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Dynamic characteristics of a hyperboloid shell of revolution with application to flexible couplings

Brian G. Towner

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Dynamic Characteristics of a Hyperboloid Shell of Revolution with Application to Flexible Couplings

By

Brian G. Towner

A Thesis Submitted in
Partial Fulfillment of the
Requirement for the

MASTER OF SCIENCE
IN
MECHANICAL ENGINEERING

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A hyperboloid shell of revolution (HSR) is proposed for implementation as a coupling, into a fully integrated driveshaft/coupling assembly. The dynamics of the coupling is not clearly understood; which prompts the need for an analytic investigation of the hyperboloid shell of revolution.

The hyperboloid shell of revolution is one in which the meridian of the shell is defined by the equation of a hyperbola. Two methods are utilized to find the first bending frequency of the HSR: Finite Element Method and the Assumed Mode Shape Method. The Finite Element Method is applied to Timoshenko Beam Theory, and Galerkin’s Assumed Mode Shape Method is applied to the Kirchoff-Love theory of thin shells. Both methods are applied to a fixed-free and fixed-fixed HSR. A parametric study is done to study the effect of the geometric parameters (the minimum radius, and the axial length under certain specifications) on the natural frequencies. These results are then compared to those found using the program ANSYS.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>CHAPTER 1: INTRODUCTION</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1 INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>1.1.1 COUPLING OVERVIEW</td>
<td>1</td>
</tr>
<tr>
<td>1.1.2 TYPES OF COUPLINGS</td>
<td>2</td>
</tr>
<tr>
<td>1.1.3 COUPLINGS AND DRIVESHAFTS</td>
<td>3</td>
</tr>
<tr>
<td>1.1.4 APPLICATION OF COMPOSITES</td>
<td>4</td>
</tr>
<tr>
<td>1.2 DRIVESHAFT COUPLING ASSEMBLIES</td>
<td>7</td>
</tr>
<tr>
<td>1.2.1 GEISLINGERGESILCO ADVANCED COMPOSITE</td>
<td>7</td>
</tr>
<tr>
<td>1.2.2 INTEGRAL COMPOSITE SHAFT-COUPLING</td>
<td>10</td>
</tr>
<tr>
<td>1.2.3 COOLING TOWER COUPLING</td>
<td>12</td>
</tr>
<tr>
<td>1.2.4 LAWRIEDRIVESHAFT COUPLING</td>
<td>14</td>
</tr>
<tr>
<td>1.3 REVIEW OF HYPERBOLA</td>
<td>16</td>
</tr>
<tr>
<td>1.4 OTHER APPLICATIONS</td>
<td>16</td>
</tr>
<tr>
<td>1.5 PARAMETRIC STUDY</td>
<td>18</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CHAPTER 2: ANSYS MODEL</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1 MODELING THE SHELL</td>
<td>20</td>
</tr>
<tr>
<td>2.2 RESULTS</td>
<td>23</td>
</tr>
<tr>
<td>2.2.1 DYNAMIC RESULTS</td>
<td>23</td>
</tr>
<tr>
<td>2.2.2 STATIC RESULTS</td>
<td>26</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CHAPTER 3: TIMOSHENKO BEAM THEORY</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1 MATHEMATICAL MODEL</td>
<td>29</td>
</tr>
<tr>
<td>3.2 FINITE ELEMENT FORMULATION</td>
<td>31</td>
</tr>
<tr>
<td>3.3 RESULTS</td>
<td>35</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CHAPTER 4: SHELL THEORY</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1 MATHEMATICAL MODEL</td>
<td>38</td>
</tr>
<tr>
<td>4.1.1 KIRCHOFF-LOVE ASSUMPTIONS</td>
<td>38</td>
</tr>
<tr>
<td>4.1.2 MATHEMATICAL MODEL OF A SHELL OF REVOLUTION</td>
<td>39</td>
</tr>
<tr>
<td>4.1.2.A KINEMATICS</td>
<td>40</td>
</tr>
<tr>
<td>4.1.2.B CONSTITUTIVE RELATIONSHIPS</td>
<td>41</td>
</tr>
<tr>
<td>Figure</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>------</td>
</tr>
<tr>
<td>Figure 1.1: Three Types of Flexibility [1]</td>
<td>1</td>
</tr>
<tr>
<td>Figure 1.2: Directions relative to Composite Fibers [4]</td>
<td>6</td>
</tr>
<tr>
<td>Figure 1.3: Geislinger Gesilco CI-design [7]</td>
<td>8</td>
</tr>
<tr>
<td>Figure 1.4: B-F Design Coupling [7]</td>
<td>8</td>
</tr>
<tr>
<td>Figure 1.5: BI-Design Coupling [7]</td>
<td>9</td>
</tr>
<tr>
<td>Figure 1.6: An integral composite drive shaft coupling [5]</td>
<td>10</td>
</tr>
<tr>
<td>Figure 1.7: Schematic of a Composite Cooling Tower Coupling [9]</td>
<td>13</td>
</tr>
<tr>
<td>Figure 1.8: Schematic of an Integrated Shaft-Coupling Assembly</td>
<td>14</td>
</tr>
<tr>
<td>Figure 1.9: Geodesic Lines of the HSR</td>
<td>15</td>
</tr>
<tr>
<td>Figure 1.10: Geometry (c &amp; d) defining a Hyperbola</td>
<td>16</td>
</tr>
<tr>
<td>Figure 1.11: Geometry of the Parametric Study</td>
<td>18</td>
</tr>
<tr>
<td>Figure 2.1: Figure 2.1: HSR Model Generated by ANSYS</td>
<td>21</td>
</tr>
<tr>
<td>Figure 2.2: The ANSYS Finite Element Mesh of HSR</td>
<td>22</td>
</tr>
<tr>
<td>Figure 2.3: Ratio c/d vs. Natural Frequency for Fixed-Free HSR (ANSYS)</td>
<td>23</td>
</tr>
<tr>
<td>Figure 2.4: Minimum Radius vs. Frequency for Fixed-Free HSR (ANSYS)</td>
<td>24</td>
</tr>
<tr>
<td>Figure 2.5: Minimum Radius vs. First Bending Frequency for Fixed-Fixed HSR (ANSYS)</td>
<td>25</td>
</tr>
<tr>
<td>Figure 2.6: Minimum Radius vs. Bending Stiffness Results</td>
<td>26</td>
</tr>
<tr>
<td>Figure 2.7: Minimum Radius vs. Torsional Stiffness</td>
<td>27</td>
</tr>
<tr>
<td>Figure 2.8: Minimum Radius vs. Axial Stiffness</td>
<td>27</td>
</tr>
<tr>
<td>Figure 3.1: Timoshenko Beam Differential Element</td>
<td>29</td>
</tr>
<tr>
<td>Figure 3.2: Fixed-Free Results by Timoshenko Beam and Finite Elements</td>
<td>36</td>
</tr>
<tr>
<td>Figure 3.3: Fixed-Fixed Results by Timoshenko Beam and Finite Elements</td>
<td>37</td>
</tr>
<tr>
<td>Figure 4.1: Coordinates of Shell [10]</td>
<td>41</td>
</tr>
<tr>
<td>Figure 4.2: Differential Element and Stress</td>
<td>41</td>
</tr>
<tr>
<td>Figure 4.3: Differential Element with Unit Forces and Moments</td>
<td>43</td>
</tr>
<tr>
<td>Figure 4.4: Displacement coordinates for the HSR</td>
<td>55</td>
</tr>
</tbody>
</table>
Figure 4.5: Bending Natural Frequencies of a Fixed-Free HSR by
  Applying the Galerkin Method to Shell Theory..........................58
Figure 4.6: Bending Natural Frequencies of a Fixed-Fixed HSR by
  Applying the Galerkin Method to Shell Theory.........................59
Figure 5.1: Fixed-Free Results Comparison: ANSYS vs. Timoshenko Beam vs.
  Shell Theory.............................................................................61
Figure 5.2: Fixed-Fixed Results Comparison: ANSYS vs. Timoshenko Beam vs.
  Shell Theory.............................................................................63
**LIST OF TABLES**

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Table 5.1: Fixed-Free Results Comparison</td>
<td>60</td>
</tr>
<tr>
<td>Table 5.2: Fixed-Fixed Results Comparison</td>
<td>62</td>
</tr>
</tbody>
</table>
**NOMENCLATURE**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Area</td>
</tr>
<tr>
<td>C</td>
<td>Membrane Stiffness</td>
</tr>
<tr>
<td>D</td>
<td>Bending Stiffness</td>
</tr>
<tr>
<td>c &amp; d</td>
<td>Parameters Defining the hyperbola</td>
</tr>
<tr>
<td>E</td>
<td>Young’s Modulus</td>
</tr>
<tr>
<td>fₙ</td>
<td>Natural Frequency (Hz)</td>
</tr>
<tr>
<td>G</td>
<td>Shear Modulus</td>
</tr>
<tr>
<td>h</td>
<td>Thickness of Shell</td>
</tr>
<tr>
<td>I</td>
<td>Moment of Inertia</td>
</tr>
<tr>
<td>Kᵢ</td>
<td>Curvature</td>
</tr>
<tr>
<td>K₆</td>
<td>Gaussian Curvature</td>
</tr>
<tr>
<td>M</td>
<td>Moment</td>
</tr>
<tr>
<td>R₀</td>
<td>Distance to the meridian; perpendicular to the centerline</td>
</tr>
<tr>
<td>R₁</td>
<td>Radius of circumferential curvature</td>
</tr>
<tr>
<td>R₂</td>
<td>Radius of meridional curvature</td>
</tr>
<tr>
<td>u, v, w</td>
<td>Local displacements of shell</td>
</tr>
<tr>
<td>u₁, u₂, z</td>
<td>Local coordinate system of shell</td>
</tr>
<tr>
<td>V</td>
<td>Shear</td>
</tr>
<tr>
<td>ε</td>
<td>Strain</td>
</tr>
<tr>
<td>ρ</td>
<td>Density</td>
</tr>
<tr>
<td>σ</td>
<td>Stress</td>
</tr>
<tr>
<td>ν</td>
<td>Poisson’s Ratio</td>
</tr>
<tr>
<td>ωₙ</td>
<td>Natural Frequency (rad/sec)</td>
</tr>
</tbody>
</table>
Chapter 1: INTRODUCTION

1.1 INTRODUCTION

1.1.1 COUPLING OVERVIEW

In most practical applications perfect alignment of couplings to machines, and/or shafts is impossible. Misalignments can occur because of several reasons including: thermal expansion, installation error, deflection caused by applied loads, and wear of bearings and machine parts. These misalignments cause high reaction forces, which result in vibration, noise, bearing failure and sometimes failure of shafts. The three main forms of misalignment are angular, axial, and lateral [1] as seen in Figure 1.1.

![Three Types of Flexibility](image)

Figure 1.1: Three Types of Flexibility [1].

The most basic way of connecting two machines or shafts is by means of a rigid coupling. Rigid couplings can transmit torque and axial thrust but are unable to handle any of the misalignments previously mentioned. If misalignments exist, rigid couplings can generate high reaction forces. These reaction forces may induce noisy operation, increased vibrations, failure of bearings, and may even cause breakage [1].
Due to misalignment the need arises for the use of flexible couplings, which must be able to transmit power and accommodate the misalignments [2]. The flexible couplings must also be able to operate at high speeds, and handle loads, caused by acceleration and deceleration, while transmitting power and torque. They must also be able to compensate for movement of the shaft, which often is in the form of vibrations [3].

1.1.2 TYPES OF COUPLINGS

According to Johnson [1], flexible couplings generally fall under the one of four categories. These categories are: Mechanical Flexible, Elastomeric, Metallic Membrane, and Miscellaneous. Mechanical flexible couplings are categorized by loose fitting parts, and/or the rolling or sliding of parts, to give the coupling its flexibility. This category of couplings includes but is not exclusive to gear couplings, chain and sprocket couplings, grid couplings, and the basic U-joint. Mechanical flexible couplings may have a high initial cost, and most require lubrication, which is a major disadvantage of this type of coupling [1, 3]. Elastomeric couplings gain their flexibility by deformation of resilient elastic materials, including plastics and rubbers. They generally fall under one of two categories: couplings that transmit torque in shear, and those that transfer torque in compression. A common compression type is a Spyder/Jaw coupling, and a common shear type is a Tyre coupling. Other Elastomeric couplings include: donut-type couplings, pin and bushing couplings, and elastomeric block coupling. Due to the elastomeric material properties, couplings that function in compression generally handle higher loads and those that function under shear (which can be considered stretching the material) generally can handle higher misalignments. While both designs offer dampening of shock and vibration, shear type couplings usually can provide higher torsional
vibration damping. A metallic membrane coupling is either made up of metallic discs or uses a metallic diaphragm to gain their flexibility. The disc type coupling uses one or more discs, which are alternately attached to each other and/or input and output flanges. The diaphragm type coupling takes advantage of a flexible metal element concentrically attached to two flanges [1, 3]. The metallic membrane coupling holds several advantages. They do not require lubrication, they are relatively tolerant to chemicals, and they have a long life and tend not to wear like other couplings because of their lack of sliding contact [1, 2]. The diaphragm type couplings accommodate axial, angular and lateral misalignments depending on the design.

Composites have been used in similar applications to metallic membrane couplings. Composite disc couplings take advantage of a pack of discs made of composite materials. The composite material allows the coupling to have a high torsional stiffness, offers higher misalignment than a similar metallic disc type coupling, and also offers good damping [1, 3].

1.1.3 COUPLINGS AND DRIVESHAFTS

Modern engines and transmissions test the capabilities of drive shafts and couplings [5]. Machines are being produced, which are operating at higher speeds and are operating much closer to the natural frequencies of the shafts and/or couplings [2]. Shaft and coupling combinations must be able to handle misalignments while being able to operate at high speeds, and they must be light in order to operate at high natural frequency requirements [3, 5]. Along with the previously mentioned requirements; they must also be aesthetically pleasing and have ease of maintenance [3].
For initial analysis of a drive shaft simply supported boundary conditions can be assumed. By simulations it is obvious that the shaft by itself does not determine the natural frequency. Motion occurs at the attachment points corresponding to the couplings [5]. These motions, often because of misalignments, cause significant reaction forces on the shaft [1]. These forces are a result of the stiffness of the coupling, and therefore the stiffness must be found.

1.1.4 APPLICATION OF COMPOSITES

The combination of two or more distinctly different materials into a new material is considered a composite material. One of the materials is usually a fiber. Fibers are often made of glass or carbon, which are considerably strong and often have much fewer defects in fiber form than in bulk form. The fibers are what give the composite material much better stiffness to weight ratios than other materials [12]. Fibers are very strong in tension and are the main contributor to the ultimate performance of the composite, but they alone are insufficient to handle compression or transverse loads. A binder, called the matrix, is used to hold the fibers together, and mold the fibers into a composite component. It is often made up of a thermosetting resin, which includes polyester and vinyl ester resins and epoxies. The matrix has both adhesive and cohesive properties that allow loads to be transferred between the fibers. The matrix is also important in protecting the fibers from the environment and giving the composite resistance to corrosion [4, 12].

The fibers and the matrix are both important to the final composite component while each has a different function [4]. By themselves neither the fibers nor the matrix are sufficient. It is when the materials are combined together they become a strong composite material [11].
Composite technology is growing rapidly in the power transmission industry, mainly because of its advantages over metals. When compared to metals, composites have higher specific strength and specific stiffness, have little to no thermal expansion, and are resistant to corrosion [12]. They also allow for engineering tailoring of a design, simply by changing the orientation angle of the fibers. This allows a part to be designed with desired dynamic characteristics. Composites' higher specific stiffness results in higher natural frequencies [6]. This allows for parts to be designed with a much larger range of operation. Due to these advantages, some of the first applications of composites were in the aerospace and cooling tower-coupling industries, and now are finding their way into commercial industry. These advantages have lead to the use of composites in drive shafts and make composites a good material for use in flexible couplings [4].

The unique properties of composites are the same properties that make their analysis difficult. Materials, such as metals, have mechanical properties that are isotropic. Materials that are isotropic have mechanical properties that are independent of the direction considered. Composites are considered to be anisotropic; therefore, the mechanical properties of a composite depend on the direction considered. This phenomenon is due to the make up of a composite material. Consider a single layer, or lamina, of a fibrous composite material, where fibers are imbedded parallel to the x-axis into a matrix. The lamina will have a much greater resistance to loading in the x-direction, than in the y or z-direction, due to the fact that the fibers will be axial loaded; therefore, the modulus of elasticity will be much higher in the x-direction (or longitudinal direction) than in the y or z-direction (or transverse direction) [13, 8, 4].
Since the strength of the lamina is much less in the transverse directions a laminate needs to be created. A laminate is created by stacking lamina with different fiber orientations. Now the composite may handle loads in the desired directions [12]. It is important to note that by design one direction may be stronger than another but for stability and rigidity of the final part, all direction will have some strength. It is the ability to arrange the fibers in different ways that makes composites unique. The strength and overall characteristics of the final product can be altered by arranging the layers of the composite at different angles [4]. These anisotropic properties that result from the different arrangements of layers are what allow us to tailor the design to meet the requirements one is looking for [11].

Laminate plate theory is the cornerstone of theory and analysis of composites. This theory relates the stiffness and strain of each layer to properties of the final laminate. A single layer or lamina can be defined by properties found analytically or by experimentation. Then by rigorous mathematical theory the different properties of the matrix can be found. The strain and stiffness of the laminate are related by matrix operations [4].
1.2 DRIVESHAFT COUPLING ASSEMBLIES

1.2.1 GEISLINGER GESILCO ADVANCED COMPOSITE

The Geislinger Gesilco advanced composite coupling, is very similar to a metallic diaphragm coupling. Geislinger was the first to produce a composite misalignment coupling to be used in the shipping industry. This application is for use on ship drive lines between engine and gears, or between gears and water-jet. Their goal was to produce a coupling that would be first torsionally stiff, as well as, be able to handle high misalignments. Also, this coupling is meant to have good sound insulation and reduce the dead weight of the coupling and in turn reduce the mass moment of inertia. If the coupling could meet these requirements it would help meet the needs of the shipping industry that thrives on weight savings. The more weight that can be saved the more payloads can be carried and the faster the ships can travel [7].

A very good design overview is given in the paper in its General Design Concept section. The composite coupling is a membrane coupling designed to meet several requirements. Some of the requirements include: being able to operate between 100 and 3000 rpm, having at least a 20000 hour service life, must be able to handle a torque of 300kNm, and must be able to handle a 3 degree angle of deflection [7]. It is important to note that this is not an integral composite shaft coupling assembly.

There are two main coupling designs, as well as, a combination of the two designs. The first design, the Cl-design, is made up of two membranes and a shaft at the inner diameter of the membranes. A diagram of the Cl-design is shown in Figure 1.3. The membranes are adhesively bonded to the shafts and then are bolted to the adjacent steel flanges.
The second design, the BF- (Butterfly)-design, consists of intermediate shafts arranged on the outer diameter of the membranes. The two halves of the coupling can then be bolted together and different sized washers can be used to help compensate for any axial misalignment.

A combination of the two couplings is called the BI-design. The main advantage of the BI-design allows the coupling to be attached to the engines flywheel by the CI-part and to a torsional elastic coupling by the BF-part [7].

Figure 1.3: Geislinger Gesileco CI-design [7].

Figure 1.4: BF-Design Coupling [7].
The composite coupling is made by means of the prepreg/autoclave manufacturing technique. This method was chosen because it offered good reproducibility and it is able to yield a high fiber volume contents. Each layer of the laminate is made up of E-glass or carbon unidirectional prepreg tapes. The design of the composite membrane used in the couplings is unique. The cross-section of the membrane is tapers towards the outer diameter and is corrugated. It was found by simulation and experimentation that this design yields a higher deflection with lower stiffness than a flat tapered membrane. The lower stiffness yields much lower reaction forces [7].

The designer of the coupling decided using finite element analysis, by means of the simulation program ANSYS, and was a suitable tool for analysis. The author did mention that there is a need for a good, reliable means of analysis for a composite membrane loaded in this manner. The results were later compared against experimental data. A couple of assumptions were used in the analysis. First, it was assumed that in the model the single layers of the laminate were being simulated correctly. Second, since the experimental data
seems to correspond closely to the FEA results it is possible to optimize the composite coupling using the FEA models [7].

Since 1993 more than 500 of these couplings have been in service. The oldest coupling has been running for over 16000 hours with no problems. This coupling is used on a fast ferry that was built in a Spanish shipping yard. It seems so far that design and development of these particular composite couplings has been a success [7].

1.2.2 INTEGRAL COMPOSITE SHAFT-COUPLING

A paper by Faust, Hogan, Margasahayam, and Hess [5] gives an overview of an integrated composite drive shaft and couplings that were in the process of being developed. The authors state that the combining of the shaft couplings is unusual and unique. The part described is first made of a shaft which is constructed using braided-fiberglass. The couplings are then integrally braided into the part. Finally the process is completed be means of resin transfer molding. The design of the integral shaft and coupling must meet several strict design criteria. Some of the most critical are as follows. The part must be able to transmit 1200 hp at 23,000 rpm, must operate in the range of 16,000 to 26,000 rpm, and must have a natural frequency of at least 31,750rpm. Figure 1.6 shows a schematic of an integral shaft coupling.

Figure 1.6: An integral composite drive shaft coupling [5].
Several equations are listed that were used in the preliminary analysis by Faust, Hogan, Margasahayam and Hess. These equations include ways of finding torsional stress, flexural strain, buckling, and bending stiffness. These equations are all based on the assumption of isotropic materials, which lead to error when used for composite materials. Thus, they are only given to provide some direction on how the part needs to be looked at. Instead, all analysis is done by means of finite element analysis using PDA/PATRAN and MSC/NASTRAN, and by testing [5].

In a parallel paper by Margasahayam and Faust [8], a detailed overview of the 3D finite element analysis done on the coupling is given. Finite element analysis was chosen mainly because of the anisotropic nature of composites, and due to the fact that the material properties differ from point to point in the coupling. Finite element analysis was also a very effective way of finding the bending, axial, and torsional stiffness in order to aid in the critical speed calculations of the shaft coupling. Using CADAM a 2D cross section was developed which was used as the basis for a 3D Finite Element model. Due to symmetry only half of the Shaft coupling was modeled. For the actual analysis, and for plotting displacement and stress PDA/PATRAN was utilized. The model was broken into 8 material zones to compensate for the varying fiber angles in each lamina. The model was broken into solid, 8-node, rectangular hexahedral elements. Solid brick elements were utilized instead of 2D shell elements for several important reasons including: braided laminate is relatively thick, interlaminar stresses were of a concern, they were used to model each layer, it makes it easier to develop an equivalent orthotropic property, and accuracy was an overriding
concern. The material properties input into the program were the calculated equivalent mechanical properties of the laminate [8].

Following the finite element analysis, tests were performed on full-scale prototypes of the shaft coupling. The shaft coupling was loaded in three separate tests in bending, axial tension, and torsion. Strain gauges were placed in the main areas of interest based on findings of the finite element analysis [8].

Both papers concerning this shaft coupling discuss the results from the finite element analysis and testing, both of which fall very close to one another. Dynamic characteristics are discussed but not in great detail; however, an important note is that the shaft alone does not control the natural frequency. Instead considerable motion due to misalignments of the attachment points of the couplings to machines has a great effect. Diaphragm bending of the couplings results in significant lateral and radial stiffness. One important notion taken from the paper is that at the time of their development they had to rely on testing and computer FEA, because of the lack of better analytical understanding [5].

1.2.3 COOLING TOWER COUPLING

Composite drive shafts and flexible couplings have also been used extensively in the cooling tower industry [9]. An example is shown on Figure 1.5. Drive shafts made of composites were first used in 1986 [11]. The couplings are used in the driveline that turns large cooling fans in the towers. A flexible element allows the couplings to handle high torque loads and high misalignments. The spacer tubes, and the flexible element, are both made by the
filament winding process. Composites were used because they yielded parts that were much lighter, stronger, stiffer, and their designs could be tailored to meet specific application requirements. The composite components also yielded higher tolerance to misalignments, are more resistant to corrosion, causing lower loads on bearings, are more resistant to fatigue, and show no thermal expansion. The composite spacer tubes are also able to span much longer distance, which allows for using fewer flexible couplings and support bearings. These couplings are also able to transmit greater torque than their metal counterparts, and therefore can be designed to meet speed requirements rather than strength requirements [9]. These couplings have proven to be a low maintenance solution to problems associated with cooling tower drive systems [4].

![Figure 1.7: Schematic of a Composite Cooling Tower Coupling [9].](image)

The paper then gives a good description of the loads the misalignment couplings must sustain. First, the coupling must be able to handle static and vibratory torque. At the same time it must handle 3 types of misalignment: axial, radial, and angular. The angular and/or radial deflections result in a very high number of load cycles over the life of the coupling.
1.2.4 LAWRIE INTEGRAL DRIVE SHAFT COUPLING ASSEMBLY

The Lawrie drive shaft consists of a shaft and two flexible couplings integrated into one unit.

The integrated shaft coupling is made from composites and is manufactured using the filament winding process. Flanges are attached to assembly after manufacturing [34]. The basic shape of the integrated shaft coupling assembly is shown in Figure 1.8.

![Figure 1.8: Schematic of an Integrated Shaft-Coupling Assembly.](image)

Similar to the previously described shaft coupling assemblies, it is critical that the assembly be able to handle high torque, while allowing flexibility in bending. The flexibility is obtained by means of the coupling portions of the assembly.

This thesis focuses on the coupling portion of the assembly, and serves to lay the foundation for future work in this area. It will consider a portion of the coupling whose shape is defined by a hyperboloid shell of revolution (HSR). Geodesic lines, made of filaments, create the shape of the coupling, in the assembly. These are straight lines that designate the shortest surface line between two points on the curved surface of the HSR. As filaments are added the resulting shape of the object will be similar to the shape of Figure 1.9.
The inherent shape of this process is that of a hyperboloid shell of revolution (HSR); whose meridian is defined by a hyperbola. Due to the inherent complexity of composites and HSR, this thesis will focus on finding the dynamic characteristics of a thin HSR (thickness = .001 in) made of an isotropic material. Specifically, the material chosen in the analysis of the shell was Aluminum; therefore, the following properties were used:

\[
\begin{align*}
\rho &= 2.55 \times 10^{-4} \text{ lbf s}^2 / \text{in}^4, \\
E &= 10 \times 10^6 \text{ psi}, \\
\nu &= .33.
\end{align*}
\]
1.3 REVIEW OF HYPERBOLA

The meridian of the hyperboloid shell is naturally defined by a hyperbola, whose equation is presented in Figure 1.10.

As can be seen from Figure 1.10, c and d are constants that define a rectangle whose diagonals form the asymptotes of the hyperbola [16]. Changing one or both of these values changes the curvature of the hyperbola. Based on this geometric relationship, the parametric study, outlined in Chapter Two, is developed.

1.4 OTHER APPLICATIONS

There are other applications to Hyperboloid Shells of Revolution, including water towers, TV towers, structural supports, factory chimneys, and designs of buildings. The most common may be the application of cooling towers. This shape was chosen for the cooling towers because of geometric features relating to geodesic lines. According to Krivoshayko, this property allows the steel reinforcements to be optimally placed within the shell that creates the cooling tower [23].
Countless papers can be found on the analysis of cooling towers. This analysis focuses mainly on the buckling and vibrations of the cooling towers. Early work, like that done by Carter [26] and Neal [25], were done by means of basic shell theory and by extensive experimentation. Much of this research was done in interest to failure of cooling towers in the 1960s [25, 26]. Carter explains that much of the early experimentation was done with insufficient boundary conditions, leading to error in results. Later analysis of cooling towers focuses on the use of Finite Element Analysis. Examples of this work include, but are not limited to, work done by Aksu [24], and Tan [27]. Also, general work that mentions application to hyperboloids includes, but is not limited to, work by Lee & Bathe [30], and by Fan and Luah [30].

The analysis in this thesis is different when compared to analysis done in the previous mentioned papers and similar literature. One of the obvious differences is that the flexible coupling is considered symmetric about what would be the x-y plain in Figure 1.10. Cooling towers are generally much longer on one side of the throat, where the minimum radius is located. More importantly no literature could be found, that studied the affects of changing the parameters on the natural frequency of the hyperboloid. This is the main focus of this thesis as is explained in Chapter Two. The analysis is conducted by means of beam theory, shell theory, and the use of a finite element computer program (ANSYS) in this investigation.
1.5 PARAMETRIC STUDY

In order to study how changing the HSR affects its natural frequency, a parametric study was set up. Of most importance the affect of changing the minimum radius of the HSR was considered for this thesis.

For this study a window was set up where $L_{\text{max}}$ and $R_{\text{max}}$ were set to 6 inches and 3 inches respectively, as shown in Figure 1.11. The value of $R_{\text{min}}$ (c in the equation of the hyperbola) was changed in .25 inch increments. This also required the value of $d$, in the equation of a hyperbola, to change according to Equation 1.1.

$$d = \frac{|c^*z_{\text{max}}|}{\sqrt{R_{0_{\text{max}}}^2 - c^2}},$$

where $z_{\text{max}} = 3\text{in}$, and $R_{0_{\text{max}}} = 3\text{in}$.  

(1.1)
A spreadsheet was created in Excel in order to efficiently find the changing values of the variable d, while changing $R_{\text{min}} (c)$. This spreadsheet can be found in Appendix A, and was utilized throughout the thesis.

The parametric study was conducted by three methods. The first is by use of the finite element program ANSYS, which was used to provide a base line for results and to give some initial insight into the problem. Second, the HSR is modeled by Timoshenko Beam Theory, and solved for by the finite element method. Third, the HSR is modeled by Shell Theory and corresponding results obtained by the Galerkin Assumed Mode Shape Method. Last, results of bending natural frequencies, from all three, are compared to one another.

The boundary conditions applied to the HSR, are fixed-free and fixed-fixed. The ends that are fixed will be considered cantilevered, and application of the boundary conditions is explained within the thesis.
CHAPTER 2: ANSYS MODEL

2.1 MODELING THE SHELL

A parametric study of the HSR was performed using the finite element program ANSYS, version 8.0. This was done to set a basis for the study. In Appendix B, batch files (log files) can be found for modeling and analyzing all the cases discussed within. The Batch files may be copied and/or modified for specific cases, and then run in ANSYS 8.0.

In order to correctly model the HSR, the hyperbola defining the meridian had to be created. This was done by defining one-half of the curve, first by 6 key-points, and then by a spline connecting the points. The key-points were plotted according to the equation of the specific meridian, where \( z \) is the independent variable, and \( R_0 \) is the dependent variable. The "spline with option" command was used to create half of the meridian by connecting the key-points, and defining the slopes at the beginning and end of the curve. The line was then reflected to create the full meridian, and extruded 360 degrees to create the HSR.

Figure 2.1 shows an example of the shell modeled in ANSYS. The Tables found in Appendix A were used to quickly find the points defining the line and the slopes at the end of the spline. These values can be taken from the spreadsheet and changed within the batch files for each specific HSR.
The element used in the ANSYS analysis was Shell93. This is an 8-Node structural shell element. Each node has six degrees of freedom, including both translation and rotation. According to the ANSYS tutorial, this element is suited for modeling curved shells.

A preliminary study was done in order to determine a sufficient number of elements to mesh the HSR. It was found that 20 elements along the length and 16 circumferential elements were sufficient to get consistent results, for a fixed-free HSR. For the fixed-fixed HSR, the number of elements along the length was increased to 30 to obtain more consistent results. Following is a Figure of a meshed HSR in ANSYS.
During the study it was also found that \( R_{\text{min}} \) values below 1 inch, for the fixed-free HSR, and values of \( R_{\text{min}} \) below 1.25 inches, for the fixed-fixed HSR, gave inconsistent results. Refining the mesh, and/or defining a completely new mesh resulted in very random results and mode shapes were unclear. This could be due to coupling between modes and/or the increase of curvature, or that the element used could not handle the high curvature. A trend can still be found when reviewing the final results of the ANSYS study.

A dynamic study was performed on the HSR in ANSYS. The first bending, longitudinal, and torsional natural frequencies for the fixed-free HSR and the first bending frequency for the fixed-fixed HSR were found.
A cylinder of length and radius, 6 inches and 3 inches respectively, was simulated in ANSYS. The Batch file can be found in Appendix B. This was done to investigate whether or not the HSR results were converging to the results of a cylinder with an increasing $R_{\text{min}}$. It was found that the HSR results did indeed converge to a cylinder with increasing $R_{\text{min}}$, as shown by the following results.

### 2.2 RESULTS

#### 2.2.1 DYNAMIC RESULTS

Below, in Figures 2.3 and 2.4, are the fixed-free HSR ANSYS results. Figure 2.3 shows the natural frequencies versus the ratio $c/d$ ($c = R_{\text{min}}$), while Figure 2.4 shows the natural frequencies versus the minimum radius. An increasing $c/d$ in Figure 2.3 corresponds to the decreasing $R_{\text{min}}$ of Figure 2.4.

![Figure 2.3: Ratio c/d vs. Natural Frequencies for Fixed-Free HSR (ANSYS).](image)
It was expected that as $R_{\text{min}}$ decreased (and $c/d$ increased), the HSR would become less stiff, and the natural frequency would decrease. This pattern is demonstrated only for the torsional frequency. In the cases of bending and longitudinal vibration the natural frequencies first increase with decreasing $R_{\text{min}}$, followed by a drop in the natural frequencies.
Figure 2.5 shows the fundamental bending frequency of a fixed-fixed HSR, from ANSYS. Consistent results were only obtained up to $R_{\text{min}}$ of 1.25 inches. The results demonstrate an increasing natural frequency with decreasing $R_{\text{min}}$. Contrast to the fixed-free results (Figure 2.4) the bending natural frequency does not decrease with decreasing $R_{\text{min}}$.

Figure 2.5: Minimum Radius vs. First Bending Frequency for Fixed-Fixed HSR (ANSYS).
2.2.2 STATIC RESULTS

A static study of the fixed-free HSR was also performed. From the static analysis the stiffness of the coupling as a function $R_{\text{min}}$ was found, which would be helpful for the design of the integrated shaft-coupling unit. Three cases were set up to find the bending, axial, and torsional stiffness of the HSR. Batch files can be found in Appendix B that can be used to find the stiffness of a HSR. Figures 2.6, 2.7 and 2.8 show the Bending Stiffness, Torsional Stiffness and Axial Stiffness as functions of $R_{\text{min}}$, respectively.

![Graph showing minimum radius vs. bending stiffness](image-url)

**Figure 2.6:** Minimum Radius vs. Bending Stiffness.
Figure 2.7: Minimum Radius vs. Torsional Stiffness.

Figure 2.8: Minimum Radius vs. Axial Stiffness.
The trends of the stiffness plots (Figures 2.6, 2.7, and 2.8) are similar to their corresponding natural frequency trends (Figure 2.4). Both the bending and axial stiffness’ initially increase with decreasing $R_{\text{min}}$, followed by a decrease in the frequencies. Similar to the torsional natural frequencies the torsional stiffness decreases continually with decreasing $R_{\text{min}}$. 
CHAPTER 3: TIMOSHENKO BEAM THEORY

3.1 MATHEMATICAL MODEL

The HSR was further studied by means of beam theory. In order to consider both shear and bending, the Timoshenko beam theory was utilized. This is in contrast to the Euler-Bernoulli beam theory that neglects shear strain, and assumes the cross-section remains plane and perpendicular to the longitudinal axis during bending [20]. According to Timoshenko’s theory the cross-section remains plane but does not remain normal to the axis. The shear angle, γ, is the difference between the angle to the normal of the cross-section, φ, and the slope of the centerline, ∂w/∂x (Figure 3.1).

Figure 3.1: Timoshenko Beam Differential Element.

Two equations of motion, one for transverse translation, w, and one for rotation, φ, were written from the Timoshenko beam differential element (Figure 3.1). Equations 3.1 and 3.2 are the equations of motion in terms of the shear, V, and moment, M.

\[ V' = -\rho A\ddot{w}. \]  
(3.1)

\[ M' - V = \rho I\ddot{\phi}. \]  
(3.2)
The moment and the shear were found from the elastic equation of the beam and are given by Equations 3.3 and 3.4.

\[ M = EI \frac{\partial \phi}{\partial x}, \]  
\[ (3.3) \]

And

\[ V = kGAY = kGA \left( \phi - \frac{\partial w}{\partial x} \right). \]  
\[ (3.4) \]

The constant \( k \) is the Timoshenko shear coefficient, \( G \) is the shear modulus, \( E \) is Young's Modulus, \( A \) is the area, and \( I \) is the moment of inertia. The shear coefficient varies depending on the shape of the cross-section. This constant is used to account for the assumption of constant shear over the cross-section [20]. Depending on the source, its value can vary. For this thesis \( k \) was set to .5 and .9. This gives a range of results for the natural frequencies. Together, with the appropriate boundary conditions, Equations 3.5 and 3.6, represent the mathematical model of the Timoshenko Beam.

\[ kGA \left( \frac{\partial \phi}{\partial x} - \frac{\partial^2 w}{\partial x^2} \right) = -\rho A \ddot{w} \]  
\[ (3.5) \]

\[ EI \frac{\partial^2 \phi}{\partial x^2} - kGA \left( \phi - \frac{\partial w}{\partial x} \right) = \rho I \ddot{\phi} \]  
\[ (3.6) \]

Equations 3.5 and 3.6 are obtained from substituting Equations 3.3 and 3.4 into Equations 3.1 and 3.2, respectively. The Equations will be solved for the natural frequencies using finite elements and Matlab.
3.2 FINITE ELEMENT FORMULATION

The finite element formulation was done in four steps:

(1) Mesh Generation and Function Approximation: The translation, \( w \), and the rotation, \( \phi \), were approximated by:

\[
\phi = \sum \Psi_j \Phi_j \quad \text{and} \quad w = \sum \Psi_j W_j.
\]  

(3.7)

Where \( W_j \) and \( \Phi_j \) are the nodal transverse displacement and rotation components, respectively, and \( \Psi_j \) is the shape function. The Hermite cubic interpolation functions, Equation 3.8, were adopted for this analysis.

\[
\begin{align*}
\Psi_1(x) &= 1 + 2 \left( \frac{x}{h} \right)^3 - 3 \left( \frac{x}{h} \right)^2, \\
\Psi_2(x) &= x - 2 \frac{x^2}{h} + \frac{x^3}{h^2}, \\
\Psi_3(x) &= 3 \left( \frac{x}{h} \right)^2 - 2 \left( \frac{x}{h} \right)^3, \\
\Psi_4(x) &= \frac{x^3}{h^2} - \frac{x^2}{h}.
\end{align*}
\]

(3.8)

(2) The Element Equation: The element equation can be written as

\[
[M]^e \ddot{U}^e + [K]^e U^e = 0
\]

(3.9)

Where \([M]^e\) and \([K]^e\) are the element mass and stiffness matrices, respectively, and \(U^e\) is the element nodal displacement vector.
The shape functions were first substituted into the equations of motion to obtain the residuals, $R$:

$$R_{j,1} = \ddot{\Psi} + \rho A \sum \psi_j \dot{\psi}_j.$$  \hspace{1cm} \text{(3.10)}

And

$$R_{j,2} = [\ddot{M} - \ddot{\Psi}] - \rho I \sum \psi_j \dot{\phi}_j.$$  \hspace{1cm} \text{(3.11)}

Where

$$\ddot{\Psi} = kGA \left( \sum \psi_j \phi_j - \sum \psi_j \dot{w}_j \right).$$  \hspace{1cm} \text{(3.12)}

And

$$\ddot{M} = E I \sum \psi_j \dot{\phi}_j.$$  \hspace{1cm} \text{(3.13)}

Next, the weighted residuals were formed by multiplying the residuals by the weight functions, $\Psi_i$. The weighted residuals were integrated over the element and set equal to zero.

$$\int_{x_1}^{x_2} \Psi_i \ddot{\Psi} dx + \int_{x_1}^{x_2} \Psi_i \rho A \sum \psi_j \dot{\psi}_j dx = 0$$  \hspace{1cm} \text{(3.14)}

$$\int_{x_1}^{x_2} \Psi_i [\ddot{M} - \ddot{\Psi}] dx - \int_{x_1}^{x_2} \Psi_i \rho l \sum \psi_j \dot{\phi}_j dx = 0$$  \hspace{1cm} \text{(3.15)}

Integrating Equations 3.14 and 3.15 by parts and expanding yields:

$$\int_{x_1}^{x_2} \Psi_i \rho A \sum \psi_j \dot{\psi}_j dx - \int_{x_1}^{x_2} \Psi_i \sum kGA \left[ \sum \psi_j \phi_j - \sum \psi_j \dot{w}_j \right] dx = - \int_{x_1}^{x_2} \Psi_i \nu dx,$$  \hspace{1cm} \text{(3.16)}

And

$$\int_{x_1}^{x_2} \Psi_i \rho l \sum \psi_j \dot{\phi}_j dx + \int_{x_1}^{x_2} \left[ \Psi_i \sum E I \psi_j \phi_j \right] dx + \int_{x_1}^{x_2} \left[ \Psi_i \sum kGA \left[ \sum \psi_j \phi_j - \sum \psi_j \dot{w}_j \right] \right] dx = \int_{x_1}^{x_2} \Psi_i M dx.$$  \hspace{1cm} \text{(3.17)}
Equations 3.16 and 3.17 can be represented as:

\[ M_{ij} \ddot{W}_j + (K_{ww})_{ij} W_j + (K_{\omega \omega})_{ij} \Phi_j = \Psi_i V_{x_1}^2, \]  
(3.18)

And

\[ J_{ij} \ddot{\Phi}_j + (K_{w\phi})_{ij} W_j + (K_{\phi \phi})_{ij} \Phi_j = \Psi_i M_{x_1}^2. \]  
(3.19)

Where:

\[ M_{ij} = \rho \int_{x_1}^{x_2} \Psi_i A \Psi_j dx, \]
\[ (K_{ww})_{ij} = kG \int_{x_1}^{x_2} \Psi_i A \Psi_j^2 dx, \]
\[ (K_{\omega \omega})_{ij} = -kG \int_{x_1}^{x_2} \Psi_i A \Psi_j dx, \]  
(3.20)

\[ J_{ij} = \rho \int_{x_1}^{x_2} \Psi_i \Gamma A \Psi_j dx, \]
\[ (K_{w\phi})_{ij} = -kG \int_{x_1}^{x_2} \Psi_i A \Psi_j^2 dx, \]
\[ (K_{\phi \phi})_{ij} = \int_{x_1}^{x_2} [\Psi_j^2 \Phi_j + \Psi_j kG A \Psi_j] dx \]

In matrix form, Equations 3.18 and 3.19, are given by 3.21. Notice that, there are no external forces, moments, or masses applied to the shell in this analysis; therefore, the force vector is set to all zeros.

\[
\begin{bmatrix}
M \\
J
\end{bmatrix}
\begin{bmatrix}
\ddot{W} \\
\ddot{\Phi}
\end{bmatrix}
+ \begin{bmatrix}
K_{ww} & K_{pw} \\
K_{wp} & K_{pp}
\end{bmatrix}
\begin{bmatrix}
W \\
\Phi
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]  
(3.21)
In Equation 3.21, \( W \) is the nodal transverse displacement vector \( (W = [W_1, W_1', W_2, W_2']^T) \) and \( \Phi \) is the nodal angular rotation vector \( (\Phi = [\Phi_1, \Phi_1', \Phi_2, \Phi_2']^T) \), where a prime denotes the first derivative with respect to \( x \).

Note that upon performing the integration in Equation 3.21, \( A \) and \( I \) are considered functions of the axial global coordinate \( X \):

\[
A(X) = \pi r_0(X)^2 - r_1(X)^2 \quad \text{and} \quad I(X) = \frac{\pi}{4} \left( r_0(X)^4 - r_1(X)^4 \right)
\]  

(3.22)

Where global coordinate \( X \) is a function of local element coordinate, \( x \):

\[
X = X_0 + x,
\]  

(3.23)

and \( X_0 \) is the global coordinate of the element coordinate \( x_1 \) (at node 1).

(3) Assembly of the Global Equation: Assembly of the local element equations into the global equation was done by using the standard finite element assembly method [32], rendering:

\[
[M]\ddot{U} + [K]U = 0
\]  

(3.24)

Where \([M]\) and \([K]\) are the assembled global mass and stiffness matrices, respectively, and \( U \) is the global nodal displacement vector. Programs were written in Matlab to assemble the global equation for the fixed-free and fixed-fixed boundary conditions (Appendix C).

(4) Solving for the Eigenvalues: A Matlab program was written to obtain the eigenvalues and natural frequencies from Equation 3.24. The program can be found in Appendix C.
3.3 RESULTS

Figures 3.2 and 3.3 are the results for a fixed-free and fixed-fixed HSR, respectively. The results were obtained by applying finite elements to the Timoshenko mathematical model with the appropriate boundary conditions.

For the fixed ends:

\[ w = \frac{dw}{dx} = \phi = 0, \quad (3.25) \]

And for the free end:

\[ kAG \left( \frac{dw}{dx} - \phi \right) = EI \frac{d\phi}{dx} = 0. \quad (3.26) \]

In Appendix C the Matlab M-files used to find the natural frequencies for varying \( R_{\text{min}} \) can be found.
Figure 3.2 shows the bending natural frequencies, for the fixed-free HSR, versus $R_{\text{min}}$. Plots are given for the shear factor, $k$, equal to .5 and .9. The results show a decreasing natural frequency with decreasing $R_{\text{min}}$. As expected the natural frequency increases with an increase of $k$. Note that the curves for the different shear coefficients seem to converge with decreasing $R_{\text{min}}$.

Figure 3.2: Fixed-Free Results by Timoshenko Beam and Finite Elements.
Figure 3.3 shows the bending natural frequencies, for the fixed-fixed HSR, versus the minimum radius. Again, plots are given for the shear factor, k, equal to .5 and .9. The results show that as $R_{\text{min}}$ decreases the natural frequency increases. Again, the higher shear coefficient results in higher natural frequencies. Notice that, contrast to the fixed-free results (Figure 3.2) the two curves ($k = .5$ and $k = .9$) diverge as $R_{\text{min}}$ decreases.

![Figure 3.3: Fixed-Fixed Results by Timoshenko Beam and Finite Elements.](image-url)
Chapter 4: SHELL THEORY

4.1 MATHEMATICAL MODEL

4.1.1 KIRCHOFF-LOVE THEORY OF SHELLS

During the second half of the 19th Century Love added assumptions to Kirchoff's assumptions for the theory of plates so that it could be extended to the theory of shells. This is sometimes called the Kirchoff-Love Theory of Shells or just Love's Theory of Shells. Later Reissner added the influence of transverse shear strains to the theory of shells to provide more accurate solutions. Many others have contributed to the mechanics of shells of revolution including but not limited to: Timoshenko, Girkann, Novozhilov, Vlasov, Lur’e, and Krauss [10].

The basic approach of shell theory is to replace 3-dimensional analysis by the analysis of hypothetical 2-dimensions. The kinematics and the kinetics are normally referred to the middle surface of the shell. This premise forms the foundation of the linear classical shell theory [5].

Four main assumptions were made by Love, and he referred to them as his “first approximation” shell theory.

(1) The shell thickness is small compared with the smallest radius of curvature of the middle surface of the shell.

(2) The deformations and displacements are so small that quantities of second order or higher are neglected.
(3) Normal stresses, transverse to the middle surface, are small when compared with other stresses, and can be neglected.

(4) Normals to the middle surface of the shell will remain normal to the middle surface in all deformed configurations of the shell, and will not be subject to deformation.

The first assumption is the basis for all thin shell theory [14]. The thickness of the shell should be several times less than the radii of the shell as well as other dimensions describing the shell. According to Novozhilov [10], the relationship $h/R < 1/20$ should be satisfied in order to achieve errors of 5% or less, where ‘$h$’ is the shell thickness and ‘$R$’ is the smallest radius of the shell. The fourth assumption is Kirchhoff’s hypothesis. According to this assumption the strains in the direction normal to the shell are zero. This greatly simplifies the development of the theory [10].

4.1.2 MATHEMATICAL MODEL OF A SHELL OF REVOLUTION

The mathematical model of a shell of revolution is developed based on the classical mechanics approach. The equilibrium equations are developed as functions of forces and displacements. Substituting the constitutive equations (stress-strain relationships) and the strain displacement relationships (Kinematics) into the force equations, the mathematical model is obtained.
4.1.2.A KINEMATICS

The relationships of the displacements to the strains, $\varepsilon_1$ and $\omega$, the changes in curvature, $\chi_\alpha$, and the twist, $\tau$, can be written as [10]:

\[
\varepsilon_1 = \frac{1}{R_0} \frac{\partial u}{\partial \theta} + \frac{1}{R_1} (v \cot \varphi - w),
\]

\[
\varepsilon_2 = \frac{1}{R_2} \left( \frac{\partial v}{\partial \phi} - w \right),
\]

\[
\omega = \frac{1}{R_0} \frac{\partial v}{\partial \theta} + \frac{1}{R_2} \frac{\partial u}{\partial \phi} - \frac{u}{R_0} \cos \varphi,
\]

\[
\chi_1 = -\frac{1}{R_0} \left( \frac{\partial X_2}{\partial \theta} + X_2 \cos \varphi \right),
\]

\[
\chi_2 = -\frac{1}{R_2} \frac{\partial X_2}{\partial \phi},
\]

\[
\tau = \frac{1}{R_0} \left( \frac{\partial X_2}{\partial \theta} - X_1 \cos \varphi \right) - \frac{1}{R_1 R_2} \frac{\partial u}{\partial \phi},
\]

where the rotations, $X_{\alpha}$, are given as:

\[
X_1 = \frac{1}{R_0} \frac{\partial w}{\partial \theta} + \frac{u}{R_1},
\]

\[
X_2 = \frac{1}{R_2} \left( \frac{\partial w}{\partial \phi} + v \right).
\]

Where $u$ is the circumferential displacement, $v$ is the displacement tangent to the meridian, $w$ is the transverse displacement normal to the surface, and the radii, $R_0$, $R_1$, and $R_2$ (Figure 4.1) are the distance normal from the center axis to the meridian, the radius of circumferential curvature, and the radius of the meridians curvature, respectively [10].
4.1.2.B CONSTITUTIVE RELATIONSHIPS

Figure 4.2 shows a differential element for a shell, with the stress components acting across the thickness of the shell. Notice, according to Love’s assumptions the stresses normal to the middle surface are neglected.
Hooke’s Law gives the stress-strain relationships. In three dimensions Hooke’s Law can be written as:

\[ \varepsilon_i = \frac{1}{E} \left[ \sigma_i - \nu(\sigma_2 + \sigma_3) \right], \]

\[ \varepsilon_2 = \frac{1}{E} \left[ \sigma_2 - \nu(\sigma_1 + \sigma_3) \right], \]

\[ \varepsilon_3 = \frac{1}{E} \left[ \sigma_3 - \nu(\sigma_1 + \sigma_2) \right], \]

\[ \varepsilon_{12} = \frac{\sigma_{12}}{G}, \]

\[ \varepsilon_{21} = \frac{\sigma_{21}}{G}, \]

\[ \varepsilon_{\alpha3} = \frac{\sigma_{\alpha3}}{G}. \]

where \( E \) is Young’s Modulus, \( G \) is the Shear Modulus, and \( \nu \) is Poisson’s ratio.

Considering Love’s third and fourth assumptions \( \varepsilon_3 = \varepsilon_{13} = \varepsilon_{23} = \sigma_{13} = \sigma_{23} = \sigma_3 = 0 \),

Hooke’s law reduces to:

\[ \varepsilon_i = \frac{1}{E} \left[ \sigma_i - \nu(\sigma_2 + \sigma_3) \right], \]

\[ \varepsilon_2 = \frac{1}{E} \left[ \sigma_2 - \nu(\sigma_1 + \sigma_3) \right], \]

\[ \varepsilon_3 = 0, \]

\[ \varepsilon_{12} = \frac{\sigma_{12}}{G}, \]

\[ \varepsilon_{21} = \frac{\sigma_{21}}{G}, \]

\[ \varepsilon_{\alpha3} = 0. \]  

Equation 4.6 can be presented in a more condensed form as:

\[ \sigma_\alpha = \frac{E}{1 - \nu^2} (\varepsilon_\alpha + \nu \varepsilon_v), \]

\[ \sigma_{\alpha v} = G \cdot \varepsilon_{\alpha v}. \]  

42
Where:
\[ \nabla = 3 - \alpha, \quad \text{with } \alpha = 1 \text{ and } \alpha = 2. \quad (4.8) \]

### 4.1.2. C FORCE AND MOMENT EXPANSION

The unit forces and moments acting on the faces of the differential element are shown in Figure 4.3.

![Figure 4.3: Unit Forces, Moments, and Torques Acting on the Differential Element.](image)

The forces \( N_\alpha, N_{\alpha V}, \) and \( Q_\alpha \) are the normal, shear and transverse forces respectively; while moments and torques are \( M_\alpha \) and \( M_{\alpha V} \). They are defined per unit length and are found by integrating the stress over the thickness of the shell.

\[
N_\alpha = \int_{-h/2}^{h/2} \sigma_\alpha (1 - zK_v) dz,
\]

\[
N_{\alpha V} = \int_{-h/2}^{h/2} \sigma_{\alpha V} (1 - zK_v) dz,
\]

\[
Q_\alpha = \int_{-h/2}^{h/2} \sigma_{\alpha 3} (1 - zK_v) dz,
\]

\[
M_\alpha = \int_{-h/2}^{h/2} \sigma_\alpha (1 - zK_v) z dz,
\]

\[
M_{\alpha V} = \int_{-h/2}^{h/2} \sigma_{\alpha V} (1 - zK_v) z dz.
\]
Where $K_v$ is the curvatures defined as $K_v = 1/R_v$.

Considering Love’s third assumption ($\sigma_{13} = \sigma_{23} = 0$), Equation 4.9 sets $Q_\alpha = 0$. However, $Q_\alpha$ is not assumed to be zero and will be solved for later using the equilibrium equations.

Carrying out the integration of Equation 4.9 and neglecting higher order terms, the final unit force, moments, and torques expressed in terms of strain components are:

\[ N_\alpha = C(\varepsilon_\alpha + \nu_v \varepsilon_v), \]

\[ N_{12} = N_{21} = C_{12} \omega, \]

\[ M_\alpha = D(\chi_\alpha + \nu_v \varepsilon_v), \]

\[ M_{12} = M_{21} = D_{12} \tau. \]

Where:

\[ D = \frac{Eh^3}{12(1-\nu^2)}, \quad C = \frac{Eh}{1-\nu^2}, \quad D_{12} = \frac{G_{12}h^3}{6}, \quad C_{12} = G_{12}h, \quad (4.11) \]

And

\[ G_{12} = \frac{E}{2(1+\nu)}, \quad (4.12) \]

where $G_{12}$ is the shear modulus.
4.1.2.D EQUILIBRIUM EQUATIONS

From the dynamic equilibrium of the differential element (Figure 4.3) the equilibrium equations can be defined as [10, 31]:

\[
\begin{align*}
R_2 \frac{\partial N_1}{\partial \vartheta} + \frac{\partial (R_0 N_{21})}{\partial \varphi} + R_2 N_{12} \cos \varphi - R_2 Q_1 \sin \varphi + R_0 R_2 f_1 &= 0, \\
R_2 \frac{\partial N_{12}}{\partial \vartheta} + \frac{\partial (R_0 N_2)}{\partial \varphi} - R_2 N_1 \cos \varphi - R_0 Q_2 + R_0 R_2 f_2 &= 0, \\
R_2 \frac{\partial Q_1}{\partial \vartheta} + \frac{\partial (R_0 Q_2)}{\partial \varphi} + R_2 N_1 \sin \varphi + R_0 N_2 + R_0 R_2 f_3 &= 0, \\
R_2 \frac{\partial M_{1}}{\partial \vartheta} + \frac{\partial (R_0 M_{21})}{\partial \varphi} + R_2 M_2 \cos \varphi - R_0 R_2 Q_1 + R_0 R_2 f_4 &= 0, \\
R_2 \frac{\partial M_{12}}{\partial \vartheta} + \frac{\partial (R_0 M_2)}{\partial \varphi} - R_2 M_1 \cos \varphi - R_0 R_2 Q_2 + R_0 R_2 f_5 &= 0, \\
N_{12} - N_{21} + \frac{M_{21}}{R_2} - \frac{M_{12}}{R_1} &= 0.
\end{align*}
\]

The values of \( f_i \), in Equation 4.13, are given by Equation 4.14 [10].

\[
\begin{align*}
f_1 &= p_1 - \rho \rho h \frac{\partial^2 u}{\partial t^2} - \lambda_1 h \frac{\partial u}{\partial t} - k_1 u, \\
f_2 &= p_2 - \rho \rho h \frac{\partial^2 v}{\partial t^2} - \lambda_2 h \frac{\partial v}{\partial t} - k_2 v, \\
f_3 &= p_3 - \rho \rho h \frac{\partial^2 w}{\partial t^2} - \lambda_3 h \frac{\partial w}{\partial t} - k_3 w, \\
f_4 &= \rho \rho h^3 \frac{\partial^2 X_1}{\partial t^2} + \frac{\lambda_1 h^3}{12} \frac{\partial X_1}{\partial t}, \\
f_5 &= -\rho \rho h^3 \frac{\partial^2 X_2}{\partial t^2} + \frac{\lambda_2 h^3}{12} \frac{\partial X_2}{\partial t}.
\end{align*}
\]

Where \( f_i \) are the auxiliary forces, \( p_i \) represents the load on the shell, \( k_i \) are coefficients describing the elasticity of a Winkler-type subgrade, and \( \lambda_1 \) represent the damping coefficient of the material [10]. The analysis of the HSR, in this thesis, was considered to be undamped,
ignored \( k_i \), and had no external loads applied to the shell; therefore, the values of \( f_i \) simplify to:

\[
\begin{align*}
    f_1 &= -\rho h \frac{\partial^2 u}{\partial t^2}, \\
    f_2 &= -\rho h \frac{\partial^2 v}{\partial t^2}, \\
    f_3 &= -\rho h \frac{\partial^2 w}{\partial t^2}, \\
    f_4 &= \rho \frac{h^3}{12} \frac{\partial^2 X_1}{\partial t^2}, \\
    f_5 &= -\rho \frac{h^3}{12} \frac{\partial^2 X_2}{\partial t^2}.
\end{align*}
\]

(4.15)

In this form \( f_1, f_2, \) and \( f_3 \) are the linear inertia forces of the shell in the \( u, v \) and \( w \) directions, respectively, and the values of \( f_4 \) and \( f_5 \) represent the rotary inertia of the shell.

The sixth equilibrium equation may not exactly be satisfied for a doubly curved shell. To avoid this inconsistency Novozhilov used an energy method to express \( N_{12} \) and \( N_{21} \) as [10]:

\[
N_{12} = C_{12} \omega - D_{12} K_2 \tau, \quad \text{and} \quad \quad N_{21} = C_{12} \omega - D_{12} K_1 \tau.
\]

(4.16)

A derivation of these relationships (Equation 4.16) is given by Leissa [14]. These relationships will be used in the development of the HSR's equations of motion, as done by Mazurkiewicz and Nagorgski [10].
4.1.3 MATHEMATICAL MODEL OF THE HYPERBOLOID SHELL OF REVOLUTION

The developed approach of the equilibrium equation of Chapter 4.1.2 is adapted and applied to the development of the mathematical model of the HSR. Three-coupled equation of motion are developed from the equilibrium equations and presented in matrix form.

4.1.3.A KINEMATICS

Considering the Equation of a hyperbola, two dimensionless coordinates were defined as:
\[ \xi = \frac{z}{d}, \quad \eta = \frac{R_0}{c}, \quad (4.18) \]

which changed the Equation of a hyperbola to:
\[ \eta^2 - \xi^2 = 1. \quad (4.19) \]

Using the three relationships:
\[ \eta = \sqrt{1 + \xi^2}, \quad \xi = \sqrt{1 + (1 + \lambda^2)\eta^2}, \quad \text{and} \quad \lambda = \frac{c}{d}, \quad (4.20) \]

The following relationships were verified [10]:
\[ \frac{dR_0}{dz} = \lambda \frac{d\eta}{d\xi}, \quad \frac{d^2R_0}{dz^2} = \frac{\lambda^2}{c} \frac{d^2\eta}{d\xi^2}, \quad \frac{d\eta}{d\xi^2} = \frac{\xi}{\eta}, \quad \frac{d^2\eta}{d\xi^2} = \frac{1}{\eta^3}, \quad (4.21) \]

\[ R_1 = c\zeta, \quad R_2 = -\frac{c\zeta^3}{\lambda^2} = -\frac{R_1^3}{\lambda^2 c^2}, \quad K_0 = K_1 K_2 \frac{\lambda^2}{c^2 \zeta^4}, \]

\[ \frac{1}{R_2} \frac{\partial}{\partial \varphi} = \frac{\lambda \eta \zeta}{c \zeta^3}, \quad \sin \varphi = \frac{\eta}{\zeta}, \quad \cos \varphi = \frac{\lambda \zeta}{\zeta}, \quad \cot \varphi = \frac{\lambda \zeta}{\eta}. \]
It is important to note, in the case of a HSR, that $R_2$ is negative. For this reason exact analytical solutions are considered difficult to obtain [10].

Substituting Equations 4.20 and 4.21 into Equations 4.1 and 4.2, the strain-displacement relationships of the HSR were verified to be:

$$
\begin{align*}
\epsilon_1 &= \frac{1}{c\eta} \left( \frac{\partial u}{\partial \theta} + \frac{\lambda \eta}{\zeta} v \right) - \frac{w}{c\zeta}, \\
\epsilon_2 &= \frac{\lambda \eta}{c\zeta} \frac{\partial v}{\partial \xi} + \frac{\lambda^2 w}{c\zeta^3}, \\
\omega &= \frac{1}{c\eta} \left( \frac{\partial v}{\partial \theta} - \frac{\lambda \xi}{\zeta} u \right) + \frac{\lambda \eta}{c\zeta} \frac{\partial u}{\partial \xi}, \\
\chi_1 &= -\frac{1}{c\eta} \left( \frac{\partial X_1}{\partial \theta} + \frac{\lambda \xi}{\zeta} X_2 \right), \\
\chi_2 &= -\frac{\lambda \eta}{c\zeta} \frac{\partial X_2}{\partial \xi}, \\
\tau &= -\frac{1}{c\eta} \left( \frac{\partial X_2}{\partial \theta} - \frac{\lambda \xi}{\zeta} X_1 \right) - \frac{\lambda \eta}{c^2 \zeta^2} \frac{\partial u}{\partial \xi},
\end{align*}
$$

(4.22)

where the rotations, $X_\alpha$, are:

$$
\begin{align*}
X_1 &= \frac{1}{c\eta} \left( \frac{\partial w}{\partial \theta} + \frac{\eta}{\zeta} u \right), \\
X_2 &= \frac{\lambda \eta}{c\zeta} \left( \frac{\partial w}{\partial \xi} - \frac{\lambda}{\eta \zeta^2} v \right).
\end{align*}
$$

(4.23)
4.1.3.B EQUILIBRIUM EQUATIONS

Considering Equations 4.10, 4.16, and 4.21 the unit forces, moments, and torques become

(They can be found in their expanded form in Appendix D):

\[ N_{\alpha} = C(\varepsilon_{\alpha} + \nu \varepsilon_{\nu}), \]
\[ N_{12} = \frac{C}{2} (1-\nu)\omega + \frac{D(1-\nu)}{c \zeta} \tau, \]
\[ N_{21} = \frac{C}{2} (1-\nu)\omega - \frac{D(1-\nu)}{c \zeta} \tau, \]
\[ M_{\alpha} = D(\chi_{\alpha} + \nu \chi_{\nu}), \]
\[ M_{12} = M_{21} = D(1-\nu)\tau. \]  

Substituting Equations 4.20 and 4.21 into Equation 4.13 the six equilibrium equations for the HSR were verified to be [10]:

\[
\begin{align*}
\frac{\partial N_1}{\partial \vartheta} + \frac{\lambda}{\zeta} \frac{\partial (\eta N_{21})}{\partial \xi} + \frac{\lambda}{\zeta} N_{12} - \frac{\eta}{\zeta} Q_1 + c \eta f_1 &= 0, \\
\frac{\partial N_{12}}{\partial \vartheta} + \frac{\lambda}{\zeta} \frac{\partial (\eta N_2)}{\partial \xi} - \frac{\lambda}{\zeta} N_{12} + \frac{\lambda^2}{\zeta^3} Q_2 + c \eta f_2 &= 0, \\
\frac{\partial Q_1}{\partial \vartheta} + \frac{\lambda}{\zeta} \frac{\partial (\eta Q_2)}{\partial \xi} + \frac{\eta}{\zeta} N_1 - \frac{\lambda^2}{\zeta^3} N_2 + c \eta f_3 &= 0, \\
\frac{\partial M_1}{\partial \vartheta} + \frac{\lambda}{\zeta} \frac{\partial (\eta M_{21})}{\partial \xi} + \frac{\lambda}{\zeta} M_{12} - c \eta Q_1 + c \eta f_4 &= 0, \\
\frac{\partial M_{12}}{\partial \vartheta} + \frac{\lambda}{\zeta} \frac{\partial (\eta M_2)}{\partial \xi} - \frac{\lambda}{\zeta} M_1 - c \eta Q_2 + c \eta f_5 &= 0, \\
N_{12} - N_{21} + \frac{M_{21}}{R_2} - \frac{M_{12}}{R_1} &= 0,
\end{align*}
\]  

(4.25)

where the variables \( f_i \) are given by Equations 4.15, and \( N_\alpha, N_{12}, N_{21}, M_\alpha, M_{12} \) and \( M_{21} \) are given by Equations 4.24.
As was previously mentioned, the transverse forces $Q_\alpha$ were not assumed to be equal to zero. They were found from the fourth and fifth equations of Equation 4.25.

\[
Q_1 = \frac{1}{c\eta} \left( \frac{\partial M_1}{\partial \theta} + \frac{\lambda \eta \partial (\eta M_{21})}{\zeta \partial \xi} + \frac{\lambda \xi}{\zeta} M_{12} \right) + f_4,
\]

\[
Q_2 = \frac{1}{c\eta} \left( \frac{\partial M_{12}}{\partial \theta} + \frac{\lambda \eta \partial (\eta M_2)}{\zeta \partial \xi} - \frac{\lambda \xi}{\zeta} M_1 \right) + f_5,
\]

where $f_4$ and $f_5$ are from Equation 4.15, and the unit forces, moments, and torques are given by Equation 4.24.

The transverse forces (Equation 4.26) were then substituted into the first three equilibrium equations; rendering the following three coupled equations of motion:

\[
\begin{align*}
\frac{\partial N_1}{\partial \theta} + \frac{\lambda \eta \partial (\eta N_{21})}{\zeta \partial \xi} + \frac{\lambda \xi}{\zeta} N_{12} - \frac{1}{c\zeta} \frac{\partial M_1}{\partial \theta} - \frac{\lambda \eta \partial (\eta M_{21})}{c \zeta^2 \partial \xi} - \frac{\lambda \xi}{c \zeta^2} M_{12} + f_1^* &= 0, \\
\frac{\partial N_{12}}{\partial \theta} + \frac{\lambda \eta \partial (\eta N_2)}{\zeta \partial \xi} - \frac{\lambda \xi}{\zeta} N_1 - \frac{\lambda \xi}{c \zeta^3 \partial \theta} + \frac{\lambda \eta \partial (\eta M_2)}{c \zeta^4 \partial \xi} - \frac{\lambda \xi}{c \zeta^4} M_1 + f_2^* &= 0, \\
1 \frac{\partial^2 M_1}{c \eta \partial \theta^2} + \frac{\lambda}{c \zeta} \frac{\partial^2 (\eta M_2)}{\partial \xi \partial \theta} + \frac{\lambda \xi}{c \eta \zeta} \frac{\partial M_{12}}{\partial \xi} + \frac{\lambda \eta \partial^2 M_{12}}{c \zeta \partial \xi \partial \theta} \left( \frac{\eta \partial (\eta M_2)}{c \zeta \partial \xi} \right) - \frac{\lambda^2 \eta}{c \zeta^2} \left( \frac{\xi}{\zeta} M_1 \right) + \frac{\eta}{\zeta} N_1 - \frac{\lambda^2 \eta}{\zeta^3} N_2 + f_3^* &= 0.
\end{align*}
\]
Where:

\[ f_1^* = c \eta f_1 - \eta \zeta f_4, \]
\[ f_2^* = \lambda^2 \eta \zeta f_5 + c \eta f_2, \]
\[ f_3^* = \frac{\partial (f_4)}{\partial \theta} + \frac{\lambda \eta}{\zeta} \frac{\partial (\eta f_5)}{\partial \xi} + c \eta f_3. \]  

(4.28)

The three-coupled equations of motion (Equation 4.27) can be written as:

\[ \sum_{i=1}^{3} (L_i u + L_{i2} v + L_{i3} w + f_i^*) = 0 \]  

(4.29)

or

\[ \begin{bmatrix} \hat{M}_{11} & \hat{M}_{12} & \hat{M}_{13} \\ \hat{M}_{21} & \hat{M}_{22} & \hat{M}_{23} \\ \hat{M}_{31} & \hat{M}_{32} & \hat{M}_{33} \end{bmatrix} \begin{bmatrix} \ddot{u} \\ \ddot{v} \\ \ddot{w} \end{bmatrix} + \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = 0. \]  

(4.30)

where \( \hat{M}_{ij} \) is constructed from \( f_i^* \). The expanded expressions for \( \hat{M}_{ij} \) and \( L_{ij} \) are found in Appendix E.

The mathematical model (Equation 4.30) can be expressed in a condensed form as:

\[ \hat{M} \ddot{U} + \hat{L} U = 0, \]  

(4.31)

Where \( U = [u \ v \ w]^T \).
4.2 SOLUTION

The equations of motion are solved for the natural frequency using the Assumed Mode Shape Method. First, an overview of the Galerkin Assumed Mode Shape Method is given; followed by the application of the method to the HSR.

4.2.1 INTRODUCTION TO THE GALERKIN ASSUMED MODE SHAPE METHOD

The Galerkin Assumed Mode Shape Method is used to approximate solutions of differential equations. Specifically, it is a method of weighted residuals, where the weight functions are equal to the assumed mode shape functions. Considering the following equation; where $L$ is a differential operator acting on the variable $u$, and $f$ is a known function.

$$L(u) = f.$$  \hfill (4.32)

The variable $u$ is approximated by the expression:

$$u_j = \sum_j \Psi_j U_j,$$  \hfill (4.33)

where $\Psi_j$ is an approximate solution that satisfies the boundary conditions of Equation 4.32.

Substitution of the approximation of $u_j$ into Equation 4.32 produces the residual, $R$.

$$L(u_j) - f = R$$  \hfill (4.34)

It is required that the residual be zero over the domain. In order to satisfy this condition, the weighted-residual is formed, integrated over the domain and set equal to zero as shown by Equation 4.35. Expanded, this Equation represents $j$ equations of $j$ unknowns, $U_j$.

$$\int_{\Omega} \Psi_j R d\Omega = 0$$  \hfill (4.35)
4.2.2 APPLICATION OF THE GALERKIN ASSUMED MODE SHAPE METHOD TO THE HYPERBOLOID SHELL OF REVOLUTION

In order to apply the Galerkin Assumed Mode Shape Method to the HSR, the mode shapes had to be approximated for the fixed-free and fixed-fixed HSR. The assumed shapes of a cylinder were used to approximate the shapes of the HSR. According to Leissa [14] the mode shape functions, for the 'mth' mode shape, of a cylinder can be approximated by:

\[ \psi_m^u = X_m(z) \sin(n \theta), \]
\[ \psi_m^v = X_m'(z) \cos(n \theta), \]
\[ \psi_m^w = X_m(z) \cos(n \theta), \]

(4.36)

where \( X_m(z) \) is the mode shape of the lateral vibration of a beam, and the corresponding coordinate of the shape function, \( \psi_m \), is given by its superscript. It was known approximating the mode shapes of the HSR by the mode shape functions of a cylinder, Equation 4.36, introduced error.

The mode shape of a lateral vibrating beam is given by:

\[ X_m(z) = C_1 \cos \beta_m z + C_2 \sin \beta_m z + C_3 \cosh \beta_m z + C_4 \sin \beta_m z, \]

(4.37)

where \( C_1, C_2, C_3, \) and \( C_4 \) are constants found by applying the appropriate boundary conditions and \( \beta_m \) is a parameter proportional to the natural frequency and depends on the boundary conditions:

\[ \beta_m = \sqrt{\omega_m \left( \frac{EI}{\rho A} \right)^{\frac{1}{4}}}, \]

(4.38)
Both the fixed-free and fixed-fixed beam’s mode shapes were found using the appropriate boundary conditions.

For a fixed end:

\[ X_m = 0, \quad \text{and} \quad \frac{\partial X_m}{\partial z} = 0. \quad \text{(4.39)} \]

For a free end the bending moment and shear are both zero; therefore:

\[ \frac{\partial^2 X_m}{\partial z^2} = 0, \quad \text{and} \quad \frac{\partial^3 X_m}{\partial z^3} = 0. \quad \text{(4.40)} \]

For both the fixed-free and fixed-fixed beam the approximate mode shape is:

\[ X_m(z) = \sin \beta_m z - \sinh \beta_m z - \alpha_m \left( \cos \beta_m z - \cosh \beta_m z \right). \quad \text{(4.41)} \]

For a fixed-free beam:

\[ \alpha_m = \frac{\sin \beta_m l + \sinh \beta_m l}{\cos \beta_m l + \cosh \beta_m l}, \quad \text{(4.42)} \]

with \( \beta_1 l = 1.875104 \) for the first natural frequency.

For a fixed-fixed beam:

\[ \alpha_m = \frac{\sinh \beta_m l - \sin \beta_m l}{\cos \beta_m l - \cosh \beta_m l}, \quad \text{(4.43)} \]

with \( \beta_1 l = 4.730041 \) for the first natural frequency.
The displacements were then approximated as:

\[ u = \Psi_m^u U = X_m(z) \sin(n \theta) U \]
\[ v = \Psi_m^v V = X'_m(z) \cos(n \theta) V, \]
\[ w = \Psi_m^w W = X_m(z) \cos(n \theta) W, \]

where \( X_m(z) \) is for a fixed-free or fixed-fixed beam, and for the HSR, \( u \) is the circumferential displacement, \( v \) is the displacement tangent to the meridian and \( w \) is the transverse displacement normal to the surface, as shown by Figure 4.4.

![Figure 4.4: Displacement coordinates for the HSR.](image)

The displacement approximations were then written in terms of the variables \( \xi \) and \( \theta \), by substituting \( z = \xi d \) into \( X_m(z) \) and \( X'_m(z) \), and then into the three equations of motion (Equation 4.30); rendering the residuals:

\[ L_1 \{ \Psi_m^u U, \Psi_m^v V, \Psi_m^w W \} + \hat{M}_1 \{ \Psi_m^u \ddot{U}, \Psi_m^v \ddot{V}, \Psi_m^w \ddot{W} \} = R_1, \]
\[ L_2 \{ \Psi_m^u U, \Psi_m^v V, \Psi_m^w W \} + \hat{M}_2 \{ \Psi_m^u \dddot{U}, \Psi_m^v \dddot{V}, \Psi_m^w \dddot{W} \} = R_2, \]
\[ L_3 \{ \Psi_m^u U, \Psi_m^v V, \Psi_m^w W \} + \hat{M}_3 \{ \Psi_m^u \dddot{U}, \Psi_m^v \dddot{V}, \Psi_m^w \dddot{W} \} = R_3, \]

The weighted residuals were formed, integrated over \( \theta \) and \( \xi \), and set equal to zero,

\[ \int \int_{\xi \theta} P_m^u R_1 d \theta d \xi = 0, \]
\[ \int \int_{\xi \theta} P_m^v R_2 d \theta d \xi = 0, \]
\[ \int \int_{\xi \theta} P_m^w R_3 d \theta d \xi = 0. \]
Where the domain of the HSR for $\vartheta$ is from 0 to $2\pi$, and the domain of $\xi$ is from $-\xi_{\max}$ to $\xi_{\max}$.

Expanded, and in matrix form, Equation 4.46 is:

\[
\begin{bmatrix}
\int \Psi_m^u L_1 \{ \Psi^u \} d \vartheta d \xi \\
\int \Psi_m^u L_2 \{ \Psi^u \} d \vartheta d \xi \\
\int \Psi_m^u L_3 \{ \Psi^u \} d \vartheta d \xi \\
\int \Psi_m^w \hat{M}_1 \{ \Psi^w \} d \vartheta d \xi \\
\int \Psi_m^w \hat{M}_2 \{ \Psi^w \} d \vartheta d \xi \\
\int \Psi_m^w \hat{M}_3 \{ \Psi^w \} d \vartheta d \xi
\end{bmatrix}
\begin{bmatrix}
U \\
V \\
W
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

(4.47)

Expressing the displacements as:

\[
U(t) = U e^{iat},
\]

\[
V(t) = V e^{iat},
\]

\[
W(t) = W e^{iat},
\]

Renders the equation:

\[
[-\omega^2 [\hat{M}]+[\hat{L}]][Q] = [0]
\]

(4.49)

where $[\hat{M}]$ and $[\hat{L}]$ are the condensed matrices of Equation 4.47 and $[Q] = [U \ V \ W]^T$.

Setting the determinant of $[-\omega^2 [\hat{M}]+[\hat{L}]]$ equal to zero forms a polynomial as a function of the natural frequency, $\omega_n$. 

56
4.4 RESULTS

An assumption was made in order to simplify the analysis. The value of the bending stiffness, \( D \) (Equation 4.11), is much less than the zero and much much less than value of the membrane stiffness, \( C \) (Equation 4.11); therefore, the bending stiffness was neglected. It should be noted that the equations of motion used in the analysis were derived using Mathematica.

The following plots, Figures 4.5 and 4.6, show the fundamental bending frequencies of a fixed-free and fixed-fixed HSR found by applying Galerkin’s Assumed Mode Shape Method to the mathematical model (Equation 4.30). The math program Mathematica was used to find the natural frequencies for the HSR (Appendix F).
Figure 4.5 shows the fundamental natural bending frequencies versus $R_{\text{min}}$ for the fixed-free HSR. The results show a steep increase in the natural frequency with decreasing $R_{\text{min}}$; followed by a decrease in the natural frequency. A concern is that the natural frequency for an HSR never falls below that of a near cylinder ($R_{\text{min}} = 2.99999$). The purpose of the coupling is to provide flexural flexibility. That is, to have less bending stiffness and consequently a lower bending natural frequency than the cylindrical shaft. This requirement is not demonstrated by the results of Figure 4.5.

![Figure 4.5: Bending Natural Frequencies of a Fixed-Free HSR by Applying the Galerkin Method to Shell Theory.](image)

58
Figure 4.6 shows the natural bending frequencies versus $R_{\text{min}}$ for the fixed-fixed HSR. The results exhibit an increasing natural frequency with decreasing $R_{\text{min}}$. In contrast to the fixed-free results (Figure 4.5), the natural frequency of the fixed-fixed HSR never decreases.
CHAPTER 5: RESULTS COMPARISON

The following Tables and Figures show direct comparison of the results of the parametric studies found in Chapter 2, 3 and 4. Table 5.1 and 5.2 present the results of the fundamental bending natural frequency as a function of $R_{\text{min}}$, for the fixed-free and fixed-fixed boundary conditions, respectively. Results of all three methods: ANSYS, Timoshenko Beam ($k = .5$ and $k = .9$) and Shell Theory, are included in the Tables. The graphical presentation of the results, Table 5.1 and 5.2, are displayed in Figure 5.1 and 5.2, respectively.

<table>
<thead>
<tr>
<th>$R_{\text{min}}$ (in)</th>
<th>ANSYS</th>
<th>Timoshenko Beam $k = .5$</th>
<th>Timoshenko Beam $k = .9$</th>
<th>Shell Theory</th>
</tr>
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<td>3778</td>
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<td>3090</td>
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</tr>
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<td>1.25</td>
<td>1695</td>
<td>2012</td>
<td>2207</td>
<td>4436.762</td>
</tr>
<tr>
<td>1</td>
<td>1484</td>
<td>1617</td>
<td>1729</td>
<td>3944.814</td>
</tr>
</tbody>
</table>

Table 5.1: Fixed-Free Results Comparison.
Figure 5.1: Fixed-Free Results Comparison: ANSYS vs. Timoshenko Beam vs. Shell Theory.

The fixed-free results from the Timoshenko Beam and ANSYS, Figure 5.1, match very closely. The overall trend of both ANSYS and Timoshenko Beam is that of decreasing natural frequency with decreasing $R_{\text{min}}$. However, ANSYS shows an initial increase in the natural frequency followed by a decrease, and it is not until $R_{\text{min}}$ is less than 2 inches that the frequency falls below that of the corresponding cylinder. On the other hand, the Shell Theory results deviate drastically from the ANSYS and Timoshenko results. At $R_{\text{min}} = 3$ inches (the HSR converges to a cylinder) the Shell Theory results predict a much lower value of the natural frequency. As $R_{\text{min}}$ decreases, the natural frequency increases to a pronounced maximum and then decreases. For $R_{\text{min}}$ smaller than 2.5 inches, the Shell Theory predicts
much higher values of the natural frequency. The big difference between results of the Shell Theory and the other two may be attributed to the following:

1. The assumed mode shape adopted is appropriate for a cylinder. The assumed mode shape method demands adopting very accurate mode shapes, and consequently adopting the mode shape of a cylinder for the HSR would introduce errors.

2. The bending stiffness, D in Equation 4.11, was neglected in the Shell Theory Analysis which eliminated many terms from the equations of motion.

3. Due to the extreme complication of the Shell Theory, some errors might have been committed during the rigorous and lengthy derivations.

<table>
<thead>
<tr>
<th>Rmin (in)</th>
<th>ANSYS fn(Hz)</th>
<th>Timoshenko Beam fn(Hz)</th>
<th>Shell Theory fn(Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.99999</td>
<td>4070</td>
<td>7260</td>
<td>9603</td>
</tr>
<tr>
<td>2.75</td>
<td>4800</td>
<td>7488</td>
<td>9900</td>
</tr>
<tr>
<td>2.5</td>
<td>6115</td>
<td>7759</td>
<td>10245</td>
</tr>
<tr>
<td>2.25</td>
<td>7367</td>
<td>8075</td>
<td>10648</td>
</tr>
<tr>
<td>2</td>
<td>7981</td>
<td>8451</td>
<td>11124</td>
</tr>
<tr>
<td>1.75</td>
<td>8473</td>
<td>8900</td>
<td>11697</td>
</tr>
<tr>
<td>1.5</td>
<td>9262</td>
<td>9446</td>
<td>12391</td>
</tr>
<tr>
<td>1.25</td>
<td>9547</td>
<td>10117</td>
<td>13248</td>
</tr>
<tr>
<td>1</td>
<td>10945</td>
<td>14310</td>
<td>12482.15</td>
</tr>
</tbody>
</table>

Table 5.2: Fixed-Fixed Results Comparison.
Figure 5.2: Fixed-Fixed Results Comparison: ANSYS vs. Timoshenko Beam vs. Shell Theory.

The results of all three analyses, for the fixed-fixed HSR, show the same overall trend: as $R_{min}$ decreases the natural frequency increases. However, the Timoshenko Beam and Shell Theory both predict higher natural frequencies, especially at larger $R_{min}$. 
CHAPTER 6: CONCLUSION/FUTURE WORK

The analysis in the thesis provides preliminary investigation into the dynamics of a proposed hyperbolic coupling; more specifically the effect of $R_{\text{min}}$ on the fundamental flexural frequency. Two boundary conditions, fixed-free and fixed-fixed, are investigated. Three Methods (ANSYS, Timoshenko Beam, and Shell Theory) are applied for finding the bending natural frequency.

For the fixed-free boundary condition Timoshenko Beam and ANSYS showed reasonable agreement. Shell Theory showed significant difference in magnitude. However, it is interesting to see that ANSYS and Shell Theory's results demonstrated an optimum $R_{\text{min}}$ where the bending natural frequency is maximum. For the fixed-fixed boundary condition, all three methods show that the bending natural frequency increases with decreasing $R_{\text{min}}$.

This thesis opens up many areas of future work. Development of the shell theory needs to be continued. Experimental work should be conducted to check the analytical results and to confirm the phenomena of the peak bending natural frequency demonstrated by ANSYS and Shell Theory, for a fixed-free HSR.

Most importantly future work must include the application of composites to the HSR. Extension of the current analysis to composites materials is important because the final goal is to produce a composite coupling, which can be integrated with a composite shaft into a single composite shaft-coupling unit. Beam Theory or a better-developed Shell
Theory could be applied to a composite coupling and/or a composite shaft-coupling unit. Experimentation may be a realistic alternative for analyzing a composite coupling.
REFERENCES


APPENDIX A: HELPFUL TABLES

The following tables were produced using Microsoft Excel. For each value of \( c(R_{\text{min}}) \), the value for \( d \) is shown. Also included are the coordinates of \( z, R_0 \), and the slope at the end of the meridian, for each specific \( R_{\text{min}} \). These values were input into the ANSYS batch files to model each HSR.

<table>
<thead>
<tr>
<th>( R_{\text{omax}} )</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z_{\text{max}} )</td>
<td>3</td>
</tr>
<tr>
<td>( dz )</td>
<td>3/5</td>
</tr>
<tr>
<td>( dc )</td>
<td>3/4</td>
</tr>
</tbody>
</table>

\[
c = 2.99999 \\
d = 1161463/519 \\
c/d = 1/387
\]

<table>
<thead>
<tr>
<th>( z )</th>
<th>( R_0 )</th>
<th>( dR_0/dz )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.99999</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>2.9999904</td>
<td></td>
</tr>
<tr>
<td>1.2</td>
<td>2.9999916</td>
<td></td>
</tr>
<tr>
<td>1.8</td>
<td>2.9999936</td>
<td></td>
</tr>
<tr>
<td>2.4</td>
<td>2.9999964</td>
<td>1/408</td>
</tr>
</tbody>
</table>

\[
c = 23/4 \\
d = 6866/983 \\
c/d = 231/578
\]

<table>
<thead>
<tr>
<th>( z )</th>
<th>( R_0 )</th>
<th>( dR_0/dz )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>3/4</td>
</tr>
<tr>
<td>0.375</td>
<td>2.412</td>
<td>0.551</td>
</tr>
<tr>
<td>1</td>
<td>2.205</td>
<td>0.259</td>
</tr>
<tr>
<td>1.4</td>
<td>2.412</td>
<td>0.489</td>
</tr>
<tr>
<td>2</td>
<td>2.417</td>
<td>0.457</td>
</tr>
<tr>
<td>2.625</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

\[
c = 1/2 \\
d = 4415/794 \\
c/d = 419/758
\]

<table>
<thead>
<tr>
<th>( z )</th>
<th>( R_0 )</th>
<th>( dR_0/dz )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>1.2</td>
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<tr>
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<td>2.274</td>
<td>0.525</td>
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<tr>
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<td>0.815</td>
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<td>2.67</td>
<td>0.97</td>
</tr>
<tr>
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<td>2.44</td>
<td>0.53</td>
</tr>
<tr>
<td>2.5</td>
<td>2.658</td>
<td>1.17</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>1.408</td>
</tr>
</tbody>
</table>

\[
c = 21/4 \\
d = 3239/595 \\
c/d = 506/765
\]

<table>
<thead>
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<th>( R_0 )</th>
<th>( dR_0/dz )</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
<tr>
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<td>2.207</td>
<td>0.727</td>
</tr>
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<td>2.93</td>
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<td>2.838</td>
<td>0.702</td>
</tr>
<tr>
<td>2</td>
<td>52.69</td>
<td>1.71</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>1.7</td>
</tr>
</tbody>
</table>

\[
c = 27/4 \\
d = 658/963 \\
c/d = 682/915
\]

<table>
<thead>
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<th>( R_0 )</th>
<th>( dR_0/dz )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>2.884</td>
<td>0.531</td>
</tr>
<tr>
<td>1</td>
<td>2.216</td>
<td>0.529</td>
</tr>
<tr>
<td>1.625</td>
<td>2.658</td>
<td>0.963</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
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<td>3</td>
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</table>

\[
c = 7/16
\]
<table>
<thead>
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<th></th>
<th>c = 1 3/4</th>
<th>c/d</th>
<th>c = 1 1/2</th>
<th>c/d</th>
</tr>
</thead>
<tbody>
<tr>
<td>d = 2 151/977</td>
<td>571/703</td>
<td>d = 1 571/780</td>
<td>181/209</td>
<td></td>
</tr>
<tr>
<td>z</td>
<td>Ro</td>
<td>dRo/dz</td>
<td>z</td>
<td>Ro</td>
</tr>
<tr>
<td>0</td>
<td>1 3/4</td>
<td></td>
<td>0</td>
<td>1 1/2</td>
</tr>
<tr>
<td>3/5</td>
<td>1 1187/229</td>
<td></td>
<td>3/5</td>
<td>1 440/749</td>
</tr>
<tr>
<td>1</td>
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<td>1 1/320</td>
<td>1</td>
<td>1 1/5</td>
</tr>
<tr>
<td>4/5</td>
<td>2 2224/799</td>
<td></td>
<td>1</td>
<td>4 4/5</td>
</tr>
<tr>
<td>2</td>
<td>2 101/163</td>
<td></td>
<td>2</td>
<td>2 2/5</td>
</tr>
<tr>
<td>3</td>
<td>3 95/144</td>
<td></td>
<td>3</td>
<td>3</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>c = 1 1/4</th>
<th>c/d</th>
<th>c = 1 33/544</th>
<th>c/d</th>
</tr>
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<tbody>
<tr>
<td>d = 1 3/8</td>
<td>10/11</td>
<td>d = 1 544/577</td>
<td></td>
<td></td>
</tr>
<tr>
<td>z</td>
<td>Ro</td>
<td>dRo/dz</td>
<td>z</td>
<td>Ro</td>
</tr>
<tr>
<td>0</td>
<td>1 1/4</td>
<td></td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3/5</td>
<td>1 1183/503</td>
<td></td>
<td>0.6</td>
<td>1 137/920</td>
</tr>
<tr>
<td>1</td>
<td>1 607/921</td>
<td></td>
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<td>1 307/602</td>
</tr>
<tr>
<td>4/5</td>
<td>2 23/389</td>
<td></td>
<td>1.8</td>
<td>1 385/397</td>
</tr>
<tr>
<td>2</td>
<td>2 427/830</td>
<td></td>
<td>2.4</td>
<td>2 417/880</td>
</tr>
<tr>
<td>3</td>
<td>3 119/144</td>
<td></td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

69
APPENDIX B: ANSYS BATCH FILES

The following sets of code are the batch files used to find the ANSYS results within this thesis. The files can be copied and/or modified and input into ANSYS, version 8.0.

```
!----------------Finding the natural frequencies of a fixed-free HSR-------------------
/BATCH
!!THIS FILE WAS CREATED BY BRIAN G. TOWNER.
!!THIS FILE WILL SETUP AND SOLVE FOR THE NATURAL FREQUENCIES
!!OF THE HYPERBOLIC SHELL OF REVOLUTION (HSR).
!!THE HSR BOUNDARY CONDITIONS ARE FIXED ON ONE END
!!AND FREE ON THE OTHER.
!/COM,ANSYS RELEASE 8.0 UP20030930 13:16:37 12/16/2004
/input, menust,tmp,,
!/GRA,POWER
!/GST,ON
!/PLO,INFO,3
!/GRO,CURL,ON
!/CPLANE,1
!/REPOL,RESIZE
WPSTYLE,,,,,,0
!/REPOL,RESIZE
/PREP7
!*

!--------------------------------------

!!SHELL TYPE...
ET,1,SHELL93
!*
!!THICKNESS OF SHELL...
R,1,.001,,
!*
!*  

!--------------------------------------

!!MATERIAL PROPERTIES...
!!MATERIAL IS ALUMINUM.
MPTEMP,,,,,,
MPTEMP,1,0
!!MODULUS OF ELASTICITY..
MPDATA,EX,1,,10e6
!!POISSON'S RATIO...
MPDATA,PRXY,1,,.33
MPTEMP,,,,,,
MPTEMP,1,0
!!DENSITY OF MATERIAL...
MPDATA,DENS,1,,2.55e-4
!*  

!!DEFINING AND CREATING ONE-HALF OF THE MERIDIAN.
!K,,RADIUS.0,.Z
```
K, 2, 0, 0,
K, 2+4/81, 0, 3/5,
K, 2+88/461, 0, 1+1/5,
K, 2+216/529, 0, 1+4/5,
K, 2+658/963, 0, 2+2/5,
K, 3, 0, 3,
K, 0, 0, 3,
K, 0, 0, -3,
!!CREATING THE MERIDIAN...
FLST, 3, 6, 3
FITEM, 3, 1
FITEM, 3, 2
FITEM, 3, 3
FITEM, 3, 4
FITEM, 3, 5
FITEM, 3, 6
!!NOTE: THE FOLLOWING LINE DEFINES THE SLOPE OF THE
!!MERIDIAN AT EACH END.
BSPLIN, P51X,. . .,0.0,-1.5,0.9,
FLST, 3, 1, 4, ORDE, 1
FITEM, 3, 1
LSYM, Z,P51X,. . .,0,0
FLST, 2, 2, 4, ORDE, 2
FITEM, 2, 1
FITEM, 2, -2
LGLUE, P51X
FLST, 2, 2, 4, ORDE, 2
FITEM, 2, 1
FITEM, 2, 3
LCOMB, P51X, , 0
FLST, 2, 1, 4, ORDE, 1
FITEM, 2, 1
FLST, 8, 2, 3
FITEM, 8, 7
FITEM, 8, 8
AROTAT, P51X, . . ., P51X, 360.,
<table>
<thead>
<tr>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>---------------------------------------------------------------------------------------------------</td>
</tr>
</tbody>
</table>
!!MESHING THE SHELL...
AATT, 1, 1, 1, 0,
FLST, 5, 4, 4, ORDE, 2
FITEM, 5, 1
FITEM, 5, -4
CM, _Y, LINE
LSEL,. . ., P51X
CM, _Y1, LINE
CMSEL,. _Y
*
!!NUMBER OF ELEMENTS ALONG THE LENGTH->20...
LESIZE, _Y1,. . ., 20,. . ., 1
*
FLST, 5, 8, 4, ORDE, 2
FITEM, 5, 5
FITEM, 5, -12
CM, _Y, LINE
LSEL,. . ., P51X
CM._Y1,L,LINE
CMSEL._Y
!*
!!NUMBER OF ELEMENTS ALONG THE CIRCUMFERENCE->4*4...
LESIZE._Y1,,4,,1
!*
MSHAPE,0,2D
MSHKEY,0
!*
FLST,5,4,5,ORDE,2
FITEM,5,1
FITEM,5,-4
CM._Y,AREA
ASEL,,P51X
CM._Y1,AREA
CHKMSH,'AREA'
CMSEL,S._Y
!*
AMESH._Y1
!*
CMDELE._Y
CMDELE._Y1
CMDELE._Y2
!*
FINISH
<table>
<thead>
<tr>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>--------------------------------------------------------------------------------</td>
</tr>
</tbody>
</table>
!!APPLYING THE BOUNDARY CONDITIONS...
/SOL
FLST,2,4,4,ORDE,4
FITEM,2,6
FITEM,2,8
FITEM,2,10
FITEM,2,12
!*
/GO
DL,P51X,,ALL,0
T
FINISH
/POST1
FINISH
/SOL
!*
<table>
<thead>
<tr>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>--------------------------------------------------------------------------------</td>
</tr>
</tbody>
</table>
!!SETTING UP THE ANALYSIS...
ANTYPE,2
!*
MSAVE,0
!*
!!DESIGNATE THE NUMBER OF NATURAL FREQUENCIES TO EXTRACT AND EXPAND...
MODOPT,LANB,100
EQLV,SPAR
MXPAND,100,,0
LUMP,0
PSTRES,0
!*  
!!DESIGNATE NUMBER OF MODES TO EXTRACT AND THE FREQUENCY RANGE...
MODOPT,LANB,100,0,0,,OFF
! /STATUS,SOLU  
/VIEW,1,1,1,1  
/ANG,1  
/REP,FAST  
E PLOT
!!SOLVE...
SOLVE  
FINISH  
/POST1  
SET,L I ST  
! LGWRITE,set_fixed_free,lgw,C:\DOCUME~1\bgt7536\DESKTOP\COMMENT
Finding the bending stiffness of a fixed-free HSR

This file was created by Brian G. Towner.

This file will setup the static bending of the coupling in order to find the absolute bending stiffness.

COM, ANSYS RELEASE 8.0 UP20030930 13:16:37 12/16/2004

GRA, POWER
GST, ON
/RO, INFO, 3
/GRO, CURL, ON
/CPLANE, 1
/REPL, RESIZE
WPSTYLE, ...., 0
/REPL, RESIZE
/REP7
*

Shell Type...
ET, 1, SHELL93
*
Thickness of the shell...
R, 1, .001, , , ,
*
*
Material Properties...
Material is Aluminum
MPTEMP, , , , , , ,
MPTEMP, 1, 0
Modulus of Elasticity
MPDATA, EX, 1, 10e6
Poisson's Ratio..
MPDATA, PRXY, 1, .33
MPTEMP, , , , , ,
MPTEMP, 1, 0
Density of Material...
MPDATA, DENS, 1, 2.55e-4

Defining and Creating Meridian of Shell...
Plotting six points defining 1/2 of the meridian...
K, Radius, 0, z
K, 2, 0, 0,
K, 2+4/81, 0, 3/5,
K, 2+48/461, 0, 1+1/5,
K, 2+16/529, 0, 1+4/5,
K, 2+65/963, 0, 2+2/5,
K, 3, 0, 3,
K, 0, 0, 9,
K, 0, 0, -9,
Creating the meridian
FLST, 3, 6, 3
FITEM, 3, 1
!!Note: The following line defines the slope of the
!!Meridan at each end.
BSPLIN, ,P51X, , ,0,0,-1,5,0,9,
FLST,3,1,4,ORDE,1
ITEM,3,1
LSYM,M,Z,P51X, , ,0,0
FLST,2,2,4,ORDE,2
ITEM,2,1
ITEM,2,-2
LGLUE,P51X
FLST,2,2,4,ORDE,2
ITEM,2,1
ITEM,2,3
LCOMB,P51X, ,0
FLST,2,1,4,ORDE,1
ITEM,2,1
FLST,8,2,3
ITEM,8,7
ITEM,8,8
AROTAT,P51X, , , ,P51X, ,360,
<table>
<thead>
<tr>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>---------------------------------------------------------------------</td>
</tr>
</tbody>
</table>

!!MESHING THE SHELL..
AATT, 1, 1, 1, 0,
FLST,5,4,4,ORDE,2
ITEM,5,1
ITEM,5,-4
CM,_Y,LINE
LSEL, , ,P51X
CM,_Y1,LINE
CMSEL,_,Y
!*
!!Number of elements along the length->20..
LESIZE,_Y1, , ,20, , ,1
!* FLST,5,8,4,ORDE,2
ITEM,5,5
ITEM,5,-12
CM,_Y,LINE
LSEL, , ,P51X
CM,_Y1,LINE
CMSEL,_,Y
!*
!!Number of elements along the circumference->4*4..
LESIZE,_Y1, , ,4, , ,1
!* MSHAPE,0,2D
MSHKEY,0
!* FLST,5,4,5,ORDE,2
ITEM,5,1

FITEM,5,-4
CM,_Y,AREA
ASEL,,P51X
CM,_Y1,AREA
CHKMSH,'AREA'
CMSEL,S,_Y
!* AMESH_Y1
!* CMDELE,_Y
CMDELE,_Y1
CMDELE,_Y2
!* FINISH
/SOL

FINISH
/POST1
FINISH
/SOL

!/STATUS,SOLU
/VIEW,1,1,1,1
/ANG,1
/REP,FAST
EPLT
<table>
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<tr>
<th></th>
</tr>
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<tbody>
<tr>
<td>-----------------------------------------------</td>
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</tbody>
</table>
!!APPLYING THE BOUNDARY CONDITIONS...
FINISH
/SOL
FLST,2,32,1,ORDE,11
FITEM,2,10
FITEM,2,50
FITEM,2,-57
FITEM,2,298
FITEM,2,338
FITEM,2,-344
FITEM,2,546
FITEM,2,586
FITEM,2,-592
FITEM,2,793
FITEM,2,-799
!* /GO
D,P51X,,ALL,,
FLST,2,32,1,ORDE,8
FITEM,2,1
FITEM,2,-9
FITEM,2,290
FITEM,2,-297
FITEM,2,538
FITEM,2,-545
!SELELCTS THE FIXED END NODES IN ORDER TO FIND REACTION FY..

```
FITEM,2,786
FITEM,2,-792
!/GO
D,P51X,,.25,,,UY,,,
FINISH
/POST1
FINISH
/SOL
!!!SOLVE....
/STATUS,SOLU
SOLVE

FLST,5,32,l,ORDE,11
FITEM,5,10
FITEM,5,50
FITEM,5,-57
FITEM,5,298
FITEM,5,338
FITEM,5,-344
FITEM,5,546
FITEM,5,586
FITEM,5,-592
FITEM,5,793
FITEM,5,-799
NSEL,S,,P51X
FINISH
/POST1
!* 
RSYS,0
AVPRIN,0,0
SHELL,TOP
!AVRES,2
!/EFACET,1
LAYER,0
FORCE,TOTAL
!* 
!PRRSOL,FY
!LGWRITE,BT.lgw,C:\DOCUMENTS~1\bgt7536\Desktop\COMMENT
```
Finding the torsional stiffness of a fixed-free HSR

/*!
!!HEX FILE WAS CREATED BY BRIAN G. TOWNER.
!!THIS FILE WILL SETUP THE TWISTING OF THE COUPLING
!!IN ORDER TO FIND THE ABSOLUTE TORSIONAL STIFFNESS.
!COM,ANSYS RELEASE 8.0 UP20030930 16:59:38 01/10/2005
/INP,menust,tmp,"l
/GRAP,POWER
/GST,ON
/PLO,INFO,3
/GRO,CURL,ON
/CPLANE,1
/REPLOT,RESIZE
WPSTYLE,0
/REPLOT,RESIZE
/REP7
*/

!!SHELL TYPE...
ET,1,SHELL93
*
!!THICKNESS OF THE SHELL...
R,1,.001,. , , , ,
*
*

!!MATERIAL PROPERTIES...
!!MATERIAL IS ALUMINUM.
MPT,TEMP,1,0
MPT,TEMP,1,0
!!MODULUS OF ELASTICITY...
MPDATA,EX,1,10e6
!!POISSON'S RATIO...
MPDATA,PRXY,1,...33
MPT,TEMP,1,0
MPT,TEMP,1,0
!!DENSITY OF MATERIAL...
MPDATA,DENS,1,...2.55e-4
*

!!DEFINING AND CREATING MERIDIAN OF THE SHELL...
!!PLOTTING SIX POINTS DEFINING 1/2 OF THE MERIDIAN...
!K,,rADIUS,0, Z
K,,1.75,0,0,
K,,1+187/229,0,3/5,
K,,2+1/320,0,1+1/5,
K,,2+224/799,0,1+4/5,
K,,2+101/163,0,2+2/5,
K,,3,0,3,
!!CREATING THE MERIDIAN...
FLST,3,6,3
FITEM,3,1
FITEM,3,2
FITEM,3,3
!!NOTE: THE FOLLOWING LINE DEFINES THE SLOPE OF THE
!!MERIDIAN AT EACH END.
BSPLIN, P51X, . . . , 0.0, -1.95, 0.144,
FLST, 3.1, 4, ORDE, 1
FITEM, 3, 1
LSYMM, Z, P51X, . . . , 0.0
FLST, 2.2, 4, ORDE, 2
FITEM, 2, 1
FITEM, 2, -2
LGLUE, P51X
FLST, 2.2, 4, ORDE, 2
FITEM, 2, 1
FITEM, 2, 3
LCOMB, P51X, 0
K, 0.0, 3,
K, 0.0, -3,
GPLOT
!/AUTO, 1
!/REP, FAST
FLST, 2.1, 4, ORDE, 1
FITEM, 2, 1
FLST, 8.2, 3
FITEM, 8, 7
FITEM, 8, 1
AROTAT, P51X, . . . , P51X, 360,

!!MESHING THE SHELL...
AATT, 1, 1, 1, 0,
!LPLOT
FLST, 5, 8, 4, ORDE, 2
FITEM, 5, 5
FITEM, 5, -12
CM, Y, LINE
LSEL, . . . , P51X
CM, Y1, LINE
CMSEL, _Y
!
!!NUMBER OF ELEMENTS ABOUT THE CIRCUMFERENCE->4*4...
LESIZE, _Y1, . . , 4, . . , 1
*
FLST, 5, 4, 4, ORDE, 2
FITEM, 5, 1
FITEM, 5, -4
CM, Y, LINE
LSEL, . . , P51X
CM, Y1, LINE
CMSEL, _Y
!
!!NUMBER OF ELEMENTS ALONG THE LENGTH->20...
LESIZE, _Y1, . . , 20, . . , 1
*
MSHAPE, 0, 2D
!!CHANGES THE ACTIVE COORDINATE SYSTEM TO CYLINDRICAL
!!AND CHANGES THE NODAL COORDINATES TO CYLINDRICAL.

CSYS, 1

!/SOL

!/STATUS,SOLU

/VIEW, 1, 1, 1, 1

/ANG, 1

/REP, FAST

Eplot

! APPL YING THE BOUNDARY CONDITIONS...

FLST, 2, 32, 1, ORDE, 11

FITEM, 2, 10

FITEM, 2, 50

FITEM, 2, -57

FITEM, 2, 298

FITEM, 2, 338

FITEM, 2, -344

FITEM, 2, 546

FITEM, 2, 586

FITEM, 2, -592

FITEM, 2, 793

FITEM, 2, -799

*/

/GO

D, P51X, 0, ... ALL, ...,

FLST, 2, 32, 1, ORDE, 8

FITEM, 2, 1

FITEM, 2, -9

FITEM, 2, 290

80
!*
/GO
!APPLIES A DISPLACEMENTS OF 5 DEGREES ON THE FREE END...
D,P51X,,5,,UY,,

!!SOLVE...
!STATUS,SOLU
SOLVE

!!SELECTS THE FIXED END NODES IN ORDER TO FIND REACTION FY
FLST,5,32,1,ORDE,8
FITEM,5,1
FITEM,5,-9
FITEM,5,290
FITEM,5,-297
FITEM,5,538
FITEM,5,-545
FITEM,5,786
FITEM,5,-792
NSELS,,P51X
FINISH
/POST1
!*
RSYS,1
AVPRIN,0,0
SHELL, TOP
! AVRES,2
!/EFACET,1
LAYER,0
FORCE,TOTAL

! LGWRITE,set_fixed_free_static_torsion.lgw,H:\thesis\Ansys\BATCHF~1\,COMMENT
Finding the axial stiffness of a fixed-free HSR

/BATCH

!!This file was created by Brian G. Towner.
!!This file will setup the static axial deflection of the coupling
!!in order to find the absolute axial stiffness.

/COM, ANSYS RELEASE 8.0 UP20030930 13:16:37 12/16/2004
/input, menust, tmp, ".......

/GRA, POWER
/GST, ON
/PLO, INFO, 3
/GRO, CURL, ON
/CPLANE, 1
/REPlot, RESIZE
WPSTYLE, ...., 0
/REPlot, RESIZE
/PREP7

!!SHELL TYPE...
ET, 1, SHELL93

!!THICKNESS OF THE SHELL...
R, 1, .001, , , ,

!!MATERIAL PROPERTIES...
!!MATERIAL IS ALUMINUM.
MPTEMP, , , , ,

!!MODULES OF ELASTICITY...
MPDATA, EX, 1, .10e6

!!POISSON'S RATIO...
MPDATA, PRXY, 1, .33

!!DENSITY OF MATERIAL...
MPDATA, DENS, 1, .2.55e-4

!!DEFINING AND CREATING THE MERIDIAN OF THE SHELL...
!!PLOTTING SIX POINTS DEFINING 1/2 OF THE MERIDIAN...
K, 1, .75, 0, 0,
K, 1+187/229, 0, 3/5,
K, 2+1/320, 0, 1+1/5,
K, 2+224/799, 0, 1+4/5,
K, 2+101/163, 0, 2+2/5,
K, 3, 0, 3,
K, 0, 0, 3,
K, 0, 0, -3,

!!CREATING THE MERIDIAN
FLST, 3, 6, 3
FITEM, 3, 1
FITEM, 3, 2
!!NOTE: THE FOLLOWING LINE DEFINES THE SLOPE OF THE 
!!MERIDIAN AT EACH END...
BSPLIN, .P51X, ..., 0.0-1.95, 0, 144,
FLST, 3, 1, 4, ORDE, 1
FITEM, 3, 1
LSYMM, Z, P51X, ..., 0, 0
FLST, 2, 2, 4, ORDE, 2
FITEM, 2, 1
FITEM, 2, -2
LGUE,P51X
FLST, 2, 2, 4, ORDE, 2
FITEM, 2, 1
FITEM, 2, 3
LCOMB, P51X, 0
FLST, 2, 1, 4, ORDE, 1
FITEM, 2, 1
FLST, 8, 2, 3
FITEM, 8, 7
FITEM, 8, 8
AROTAT, P51X, ..., P51X, 360, ,

!!MESHING THE SHELL...
AATT, 1, 1, 1, 0,
FLST, 5, 4, 4, ORDE, 2
FITEM, 5, 1
FITEM, 5, -4
CM, Y, LINE
LSEL, ..., P51X
CM, Y1, LINE
CMSEL, Y
!*
!!NUMBER OF ELEMENTS ALONG THE LENGTH->20...
LESIZE, Y1, ..., 20, ..., 1
!* FLST, 5, 8, 4, ORDE, 2
FITEM, 5, 5
FITEM, 5, -12
CM, Y, LINE
LSEL, ..., P51X
CM, Y1, LINE
CMSEL, Y
!* !*
!!NUMBER OF ELEMENTS ABOUT THE CIRCUMFERENCE->4*4...
LESIZE, Y1, ..., 4, ..., 1
!* MSHAPE, 0, 2D
MSHKEY, 0
!* FLST, 5, 4, 5, ORDE, 2
FITEM, 5, 1
FITEM, 5, -4

83
CM, Y, AREA
ASEL,, P51X
CM, Y1, AREA
CHKMSH, AREA
CMSEL.S, _Y
!* AMESH, Y1
!* CMDELE, _Y
CMDELE, _Y1
CMDELE, _Y2
!* FINISH
/SOL

T
FINISH
/POST1
FINISH
/SOL

!/STATUS,SOLU
/VIEW,1,1,1,1
/ANG,1
/REP,FAST
EPLLOT

!____________________________________________________________________________
!____________________________________________________________________________
!!APPLYING THE BOUNDARY CONDITIONS...
/SOL
CSYS,1
FINISH
/PREP7
FLST, 2, 992, 1, ORDE, 2
FTITEM, 2, 1
FTITEM, 2, -992
NROTAT, P51X
FINISH
/SOL
FLST, 2, 32, 1, ORDE, 11
FTITEM, 2, 10
FTITEM, 2, 50
FTITEM, 2, -57
FTITEM, 2, 298
FTITEM, 2, 338
FTITEM, 2, -344
FTITEM, 2, 546
FTITEM, 2, 586
FTITEM, 2, -592
FTITEM, 2, 793
FTITEM, 2, -799
!* /GO
D,P51X, ,0,..., ALL, ..., FLST, 2, 32, 1, ORDE, 8

84
FITEM,2,1
FITEM,2,-9
FITEM,2,290
FITEM,2,-297
FITEM,2,538
FITEM,2,-545
FITEM,2,786
FITEM,2,-792
!*
/GO
D,P51X,0,...UX,...

FLST,2,32,1,ORDE,8
FITEM,2,1
FITEM,2,-9
FITEM,2,290
FITEM,2,-297
FITEM,2,538
FITEM,2,-545
FITEM,2,786
FITEM,2,-792
!*
/GO
D,P51X,.25,...UZ,...
!!SOLVE...
!/STATUS,SOLU
SOLVE
! LGWRITE,fun,lgw,C:\DOCUME~1\bgt7536\Desktop\,COMMENT
/BATCH
!!THIS FILE WAS CREATED BY BRIAN G. TOWNER.
!!THIS FILE WILL SETUP AND SOLVE FOR THE NATURAL FREQUENCIES
!!OF THE HYPERBOLIC SHELL OF REVOLUTION (HSR).
!!THE HSR BOUNDARY CONDITIONS ARE FREE ON BOTH ENDS.
! /COM,ANSYS RELEASE 8.0 UP20030930 13:16:37 12/16/2004
/input,menust.tmp,"..............,1
! /GRA,POWER
! /GST,ON
! /PLO,INFO,3
! /GRO,CURL,ON
! /CPLANE,1
! /REPL,RESIZE
/WPSY,T,00,0
! /REPL,RESIZE
/PREP7
*

!!SHELL TYPE...
ET,1,SHELL93
*
!!THICKNESS OF SHELL...
R,1,001,,
*
!
!!MATERIAL PROPERTIES...
!!MATERIAL IS ALUMINUM.
MPTEMP,,,,,
MPTEMP,1,0
!!MODULUS OF ELASTICITY..
MPDATA,EX,1,106
!!POISSON'S RATIO...
MPDATA,PRXY,1,33
MPTEMP,,,,,
MPTEMP,1,0
!!DENSITY OF MATERIAL...
MPDATA,DENS,1,2.55E-4
*

!!DEFINING AND CREATING MERIDIAN OF SHELL...
!!PLOTTING SIX POINTS DEFINING 1/2 OF THE MERIDIAN.
!K,,RADIUS,0,Z
K,,1.75,0,0,
K,,1+187/229,0,3,5,
K,,2+1/130,0,1,5,
K,,2+224/799,0,1,0,4,5,
K,,2+101/163,0,2,2,5,
K,,3,0,3,
K,,0,0,3,
K,,0,0,-3,
!!CREATING THE MERIDIAN...
FLST,3,6,3
FITEM,3,1
!!NOTE: THE FOLLOWING LINE DEFINES THE SLOPE OF THE MERIDIAN AT EACH END.
BSPLIN, P51X, , , 0,0,-1,95,0,144,
FLST,3,1,4,ORDE,1
FLST,3,3
FLST,3,5
FLST,3,6
LSYMM,Z,P51X, , , 0,0
FLST,2,2,4,ORDE,2
FLST,2,4,ORDE,2
LGLUE,P51X
FLST,2,2,4,ORDE,2
FLST,2,1
FLST,2,-2
LGLUE,P51X
FLST,2,2,4,ORDE,2
FLST,2,1
FLST,2,3
LGLUE,P51X, ,
FLST,2,1,4,ORDE,1
FLST,2,1
FLST,8,2,3
FLST,8,7
FLST,8,8
AROTAT,P51X, , , P51X, ,360, ,
!----------------------------------------------------------------------------------------------------------------------------------

!!MESHING THE SHELL...
AATT, 1, 1, 1, 0,
FLST,5,4,4,ORDE,2
FLST,5,8,4,ORDE,2
FLST,5,5
FLST,5,-12
CM,_Y,LINE
LSEL, , , P51X
CM,_Y1,LINE
CMSEL,_Y

!!NUMBER OF ELEMENTS ALONG THE LENGTH->20...
LESIZE,_Y1, ,20, , , ,1
FLST,5,8,4,ORDE,2
FLST,5,5
FLST,5,-12
CM,_Y,LINE
LSEL, , , P51X
CM,_Y1,LINE
CMSEL,_Y

!!NUMBER OF ELEMENTS ALONG THE CIRCUMFERENCE->4*4...
LESIZE,_Y1, ,4, , , ,1
MSHAPE,0,2D
MSHKEY,0
FLST,5,4,5,ORDE,2
FLST,5,1
FITEM, 5, -4  
CM._Y, AREA  
ASEL, , , PSIX  
CM._Y1, AREA  
CHKMSH,'AREA'  
CMSEL,S, _Y  
!*  
AMESH,_Y1  
!*  
CMDELE,_Y  
CMDELE,_Y1  
CMDELE,_Y2  
!*  

FINISH  
/SOL  
!----------------------------------------------------------------------  
!----------------------------------------------------------------------  
!!SETTING UP THE ANALYSIS...  
ANTYPE, 2  
!*  
MSAVE, 0  
!*  
!!DESIGNATE THE NUMBER OF NATURAL FREQUENCIES TO EXTRACT AND EXPAND...  
MODOPT, LANB, 100  
EQSLV, SPAR  
MXPAND, 100, , 0  
LUMPM, 0  
PSTRES, 0  
!*  
!!DESIGNATE NUMBER OF MODES TO EXTRACT AND THE FREQUENCY RANGE...  
MODOPT, LANB, 100, 4800, 6000, , OFF  
! /STATUS, SOLU  
/VVIEW, 1, 1, 1, 1  
/ANG, 1  
/REP, FAST  
EPLT  
!!SOLVE...  
SOLVE  
FINISH  
/POST1  
SET, LIST  
! LGWRITE, set_fixed_free, lgw, C:\DOCUME-_1\bgt7536\DESKTOP\COMMENT
Finding the natural frequencies of a fixed-fixed HSR

/BATCH
!!THIS FILE WAS CREATED BY BRIAN G. TOWNER.
!!THIS FILE WILL SETUP AND SOLVE FOR THE NATURAL FREQUENCIES
!!OF THE HYPERBOLIC SHELL OF REVOLUTION (HSR).
!!THE HSR BOUNDARY CONDITIONS ARE FIXED ON BOTH ENDS.
!/COM,ANSYS RELEASE 8.0 UP20030930 13:16:37 12/16/2004
/input,menust,tmp,"..........,1
!/GRA,POWER
!/GST,ON
!/PLO,INFO,3
!/GRO,CURL,ON
!/CPLANE,1
!/REPLIT,RESIZE
WPSTYLE,.....,0
!/REPLIT,RESIZE
/PREP7
!*  

SHELL TYPE...
ET,1,SHELL93
!*  
!!THICKNESS OF SHELL...
R,1,.001,.......
!*  
!*  

MATERIAL PROPERTIES...
!!MATERIAL IS ALUMINUM.
MPTEMP,....... MPTEMP,1,0
!!MODULUS OF ELASTICITY...
MPDATA,EX,1,1.0e6
!!POISSON'S RATIO...
MPDATA,PRXY,1,33
MPTEMP,....... MPTEMP,1,0
!!DENSITY OF MATERIAL...
MPDATA,DENS,1,2.55e-4
!*  
!*  
!!DEFINING AND CREATING ONE-HALF OF THE MERIDIAN.
!K,RADIUS,0,Z
K,1,1.25,0,0,
K,1,1+13/503,0,3/5,
K,1,1+607/921,0,1+1/5,
K,1,2+23/389,0,1+4/5,
K,1,2+427/830,0,2+2/5,
K,1,3,0,3,
K,1,0,0,3,
K,1,0,0,-3,
!!CREATING THE MERIDIAN...
FLST,3,6,3
FITEM,3,1
FITEM,3,2
FITEM,3,3
FITEM,3,4
FITEM,3,5
FITEM,3,6
!!NOTE: THE FOLLOWING LINE DEFINES THE SLOPE OF THE
!!MERIDIAN AT EACH END.
BSPLIN,JP51X, , , ,0,0,119,0,144,
FLST,3,1,4,ORDE,1
FITEM,3,1
LSYM, Z,JP51X, , ,0,0
FLST,2,2,4,ORDE,2
FITEM,2,1
FITEM,2-2
LGLUE,JP51X
FLST,2,2,4,ORDE,2
FITEM,2,1
FITEM,2,3
LCOMB,JP51X,0
FLST,2,1,4,ORDE,1
FITEM,2,1
FLST,8,2,3
FITEM,8,7
FITEM,8,8
AROTAT,JP51X, , , , , ,P51X,360,

!------------------------------

!!MESHING THE SHELL...
AATT,1,1,1,0,
FLST,5,4,4,ORDE,2
FITEM,5,1
FITEM,5-4
CM,Y,LINE
LSEL,,JP51X
CM,Y1,LINE
CMSEL,Y
!*!!NUMBER OF ELEMENTS ALONG THE LENGTH->30...
LESIZE,Y1,,30,,1
!*FLST,5,8,4,ORDE,2
FITEM,5,5
FITEM,5,-12
CM,Y,LINE
LSEL,,JP51X
CM,Y1,LINE
CMSEL,,Y
!*!!NUMBER OF ELEMENTS ALONG THE CIRCUMFERENCE->4*4...
LESIZE,Y1,,4,,1
!*MSHAPE,0,2D
MSHKEY,0
!*FLST,5,4,5,ORDE,2
FITEM,5,1
FITEM,5,-4
CM_,Y,AREA
ASEL,,P51X
CM_,Y1,AREA
CHKMSH,'AREA'
CMSEL,S_,Y
!* 
AMESH_,Y1
!* 
CMDELE_,Y
CMDELE_,Y1
CMDELE_,Y2
!* 

FINISH
!----------------------------------------------------------
!----------------------------------------------------------
!!APPLYING THE BOUNDARY CONDITIONS...
/SOL
FLST,2,4,4,ORDE,4
FITEM,2,6
FITEM,2,8
FITEM,2,10
FITEM,2,12
!* 
/GO
DL,P51X,,ALL,0
/SOL
/SOL
FLST,2,44,1,ORDE,20
FITEM,2,1
FITEM,2,-9
FITEM,2,166
FITEM,2,254
FITEM,2,342
FITEM,2,430
FITEM,2,-437
FITEM,2,534
FITEM,2,622
FITEM,2,710
FITEM,2,798
FITEM,2,-805
FITEM,2,902
FITEM,2,990
FITEM,2,1078
FITEM,2,1166
FITEM,2,-1172
FITEM,2,1209
FITEM,2,1297
FITEM,2,1385
!* 
/GO
D,P51X,,0,...ALL..., 
------------------------
FINISH
/POST1
FINISH
/SOL
!* 
!----------------------------------------------------------------------------------------------------------
!----------------------------------------------------------------------------------------------------------
!!SETTING UP THE ANALYSIS...
ANTYPE,2
!* 
MSAVE,0
!* 
!!DESIGNATE THE NUMBER OF NATURAL FREQUENCIES TO EXTRACT AND EXPAND... 
MODOPT,LANB,100
EQSLV,SPAR
MXPAND,100, ,0
LUMP,0
PSTRES,0 
!* 
!!DESIGNATE NUMBER OF MODES TO EXTRACT AND THE FREQUENCY RANGE... 
MODOPT,LANB,100,9200,0, OFF 
/STATUS,SOLU
/VIEW,1,1,1,1
/ANG,1
/REP,FAST
EPLOT 
!!SOLVE...
SOLVE 
FINISH 
/POST1
SET,LIST 
! LGWRITE,set_fixed_free,lgw,C:\DOCUME~1\bgt7536\DESKTOP,COMMENT
APPENDIX C: MAPLE AND MATLAB FILES

The following Maple 6 Worksheet, and Matlab 6.5, programs were used to solve the Timoshenko beam problem as applied to the HSR. The Maple program solves for each of the element mass and stiffness matrix elements. The Matlab M-file, Element.m, assembles the element matrices. The Matlab files Fixedfree.m, and FixedFixed.m, assemble the global matrices and solve for the natural frequencies of the HSR.

> #Element
> #This is a Maple 6 Worksheet used to find the expressions for the
> #elements of the mass and stiffness matrix for one element.
> Xg:=X0+x;
> Phi1:=1+2*(x/h)^3-3*(x/h)^2;
> Phi2:=-2*(x^2)/h+(x^3)/(h^2);
> Phi3:=-2*(x/h)^2-2*(x/h)^3;
> Phi4:=(x^3)/(h^2)-(x^2)/h;
> f:=sqrt(1+(Xg/d)^2);
> A:=pi*f^2*(c0^2-ci^2);
> Ix:=f^4*(c0^4-ci^4)*pi/4;
> #Finding the Mass Matrix Elements
> Mll:=int(Phi1*A*Phi1,x=0..h);
> M12:=int(Phi1*A*Phi2,x=0..h);
> M13:=int(Phi1*A*Phi3,x=0..h);
> M14:=int(Phi1*A*Phi4,x=0..h);
> M22:=int(Phi2*A*Phi2,x=0..h);
> M23:=int(Phi2*A*Phi3,x=0..h);
> M24:=int(Phi2*A*Phi4,x=0..h);
> M33:=int(Phi3*A*Phi3,x=0..h);
> M34:=int(Phi3*A*Phi4,x=0..h);
> M44:=int(Phi4*A*Phi4,x=0..h);
> #Finding the Stiffness Elements
> #Elements for Kww
> Kww11:=k*G*int(diff(Phi1,x)*A*diff(Phi1,x),x=0..h);
> Kww12:=k*G*int(diff(Phi1,x)*A*diff(Phi2,x),x=0..h);
\[ K_{ww13} := k \cdot G \cdot \int (\frac{\partial \Phi_1}{\partial x}) A \cdot (\frac{\partial \Phi_3}{\partial x}), x=0..h \]
\[ K_{ww14} := k \cdot G \cdot \int (\frac{\partial \Phi_1}{\partial x}) A \cdot (\frac{\partial \Phi_4}{\partial x}), x=0..h \]
\[ K_{ww22} := k \cdot G \cdot \int (\frac{\partial \Phi_2}{\partial x}) A \cdot (\frac{\partial \Phi_2}{\partial x}), x=0..h \]
\[ K_{ww23} := k \cdot G \cdot \int (\frac{\partial \Phi_2}{\partial x}) A \cdot (\frac{\partial \Phi_3}{\partial x}), x=0..h \]
\[ K_{ww24} := k \cdot G \cdot \int (\frac{\partial \Phi_2}{\partial x}) A \cdot (\frac{\partial \Phi_4}{\partial x}), x=0..h \]
\[ K_{ww33} := k \cdot G \cdot \int (\frac{\partial \Phi_3}{\partial x}) A \cdot (\frac{\partial \Phi_3}{\partial x}), x=0..h \]
\[ K_{ww34} := k \cdot G \cdot \int (\frac{\partial \Phi_3}{\partial x}) A \cdot (\frac{\partial \Phi_4}{\partial x}), x=0..h \]
\[ K_{ww44} := k \cdot G \cdot \int (\frac{\partial \Phi_4}{\partial x}) A \cdot (\frac{\partial \Phi_4}{\partial x}), x=0..h \]

# Element For \( K_{pw} \), Note: \( K_{pw} = \text{Transpose}[K_{pw}] \)
\[ K_{pw11} := -k \cdot G \cdot \int (\frac{\partial \Phi_1}{\partial x}) A \cdot \Phi_1, x=0..h \]
\[ K_{pw12} := -k \cdot G \cdot \int (\frac{\partial \Phi_1}{\partial x}) A \cdot \Phi_2, x=0..h \]
\[ K_{pw13} := -k \cdot G \cdot \int (\frac{\partial \Phi_1}{\partial x}) A \cdot \Phi_3, x=0..h \]
\[ K_{pw14} := -k \cdot G \cdot \int (\frac{\partial \Phi_1}{\partial x}) A \cdot \Phi_4, x=0..h \]
\[ K_{pw21} := -k \cdot G \cdot \int (\frac{\partial \Phi_2}{\partial x}) A \cdot \Phi_1, x=0..h \]
\[ K_{pw22} := -k \cdot G \cdot \int (\frac{\partial \Phi_2}{\partial x}) A \cdot \Phi_2, x=0..h \]
\[ K_{pw23} := -k \cdot G \cdot \int (\frac{\partial \Phi_2}{\partial x}) A \cdot \Phi_3, x=0..h \]
\[ K_{pw24} := -k \cdot G \cdot \int (\frac{\partial \Phi_2}{\partial x}) A \cdot \Phi_4, x=0..h \]
\[ K_{pw31} := -k \cdot G \cdot \int (\frac{\partial \Phi_3}{\partial x}) A \cdot \Phi_1, x=0..h \]
\[ K_{pw32} := -k \cdot G \cdot \int (\frac{\partial \Phi_3}{\partial x}) A \cdot \Phi_2, x=0..h \]
\[ K_{pw33} := -k \cdot G \cdot \int (\frac{\partial \Phi_3}{\partial x}) A \cdot \Phi_3, x=0..h \]
\[ K_{pw34} := -k \cdot G \cdot \int (\frac{\partial \Phi_3}{\partial x}) A \cdot \Phi_4, x=0..h \]
\[ K_{pw41} := -k \cdot G \cdot \int (\frac{\partial \Phi_4}{\partial x}) A \cdot \Phi_1, x=0..h \]
\[ K_{pw42} := -k \cdot G \cdot \int (\frac{\partial \Phi_4}{\partial x}) A \cdot \Phi_2, x=0..h \]
\[ K_{pw43} := -k \cdot G \cdot \int (\frac{\partial \Phi_4}{\partial x}) A \cdot \Phi_3, x=0..h \]
\[ K_{pw44} := -k \cdot G \cdot \int (\frac{\partial \Phi_4}{\partial x}) A \cdot \Phi_4, x=0..h \]

# Element For \( K_{pp} \)
\[ K_{pp11} := \int (\frac{\partial \Phi_1}{\partial x}) E_1 L \cdot x \cdot (\frac{\partial \Phi_1}{\partial x}) + \Phi_1 k \cdot G \cdot A \cdot \Phi_1, x=0..h \]
\[ K_{pp12} := \int (\frac{\partial \Phi_1}{\partial x}) E_1 L \cdot x \cdot (\frac{\partial \Phi_2}{\partial x}) + \Phi_1 k \cdot G \cdot A \cdot \Phi_2, x=0..h \]
\[ K_{pp13} := \int (\frac{\partial \Phi_1}{\partial x}) E_1 L \cdot x \cdot (\frac{\partial \Phi_3}{\partial x}) + \Phi_1 k \cdot G \cdot A \cdot \Phi_3, x=0..h \]
\[ K_{pp14} := \int (\frac{\partial \Phi_1}{\partial x}) E_1 L \cdot x \cdot (\frac{\partial \Phi_4}{\partial x}) + \Phi_1 k \cdot G \cdot A \cdot \Phi_4, x=0..h \]
\[ K_{pp22} := \int (\frac{\partial \Phi_2}{\partial x}) E_1 L \cdot x \cdot (\frac{\partial \Phi_2}{\partial x}) + \Phi_2 k \cdot G \cdot A \cdot \Phi_2, x=0..h \]
\[ K_{pp23} := \int (\frac{\partial \Phi_2}{\partial x}) E_1 L \cdot x \cdot (\frac{\partial \Phi_3}{\partial x}) + \Phi_2 k \cdot G \cdot A \cdot \Phi_3, x=0..h \]
\[ K_{pp24} := \int (\frac{\partial \Phi_2}{\partial x}) E_1 L \cdot x \cdot (\frac{\partial \Phi_4}{\partial x}) + \Phi_2 k \cdot G \cdot A \cdot \Phi_4, x=0..h \]
\[ K_{pp33} := \int (\frac{\partial \Phi_3}{\partial x}) E_1 L \cdot x \cdot (\frac{\partial \Phi_3}{\partial x}) + \Phi_3 k \cdot G \cdot A \cdot \Phi_3, x=0..h \]
\[ K_{pp34} := \int (\frac{\partial \Phi_3}{\partial x}) E_1 L \cdot x \cdot (\frac{\partial \Phi_4}{\partial x}) + \Phi_3 k \cdot G \cdot A \cdot \Phi_4, x=0..h \]
\[ K_{pp44} := \int (\frac{\partial \Phi_4}{\partial x}) E_1 L \cdot x \cdot (\frac{\partial \Phi_4}{\partial x}) + \Phi_4 k \cdot G \cdot A \cdot \Phi_4, x=0..h \]
%ElementMatrix.m
%Brian Towner
%Thesis: Dynamic Characteristics of a Hyperboloid Shell of Revolution
%This Matlab M-file is used to assemble the local mass and stiffness
%matrices. Each element was copied from the solution of the
%previous Maple Worksheet.

d = (c*Zmax)/sqrt(Rmax^2-c^2); %inches
thickness = .001; %inches
c0 = c + thickness/2; %inches
ci = c - thickness/2; %inches

E1 = 10e6; %Young’s Modulus
nu = .33; %Poisson’s Ratio
density = .000255; %Density of Material.
G = E1/(2*(1+nu));
k = .9; %Shear Correction factor.

%The expression of the results from the Maple worksheet should be imported into this M-
%file. The following will assemble the appropriate local matrices for the HSR.

%Mass Elements:

M = density*[M11, M12, M13, M14; M12, M22, M23, M24; M13, M23, M33, M34; M14,
M24, M34, M44];

J = density*[J11, J12, J13, J14; J12, J22, J23, J24; J13, J23, J33, J34; J14, J24, J34, J44];

%Stiffness Elements:

Kpw = [Kpw11, Kpw12, Kpw13, Kpw14; Kpw21, Kpw22, Kpw23, Kpw24; Kpw31, Kpw32,
Kpw33, Kpw34; Kpw41, Kpw42, Kpw43, Kpw44];

Kwp = Kpw';

Kww = [Kww11, Kww12, Kww13, Kww14; Kww12, Kww22, Kww23, Kww24; Kww13,
Kww23, Kww33, Kww34; Kww14, Kww24, Kww34, Kww44];

%Total Local Mass and Stiffness:

Ml = [M, zeros(4,4); zeros(4,4), J];

Kl = [Kww, Kpw; Kwp, Kpp];
%Rearranging local matrices for ease of assembling global matrices:

\[
\text{Melement} = [M_{l(1,1)}, M_{l(1,5)}, M_{l(1,2)}, M_{l(1,6)}, M_{l(1,3)}, M_{l(1,7)}, M_{l(1,4)}, M_{l(1,8)}; \\
M_{l(1,5)}, M_{l(5,5)}, M_{l(2,5)}, M_{l(5,6)}, M_{l(3,5)}, M_{l(5,7)}, M_{l(4,5)}, M_{l(5,8)}; \\
M_{l(1,2)}, M_{l(2,5)}, M_{l(2,2)}, M_{l(2,6)}, M_{l(2,3)}, M_{l(2,7)}, M_{l(2,4)}, M_{l(2,8)}; \\
M_{l(1,6)}, M_{l(5,6)}, M_{l(2,6)}, M_{l(6,6)}, M_{l(3,6)}, M_{l(6,7)}, M_{l(4,6)}, M_{l(6,8)}; \\
M_{l(1,3)}, M_{l(3,5)}, M_{l(2,3)}, M_{l(3,6)}, M_{l(3,3)}, M_{l(3,7)}, M_{l(3,4)}, M_{l(3,8)}; \\
M_{l(1,7)}, M_{l(5,7)}, M_{l(2,7)}, M_{l(6,7)}, M_{l(3,7)}, M_{l(7,7)}, M_{l(4,7)}, M_{l(7,8)}; \\
M_{l(1,4)}, M_{l(4,5)}, M_{l(2,4)}, M_{l(4,6)}, M_{l(3,4)}, M_{l(4,7)}, M_{l(4,4)}, M_{l(4,8)}; \\
M_{l(1,8)}, M_{l(5,8)}, M_{l(2,8)}, M_{l(6,8)}, M_{l(3,8)}, M_{l(7,8)}, M_{l(4,8)}, M_{l(8,8)}];
\]

\[
\text{Kelement} = [K_{l(1,1)}, K_{l(1,5)}, K_{l(1,2)}, K_{l(1,6)}, K_{l(1,3)}, K_{l(1,7)}, K_{l(1,4)}, K_{l(1,8)}; \\
K_{l(1,5)}, K_{l(5,5)}, K_{l(2,5)}, K_{l(5,6)}, K_{l(3,5)}, K_{l(5,7)}, K_{l(4,5)}, K_{l(5,8)}; \\
K_{l(1,2)}, K_{l(2,5)}, K_{l(2,2)}, K_{l(2,6)}, K_{l(2,3)}, K_{l(2,7)}, K_{l(2,4)}, K_{l(2,8)}; \\
K_{l(1,6)}, K_{l(5,6)}, K_{l(2,6)}, K_{l(6,6)}, K_{l(3,6)}, K_{l(6,7)}, K_{l(4,6)}, K_{l(6,8)}; \\
K_{l(1,3)}, K_{l(3,5)}, K_{l(2,3)}, K_{l(3,6)}, K_{l(3,3)}, K_{l(3,7)}, K_{l(3,4)}, K_{l(3,8)}; \\
K_{l(1,7)}, K_{l(5,7)}, K_{l(2,7)}, K_{l(6,7)}, K_{l(3,7)}, K_{l(7,7)}, K_{l(4,7)}, K_{l(7,8)}; \\
K_{l(1,4)}, K_{l(4,5)}, K_{l(2,4)}, K_{l(4,6)}, K_{l(3,4)}, K_{l(4,7)}, K_{l(4,4)}, K_{l(4,8)}; \\
K_{l(1,8)}, K_{l(5,8)}, K_{l(2,8)}, K_{l(6,8)}, K_{l(3,8)}, K_{l(7,8)}, K_{l(4,8)}, K_{l(8,8)}];
\]
%Fixedfree.m
%Brian Towner
%Thesis: Dynamic Characteristics of a Hyperboloid Shell of Revolution
%This Matlab M-file is used to assemble the global mass and stiffness
%matrices, and to solve for the bending, natural frequencies of an HSR.
%The boundary conditions are fixed-free.

Lmax = 6; %inches
Rmax = 3; %inches
c = 1.00; %inches
Zmax = Lmax/2; %inches

Ne = 8; %Number of Elements.
Nn = Ne + 1; %Number of Nodes.
h = Lmax/Ne; %Length of Each Element.

Mglobal = zeros(4*Nn,4*Nn);
Kglobal = zeros(4*Nn,4*Nn);

%Assembling the Global Matrices:
for i = 0:4:Ne*4-4;
    X0 = -Zmax+i*h/4; %Sets the starting coordinate of each element.
    ElementMatrix2test; %Inputs local mass and stiffness matrices.

    for j = 1:8
        for p = 1:8
            Mglobal(i+j,i+p) = Mglobal(i+j,i+p) + Melement(j,p);
            Kglobal(i+j,i+p) = Kglobal(i+j,i+p) + Kelement(j,p);
        end
    end
end

%Dealing with the Fixed-Free Boundary Conditions:
NewM = Mglobal(3:4*Nn,3:4*Nn);
NewK = Kglobal(3:4*Nn,3:4*Nn);

%Solving for the Natural Frequencies:
lambda = eig(NewK,NewM);
wn = sqrt(lambda); %rad/sec
fn = (wn)/(2*pi) %Hz
%FixedFixed.m
%Brian Towner
%Thesis: Dynamic Characteristics of a Hyperboloid Shell of Revolution
%This Matlab M-file is used to assemble the global mass and stiffness
%matrices, and to solve for the bending, natural frequencies of an HSR.
%The boundary conditions are fixed-fixed.

Lmax = 6;       % inches
Rmax = 3;       % inches
c =1.00;        % inches
Zmax = Lmax/2;  % inches

Ne=8;          % Number of Elements.
Nn=Ne + 1;      % Number of Nodes.
h = Lmax/Ne;    % Length of Each Element.

Mglobal = zeros(4*Nn,4*Nn);
Kglobal = zeros(4*Nn,4*Nn);

for i = 0:4:Ne*4-4;
    X0 = -Zmax+i*h/4;
    ElementMatrix;

    for j = 1:8
        for p = 1:8
            Mglobal(i+j,i+p) = Mglobal(i+j,i+p) + Melement(j,p);
            Kglobal(i+j,i+p) = Kglobal(i+j,i+p) + Kelement(j,p);
        end
    end

end

%Rearranging for the Fixed-Fixed Boundary Conditions
NewM = zeros(4*Nn-6,4*Nn-6);
NewK = zeros(4*Nn-6,4*Nn-6);

NewM(1:4*Nn-7,1:4*Nn-7)=NewM(1:4*Nn-7,1:4*Nn-7)+Mglobal(4:4*Nn-4,4:4*Nn-4);
NewK(1:4*Nn-7,1:4*Nn-7)=NewK(1:4*Nn-7,1:4*Nn-7)+Kglobal(4:4*Nn-4,4:4*Nn-4);

NewK(1:4*Nn-7,4*Nn-6) = NewK(1:4*Nn-7,4*Nn-6) + Kglobal(4:4*Nn-4,4*Nn-6);
NewM(1:4*Nn-7,4*Nn-6) = NewM(1:4*Nn-7,4*Nn-6) + Mglobal(4:4*Nn-4,4*Nn-6);

NewK(4*Nn-6,1:4*Nn-7) = NewK(4*Nn-6,1:4*Nn-7) + Kglobal(4*Nn,4:4*Nn-4);
NewM(4*Nn-6,1:4*Nn-7) = NewM(4*Nn-6,1:4*Nn-7) + Mglobal(4*Nn,4:4*Nn-4);

NewK(4*Nn-6,4*Nn-6) = NewK(4*Nn-6,4*Nn-6) + Kglobal(4*Nn,4*Nn);
NewM(4*Nn-6,4*Nn-6) = NewM(4*Nn-6,4*Nn-6) + Mglobal(4*Nn,4*Nn);
Solving for the Natural Frequencies:

\[
\lambda = \text{eig}(\text{NewK}, \text{NewM});
\]

\[
wn = \sqrt{\lambda}; \quad \text{rad/sec}
\]

\[
fn = \frac{wn}{(2\pi)} \quad \text{Hz}
\]
Appendix D: Expanded Forces and Moments

\[ N_1 = C \left[ \frac{1}{c \eta} \frac{\partial v}{\partial \theta} + \left( \frac{\nu \lambda \eta}{c \xi} \frac{\partial}{\partial \xi} + \frac{\lambda \xi}{c \eta} \right) v + \left( \frac{\lambda^2 v}{c \xi^3} - \frac{1}{c \xi} \right) w \right] \]

\[ N_2 = C \left[ \frac{v}{c \eta} \frac{\partial v}{\partial \theta} + \left( \frac{\lambda \eta}{c \xi} \frac{\partial}{\partial \xi} + \frac{\nu \lambda \xi}{c \eta} \right) v + \left( \frac{\lambda^2}{c \xi^3} - \frac{v}{c \xi} \right) w \right] \]

\[ N_{12} = \frac{(1-v)C}{2} \left[ \left( \frac{\lambda \eta}{c \xi} \frac{\partial}{\partial \xi} - \frac{\lambda \xi}{c \eta} \right) u + \frac{1}{c \eta} \frac{\partial v}{\partial \theta} \right] + (1-v)D \left[ \frac{\lambda^3}{c^2 \xi^5} \left( \frac{\xi}{\eta} - \frac{\eta}{\partial \xi} \right) u + \frac{\lambda^4}{c^2 \xi^4} \frac{\partial v}{\partial \theta} \right] \]

\[ N_{21} = \frac{(1-v)C}{2} \left[ \left( \frac{\lambda \eta}{c \xi} \frac{\partial}{\partial \xi} - \frac{\lambda \xi}{c \eta} \right) u + \frac{1}{c \eta} \frac{\partial v}{\partial \theta} \right] + (1-v)D \left[ \frac{\lambda}{c^3 \xi^2} \left( \eta \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\eta} \right) u + \frac{\lambda^2}{c^3 \xi^4} \frac{\partial v}{\partial \theta} \right] \]

\[ M_{12} = M_{21} = -D(1-v) \left[ \left( \frac{\lambda \eta}{c^2 \xi^2} \frac{\partial}{\partial \xi} - \frac{\lambda \xi}{c^2 \xi^2 \eta} \right) u - \frac{\lambda^2}{c^2 \xi^3 \eta} \frac{\partial v}{\partial \theta} \right] + \left( \frac{\lambda \eta}{c^2 \xi^2} \frac{\partial}{\partial \xi} - \frac{\lambda \xi}{c^2 \xi^2 \eta} \right) \frac{\partial v}{\partial \theta} \right] \]

\[ M_1 = -D \left[ \frac{1}{c^2 \eta \xi} \frac{\partial u}{\partial \theta} - \left( \frac{\lambda^3 \xi}{c^2 \xi^4} - \frac{3 \nu \lambda \eta \xi}{c^2 \xi^6} \left( \lambda^2 + 1 \right) \right) v - \frac{\nu \lambda \eta}{c^2 \xi^4} \frac{\partial v}{\partial \xi} + \left( \frac{\lambda^2 \xi}{c^2 \xi^2} - \frac{\nu \lambda \xi}{c^2 \xi^4} \right) \frac{\partial w}{\partial \xi} \right] \]

\[ M_1 = -D \left[ \frac{v}{c^2 \eta \xi} \frac{\partial u}{\partial \theta} - \left( \frac{\nu \lambda^3 \xi}{c^2 \xi^4} - \frac{3 \lambda \eta \xi}{c^2 \xi^6} \left( \lambda^2 + 1 \right) \right) v - \frac{\lambda^3 \eta}{c^2 \xi^4} \frac{\partial v}{\partial \xi} + \left( \frac{\nu \lambda^2 \xi}{c^2 \xi^2} - \frac{\lambda^4 \xi}{c^2 \xi^4} \right) \frac{\partial w}{\partial \xi} \right] \]
APPENDIX E: EXPANDED MATRIX ELEMENTS

Mass Matrix Elements:

\[
\begin{align*}
\hat{M}_{11} &= -\frac{1}{C} \left( \eta \rho ch + \frac{\eta \rho h^3}{12 c \zeta} \right), \\
\hat{M}_{12} &= 0, \\
\hat{M}_{13} &= -\frac{1}{C} \left( \frac{\rho h^3}{12 c \zeta} \right) \frac{\partial}{\partial \theta}, \\
\hat{M}_{21} &= 0, \\
\hat{M}_{22} &= \frac{1}{C} \left( \frac{\lambda^2 \eta^2 \rho h^3}{c \zeta^4} - c \eta \rho h \right), \\
\hat{M}_{23} &= -\frac{1}{C} \left( \frac{\lambda^2 \eta^2 \rho h^3}{c \zeta^4} \right) \frac{\partial}{\partial \xi}, \\
\hat{M}_{31} &= \frac{1}{C} \left( \frac{\rho h^3}{12 c \eta \zeta} \right) \frac{\partial}{\partial \theta}, \\
\hat{M}_{32} &= \frac{1}{C} \left( \frac{\rho h^3 \lambda^2 \eta}{12 c \zeta} \right) \left( \frac{3 \xi \lambda \eta (\lambda^2 + 1)}{\zeta^5} - \frac{\lambda \xi}{\eta \xi^3} - \frac{\lambda \eta}{\xi^3} \frac{\partial}{\partial \xi} \right), \\
\hat{M}_{33} &= \frac{1}{C} \left( \frac{\rho h^3}{12 c \eta \partial \theta^2} - \frac{\lambda^2 \eta \rho h^3}{12 c \zeta} \left[ \frac{n^2}{\zeta} \frac{\partial^2}{\partial \xi^2} + \left[ \frac{2 \xi}{\zeta} - \frac{\eta^2 \xi}{\xi^3} (\lambda^2 + 1) \right] \frac{\partial}{\partial \xi} \right] - \rho hc \eta \right).
\end{align*}
\]
Expanded Stiffness Matrix Elements:

\[ L_{11} = \frac{1}{2c^3}\eta^6 \left( \lambda^2(\nu-1)c^2\xi^2(1+(\nu+1)^2)c^2\xi^2+3(1+\lambda^2)\xi^4+(1+\lambda^2)^2\xi^6-4D(-1+\xi^4+\lambda^2\xi^2(2+\xi^2)) \right) \]

\[ +2\xi^4(D+c^2C\xi^2)\frac{\partial^2}{\partial\theta^2} - \lambda^2(\nu-1)\eta^2 \left( \xi(c^2C(\lambda^2\xi^2(\xi^2-1)+\eta^4+\lambda^2(-1+\xi^2+2\xi^4)) \right) \]

\[ = \frac{1}{2c^3}\eta^7 \left( \lambda^2(\nu-1)c^2\xi^2(1+(\nu+1)^2)c^2\xi^2+3(1+\lambda^2)\xi^4+(1+\lambda^2)^2\xi^6-4D(-1+\xi^4+\lambda^2\xi^2(2+\xi^2)) \right) \]

\[ +2\xi^4(D+c^2C\xi^2)\frac{\partial^2}{\partial\theta^2} - \lambda^2(\nu-1)\eta^2 \left( \xi(c^2C(\lambda^2\xi^2(\xi^2-1)+\eta^4+\lambda^2(-1+\xi^2+2\xi^4)) \right) \]

\[ L_{12} = \frac{1}{2c^3}\eta^7 \left( \lambda^2(\nu-1)c^2\xi^2(1+(\nu+1)^2)c^2\xi^2+3(1+\lambda^2)\xi^4+(1+\lambda^2)^2\xi^6-4D(-1+\xi^4+\lambda^2\xi^2(2+\xi^2)) \right) \]

\[ +2\xi^4(D+c^2C\xi^2)\frac{\partial^2}{\partial\theta^2} - \lambda^2(\nu-1)\eta^2 \left( \xi(c^2C(\lambda^2\xi^2(\xi^2-1)+\eta^4+\lambda^2(-1+\xi^2+2\xi^4)) \right) \]

\[ L_{13} = \frac{1}{2c^3}\eta^7 \left( \lambda^2(\nu-1)c^2\xi^2(1+(\nu+1)^2)c^2\xi^2+3(1+\lambda^2)\xi^4+(1+\lambda^2)^2\xi^6-4D(-1+\xi^4+\lambda^2\xi^2(2+\xi^2)) \right) \]

\[ +2\xi^4(D+c^2C\xi^2)\frac{\partial^2}{\partial\theta^2} - \lambda^2(\nu-1)\eta^2 \left( \xi(c^2C(\lambda^2\xi^2(\xi^2-1)+\eta^4+\lambda^2(-1+\xi^2+2\xi^4)) \right) \]

\[ +2\xi^4(D+c^2C\xi^2)\frac{\partial^2}{\partial\theta^2} - \lambda^2(\nu-1)\eta^2 \left( \xi(c^2C(\lambda^2\xi^2(\xi^2-1)+\eta^4+\lambda^2(-1+\xi^2+2\xi^4)) \right) \]
\[ L_{21} = \frac{1}{2c^3 \eta \zeta^5} \left( \lambda \left( \frac{(2D \lambda^2 \zeta - 2D \lambda^2 \nu \zeta)}{\partial \theta} + (-3c^2 \zeta \nu \zeta^4 + c^2 \nu \zeta^4) \frac{\partial}{\partial \theta} \right) \\
-2D \lambda^2 \eta^2 \frac{\partial^2}{\partial \phi \partial \zeta} + 2D \lambda^2 \eta^2 \frac{\partial^2}{\partial \phi \partial \zeta} + c^2 C \eta^2 \zeta^4 \frac{\partial^2}{\partial \phi \partial \zeta} + c^2 C \eta^2 \zeta^4 \frac{\partial^2}{\partial \phi \partial \zeta} \right) \]

\[ L_{22} = \frac{1}{2c^3 \eta} \left( \begin{array}{c}
\frac{c^2 C}{\partial \theta^2} - c^2 C \nu \frac{\partial^2}{\partial \theta^2} + \frac{2D \lambda^2}{\xi^4} \frac{\partial^2}{\partial \theta^2} - \frac{2D \nu \lambda^4}{\xi^6} \frac{\partial^2}{\partial \theta^2} - \frac{2c^3 C \lambda^2 \xi}{\xi^4} \left( \frac{\xi V + \nu (1 + \xi^2)}{\zeta^2} \right) \\
\frac{\xi \left( (1 + \nu) \eta^2 + \lambda^2 (-1 + (1 + \nu) \xi^2) \right)}{\partial \xi} + \eta^2 \xi^2 \frac{\partial^2}{\partial \xi^2} \end{array} \right) \]

\[ + \frac{1}{\xi^{12}} \left( 2D \lambda^4 \right) \left( \begin{array}{c}
\xi^2 \left( -3 + \nu + 8 \xi^2 \right) \\
+ \lambda^2 \xi^2 \left( 15 + 24 \xi^2 + 8 \xi^4 \right) + \lambda^2 \eta^2 \left( -3 + (21 + \nu) \xi^2 + 16 \xi^4 \right) \\
-\xi \left( (1 + \nu) \xi^2 \right) \frac{\partial^2}{\partial \theta^2} \\
+ \lambda^2 \eta^2 \left( -\xi \xi^2 \right) + \lambda^2 \left( 7 + 5 \xi^2 \right) \frac{\partial^2}{\partial \xi^2} + \eta^2 \xi^2 \frac{\partial^2}{\partial \xi^2} \end{array} \right) \]
$$L_{23} = -\frac{1}{c^3 \eta^2 \zeta^{10}} \left( D \lambda^2 (\nu - 3) \xi \frac{\partial^2}{\partial \vartheta^2} \xi^6 - c^2 C (\lambda^2 + 1) \xi \eta^4 ((\xi^2 - 3) \lambda^2 + \xi^2 + 1) \xi^4 \right)$$
\[ L_{31} = \frac{1}{c^3 \eta^2 C \zeta^7} \left( \begin{array}{c} 
 -c^2 C \eta^2 \xi^4 \left( 1 + \xi^2 + \lambda^2 (- \nu + \xi^2) \right) \\
 + D \lambda^2 \left( -3 - 2 \xi^2 + (5 + 8 \lambda^2 + 3 \lambda^4) \xi^4 + 4 (1 + \lambda^2)^2 \xi^6 \right) \frac{\partial}{\partial \vartheta} \\
 + \nu \eta^2 (1 + \xi^2 + 3 \lambda^2 \xi^2 + \lambda^2 (-1 + 4 \xi^2)) \end{array} \right) \]
\[
L_{33} = \frac{1}{c^3 C \eta^4 \xi^{10}} - D \left( -2 \lambda^2 \left( \nu - \xi^2 \right) \eta^4 + \eta^4 + \lambda^4 \left( 1 - 2 \nu \xi^2 + \xi^4 \right) \right) + \lambda^2 \eta^4 \left( -3 + \xi^2 + 4(1 + \lambda^2) \xi^4 + \nu \eta^2 \right) \frac{\partial^2}{\partial \theta^2} + \xi^{10} \frac{\partial^4}{\partial \theta^4} \right)
\]
APPENDIX F: MATHEMATICA FILES

The following programs can be copied and/or modified in Mathematica to solve for the natural frequencies of the HSR by shell theory. Each program sets the constants, includes the appropriate assumed shapes, assembles the stiffness and matrix matrices, and then solves for the natural frequencies.

(*----------FixedFreeFrequencies.nb----------*)
Rmax = 3;
Lmax = 6;
Zmax = Lmax/2;
h = 0.001;
E1 = 10000000;
v = 0.33;
\(\rho = 0.000255;\)
L = Lmax;
\(\eta = \text{Sqrt}[1 + \xi^2];\)
\(\lambda = c/d;\)
\(\zeta = \text{Sqrt}[1 + (1 + \lambda^2)*\xi^2];\)
C1 = (E1*h)/(1 - v^2);
D1 = 0;
c = 1.00;

\(d = (Zmax*c)/\text{Sqrt}[Rmax^2 - c^2];\)
Intmin = -(Zmax/d);
Intmax = Zmax/d;
\(dR = \lambda*D[\eta, \xi];\)
Angle = ArcTan[dR];

n = 1;
\(\beta1 = 1.875104;\)
\(\alpha1 = (\sin[\beta1] + \sinh[\beta1])/(\cosh[\beta1] + \cos[\beta1]);\)
Shape = \(\sin[(\beta1/L)*(\xi*d + L/2)] - \sinh[(\beta1/L)*(\xi*d + L/2)] - \alpha1*(\cosh[(\beta1/L)*(\xi*d + L/2)] - \cosh[(\beta1/L)*(\xi*d + L/2)]);\)

V1 = D[Shape, \(\xi\)]*Cos[n*\(\phi\)];
U1 = Shape*Sin[n*\(\phi\)];
W1 = Shape*Cos[n*\(\phi\)];

u = U1;
v = 0;
w = 0;
\(\eta = \text{Sqrt}[1 + \xi^2];\)
\(\zeta = \text{Sqrt}[1 + (1 + \lambda^2)*\xi^2];\)
X1 = \((1/(c*\eta)))*(D[w, \phi] + (\eta/\xi)*u);
X2 = \((\lambda/\eta)/c*(\xi)*(D[w, \xi] - (\lambda/(\eta*\xi^2))*v);\)
\[
\begin{align*}
\epsilon_1 &= (1/(c\eta)) \cdot (D[u, \phi] + ((\lambda \xi)/(\zeta v)) - (1/(c\zeta)) w; \\
\epsilon_2 &= ((\lambda \eta)/(c\zeta)) \cdot (D[v, \xi] + (\lambda^2/(c\zeta^3)) w; \\
\omega &= (1/(c\eta)) \cdot (D[v, \phi] - ((\lambda \xi)/(\zeta u)) + ((\lambda \eta)/(c\zeta)) \cdot (D[u, \xi]); \\
\kappa_1 &= -((1/(c\eta)) \cdot (D[X1, \phi] + ((\lambda \xi)/(\zeta X2)); \\
\kappa_2 &= -((\lambda \eta)/(c\zeta)) \cdot (D[X2, \xi]); \\
\tau &= \text{Expand}((-1/(c\eta)) \cdot (D[X2, \phi] - ((\lambda \xi)/(\zeta X1)) - ((\lambda \eta)/(c\zeta^2)) \cdot (D[u, \xi])); \\

f_1 &= (-p) \cdot h \cdot u; \\
f_2 &= (-p) \cdot h \cdot v; \\
f_3 &= (-p) \cdot h \cdot w; \\
f_4 &= \rho \cdot (h^{3/12}) \cdot X1; \\
f_5 &= (-p) \cdot (h^{3/12}) \cdot X2; \\

N1 &= \text{Expand}[C1 \cdot (\epsilon_1 + v \cdot \epsilon_2)]; \\
N2 &= \text{Expand}[C1 \cdot (\epsilon_2 + v \cdot \epsilon_1)]; \\
N12 &= \text{Expand}[(C1/2) \cdot (1 - v) \cdot \omega + ((D1 \cdot (1 - v) \cdot \lambda^2)/(c\zeta^3)) \cdot \tau]; \\
N21 &= \text{Expand}[(C1/2) \cdot (1 - v) \cdot \omega - ((D1 \cdot (1 - v))/(c\zeta)) \cdot \tau]; \\
M1 &= \text{Expand}[D1 \cdot (\kappa_1 + v \cdot \kappa_2)]; \\
M2 &= \text{Expand}[D1 \cdot (\kappa_2 + v \cdot \kappa_1)]; \\
M12 &= M21 = \text{Expand}[D1 \cdot (1 - v) \cdot \tau]; \\
Q1 &= (1/(c\eta)) \cdot (D[M1, \phi] + ((\lambda \eta)/(c\zeta)) \cdot (D[\eta \cdot M21, \xi] + ((\lambda \xi)/(\zeta \cdot M12)); \\
Q2 &= \text{Expand}[(1/(c\eta)) \cdot (D[M12, \phi] + ((\lambda \eta)/(c\zeta)) \cdot (D[\eta \cdot M2, \xi] - ((\lambda \xi)/(\zeta \cdot M1))]; \\

LB1 &= \text{Expand}[D[N1, \phi] + ((\lambda \eta)/(c\zeta)) \cdot (D[\eta \cdot N21, \xi] + ((\lambda \xi)/(\zeta \cdot N12) - (\eta \cdot \xi)/(\zeta \cdot Q1)]; \\
LB2 &= \text{Expand}[D[N12, \phi] + ((\lambda \eta)/(c\zeta)) \cdot (D[\eta \cdot N2, \xi] - ((\lambda \xi)/(\zeta \cdot N1) + ((\lambda^2 \eta)/(c\zeta^3)) \cdot Q2)]; \\
LB3 &= \text{Expand}[D[Q1, \phi] + ((\lambda \eta)/(c\zeta)) \cdot (D[\eta \cdot Q2, \xi] + (\eta \cdot \xi)/(\zeta \cdot N1) - ((\lambda^2 \eta)/(c\zeta^3)) \cdot N2)]; \\
Mass1 &= (\eta \cdot \xi)/(\zeta \cdot M1); \\
Mass2 &= ((\lambda^2 \eta)/(c\zeta^3)) \cdot f5 + c \cdot \eta \cdot f2; \\
Mass3 &= D[f4, \phi] + ((\lambda \eta)/(c\zeta)) \cdot (D[\eta \cdot f5, \xi] + c \cdot \eta \cdot f3; \\
L11 &= LB1; L21 = LB2; L31 = LB3; Mass1 = Mass1; Mass2 = Mass2; Mass3 = Mass3; \\
Clear[\epsilon_1, \epsilon_2, \tau, \omega, \kappa_1, \kappa_2, X1, X2, N1, N2, M1, M2, N12, N21, M12, M21, LB1, LB2, LB3, Q1, Q2, u, v, Mass1, Mass2, Mass3, f1, f2, f3, f4, f5]; \\

u &= 0; \\
v &= V1; \\
w &= 0; \\
\eta &= \text{Sqrt}[1 + \xi^2]; \xi = \text{Sqrt}[1 + (1 + \lambda^2) \cdot \xi^2]; \\
X1 &= \text{Expand}[(1/(c\eta)) \cdot (D[w, \phi] + (\eta \cdot \zeta)) \cdot u]; \\
X2 &= \text{Expand}[(\lambda \eta)/(c\zeta) \cdot (D[w, \xi] - (\lambda/(\eta \cdot \zeta^2)) \cdot \nu)]; \\
\epsilon_1 &= (1/(c\eta)) \cdot (D[u, \phi] + ((\lambda \xi)/(\zeta \cdot \nu) - (1/(c\zeta)) w; \\

108
\end{align*}
\]
\[ \varepsilon_2 = (\lambda^*\eta)/(c^*\zeta^3) * D[v, \xi] + (\lambda^*\eta)/(c^*\zeta^3) * w; \]
\[ \omega = (1/(c^*\eta)) * (D[v, \varphi] - ((\lambda^*\eta)/(c^*\zeta)) * u) + ((\lambda^*\eta)/(c^*\zeta)) * D[u, \xi]; \]
\[ \kappa_1 = (-(1/(c^*\eta)) * (D[X1, \varphi]) + ((\lambda^*\eta)/(c^*\zeta)) * X2); \]
\[ \kappa_2 = (-(\lambda^*\eta)/(c^*\zeta)) * D[X2, \xi]; \]
\[ \tau = (-(1/(c^*\eta)) * (D[X2, \varphi]) - ((\lambda^*\eta)/(c^*\zeta)) * X1) - ((\lambda^*\eta)/(c^*\zeta)) * D[u, \xi]; \]
\[ f_1 = (-\rho) * h * u; \]
\[ f_2 = (-\rho) * h * v; \]
\[ f_3 = (-\rho) * h * w; \]
\[ f_4 = \rho * (h^3/12) * X1; \]
\[ f_5 = (-\rho) * (h^3/12) * X2; \]
\[ N_1 = \text{Expand}[C1 \ast (\varepsilon_1 + \varphi \varepsilon_2)]; \]
\[ N_2 = \text{Expand}[C1 \ast (\varepsilon_2 + \varphi \varepsilon_1)]; \]
\[ N_{12} = \text{Expand}[(C1/2) \ast (1 - \nu) \ast \omega + ((D1 \ast (1 - \nu) \ast \lambda^2) / (c^*\zeta^3)) \ast \tau]; \]
\[ N_{21} = \text{Expand}[(C1/2) \ast (1 - \nu) \ast \omega - ((D1 \ast (1 - \nu) / (c^*\zeta)) \ast \tau]; \]
\[ M_1 = \text{Expand}[D1 \ast (\kappa_1 + \nu \kappa_2)]; \]
\[ M_2 = \text{Expand}[D1 \ast (\kappa_2 + \nu \kappa_1)]; \]
\[ M_{12} = M_{21} = \text{Expand}[D1 \ast (1 - \nu) \ast \tau]; \]
\[ Q_1 = (1/(c^*\eta)) * (D[M1, \varphi] + ((\lambda^*\eta)/(c^*\zeta)) \ast D[\eta \ast M21, \xi]) + ((\lambda^*\eta)/(c^*\zeta)) \ast M_{12}; \]
\[ Q_2 = (1/(c^*\eta)) * (D[M12, \varphi] + ((\lambda^*\eta)/(c^*\zeta)) \ast D[\eta \ast M2, \xi]) - ((\lambda^*\eta)/(c^*\zeta)) \ast M_{12}; \]
\[ L_{B1} = D[N1, \varphi] + ((\lambda^*\eta)/(c^*\zeta)) \ast D[\eta \ast N21, \xi] + ((\lambda^*\eta)/(c^*\zeta)) \ast N_{12} - (\eta \ast \zeta) \ast Q_1; \]
\[ L_{B2} = D[N12, \varphi] + ((\lambda^*\eta)/(c^*\zeta)) \ast D[\eta \ast N2, \xi] - ((\lambda^*\eta)/(c^*\zeta)) \ast N_1 + ((\lambda^2 \ast \eta)/(c^*\zeta^3)) \ast Q_2; \]
\[ L_{B3} = D[Q1, \varphi] + ((\lambda^*\eta)/(c^*\zeta)) \ast D[\eta \ast Q2, \xi] + (\eta \ast \zeta) \ast N_1 - ((\lambda^2 \ast \eta)/(c^*\zeta^3)) \ast N_2; \]
\[ \text{Mass}_1 = (-\eta \ast \zeta) \ast f_4 + c \ast \eta \ast f_1; \]
\[ \text{Mass}_2 = ((\lambda^2 \ast \eta)/(c^*\zeta^3)) \ast f_5 + c \ast \eta \ast f_2; \]
\[ \text{Mass}_3 = D[f4, \varphi] + ((\lambda^*\eta)/(c^*\zeta)) \ast D[\eta \ast f5, \xi] + c \ast \eta \ast f_3; \]
\[ L_{12} = L_{B1}; L_{22} = L_{B2}; L_{32} = L_{B3}; \text{Mass}_1 = \text{Mass}_1; \text{Mass}_2 = \text{Mass}_2; \]
\[ \text{Mass}_3 = \text{Mass}_3; \]
\[ \text{Clear}[\varepsilon_1, \varepsilon_2, \tau, \omega, \kappa_1, \kappa_2, X1, X2, N1, N2, M1, M2, N12, N21, M12, M21, LB1, LB2, LB3, Q1, Q2, u, v, w, Mass1, Mass2, Mass3, f1, f2, f3, f4, f5]; \]
\[ u = 0; \]
\[ v = 0; \]
\[ w = W1; \]
\[ \eta = \text{Sqrt}[1 + \xi^2]; \]
\[ \zeta = \text{Sqrt}[1 + (1 + \lambda^2) \ast \zeta^2]; \]
\[ X1 = \text{Expand}[(1/(c^*\eta)) * (D[w, \varphi] + (\eta \ast \zeta) \ast u)]; \]
\[ X2 = \text{Expand}[(\lambda^*\eta)/(c^*\zeta)) \ast (D[w, \xi] - (\lambda/(\eta \ast \zeta^2)) \ast \nu)]; \]
\[ \varepsilon_1 = (1/(c^*\eta)) \ast (D[u, \varphi] + ((\lambda^*\eta)/(c^*\zeta) \ast \nu) - (1/(c^*\zeta)) \ast w; \]
\[ \varepsilon_2 = ((\lambda^*\eta)/(c^*\zeta)) \ast D[v, \xi] + (\lambda^2/(c^*\zeta^3)) \ast w; \]
$$\omega = \left(1/(c^2\eta)\right)\left(D[\nu, \phi] - (\lambda*\xi)/(\zeta*u) + (\lambda*\eta)/(c^2\xi)\right)D[u, \xi];$$
$$\kappa_1 = \left(-1/(c^2\eta)\right)\left(D[X1, \phi] + (\lambda*\xi)/(\zeta*X2)\right);$$
$$\kappa_2 = \left(-\left((\lambda*\eta)/(c^2\xi)\right)\right)\left(D[X2, \xi] \right);$$
$$\tau = \text{Expand}\left(-\left(1/(c^2\eta)\right)\left(D[X2, \phi] - (\lambda*\xi)/(\zeta*X1) - (\lambda*\eta)/(c^2\xi^2)\right)\left(D[u, \xi] \right);\right)$$

$$f_1 = (-\rho)*h*u;$$
$$f_2 = (-\rho)*h*v;$$
$$f_3 = (-\rho)*h*w;$$
$$f_4 = \rho*(h^3/12)*X1;$$
$$f_5 = (-\rho)*(h^3/12)*X2;$$

$$N_1 = \text{Expand}\left[C1*(\epsilon_1 + \nu*\epsilon_2)\right];$$
$$N_2 = \text{Expand}\left[C1*(\epsilon_2 + \nu*\epsilon_1)\right];$$
$$N_{12} = \text{Expand}\left[(C1/2)*(1 - \nu)*\omega + (D1*(1 - \nu)*\lambda^2)/(c^2\zeta^3)\right]$$

$$Q_1 = \left(1/(c^2\eta)\right)\left(D[M1, \phi] + (\lambda*\eta)/(\zeta)\right)\left(D[\eta*M21, \xi] + (\lambda*\xi)/(\zeta)\right)M12;$$

$$Q_2 = \text{Expand}\left[\left(1/(c^2\eta)\right)\left(D[M12, \phi] + (\lambda*\eta)/(\zeta)\right)D[\eta*M21, \xi] - (\lambda*\xi)/(\zeta)\right]M1;$$

$$L_{B1} = \text{Expand}\left[D[N1, \phi] + (\lambda*\eta)/(\zeta)\right)\left(D[\eta*N21, \xi] + (\lambda*\xi)/(\zeta)\right)N12 - (\eta)/(\zeta)Q1;$$

$$L_{B2} = \text{Expand}\left[D[N12, \phi] + (\lambda*\eta)/(\zeta)\right)\left(D[\eta*N2, \xi] - (\lambda*\xi)/(\zeta)\right)N1 + (\lambda^2*\eta)/(\zeta^3)Q2;$$

$$L_{B3} = \text{Expand}\left[D[Q1, \phi] + (\lambda*\eta)/(\zeta)\right)\left(D[\eta*Q2, \xi] - (\eta)/(\zeta)\right)N1 - (\lambda^2*\eta)/(\zeta^3)N2;$$

$$\text{Mass}_1 = (-\eta)/(\zeta)Q4 + c^*\eta*f1;$$

$$\text{Mass}_2 = (\lambda^2*\eta)/(\zeta^3)Q5 + c^*\eta*f2;$$

$$\text{Mass}_3 = D[f4, \phi] + (\lambda*\eta)/(\zeta)\left(D[\eta*Q5, \xi] + c^*\eta*f3; \right.$$}

$$L_{13} = L_{B1};$$
$$L_{23} = L_{B2};$$
$$L_{33} = L_{B3};$$
$$L_{33} = \text{Mass}_1;$$
$$\text{Mass}_2 = \text{Mass}_2;$$

$$\text{Mass}_3 = \text{Mass}_3;$$

$$\text{Clear}[\epsilon_1, \epsilon_2, \tau, \omega, \kappa_1, \kappa_2, X1, X2, N1, N2, M1, M2, N12, N21, M12, M21, LB1, LB2, LB3, Q1, Q2, u, v, w, Mass1, Mass2, Mass3, f_1, f_2, f_3, f_4, f_5];$$

$$V = V1;$$
$$U = U1;$$
$$W = W1;$$
$$v = V;$$
$$\Psi = \Psi;$$
$$\Psi = \Psi;$$

$$\text{MN1} = (-w^2)\text{NIntegrate}\left[\text{Integrate}\left[\text{Mass1*Psi}, \{\phi, 0, 2*Pi\}\right], \{\xi, \text{Intmin}, \text{Intmax}\}\right];$$
MN12 = (-\(w^2\)) \[\text{NIntegrate} \int_{\phi}^{2\pi} \text{Mass12*}\Psi_{u}, \{\phi, 0, 2*\pi}\]\[\{\xi, \text{Intmin, Intmax}\}\];
MN13 = (-\(w^2\)) \[\text{NIntegrate} \int_{\phi}^{2\pi} \text{Mass13*}\Psi_{u}, \{\phi, 0, 2*\pi}\]\[\{\xi, \text{Intmin, Intmax}\}\];
MN21 = (-\(w^2\)) \[\text{NIntegrate} \int_{\phi}^{2\pi} \text{Mass21*}\Psi_{v}, \{\phi, 0, 2*\pi}\]\[\{\xi, \text{Intmin, Intmax}\}\];
MN22 = (-\(w^2\)) \[\text{NIntegrate} \int_{\phi}^{2\pi} \text{Mass22*}\Psi_{v}, \{\phi, 0, 2*\pi}\]\[\{\xi, \text{Intmin, Intmax}\}\];
MN23 = (-\(w^2\)) \[\text{NIntegrate} \int_{\phi}^{2\pi} \text{Mass23*}\Psi_{w}, \{\phi, 0, 2*\pi}\]\[\{\xi, \text{Intmin, Intmax}\}\];
MN31 = (-\(w^2\)) \[\text{NIntegrate} \int_{\phi}^{2\pi} \text{Mass31*}\Psi_{w}, \{\phi, 0, 2*\pi}\]\[\{\xi, \text{Intmin, Intmax}\}\];
MN32 = (-\(w^2\)) \[\text{NIntegrate} \int_{\phi}^{2\pi} \text{Mass32*}\Psi_{w}, \{\phi, 0, 2*\pi}\]\[\{\xi, \text{Intmin, Intmax}\}\];
MN33 = (-\(w^2\)) \[\text{NIntegrate} \int_{\phi}^{2\pi} \text{Mass33*}\Psi_{w}, \{\phi, 0, 2*\pi}\]\[\{\xi, \text{Intmin, Intmax}\}\];
MN = \{\{\text{MN11, MN12, MN23}\}, \{\text{MN21, MN22, MN23}\}, \{\text{MN31, MN32, MN33}\}\}

LN11 = \[\text{NIntegrate} \int_{\phi}^{2\pi} \text{L11*}\Psi_{u}, \{\phi, 0, 2*\pi}\]\[\{\xi, \text{Intmin, Intmax}\}\];
LN12 = \[\text{NIntegrate} \int_{\phi}^{2\pi} \text{L12*}\Psi_{u}, \{\phi, 0, 2*\pi}\]\[\{\xi, \text{Intmin, Intmax}\}\];
LN13 = \[\text{NIntegrate} \int_{\phi}^{2\pi} \text{L13*}\Psi_{u}, \{\phi, 0, 2*\pi}\]\[\{\xi, \text{Intmin, Intmax}\}\];
LN21 = \[\text{NIntegrate} \int_{\phi}^{2\pi} \text{L21*}\Psi_{v}, \{\phi, 0, 2*\pi}\]\[\{\xi, \text{Intmin, Intmax}\}\];
LN22 = \[\text{NIntegrate} \int_{\phi}^{2\pi} \text{L22*}\Psi_{v}, \{\phi, 0, 2*\pi}\]\[\{\xi, \text{Intmin, Intmax}\}\];
LN23 = \[\text{NIntegrate} \int_{\phi}^{2\pi} \text{L23*}\Psi_{w}, \{\phi, 0, 2*\pi}\]\[\{\xi, \text{Intmin, Intmax}\}\];
LN31 = \[\text{NIntegrate} \int_{\phi}^{2\pi} \text{L31*}\Psi_{w}, \{\phi, 0, 2*\pi}\]\[\{\xi, \text{Intmin, Intmax}\}\];
LN32 = \[\text{NIntegrate} \int_{\phi}^{2\pi} \text{L32*}\Psi_{w}, \{\phi, 0, 2*\pi}\]\[\{\xi, \text{Intmin, Intmax}\}\];
LN33 = \[\text{NIntegrate} \int_{\phi}^{2\pi} \text{L33*}\Psi_{w}, \{\phi, 0, 2*\pi}\]\[\{\xi, \text{Intmin, Intmax}\}\];
LN = \{\{\text{LN11, LN12, LN13}\}, \{\text{LN21, LN22, LN23}\}, \{\text{LN31, LN32, LN33}\}\}

A = \{\{\text{MN11 + LN11, MN12 + LN12, MN13 + LN13}\}, \{\text{MN21 + LN21, MN22 + LN22, MN23 + LN23}\}, \{\text{MN31 + LN31, MN32 + MN32, MN33 + LN33}\}\}

\text{Adet} = \text{Det}[A]
\text{wn} = \text{Solve}[\text{Adet} == 0, w]
\text{fn} = \text{wn}/(2*\pi)

\text{Clear[Rmax, Lmax, Zmax, h, E1, v, \rho, L, z, \eta, \lambda, \xi, C1, D1, c, d, \text{Intmin, Intmax, R0, dR, Angle, m, a, V1, U1, W1, \alpha1, \beta1, \PsiV, \PsiW, mm, mn, u, v, w, \PsiV, \PsiW, \Psiu, \xi, Shape, L11, L12, L13, L21, L22, L23, L31, L32, L33, e1, e2, \tau, \omega, k1, k2, X1, X2, N1, N2, M1, M2, N12, N21, M12, M21, LB1, LB2, LB3, Q1, Q2, LN, LN11, LN12, LN13, LN21, LN22, LN23, LN31, LN32, LN33, MN, MN11, MN12, MN23, MN21, MN22, M23, MN31, MN32, MN33, Mass11, Mass12, Mass13, Mass21, Mass22, Mass23, Mass31, Mass32, Mass33, Mass1, Mass2, Mass3\]};}
(*---FixedFixedFrequencies.nb---*)

Rmax = 3;
Lmax = 6;
Zmax = Lmax/2;
h = 0.001;
E1 = 100000000;
v = 0.33;
\rho = 0.000255;
L = Lmax;
\eta = Sqrt[1 + \xi^2];
\lambda = c/d;
\zeta = Sqrt[1 + (1 + \lambda^2)*\xi^2];
Cl = (E1*h)/(1 - v^2);
D1 = 0;
c = 1.00;

d = (Zmax*c)/Sqrt[Rmax^2 - c^2];
Intmin = -(Zmax/d); Intmax = Zmax/d;
dR = \lambda*D[\eta, \xi];
Angle = ArcTan[dR];
n = 1;
\beta = 4.730041;
\alpha = (-Sin[\beta1] + Sinh[\beta1])/(-Cosh[\beta1] + Cos[\beta1]);
Shape = -Sin[(\beta1/L)*(\xi*d + L/2)] + Sinh[(\beta1/L)*(\xi*d + L/2)] +
\alpha1*(-Cos[(\beta1/L)*(\xi*d + L/2)] + Cosh[(\beta1/L)*(\xi*d + L/2)]);

V1 = D[Shape, \xi]*Cos[n*\phi];
U1 = Shape*Sin[n*\phi];
W1 = Shape*Cos[n*\phi];

u = U1;
v = 0;
w = 0;
\eta = Sqrt[1 + \xi^2]; \zeta = Sqrt[1 + (1 + \lambda^2)*\xi^2];
X1 = (1/(c*\eta))*(D[w, \varphi] + (\eta/\zeta)*u);
X2 = ((\lambda*\eta)/(c*\zeta))*D[w, \xi] - (\lambda/(\eta*\xi^2))*v);
\epsilon1 = (1/(c*\eta))*(D[u, \varphi] + ((\lambda*\xi)/\zeta)*v) - (1/(c*\zeta))*w;
\epsilon2 = ((\lambda*\eta)/(c*\zeta))*D[v, \xi] + (\lambda^2/(c*\xi^3))*w;
\[
\begin{align*}
\omega &= \frac{1}{(c^*\eta)} \cdot (D[v, \varphi] - ((\lambda*\xi)/\zeta) \cdot u) + ((\lambda*\eta)/(c^*\zeta)) \cdot D[u, \xi]; \\
\kappa_1 &= -\left(\frac{1}{(c^*\eta)}\right) \cdot (D[X1, \varphi] + ((\lambda*\xi)/\zeta) \cdot X2); \\
\kappa_2 &= -\left((\lambda*\eta)/(c^*\zeta)\right) \cdot D[X2, \xi]; \\
\tau &= \text{Expand} \left( -\left(\frac{1}{(c^*\eta)}\right) \cdot (D[X2, \varphi] - ((\lambda*\xi)/\zeta) \cdot X1) - ((\lambda*\eta)/(c^2*\zeta^2)) \cdot D[u, \xi] \right); \\
f_1 &= (-\rho) \cdot h^* u; \\
f_2 &= (-\rho) \cdot h^* v; \\
f_3 &= (-\rho) \cdot h^* w; \\
f_4 &= \rho \cdot (h^* 3/12) \cdot X1; \\
f_5 &= (-\rho) \cdot (h^* 3/12) \cdot X2; \\
N_1 &= \text{Expand} \left[ (C1^* (\epsilon_1 + v^* \epsilon_2)) \right]; \\
N_2 &= \text{Expand} \left[ (C1^* (\epsilon_2 + v^* \epsilon_1)) \right]; \\
N_{12} &= \text{Expand} \left[ (C1/2)^*(1 - v^*) \cdot \omega + ((D1^*(1 - v^*) \cdot (c^*\zeta^3))^* \tau) \right]; \\
N_{21} &= \text{Expand} \left[ (C1/2)^*(1 - v^*) \cdot \omega - ((D1^*(1 - v^*))^* \tau) \right]; \\
M_1 &= \text{Expand} \left[ (D1^* (\kappa_1 + v^* \kappa_2)) \right]; \\
M_2 &= \text{Expand} \left[ (D1^* (\kappa_2 + v^* \kappa_1)) \right]; \\
M_{12} &= \text{Expand} \left[ (D1^* (1 - v^*) \cdot \tau) \right]; \\
Q_1 &= \left(\frac{1}{(c^*\eta)}\right) \cdot (D[M1, \varphi] + ((\lambda*\eta)/\zeta) \cdot D[\eta \cdot M21, \xi] + ((\lambda*\xi)/\zeta) \cdot M12); \\
Q_2 &= \text{Expand} \left[ (1/(c^*\eta)) \cdot (D[M12, \varphi] + ((\lambda*\eta)/\zeta) \cdot D[\eta \cdot M2, \xi] - ((\lambda*\xi)/\zeta) \cdot M1) \right]; \\
L_{11} &= \text{Expand} \left[ (D[N1, \varphi] + ((\lambda*\eta)/\zeta) \cdot D[\eta \cdot N21, \xi] + ((\lambda*\xi)/\zeta) \cdot N12 - (\eta/\zeta) \cdot Q1) \right]; \\
L_{21} &= \text{Expand} \left[ (D[N12, \varphi] + ((\lambda*\eta)/\zeta) \cdot D[\eta \cdot N2, \xi] - ((\lambda*\xi)/\zeta) \cdot N1 + ((\lambda^2*\eta)/\zeta^3) \cdot Q2) \right]; \\
L_{31} &= \text{Expand} \left[ (D[L1, \varphi] + ((\lambda*\eta)/\zeta) \cdot D[\eta \cdot Q2, \xi] + (\eta/\zeta) \cdot N1 - ((\lambda^2*\eta)/\zeta^3) \cdot N2) \right]; \\
\text{Mass}_1 &= (\eta/\zeta) \cdot f_4 + c^* \eta \cdot f_1; \\
\text{Mass}_2 &= ((\lambda^2*\eta)/\zeta^3) \cdot f_5 + c^* \eta \cdot f_2; \\
\text{Mass}_3 &= D[f4, \varphi] + ((\lambda*\eta)/\zeta) \cdot D[\eta \cdot f5, \xi] + c^* \eta \cdot f_3; \\
L_{11} &= L_{12}; \\
L_{21} &= L_{22}; \\
L_{31} &= L_{32}; \\
L_{12} &= L_{21}; \\
\text{Mass} &= \text{Mass}_1; \\
\text{Mass} &= \text{Mass}_2; \\
\text{Mass} &= \text{Mass}_3; \\
\text{Clear} &= \epsilon_1, \epsilon_2, \tau, \omega, \kappa_1, \kappa_2, X1, X2, N1, N2, M1, M2, N12, N21, M12, M21, LB1, LB2, LB3, Q1, Q2, u, v, Mass1, Mass2, Mass3, f1, f2, f3, f4, f5]; \\
u &= 0; \\
v &= V1; \\
w &= 0; \\
\eta &= \text{Sqrt}[1 + \xi^2]; \\
\zeta &= \text{Sqrt}[1 + (1 + \lambda^2) \cdot \xi^2]; \\
X_1 &= \text{Expand} \left[ (1/(c^*\eta)) \cdot (D[w, \varphi] + (\eta/\zeta) \cdot u) \right]; \\
X_2 &= \text{Expand} \left[ (\lambda*\eta)/(c^*\zeta) \cdot (D[w, \xi] - (\lambda/(\eta \cdot \xi^2))^* v) \right]; \\
\epsilon_1 &= (1/(c^*\eta)) \cdot (D[u, \varphi] + ((\lambda*\xi)/\zeta) \cdot v) - (1/(c^*\zeta)) \cdot w; \\
\epsilon_2 &= ((\lambda*\eta)/(c^*\zeta)) \cdot D[v, \xi] + (\lambda^2/(c^*\zeta^3)) \cdot w; \\
\omega &= (1/(c^*\eta)) \cdot (D[v, \varphi] - ((\lambda*\xi)/\zeta) \cdot u + ((\lambda*\eta)/(c^*\zeta)) \cdot D[u, \xi] \right); \\
\kappa_1 &= (((1/(c^*\eta))) \cdot (D[X1, \varphi] + ((\lambda*\xi)/\zeta) \cdot X2); \\
\kappa_2 &= (((\lambda*\eta)/(c^*\zeta)) \cdot D[X2, \xi]; \\
\tau &= (((1/(c^*\eta))) \cdot (D[X2, \varphi] - ((\lambda*\xi)/\zeta) \cdot X1) - ((\lambda*\eta)/(c^2*\zeta^2)) \cdot D[u, \xi] \right); \\
f_1 &= (-\rho) \cdot h^* u; \\
f_2 &= (-\rho) \cdot h^* v; \\
f_3 &= (-\rho) \cdot h^* w; \\
f_4 &= \rho \cdot (h^* 3/12) \cdot X1; \\
\end{align*}
\]
\[f_5 = (-\rho)*(h^{3/12})*X_2;\]
\[N_1 = \text{Expand}[C_1*(\epsilon_1 + v^*\epsilon_2)];\]
\[N_2 = \text{Expand}[C_1*(\epsilon_2 + v^*\epsilon_1)];\]
\[N_{12} = \text{Expand}[(C_1/2)*(1 - v)^*\omega + ((D_1*(1 - v)^*\lambda^2)/(c^*\zeta^3))^*\tau];\]
\[N_{21} = \text{Expand}[(C_1/2)*(1 - v)^*\omega - ((D_1*(1 - v))/(c^*\zeta))^*\tau];\]
\[M_1 = \text{Expand}[D_1*(\kappa_1 + v^*\kappa_2)];\]
\[M_2 = \text{Expand}[D_1*(\kappa_2 + v^*\kappa_1)];\]
\[M_{12} = M_{21} = \text{Expand}[D_1*(1 - v)^*\tau];\]
\[Q_1 = (1/(c^*\zeta))*(D[w, \varphi] + ((\lambda^*\eta)/(c^*\zeta))*D[\eta]*M_{21}, \xi] + ((\lambda^*\xi)/(c^*\zeta))*M_{12});\]
\[Q_2 = (1/(c^*\zeta))*(D[w, \varphi] + ((\lambda^*\eta)/(c^*\zeta))*D[\eta]*M_{21}, \xi] - ((\lambda^*\xi)/(c^*\zeta))*M_{12};\]
\[L_{12} = L_{21} = L_{22}; L_{32} = L_{33} = \text{Mass}_{12} = \text{Mass}_{11}; \text{Mass}_{22} = \text{Mass}_{22};\]
\[\text{Mass}_{32} = \text{Mass}_{33};\]
\[\text{Clear}[\epsilon_1, \epsilon_2, \tau, \omega, \kappa_1, \kappa_2, X_1, X_2, N_1, N_2, M_1, M_2, N_{12}, N_{21}, M_{12}, M_{21}, L_{11}, L_{22}, L_{33}, Q_{11}, Q_{22}, u, v, w, \text{Mass}_{11}, \text{Mass}_{22}, \text{Mass}_{33}, f_1, f_2, f_3, f_4, f_5];\]
\[u = 0;\]
\[v = 0;\]
\[w = W_1;\]
\[\eta = \text{Sqrt}[1 + \xi^2]; \zeta = \text{Sqrt}[1 + (1 + \lambda^2)*\xi^2];\]
\[X_1 = \text{Expand}[(1/(c^*\zeta))*(D[w, \varphi] + (\eta^*\zeta))^*u)];\]
\[L_{22} = \text{Expand}[(\lambda^*\eta)/(c^*\zeta))^*D[w, \varphi] - (\lambda^*\xi)/(c^*\zeta))^*u)];\]
\[\epsilon_2 = (1/(c^*\zeta))^*D[\lambda^*\eta]/(c^*\zeta)^*v] - (1/(c^*\zeta))^*w];\]
\[\epsilon_2 = ((\lambda^*\eta)/(c^*\zeta))^*D[\lambda^*\eta]/(c^*\zeta)^*v] + ((\lambda^*\xi)/(c^*\zeta))^*w];\]
\[\omega = (1/(c^*\zeta))^*D[\lambda^*\eta]/(c^*\zeta)^*u] + ((\lambda^*\eta)/(c^*\zeta))^*D[\lambda^*\eta]/(c^*\zeta)^*u];\]
\[\kappa_2 = ((\lambda^*\eta)/(c^*\zeta))^*D[X_1, \varphi] + ((\lambda^*\xi)/X^2);\]
\[\tau = \text{Expand}[(1/(c^*\zeta))^*D[X_2, \varphi] - ((\lambda^*\eta)/(c^*\zeta))^*X_1] - ((\lambda^*\eta)/(c^*\zeta))^*D[u, \xi]];\]
\[f_1 = (-\rho)*h^*u;\]
\[f_2 = (-\rho)*h^*v;\]
\[f_3 = (-\rho)*h^*w;\]
\[f_4 = \rho^*(h^{3/12})*X_1;\]
\[f_5 = (-\rho)^*(h^{3/12})*X_2;\]
\[N_1 = \text{Expand}[C_1*(\epsilon_1 + v^*\epsilon_2)];\]
\[N_2 = \text{Expand}[C_1*(\epsilon_2 + v^*\epsilon_1)];\]
\[N_{12} = \text{Expand}[(C_1/2)*(1 - v)^*\omega + ((D_1*(1 - v)^*\lambda^2)/(c^*\zeta^3))^*\tau];\]
\[N_{21} = \text{Expand}[(C_1/2)*(1 - v)^*\omega - ((D_1*(1 - v))/(c^*\zeta))^*\tau];\]
\[M_1 = \text{Expand}[D_1*(\kappa_1 + v^*\kappa_2)];\]
\[M_2 = \text{Expand}[D_1*(\kappa_2 + v^*\kappa_1)];\]
\[M_{12} = M_{21} = \text{Expand}[D_1*(1 - v)^*\tau];\]
\[
Q_1 = \left(1/(\eta^2)\right) \cdot (D[M1, \varphi] + (\lambda^2 \eta / \xi) \cdot (D[\eta^2 \cdot M21, \xi] + ((\lambda^2 \eta^2) / \xi^2) \cdot M21);
\]
\[
Q_2 = \text{Expand}\left[\left(1/(\eta^2)\right) \cdot (D[M12, \varphi] + (\lambda^2 \eta / \xi) \cdot (D[\eta^2 \cdot M12, \xi] - ((\lambda^2 \eta^2) / \xi^2) \cdot M12)\right];
\]
\[
L_{B1} = \text{Expand}\left[\left(1/(\eta^2)\right) \cdot (D[Q1, \varphi] + (\lambda^2 \eta / \xi) \cdot (D[Q1, M21, \xi] + ((\lambda^2 \eta^2) / \xi^2) \cdot Q2)\right];
\]
\[
L_{B3} = \text{Expand}\left[\left(1/(\eta^2)\right) \cdot (D[Q1, \varphi] + (\lambda^2 \eta / \xi) \cdot (D[Q1, Q2, \xi] + ((\lambda^2 \eta^2) / \xi^2) \cdot N1)\right] - ((\lambda^2 \eta^2) / \xi^2) \cdot N2;\]
\]
\[
\text{Mass}_1 = -((\lambda^2 \eta^2) / \xi^2) \cdot f4 + c^2 \cdot \eta^2 \cdot f1;\]
\]
\[
\text{Mass}_2 = ((\lambda^2 \eta^2) / \xi^2) \cdot f5 + c^2 \cdot \eta^2 \cdot f2;\]
\]
\[
\text{Mass}_3 = D[f4, \varphi] + ((\lambda^2 \eta^2) / \xi^2) \cdot (D[\eta^2 \cdot f5, \xi] + c^2 \cdot \eta^2 \cdot f3);\]
\]
\[
L_{13} = L_{B1};\]
\]
\[
L_{23} = L_{B2};\]
\]
\[
L_{33} = L_{B3};\]
\]
\[
\text{Clear}[\phi, \psi, \tau, \omega, \kappa_1, \kappa_2, X1, X2, N1, N2, M1, M2, N12, N21, M12, M21, L1, L2, L3, Q1, Q2, u, v, w, Mass1, Mass2, Mass3, f1, f2, f3, f4, f5];
\]
\[
\Psi_V = V1;\]
\[
\Psi_U = U1;\]
\]
\[
\Psi_W = W1;\]
\]
\[
\Psi_v = \Psi_V;\]
\]
\[
\Psi_w = \Psi_W;\]
\]
\[
\Psi_u = \Psi_U;\]
\]
\[
MN_{11} = (-w^2) \cdot N\text{Integrate}[\text{Integrate}[\text{Mass11} \cdot \Psi_u, \{\varphi, 0, 2 \cdot Pi\}], \{\xi, \text{Intmin}, \text{Intmax}\}];\]
\]
\[
MN_{12} = (-w^2) \cdot N\text{Integrate}[\text{Integrate}[\text{Mass12} \cdot \Psi_u, \{\varphi, 0, 2 \cdot Pi\}], \{\xi, \text{Intmin}, \text{Intmax}\}];\]
\]
\[
MN_{13} = (-w^2) \cdot N\text{Integrate}[\text{Integrate}[\text{Mass13} \cdot \Psi_u, \{\varphi, 0, 2 \cdot Pi\}], \{\xi, \text{Intmin}, \text{Intmax}\}];\]
\]
\[
MN_{21} = (-w^2) \cdot N\text{Integrate}[\text{Integrate}[\text{Mass21} \cdot \Psi_v, \{\varphi, 0, 2 \cdot Pi\}], \{\xi, \text{Intmin}, \text{Intmax}\}];\]
\]
\[
MN_{22} = (-w^2) \cdot N\text{Integrate}[\text{Integrate}[\text{Mass22} \cdot \Psi_v, \{\varphi, 0, 2 \cdot Pi\}], \{\xi, \text{Intmin}, \text{Intmax}\}];\]
\]
\[
MN_{23} = (-w^2) \cdot N\text{Integrate}[\text{Integrate}[\text{Mass23} \cdot \Psi_v, \{\varphi, 0, 2 \cdot Pi\}], \{\xi, \text{Intmin}, \text{Intmax}\}];\]
\]
\[
MN_{31} = (-w^2) \cdot N\text{Integrate}[\text{Integrate}[\text{Mass31} \cdot \Psi_w, \{\varphi, 0, 2 \cdot Pi\}], \{\xi, \text{Intmin}, \text{Intmax}\}];\]
\]
\[
MN_{32} = (-w^2) \cdot N\text{Integrate}[\text{Integrate}[\text{Mass32} \cdot \Psi_w, \{\varphi, 0, 2 \cdot Pi\}], \{\xi, \text{Intmin}, \text{Intmax}\}];\]
\]
\[
MN_{33} = (-w^2) \cdot N\text{Integrate}[\text{Integrate}[\text{Mass33} \cdot \Psi_w, \{\varphi, 0, 2 \cdot Pi\}], \{\xi, \text{Intmin}, \text{Intmax}\}];\]
\]
\[
MN = \{\{MN_{11}, MN_{12}, MN_{23}\}, \{MN_{21}, MN_{22}, MN_{23}\}, \{MN_{31}, MN_{32}, MN_{33}\}\};\]
\]
\[
LN_{11} = N\text{Integrate}[\text{Integrate}[L11 \cdot \Psi_u, \{\varphi, 0, 2 \cdot Pi\}], \{\xi, \text{Intmin}, \text{Intmax}\}];\]
\]
\[
LN_{12} = N\text{Integrate}[\text{Integrate}[L12 \cdot \Psi_u, \{\varphi, 0, 2 \cdot Pi\}], \{\xi, \text{Intmin}, \text{Intmax}\}];\]
\]
\[
LN_{13} = N\text{Integrate}[\text{Integrate}[L13 \cdot \Psi_u, \{\varphi, 0, 2 \cdot Pi\}], \{\xi, \text{Intmin}, \text{Intmax}\}];\]
\]
\[
LN_{21} = N\text{Integrate}[\text{Integrate}[L21 \cdot \Psi_v, \{\varphi, 0, 2 \cdot Pi\}], \{\xi, \text{Intmin}, \text{Intmax}\}];\]
\]
\[
LN_{22} = N\text{Integrate}[\text{Integrate}[L22 \cdot \Psi_v, \{\varphi, 0, 2 \cdot Pi\}], \{\xi, \text{Intmin}, \text{Intmax}\}];\]
\]
\[
LN_{23} = N\text{Integrate}[\text{Integrate}[L23 \cdot \Psi_v, \{\varphi, 0, 2 \cdot Pi\}], \{\xi, \text{Intmin}, \text{Intmax}\}];\]
\]
\[
LN_{31} = N\text{Integrate}[\text{Integrate}[L31 \cdot \Psi_w, \{\varphi, 0, 2 \cdot Pi\}], \{\xi, \text{Intmin}, \text{Intmax}\}];\]
\]
\[
LN_{32} = N\text{Integrate}[\text{Integrate}[L32 \cdot \Psi_w, \{\varphi, 0, 2 \cdot Pi\}], \{\xi, \text{Intmin}, \text{Intmax}\}];\]
\]
\[
LN_{33} = N\text{Integrate}[\text{Integrate}[L33 \cdot \Psi_w, \{\varphi, 0, 2 \cdot Pi\}], \{\xi, \text{Intmin}, \text{Intmax}\}];\]
\]
\[
LN = \{\{LN_{11}, LN_{12}, LN_{13}\}, \{LN_{21}, LN_{22}, LN_{23}\}, \{LN_{31}, LN_{32}, LN_{33}\}\};\]
A = \{\{MN11 + LN11, MN12 + LN12, MN13 + LN13\}, \{MN21 + LN21, MN22 + LN22, MN23 + MN23\}, \{MN31 + LN31, MN32 + MN32, MN33 + LN33\}\}

\text{Adet} = \text{Det}[A]
\text{wn} = \text{Solve}[\text{Adet} == 0, w]
\text{fn} = \text{wn}/(2*\text{Pi})

\text{Clear}[R\text{max}, \text{Lmax}, \text{Zmax}, h, E1, v, \rho, L, \eta, \lambda, \zeta, C1, D1, c, d, \text{Intmin}, \text{Intmax}, R0, dR, \text{Angle}, m, a, V1, U1, W1, \alpha1, \beta1, \PsiV, \PsiW, mm, nn, u, v, w, \PsiV, \PsiW, \Psiu, \xi, \text{Shape}, L11, L12, L13, L21, L22, L23, L31, L32, L33, \epsilon1, \epsilon2, \tau, \omega, \kappa1, \kappa2, X1, X2, N1, N2, M1, M2, N12, N21, M12, M21, LB1, LB2, LB3, Q1, Q2, LN, LN11, LN12, LN13, LN21, LN22, LN23, LN31, LN32, LN33, MN, MN11, MN12, MN23, MN21, MN22, M23, MN31, MN32, MN33, Mass11, Mass12, Mass13, Mass21, Mass22, Mass23, Mass31, Mass32, Mass33, Mass1, Mass2, Mass3\];
APPENDIX G: EXAMPLES USING ASSUMED MODE SHAPES

Two short examples of using the Assumed Mode Method are shown below. They were done to gain further understanding of the basic method, leading up to the use of the Galerkin Method outlined in the Thesis. The method was first applied to find the relationship for the frequency of a laterally vibrating, simply supported beam shown in Figure G.1.

![Simply Supported Beam](image)

Figure G.1: Simply Supported Beam.

The equation of a laterally vibrating beam can be shown to be:

\[ \rho A u^{\cdot\cdot} + EI \frac{d^4 u}{dx^4} = 0, \]  

(G.1)

where:

\( \rho = \) density, \( A = \) cross sectional area, \( I = \) moment of inertia, 
\( E = \) modulus of elasticity.

For a simply supported beam the boundary conditions are:

\[ u(0) = 0, \quad \text{and} \quad u(L) = 0. \]
The chosen assumed mode shape must satisfy the boundary conditions. The assumed mode shape chosen for the simply supported beam is:

\[ u = \sum_{m=1}^{\infty} U_m e^{im\alpha} \sin \left( \frac{m\pi x}{L} \right) \]  

(G.2)

Plugging in the assumed mode shape into the equation of the beam yields the following relationship for the natural frequency:

\[ w_m = \frac{m^2 \pi^2}{L^2} \sqrt{\frac{EI}{\rho L}} \]  

(G.3)

This expression matches the derived expression given in Soedel [18], for the natural frequency of a simply supported beam.
Another example of the use of the assumed mode method was applied to a simply supported rectangular plate. The plate is made of 6061-T6 Aluminum and its dimensions are shown in Figure G.2.

![Simply Supported Plate](image)

Figure G.2: Simply Supported Plate.

The equation of a laterally vibrating rectangular plate can be shown to be:

\[
D \left[ \frac{\partial^4 u_3}{\partial x^4} + 2 \frac{\partial^4 u_3}{\partial x^2 \partial y^2} + \frac{\partial^4 u_3}{\partial y^4} \right] + \rho \ddot{u} = 0; \tag{G.4}
\]

Where:

\[
D = \frac{Eh^3}{12(1-\nu^2)}.
\]

For a simply supported rectangular plate the boundary conditions are:

\[
u_3 (x = 0, a) = 0, \quad \text{and} \quad u_3 (y = 0, b) = 0.
\]
The chosen assumed mode shape must again satisfy the boundary conditions. The assumed mode shape chosen for the simply supported rectangular beam is:

$$u = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} U_n e^{ix} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right)$$

(G.5)

Plugging in the assumed mode shape into the equation of the plate yields the following expression for the natural frequency:

$$\omega_{mn} = \pi^2 \sqrt{\frac{D}{\rho h} \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)}$$

(G.6)

This expression matches the derived expression given by Soedel [18], for a simply supported rectangular plate.