 1.20. Axial Spacing 0.90.
Figure 89. Anemometer Output vs. Time. Pressure Ratio 1.20. Axial Spacing 1.80.
Figure 90. Anemometer Output vs. Time. Pressure Ratio 1:10. Axial Spacing 0.45.
Figure 91. Anemometer Output vs. Time. Pressure Ratio 1.10. Axial Spacing 0.90.
Figure 92. Anemometer Output vs. Time. Pressure Ratio 1.10. Axial Spacing 1.80.
Figure 93. Anemometer Output vs. Time. Pressure Ratio 1.05. Axial Spacing 0.45.
Figure 94. Anemometer Output vs. Time. Pressure Ratio 1.05. Axial Spacing 0.90.
Figure 95. Anemometer Output vs. Time. Pressure Ratio 1.05. Axial Spacing 1.80.
Figure 96. Anemometer Output Voltage vs. Flow Velocity
volts as shown in figure 97. Below this range the relationship between voltage and velocity is highly nonlinear. The minimum velocities associated with a pressure ratio of 1.1 and all velocities associated with a pressure ratio of 1.05 lie in the nonlinear range. Great care must therefore be taken in comparing these plots. The magnitude of the free stream and wake velocities decreases from a maximum at the highest pressure ratio to a minimum at the lowest pressure ratio at all three axial spacings.

The time plot data was reduced by reading the average minimum voltage in the wake and the average maximum voltage in the free stream from the time plots and converting these numbers to minimum and maximum velocities using figures 96 and 97. The ratio of the average minimum velocity to the average maximum velocity for each pressure ratio and each axial spacing was then determined. The values of the velocity defect ratio are given in table 14.

The high frequency components superimposed on the voltage traces are due to flow turbulence. The probe was therefore held stationary in both the free stream and the wake at all three axial spacings while a spectral analysis was performed on the output. The spectral plots obtained in this manner, figures 98 through 115, give the peak amplitudes of each frequency component in the raw data. The power spectral density (PSD) on each plot gives an indication of the turbulence level in the flow, i.e., a high PSD implies a high turbulence level. It is evident from these plots that a higher level of turbulence exists in the wake than in the free stream at all pressure ratios and at all axial spacings. The turbulence level in both the wake and the free stream at a given axial spacing decreases from a maximum at the highest pressure ratio to a minimum at the lowest pressure ratio. The turbulence level in the wake at a given pressure ratio decreases as the axial spacing is increased while the free stream turbulence level increases with increasing axial spacing. This is due to the mixing between the wake and the free stream which causes the wake to decay.
Figure 97. Anemometer Output Voltage vs Flow Velocity in Linear Range.

<table>
<thead>
<tr>
<th>Pressure Ratio</th>
<th>Axial Spacing</th>
<th>Average Minimum Velocity ( V_{\text{min}} ) (ips)</th>
<th>Average Maximum Velocity ( V_{\text{max}} ) (ips)</th>
<th>( \frac{V_{\text{min}}}{V_{\text{max}}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.20</td>
<td>0.45</td>
<td>4.44</td>
<td>6.63</td>
<td>0.67</td>
</tr>
<tr>
<td>1.20</td>
<td>0.90</td>
<td>4.36</td>
<td>5.36</td>
<td>0.81</td>
</tr>
<tr>
<td>1.20</td>
<td>1.80</td>
<td>5.42</td>
<td>6.88</td>
<td>0.79</td>
</tr>
<tr>
<td>1.10</td>
<td>0.45</td>
<td>1.92</td>
<td>4.28</td>
<td>0.45</td>
</tr>
<tr>
<td>1.10</td>
<td>0.90</td>
<td>2.81</td>
<td>4.07</td>
<td>0.69</td>
</tr>
<tr>
<td>1.10</td>
<td>1.80</td>
<td>4.85</td>
<td>6.31</td>
<td>0.77</td>
</tr>
<tr>
<td>1.05</td>
<td>0.45</td>
<td>1.56</td>
<td>4.20</td>
<td>0.37</td>
</tr>
<tr>
<td>1.05</td>
<td>0.90</td>
<td>1.25</td>
<td>1.84</td>
<td>0.68</td>
</tr>
<tr>
<td>1.05</td>
<td>1.80</td>
<td>1.20</td>
<td>1.92</td>
<td>0.63</td>
</tr>
</tbody>
</table>
Figure 98. Peak Nonsteady Anemometer Output Amplitude vs. Frequency. Probe Stationary in Free Stream. Pressure Ratio 1.20. Axial Spacing 0.45.
Figure 99. Peak Nonsteady Anemometer Output Amplitude vs. Frequency. Probe Stationary in Wake. Pressure Ratio 1.20. Axial Spacing 0.45.
Figure 100. Peak Nonsteady Anemometer Output Amplitude vs. Frequency. Probe Stationary in Free Stream. Pressure Ratio 1.20. Axial Spacing 0.90.
Figure 101. Peak Nonsteady Anemometer Output Amplitude vs. Frequency. Probe Stationary in Wake. Pressure Ratio 1.20. Axial Spacing 0.90.
Figure 102. Peak Nonsteady Anemometer Output Amplitude vs. Frequency. Probe Stationary in Free Stream. Pressure Ratio 1.20. Axial Spacing 1.80.
Figure 103. Peak Nonsteady Anemometer Output Amplitude vs. Frequency. Probe Stationary in Wake. Pressure Ratio 1.20. Axial Spacing 1.80.
Figure 104. Peak Nonsteady Anemometer Output Amplitude vs. Frequency. Probe Stationary in Free Stream. Pressure Ratio 1.10. Axial Spacing 0.45.
Figure 105. Peak Nonsteady Anemometer Output Amplitude vs. Frequency. Probe Stationary in Wake. Pressure Ratio 1.10. Axial Spacing 0.45.
Figure 106. Peak Nonsteady Anemometer Output Amplitude vs. Frequency. Probe Stationary in
Free Stream. Pressure Ratio 1.10. Axial Spacing 0.90.
Figure 107. Peak Nonsteady Anemometer Output Amplitude vs. Frequency. Probe Stationary in Wake. Pressure Ratio 1.10. Axial Spacing 0.90.
Figure 108. Peak Nonsteady Anemometer Output Amplitude vs. Frequency. Probe Stationary in Free Stream. Pressure Ratio 1.10. Axial Spacing 1.80.
Figure 109. Peak Nonsteady Anemometer Output Amplitude vs. Frequency. Probe Stationary in Wake. Pressure Ratio 1.10. Axial Spacing 1.80.
Figure 110. Peak Nonsteady Anemometer Output Amplitude vs. Frequency. Probe Stationary in Free Stream. Pressure Ratio 1.05. AxialSpacing 0.45.
Figure 111. Peak Nonsteady Anemometer Output Amplitude vs. Frequency. Probe Stationary in Wake. Pressure Ratio 1.05. Axial Spacing 0.45.
Figure 112. Peak Nonsteady Anemometer Output Amplitude vs. Frequency. Probe Stationary in Free Stream. Pressure Ratio 1.05. Axial Spacing 0.90.
Figure 113. Peak Nonsteady Anemometer Output Amplitude vs. Frequency. Probe Stationary in Wake. Pressure Ratio 1.05. Axial Spacing 0.90.
Figure 114. Peak Nonsteady Anemometer Output Amplitude vs. Frequency. Probe Stationary in Free Stream. Pressure Ratio 1.05. Axial Spacing 1.80.
Figure 115. Peak Nonsteady Anemometer Output Amplitude vs. Frequency. Probe Stationary in Wake. Pressure Ratio 1.05. Axial Spacing 1.80.
Spectral Plots: Figures 116 through 118

The spectra obtained for all three pressure ratios at the nominal axial spacing (0.90) consist of a strong fundamental component (NPF) and little or no higher harmonic components, indicating that the velocity variation downstream of the stator row is nearly a pure sinusoid. The magnitude of the second harmonic of NPF is 25 percent of the magnitude of the NPF component at a pressure ratio of 1.2 and 19 percent of the fundamental magnitude at a pressure ratio of 1.05. No significant second harmonic of NPF is evident at a pressure ratio of 1.1. Similar comments apply to the spectra obtained at the other axial spacings. The instrumentation is incapable of determining the phase relationship between these spectral components. Care should be taken in any attempt to reconstruct the velocity profile using the harmonic components from these spectra.
Figure 116. Nonsteady Anemometer Output Amplitude vs. Frequency. Pressure Ratio 1.2.
Axial Spacing 0.90.
Figure 117. Nonsteady Anemometer Output Amplitude vs. Frequency. Pressure Ratio 1.1. Axial Spacing 0.90.
Figure 118. Nonsteady Anemometer Output Amplitude vs Frequency. Pressure Ratio 1.05. Axial spacing 0.90.
12.0 COMPARISON OF WATER TABLE RESULTS WITH THEORETICAL RESULTS

12.1 MODELING CONDITIONS FOR KEMP-SEARS ANALYSIS

To apply the Kemp-Sears analyses [2] [3] to an actual turbine stage it is necessary to redefine the actual stage parameters shown in figure 119 so that they are compatible with the Kemp-Sears parameters shown in figure 1. The geometry of a low-pressure turbine stage may be made to correspond to that of the elementary Kemp-Sears stage, within the basic assumptions and limitations of the analysis. The velocity triangle associated with the prototype stage may differ considerably from that of the Kemp-Sears stage, and these velocity triangles must be matched in order for the Kemp-Sears stage theory to apply. This matching is achieved by assuming that the axial velocity component \( V_a \) is constant through the stage. First consider the row of stator blades, shown in figure 119. The Kemp-Sears analysis requires a single velocity \( V_s \) to be associated with the stator row. Since the axial components of the inlet velocity \( V_1 \) and the exit velocity \( V_2 \) are assumed to be equal, the stator inlet and exit velocities can differ only in the magnitude and sign of their respective tangential components, \( V_{t1} \) and \( V_{t2} \). The tangential component of the velocity \( V_{ts} \) is the average of \( V_{t1} \) and \( V_{t2} \), i.e.

\[
V_{ts} = \frac{1}{2} (V_{t2} + V_{t1})
\]

It must also be assumed that the stator inlet velocity \( V_1 \) is equal to the stage exit velocity \( V_5 \) so that:

\[
V_{t1} = V_{t5}
\]

and

\[
V_{ts} = \frac{1}{2} (V_{t2} + V_{t5}) \tag{12.1}
\]

Even though this limitation may not exist in an actual stage, it is required in the analytical model or a net circulation will exist across the stage which is not included in the Kemp-Sears analysis. The magnitude of the velocity \( V_s \) is given by:
Figure 119. Velocity Notation for Turbine Stage.
By a similar process, the magnitude of the velocity $V_r$ can be shown to be:

$$V_r = [(V_{tr})^2 + (V_a)^2]^{1/2} \quad (12.3)$$

where

$$V_{tr} = 1/2 (V_{t3} + V_{t4}). \quad (12.4)$$

The stator exit flow angle $\alpha$ can be approximated as:

$$\alpha = \sin^{-1} \left( \frac{s}{d_s} \right) \quad (12.5)$$

where $s$ is the stator throat width and $d_s$ is the stator pitch. The stage exit flow angle $\beta$ may be approximated by the rotor blade trail edge angle. The following trigonometric relations exist between the components of the velocity triangle as shown in figure 119:

$$V_a = V_2 \sin \alpha \quad (12.6)$$

$$V_{t1} = V_{t3} = -V_2 \cot \beta \sin \alpha + U$$

$$V_{t2} = V_2 \cos \alpha \quad (12.7)$$

$$V_{t3} = V_2 \cos \alpha - U$$

$$V_{t4} = V_2 \cot \beta \sin \alpha$$
Angles $\alpha_s$ and $\alpha_r$ shown in figure 1 may be expressed in terms of the velocities as follows.

$$\alpha_s = \begin{cases} + \cos^{-1} \frac{V_a}{V_s} & ; \quad V_{t2} + V_{t5} \geq 0 \\ - \cos^{-1} \frac{V_a}{V_s} & ; \quad V_{t2} + V_{t5} < 0 \end{cases}$$

$$\alpha_r = \begin{cases} + \cos^{-1} \frac{V_a}{V_r} & ; \quad V_{t3} + V_{t4} \leq 0 \\ - \cos^{-1} \frac{V_a}{V_r} & ; \quad V_{t3} + V_{t4} > 0 \end{cases}$$

RELATIONS BETWEEN NON-STEADY LIFT AND DRAG AND NON-STEADY FORCES

The values of the above variables are functions of the rotor speed $U$. The stator exit velocity $V_2$ was measured using a hot wire anemometer. The computer program given in Section 7 was used to calculate the ratio of unsteady lift to the steady lift, at space/chord ratios from 0.5 to 1.75, for velocity ratios of 1.23 and 1.14. The data from the water table test described in Section 11 was obtained in the tangential and axial directions, as opposed to the lift direction (perpendicular to the chord). In order to compare the Kemp-Sears results with experimental results, the following procedure was employed.

For the flat plate shown in figure 120, let the total steady load be represented by the vector $R$. This resultant vector can be expressed in terms of the vector sum of the lift $L$ and the drag $D$, or by the vector sum of the steady axial force $A$ and the steady tangential force $F$. The lift vector $L$ can be divided into its axial and tangential components $L_A$ and $L_T$ whose magnitudes are given by:

$$L_A = L \sin \phi$$
$$L_T = L \cos \phi$$

(12.10)
Figure 120. Loading on an Airfoil in Terms of: Lift and Drag Forces; and Tangential and Axial Forces.
Similarly the drag force has components

\[ D_A = D \cos \phi \]  
\[ D_T = D \sin \phi \]  

(12.11)

The steady axial and tangential forces can then be expressed in terms of the lift and drag forces

\[ A = L_A + D_A \]

and

\[ F = L_T - D_T \]  

(12.12)

The magnitudes of the steady lift and drag forces are given by

\[ L = C_L \rho c v^2 r \]
\[ D = C_D c v^2 r \]  

(12.13)

per unit depth. The unsteady forces are treated in a similar manner. The magnitudes of the unsteady axial and tangential forces are:

\[ \Delta A = \Delta L \sin \phi + \Delta D \cos \phi \]
\[ \Delta F = \Delta L \cos \phi - \Delta D \sin \phi \]  

(12.14)

per unit length, where the value of \( \Delta L \) is calculated by multiplying the result of the Kemp-Sears analysis \( (\Delta L/L) \) by the calculated steady lift. Considerable difficulty exists in the determination of the unsteady drag \( \Delta D \). As a first approximation, it was assumed that the ratio of the unsteady drag to the steady drag is equal to the ratio of the unsteady lift to the steady lift i.e.

\[ \frac{\Delta D}{L} = \frac{\Delta L}{L} \]  

(12.15)

Solving the above expression for the unsteady drag gives:

\[ \Delta D = (\frac{D}{L}) \Delta L \]  

(12.16)
The ratio of drag to lift is given by:

\[
\frac{D}{L} = \frac{C_D}{C_L}
\]  

(12.17)

For the lift and drag coefficients occurring in this instance, the value of unsteady drag is given by:

\[
\Delta D = (0.09) \Delta L
\]  

(12.18)

Changes in the lift and drag coefficients will change the magnitude of the resulting curves, but the curve shape will remain the same.

12.3 Sample Calculations

The procedure described above will be demonstrated using conditions which were set for a single water table run as listed below.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotor Solidity Ratio (RSIGR)</td>
<td>1.4246</td>
</tr>
<tr>
<td>Stator Solidity Ratio (RSIGS)</td>
<td>1.8652</td>
</tr>
<tr>
<td>Pitch Ratio ( r/d_s ) (RATIO 1)</td>
<td>0.6408</td>
</tr>
<tr>
<td>Stator Exit Velocity ( V_2 )</td>
<td>18.8 ips</td>
</tr>
<tr>
<td>Stator Pitch ( d_s )</td>
<td>2.038 in.</td>
</tr>
<tr>
<td>Stator Throat Width ( s )</td>
<td>0.349 in.</td>
</tr>
<tr>
<td>Axial Spacing Ratio (RATIO2)</td>
<td>0.50</td>
</tr>
<tr>
<td>Rotor Speed ( RU )</td>
<td>23.06 ips</td>
</tr>
<tr>
<td>Lift Coefficient (RCLR)</td>
<td>1.2</td>
</tr>
<tr>
<td>Drag Coefficient (RCDS)</td>
<td>0.11</td>
</tr>
<tr>
<td>Rotor Stagger Angle ( \beta )</td>
<td>1.117 rad.</td>
</tr>
</tbody>
</table>

The stator exit angle \( \alpha \) is calculated using equation (12.5) above.

\[
\alpha = \sin^{-1}(\frac{s}{d_s}) = \sin^{-1}(\frac{0.349}{2.038}) = 0.172 \text{ rad}
\]

The axial velocity component from equation (12.6) is given by

\[
V_a = V_2 \sin \alpha = 18.8 \sin(0.172) = 3.218
\]
The tangential components of the velocities $V_1$ to $V_5$ are calculated by equations (12.7).

\[
V_{t1} = V_{t5} = V_2 \cot \beta \sin \alpha + U = (18.8) \cot (1.117) \sin (0.172) + 23.06 = 21.49
\]

\[
V_{t2} = V_2 \cos \alpha = (18.8) \cos (0.172) = 18.52
\]

\[
V_{t3} = V_2 \cos \alpha - U = (18.8) \cot (0.172) - 23.06 = -4.54
\]

\[
V_{t4} = V_2 \cot \beta \sin \alpha = (18.8) \cot (1.117) \sin (0.172) = 1.57
\]

The tangential components of the velocities $V_s$ and $V_r$ are found using equations (12.1) and (12.4).

\[
V_{ts} = \frac{1}{2} (V_{t5} + V_{t2}) = \frac{1}{2} (21.49 + 18.52) = 20.01
\]

\[
V_{tr} = \frac{1}{2} (V_{t3} + V_{t4}) = \frac{1}{2} (-4.54 + 1.57) = -1.49
\]

The velocities $V_s$ and $V_r$ which are input to the program are found using equations (12.2) and (12.3).

\[
V_s = \left[ (V_{ts})^2 + (V_r)^2 \right]^{1/2} = 20.26 = RVS
\]

\[
V_r = \left[ (V_{tr})^2 + (V_r)^2 \right]^{1/2} = 3.57 = RVR
\]

The angles $\alpha_s$ and $\alpha_r$ are calculated from equations (12.8) and (12.9) and are both positive.

\[
\alpha_s = \cos^{-1} \frac{V_a}{V_s} = 1.41 = \text{RAPHAS}
\]

\[
\alpha_r = \cos^{-1} \frac{V_a}{V_r} = 0.44 = \text{RAPHAR}
\]
Using the data, the program calculated the magnitude of the unsteady lift ratio to be:

\[ \frac{\Delta L}{L} = 0.2057 \]

The magnitudes of the steady lift and drag can be determined from equations (12.13).

\[ L = C_L \rho c V_r^2 = 1.2 \times (2.1 \times 10^{-5})(0.93)(3.57)^2 = 2.99 \times 10^{-4} \]

\[ D = C_D \rho c V_r^2 = 0.11 \times (2.1 \times 10^{-5})(0.93)(3.57)^2 = 2.74 \times 10^{-5} \]

The unsteady lift is given by

\[ \Delta L = \frac{\Delta L}{L} L = 0.2057(2.99 \times 10^{-4}) = 6.15 \times 10^{-5} \]

The unsteady drag is calculated using equation (12.16) to be

\[ \Delta D = (0.09) \Delta L = 5.81 \times 10^{-6} \]

Components of the steady lift in the axial and tangential directions are calculated by equations (12.10);

\[ L_A = L \sin \phi = 2.687 \times 10^{-4} \]

\[ L_T = L \cos \phi = 1.311 \times 10^{-4} \]
where $\phi$ is the stagger angle. Similarly the drag components from equations (12.11) are

$$D_A = 1.202 \times 10^{-5}$$
$$D_T = 2.46 \times 10^{-5}$$

The steady axial and tangential forces are given by equation (12.12) to be

$$A = 2.807 \times 10^{-4}$$
$$F = 1.065 \times 10^{-4}$$

The unsteady forces in the axial and tangential directions from equations (12.14) are

$$\Delta A = 5.78 \times 10^{-5}$$
$$\Delta F = 2.17 \times 10^{-5}$$

Using these results the magnitudes of the nonsteady force ratios in the axial and tangential directions can be calculated.

12.4 Comparison of Kemp-Sears Results with Water Table Results

Figure 121 is a plot of lift ratio ($\Delta L/L$) versus the axial spacing ratio ($b'/c_r$) for the two velocity ratios considered (Kemp-Sears notation). The lift ratio exhibits an exponential decrease with increasing axial spacing ratio. These results are also expressed in terms of their axial components and the tangential components by the above procedure. Curves of results are plotted with experimental data from the water table results given in
Figure 121. Unsteady Lift Ratio vs. Axial Spacing Ratio Predicted by Kemp-Sears Theory [2] [3].
Section 11, in figures 122 through 125. Theoretical curves for $\Delta D = 0.0 (\Delta L)$, and $\Delta D = 0.09 (\Delta L)$ are given to show that the influence of the non-steady drag coefficient is negligible. The experimental data for the axial force ratio increases, and then decreases with increasing axial spacing ratio, while the theoretical data predicts a steady exponential decrease with increasing axial spacing ratio. The experimental data for the tangential force ratio decreases with increasing spacing ratio, as shown in figures 123 and 125, though this increase is more gradual than that predicted by the Kemp-Sears theory. In general, numerical agreement between the experimental and theoretical results is reasonably good for axial spacing ratios greater than 1.0.

It is felt that the four major factors listed below influenced the correlation achieved between the Kemp-Sears theoretical results and the water table results.

1. **Supersonic Flow Conditions.** The water table data used here for comparison purposes was obtained from test conditions which model realistic LP blade flows, with a stage pressure ratio $P_r = 2.76$ ie supersonic inlet flow. These test conditions were established to obtain non-steady excitation data for an L.P. stage and were later utilized for the comparison described in this section. The Kemp-Sears theory applies for incompressible flow conditions. Some lack of correlation must therefore be expected, especially for low spacing ratio values.

2. **Isolated Airfoil Theory.** The Kemp-Sears analysis applies to an isolated airfoil. Errors will occur in the results to the extent that interactions occur from adjacent blades. Both the test conditions and the restrictions of the theory used will therefore influence the observed correlation.

3. **Stator Exit Flow Velocity and Direction.** The stator exit velocity magnitude and direction is apparently dependent on axial clearance and rotor speed. In the absence of other information, during testing it was assumed that the magnitude and direction of this velocity did not vary as a function of axial spacing. This could account for the extremely low values obtained for the force ratios in
Velocity Ratio $- V_r = 1.14$

Water Table Test Results

Figure 122. Unsteady Axial Force Ratio vs. Axial Spacing Ratio Obtained by Kemp-Sears Analysis and From Water Table Tests. Velocity Ratio 1.14.
Figure 123. Unsteady Tangential Force Ratio vs. Axial Spacing Ratio Obtained by Kemp-Seers Analysis and From Water Table Tests. Velocity Ratio 1.14.
Figure 124. Unsteady Axial Force Ratio vs. Axial Spacing Ratio Obtained by Kemp-Sears Analysis and From Water Table Tests. Velocity Ratio 1.23.
Figure 125. Unsteady Tangential Force Ratio vs. Axial Spacing Ratio Obtained by Kemp-Sears Analysis and From Water Table Tests. Velocity Ratio $V_r = 1.23$. 

Velocity Ratio - $V_r = 1.23$

$\Delta D = (0.0) \Delta L$

$\Delta D = (0.09) \Delta L$

Water Table Test Results
both the tangential and axial directions, at the lowest spacing ratio.

4. Viscous Stator Wakes. The Kemp-Sears analysis assumes the viscous wakes behind the stator blades to be identical in form to that behind an isolated airfoil. The correlation achieved between the theoretical results and the water table results depends on the extent to which this assumption is valid.

Of these factors, the third and fourth appear to be of primary importance. For this reason, phase two of the test program was implemented.

12.5 COMPARISON OF VELOCITY DEFECT DATA WITH THEORY

Several theories for determining the velocity defect in the wake downstream of an airfoil are available in the literature. Kemp and Sears [3] used the semi-empirical relationship obtained by Silverstein, Katzoff and Bullivant [43] to define the viscous wakes in their analysis. The expression for the ratio of the minimum velocity in the wake to the free stream velocity is

\[ \frac{u_c}{V} = 1 - \frac{1.21 \frac{C_D}{x/c + 0.3}}{1 - x/c + 0.3} \]  

(12.15)

where \( u_c \) is the minimum velocity in the wake, \( V \) is the free stream velocity, \( C_D \) is the drag coefficient of the airfoil, \( x \) is the coordinate in the direction of the free stream velocity as shown in figure 126 and \( c \) is the blade chord. This expression was obtained for an uncambered isolated airfoil. Spence [91] observed that the velocity recovery in the wake is actually more rapid than that predicted by Silverstein et al. and gave the following expression for the velocity defect.

\[ \frac{u_c}{V} = 1 - 0.1265(x/c + 0.025)^{-\frac{1}{2}} \]  

(12.16)

This expression is independent of the airfoil drag coefficient but is again
Figure 126. Wake Notation of Silverstein, Katzoff and Bullivant [43].
applicable to isolated airfoils. Raj and Lakshminarayana [56] give one of the few expressions for the maximum velocity defect downstream of a cascade of airfoils. For the near wake $(x/c \leq 0.05)$ the velocity defect is given by

$$\frac{u_c}{V} = 1 - \frac{1.25 C_D^{0.2}}{(x/c + 0.02)^{0.46}}$$

(12.17)

and for the far wake $(x/c > 0.05)$

$$\frac{u_c}{V} = 1 - \frac{0.4 C_D^{0.2}}{c/s} \cdot \frac{1}{x/c}$$

(12.18)

where $s$ is the pitch. The authors have reported excellent agreement between this theory and experimental results. The drag coefficient for airfoils in cascade is typically in the range $0.01 \leq C_D \leq 0.08$. Figure 127 is a plot of the minimum velocity in the wake expressed as a percent of the free stream velocity versus spacing ratio obtained by the three theories above for an airfoil of drag coefficient 0.02. It is evident that the effect of cascading airfoils (dashed curve) is to increase the rate of velocity recovery in the wakes from the rate observed for isolated airfoils. Figure 128 is a comparison of experimental data obtained from a water table test of a stator cascade and data obtained from an air test rig for the same cascade [92] with the theory in references [56] and [43]. The good agreement between the air test data and the theoretical curve from reference [56] indicates that the drag coefficient used in calculating the theoretical results (0.02) is close to the real value. The percent difference between the theoretical values of reference [56] and those obtained from the water table is largest at the smallest spacing ratio (34%) and decreases to 18% at the largest spacing ratio. The percent difference between the theoretical values from reference [43] and experimental values from the water table varies from 19% at the smallest spacing ratio to 9% at the largest spacing ratio. This could
Figure 127. Minimum Wake Velocity vs. Axial Spacing Ratio Predicted by Silverstein et. al. [43], Spence [91] and Raj and Lakshminarayana [56].
Figure 128. Comparison of Velocity Defect Measured on Water Table and in an Air Test Rig with Theoretical Values from Silverstein et.al. [43] and Raj and Lakshminarayana [56].
account for the poor agreement between the Kemp-Sears results and the water table results at low values of spacing ratio.

Figure 129 is a plot of the velocity field measured downstream of the stator cascade in an air turbine test rig [92]. Figure 130 is a plot obtained from the water table model of the same cascade. From a comparison of these figures, it is evident that the wake observed on the water table is significant wider than the wake observed in the air test rig. Since the wake observed on the water table is so wide, the velocity profile becomes nearly sinusoidal as shown in Section 11.5.2. If the velocity field is defined in terms of a Fourier series to be used as input to the computer programs as suggested in Section 9, the water table wake would consist of a strong fundamental component and very small higher harmonic components as shown in Section 11.5.3. The wake observed in the air test rig would also consist of a strong fundamental component but would also consist of a strong fundamental component but would have additional higher harmonic components not present in the water table wake, as demonstrated by the following.

If the wakes shown in figure 129 are idealized to the form shown in figure 131 (a) the Fourier series expansion given in reference [93] is

\[ f(x) = \frac{1}{2} (1+\alpha) + \frac{2}{\pi^2 (1-\alpha)} \sum_{n=1}^{\infty} \frac{1}{n^2} \left[ (-1)^n \cos n\pi \alpha - 1 \right] \cos n\pi w x \] (12.19)

where \( \alpha = \frac{c}{2l} \) and \( w = \frac{n}{L} \) is nozzle passing frequency. Figure 131 (b) shows the amplitudes of the harmonics of nozzle passing frequency expressed as a percentage of the amplitude of the nozzle passing frequency component. Strong harmonics are evident throughout the frequency range considered.

For subsonic (incompressible) flow conditions the width of the downstream wake as well as the magnitude of the velocity defect are strongly dependent on the drag coefficient and therefore on the Reynolds number of the flow. The
Figure 129. Velocity vs. Circumferential Position Downstream of a Stator Cascade in an Air Test Rig.
From Reference [92].

AXIAL POSITION: 0.300 IN. D/S NOZZLE DISCH.
RADIAL POSITION: MEAN DIAMETER = 10.07 IN.
TEST DATE: 4/17/78
FREE STREAM MACH NO. ~ 0.33

TURBINE NOZZLE WITHOUT STRUTS
AIR RIG TESTS
YAW PROBE MOD 1
Figure 130. Anemometer Output vs. Time. Pressure Ratio 1.10. Axial Spacing 0.90.
Figure 131. Spectral Components of Idealized Wake Behind an Isolated Airfoil.
highest nozzle exit Mach number of the stator cascade used in the phase
two testing was \( M = 0.46 \) implying sensibly incompressible flow. It would
appear that Reynolds number modeling would be more suitable for tests of
this nature. The phase one water table test was performed at a nozzle
exit Mach number of 1.16 ie transonic flow. The Mach/Froude number
modeling procedure appears to be adequate for such compressible flows
though the wake properties near the trailing edge of the stator row are
evidently dissimilar to those in the prototype machine. The difficulties
associated with Reynolds number modeling have been discussed by several
authors [70] [78]. By the very nature of the analogy the prototype gas
velocity is nearly 1000 times that of the water velocity. Typically the
kinematic viscosity of the prototype gas is 30 times that of the water.
If the Reynolds numbers are to be matched, the water table model would
have to be 33 times that of the prototype. Adams [78] has suggested
heating the water to decrease the viscosity and thereby improve the
Reynolds number similarity. If the water is heated to 120°F, the size
of the water table model can be reduced to 20 times that of the prototype.
At present heating the water appears to be an impractical
solution. The use of liquids other than water has also been proposed
by Adams and has been done by Bromelburg [90]. Unfortunately, liquids with
viscosities lower than that of water which are commercially available in the
large quantities required are generally expensive, usually volatile and
frequently explosive. The need for further studies of Reynolds number
modeling is obvious. If it is found that Reynolds number modeling is
impractical due to cost, size or safety factors, it must be concluded that
the water table is limited to modeling stages in which the flow is
compressible ie for Mach numbers above 0.7.
13. CONCLUSIONS

1. Several analytical methods for the determination of unsteady forces on turbomachine blades under subsonic flow conditions have been described and a review of the state-of-the-art has been given.

2. The limitations and range of applicability of each analysis have been defined and compared with other similar analyses.

3. A library of computer programs based on several of the above analyses has been developed and each program has been verified by comparing the results with sample results given by the authors.

4. Input and output instructions for each of the programs in the library have been given.

5. Guidelines and suggestions for further development of the program library have been given.

6. The theory of the hydraulic analogy has been presented with specific reference to the assumptions which are made in the mathematical formulation of the analogy.

7. Specific heat ratio correction factors which are used in hydraulic analogy studies have been described.

8. A two phase water table test program, intended to provide experimental verification of the Kemp-Sears theory and of the quantitative accuracy of the hydraulic analogy to subsonic gas flow, has been described.

9. Agreement between the dimensionless unsteady force obtained from a water table model stage and the dimensionless unsteady force predicted by the Kemp-Sears theory has been shown to be good when the axial spacing between the trailing edge of the stator and the leading edge of the rotor blades is greater than one-half of the blade chord length.
10. The empirical model of the stator viscous wakes on which the Kemp-Sears analysis is based has been shown to be inaccurate for airfoils in cascade. An improved model for the wakes has been discussed.

11. A comparison of the wakes downstream of a stator cascade on the water table and the wakes downstream of the analogous cascade in an air test rig under low subsonic flow conditions has been made. The agreement is sufficiently poor as to raise questions regarding the applicability of Mach/Froude number modeling in such cases.

12. The problems of Reynolds number modeling and the feasibility of using this modeling procedure for water table tests are discussed.
14. **RECOMMENDATIONS**

1. A reformulation of the vortex theories of Kemp-Sears [3] and Osborne [4], using the viscous wake model of Raj and Lakshminarayana [56], is recommended.

2. The Kemp-Sears [2] [3] and Osborne [4] stage analyses should be extended, using the Horlock [13] and Holmes [4] functions, to include the effects of convecting and non-convecting streamwise gusts on the unsteady blade forces. This will require the downwash velocity at the blades to be calculated in the analysis.

3. The Kemp-Sears program (LERAO) should be modified to reflect the changes indicated in items 1 and 2 above.

4. The analysis of Mukhopadhyay [18] should be extended to include the effects of a general convecting gust on the unsteady airfoil forces, and to include stage effects.

5. The existing computer program library should be extended to include the modified Osborne analysis (item 2 above), the modified Mukhopadhyay analysis (item 4 above), the Smith [32] analysis and the Whitehead [27] analysis.

6. An experimental program should be undertaken to provide data for comparison to the results of the actuator disk programs. The blading used in such a program would preferably have been tested in an air test rig, thereby allowing a comparison of the water table data with air test data.

7. A study of the feasibility of Reynolds number modeling on the water table should be conducted and, if such modeling is possible, a study of the quantitative accuracy of the wakes downstream of a cascade of stator blades on the water table under these modeling conditions should be conducted.
8. An experimental investigation should be conducted to the effect of surface tension reduction and undisturbed depth changes on the propagation velocity of gravity shallow water. This investigation should be conducted in a rectilinear channel and a circular test section.

9. An experiment identical to that of Harleman and Ipp should be conducted using existing techniques for surface tension reduction. Data obtained in a rectilinear flow channel and a circular test section should be compared to results obtained by Harleman and Ipp.

10. An experiment similar to that of Holmes [14] should be conducted in a rectilinear flow channel and on the water wave flume. A comparison of the data obtained by hydraulic analysis should be made with the results obtained by Holmes.
15. REFERENCES


