Vibration of hollow cylindrical shells with partial constrained layer damping

Nathan E. Smith
VIBRATION OF HOLLOW CYLINDRICAL SHELLS WITH PARTIAL CONSTRAINED LAYER DAMPING

by

Nathan E. Smith

A thesis submitted in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE
IN
MECHANICAL ENGINEERING

Dr. Hany Ghoneim
Department of Mechanical Engineering

Dr. Stephen Boedo
Department of Mechanical Engineering

Dr. Josef Török
Department of Mechanical Engineering

Dr. Edward C. Hensel
Department of Mechanical Engineering

Hany Ghoneim
(Thesis advisor)

Stephen Boedo

J. S. Török

Edward Hensel

DEPARTMENT OF MECHANICAL ENGINEERING
ROCHESTER INSTITUTE OF TECHNOLOGY

8, 2004
Vibration of Hollow Cylindrical Shells with Partial Constrained Layer Damping

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VIBRATION OF HOLLOW CYLINDRICAL SHELLS WITH PARTIAL CONSTRAINED LAYER DAMPING

by Nathan E. Smith

Chairperson of the Supervisory Committee: Dr. Hany Ghoneim

Department of Mechanical Engineering

A theoretical analysis is presented for determining the natural frequencies and damping factors for isotropic circular cylindrical shell type structures. Equations of motion for a cylindrical shell and a three-layer cylindrical shell fully and partially treated, with a viscoelastic core, are derived from equilibrium. The assumed mode method or Galerkin method are used to find the equivalent mass and stiffness matrices from which the natural frequencies can then be obtained. The effects on the natural frequency and damping factor due to various viscoelastic core thicknesses, viscoelastic shear moduli, constraining layer thickness, constraining layer Young’s moduli, and viscoelastic coverage length are discussed. The results reveal for the cylinder studied, that an optimal coverage length exists for achieving maximum damping, along with an optimal viscoelastic shear modulus.
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ACKNOWLEDGMENTS

The author wishes to thank his thesis advisor, Dr. Hany Ghoneim, for all his help and knowledge. Also, the author extends his gratefulness to the committee members Stephen Boedo, and Josef Török for their help and comments.
Nomenclature

$E_1, E=E_3$ Young's modulus of constraining layer and shell

$G_2$ Shear Modulus of viscoelastic core

$h_1, h_2, h=h_3$ Thickness of constraining layer, viscoelastic layer, and shell

$R_1, R_2, R=R_3$ Mean radius constraining layer, viscoelastic layer, and shell

$R_o$ Outer radius of shell

$L$ Length of shell

$DF$ Damping Factor

$CV$ Coverage length

$\eta$ Loss factor of VEM

$f$ Natural frequency (Hz)

$\rho_1, \rho_2, \rho_3,$ Density of constraining layer, viscoelastic layer, and shell

$\nu_1, \nu_2, \nu_3$ Poisons ratio of constraining layer, viscoelastic layer and shell
Chapter 1: Introduction

1.1 Introduction

Vibration of cylindrical shafts has been studied since the late 1950's, which is important in design. In spacecraft and planes it is usually beneficial to make components lighter to save weight and therefore shafts are becoming thinner and more shell-like. Many engineers have researched this subject to aid in the design process.

Early work on thin beams, shells of revolution and shell theory can be found in such publications as Timoshenko's Theory of Plates and Shells [1], Flugge's Stresses in Shells [2], Markus's The Mechanics of Vibrations of Cylindrical Shells [3], Vinson's The Behavior of Thin Walled Structures [4], Soedel's Vibrations of Shells and Plates [5], Leissa's Vibration of Shells [6], along with Love [7], Sanders [8] and Warburton [9].

Timoshenko derived the equations of motion for a single-layer cylindrical shell in 1959 and is one of the earliest publications on shells. Flugge's Stresses in Shells was written in 1966 where he explains why unit forces are used instead of stresses along with the relationship between stresses and unit forces. Flugge also explains what membrane theory entails, and derives kinematics along with strain-displacement relations.

Markus's The Mechanics of Vibrations of Cylindrical Shells also covered unit forces for single-layer cylindrical shells, membrane theory, and a two layered cylindrical shell fully
treated with a viscoelastic layer. Gorman [10] published a book dealing with free vibrations of beams and shells and is considered a good introduction for shell theory.

In the late 1960’s a technique known as constrained layer damping (CLD) was introduced to beam type structures to attenuate vibration. Work on constrained layer damping for beams can be traced back to Mead and Markus [11] around 1969.


Forced vibrations of a three-layer sandwich ring with a complete viscoelastic core was investigated by Hu and Huang [14] in the 1990’s. In their paper Love’s assumption was applied and equations of motion in terms of two tangential displacements and one transverse displacement are derived using the Hamilton’s principle. Stationary harmonic points loads are applied to the ring along with traveling harmonic point loads, and the frequency response function (FRF) is determined. The change in damping factor due to varying VEM thickness is then investigated and the optimal VEM thickness is obtained.

Hu and Huang [15] also published “A Linear Theory of Three-Layer Damped Sandwich Shell Vibration” in 1996. In this paper a general shell with a full CLD treatment was modeled. The Hamilton’s principle is again used to derive the equations of motion along with the Donnell-Mushtari-Vlasov simplifications. The approach starts with Love’s thin shell theory and assumes that the VEM undergoes pure shear. The transverse shear deformation of both the
shell and constraining layer are assumed to be negligible along with the normal stress in the radial direction. Kinematics for the general shell of revolution are shown along with the five generic equations of motion for the three-layer structure. The damping factor at the first mode is then investigated, which gives rise to an expectation of an optimal thickness for the VEM and constraining layer.

In 1999 Hu and Huang [16] published a second paper on a generic three-layered shell where examples are shown for a three-layered cylindrical shell and a three-layered plate. FRF’s are given along with plots showing the relationship between damping factor and VEM thickness.

Most of the current work dealing with CLD, for shells of revolution, consider fully constrained shells, as in both of Hu and Huang’s papers mentioned above. Work has been done considering discrete damping strips by Chen and Huang [17], using energy methods. Frequency response functions are shown along with graphs examining the effects of the thickness of the VEM, and the constraining layer on the damping factor.

Lastly, Saravanan [18] and Ramasamy [19] in the late 1990s studied vibration and damping effects of a CLD cylindrical filled with a fluid. Figures of loss factor and natural frequency are shown with various fluid fill levels and mode number.

In this thesis circumferential, longitudinal, and radial vibrations for a single-layer cylindrical shell with no CLD treatment will be investigated along with a fully and partially CLD treated cylinder.
The contribution, which this thesis makes, to this field of research comes from introducing a cylindrical CLD treatment of partial coverage to a cylindrical shell (see Figure 4.1). For the partially treated shell the effects on the natural frequency and damping factor due to various materials, layer thicknesses, and coverage length will be explored. The aim of this study is to determine if it is possible to dampen the lowest mode of vibration so that the CLD shaft can be operated through and above its first critical speed. Partial differential equations are derived from equilibrium and solved to produce natural frequencies using the assumed mode method and Galerkin approach. Results will be compared with results from present publications and from an equivalent finite element model using ANSYS.

1.2 Modes

It is helpful to understand how a cylindrical shell vibrates; therefore explaining the various modes shapes will be done first. Throughout this thesis a cylindrical shell of simple supports will be analyzed, with three degrees of freedom, u, v and w, which are shown below:

Figure 1.1: Degrees of freedom for cylindrical shell
where \( u, v, \) and \( w \) are the displacements in the \( x \) (longitudinal), \( \theta \) (circumferential), and \( z \) (radial) directions respectively.

The mode shapes of a cylindrical shell seem somewhat complicated at first because of the cylindrical coordinate system and it being a continuous system, but they can be plotted in three-dimensional space for clarification. For a simply supported cylindrical shell a valid set of assumed mode shapes are:

\[
\begin{align*}
  u &= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} U_{mn} e^{i\lambda x} \cos(\lambda x) \cos(n\theta) \\
  v &= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} V_{mn} e^{i\lambda x} \sin(\lambda x) \sin(n\theta) \\
  w &= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} W_{mn} e^{i\lambda x} \sin(\lambda x) \cos(n\theta)
\end{align*}
\]

where \( m \) and \( n \) are referred to as the longitudinal and circumferential wave numbers respectively. It must be noted that for every given \( m \) and \( n \) there will be three associated modes; radial, longitudinal, and circumferential \((u, v, \) and \( w)\). It should also be pointed out that these assumed modes are orthogonal which becomes important when deciding which method to use to solve the partial differential equations.

The boundary conditions are incorporated into the assumed mode shapes and can be determined by letting \( x \) equal zero and \( L \) (the length of the shell). For the longitudinal direction, at \( x \) equals to zero and \( L \), free\(^1\) boundary conditions exist. Similarly, one can determine that the circumferential and radial degrees of freedoms are fixed at the ends of the shell. To determine

\(^1\) This permits a rigid body motion therefore \( m \) cannot be equal to zero
the rotational degree of freedoms the derivative of the assumed modes shaped can be taken, and the same logic can be applied. In this section the following three modes will be discussed: m=1, n=0; m=1, n=1; m=1, n=2.

1.2.1 Radial Modes

The radial modes are probably the easiest to visualize and therefore will be explained first. The assumed displacement in the z direction, w, is a function of x and θ and varies with values of m and n. Let's first examine the assumed mode in the x direction alone.

Let m=1 and ignore the variation in w with respect to θ (i.e. let n=0), therefore the displacement in the z direction simplifies to:

\[ w = W_0 e^{iωt} \sin\left(\frac{π}{L} x\right) \]

which can be visualized as a half sin wave oscillation at ω. Therefore if n=0 there is no variation in the theta direction because \( \cos(0 \ θ)=1 \) at all values of θ, and this mode is commonly referred to as the breathing mode. A simplified way to illustrate the breathing mode is to imagine a circle with a radius which varies in time, but yet the circle remains a circle at all times. Figure 1.2 shows the m=1, n=0 mode at two different instants in time.
For $m=1$, $n=1$ the $z$ displacement will vary around the circumference with a single wave, and is usually refereed to the first bending mode. With values of $n$ above 1 the circular cross section of the original cylindrical shell becomes distorted. For example, when $n=2$ the cross section will vary from being a circle to an ellipse, with respect to time. This mode is a result of having 2 waves in the $\theta$ direction. For modes where $n$ is greater than 1 the “peaks” around the circumference can be counted and will be equal to the value of $n$ for that mode. To help visualize this approach think of superimposing a circle onto the deformed cross sectional shape and counting the number of “humps” outside of the circle.
1.2.2 Longitudinal Modes

The modes in the $x$ and $\theta$ directions, longitudinal and circumferential, can be difficult to visualize for a continuous system. One way to help visualize these modes is to imagine the continuous model of being a finite number of discrete masses and springs. A software package known as MATLAB was used to create three dimensional plots to help visualize these types of modes. The files created plot points in 3D space and play a movie to show the variation in displacement with respect to time. Figures 1.4, 1.5, nd 1.6 represent some longitudinal modes for the cylindrical shaft. The figures contain a complete cycle of the motion for a particular mode.

The first longitudinal mode, where $m = 1$ and $n = 0$, is shown below in Figure 1.4:
Figure 1.4: Plot of the \( m=1 \) \( n=0 \) longitudinal mode
To help visualize the effect of having a circumferential wave number Figure 1.5 and 1.6 show the \( m=1 \) \( n=1 \) mode and \( m=1 \) \( n=2 \) mode shape, respectively.

Figure 1.5: Plot of the \( m=1 \) \( n=1 \) longitudinal mode
Figure 1.6: Plot of the $m=1$ $n=2$ longitudinal mode
1.2.3 Circumferential Modes

Circumferential modes can be viewed much like the longitudinal modes except that they vary in the $\theta$ direction. The following Figures represent some longitudinal modes for a cylindrical shaft of simple supports. For the $m=1 \ n=0$ mode the circumferential mode does not exist with the assumed modes shapes because $\sin(0)=0$, therefore the only modes shown in the following Figures are the $m=1 \ n=1$, and $m=1 \ n=2$ modes.
Figure 1.7: Plot of the \( m=1 \) \( n=1 \) Circumferential mode
Figure 1.8: Plot of the $m=1 \ n=2$ Circumferential mode
Although three mode shapes exist for any given m and n it must be mentioned that they do not all exist in the same amount. The three coefficients of the assumed mode shapes, $U_{mn}$, $V_{mn}$, and $W_{mn}$, will determine the amplitude of its corresponding mode, and usually one type of mode, weather radial, longitudinal, or circumferential, is dominant at a given frequency. A good example of this is for the breathing mode $(m=1, n=0)$ where the length of the cylindrical shell changes to account for the expanding radius and there is no circumferential mode. If the cylinder is excited at a different frequency the radial mode may be suppressed and the longitudinal might be amplified. This will become important in later sections when discussing the results and understanding which modes are present at certain frequencies. References [20-24] may help understand the vibrational modes. Appendix A contains illustrations of various radial modes along with the programs used to create animations of modes.

Chapter 2 : Single-Layer Cylindrical Shaft

2.1 Overview of approach

The steps used in this thesis to determining the coupled partial differential equations of motion for the cylindrical shells are shown below:

- Kinetics/Kinematics
  - Define displacement field
  - Define stress-strain relations
Define strain-displacement relations

- Derive the equations of motion in terms of unit forces
- Substitute the unit forces into the equations of motion

After some management in doing these steps they will yield three partial differential equations in terms of the displacements $u$, $v$, and $w$. The next step will be to apply the assumed mode method or apply the Galerkin method to discretize the equations of motion to get the equations in the form of:

$$[M](\ddot{\xi})+[K](\xi)=(0)$$

where $M$ and $K$ are the equivalent mass and stiffness matrices and $\xi$ is the generalized coordinates.

### 2.2 Kinetics/Kinematics

Figure 2.1 illustrates a cylindrical shell with the coordinate system used throughout this thesis. It is assumed that the material is homogeneous and isotropic. The displacements in the $x$ direction, $\theta$ direction, and $z$ direction are represented as $u$, $v$, and $w$ respectively, shown in Figure 2.1.
The sign convention used throughout this thesis is described as follows:

1. Tension is positive and compression is negative.

2. A positive moment creates a positive stress on the positive face in the positive z direction. This convention is shown for a simple 2D problem below:
For most shell theory, the radial displacement, $w$, is assumed to be the same for points across the thickness of the shell; therefore $w$ is a function of $x$ and $\theta$ only, and not a function of $z$. With this assumption the radial strains are neglected. Also the displacements in the $x$ and $\theta$ directions ($u$ and $v$) of the shell are assumed to vary linearly in the radial direction, thus the displacement field can be described as:

$$
\begin{align*}
    u &= u^0 + z \beta_x \\
    v &= v^0 + z \beta_\theta
\end{align*}
$$

(2.1)

where $u^0$ and $v^0$ represent the displacement of the mid-plane, and $\beta_x$ and $\beta_\theta$ are the rotation angles in the $x$ and $\theta$, directions respectively. Linear strain-displacement equations for cylindrical coordinates can be found in Budynas[25], and applied to the above displacement field to derive the variation of strain across the thickness of the shell. The standard linear stress-strain relationships for cylindrical coordinates are shown below along with the strain-displacement equations:

$$
\begin{align*}
    \sigma_x &= \frac{E}{1-\nu^2} (\varepsilon_x + \nu \varepsilon_\theta) \\
    \sigma_\theta &= \frac{E}{1-\nu^2} (\varepsilon_\theta + \nu \varepsilon_x) \\
    \sigma_{x\theta} &= G \varepsilon_{x\theta}
\end{align*}
$$

(2.2)

---

2 For isotropic materials $G=E/(2(1+\nu))$
The rotation angles are defined as:

\[
\beta_x = -\frac{\partial w}{\partial x}, \quad \beta_\theta = \frac{\nu}{R} - \frac{1}{R} \frac{\partial w}{\partial \theta}
\]

Applying strain-displacement equations (2.3) to the displacement field (2.1), and using equations (2.4), one obtains the strain-displacement relations as:

\[
\varepsilon_x = \frac{\partial u}{\partial x} - z \frac{\partial^2 w}{\partial x^2}
\]

\[
\varepsilon_\theta = \frac{1}{R} \frac{\partial v}{\partial \theta} + \frac{w}{R} + z \left( \frac{\partial v}{\partial \theta} \frac{\partial^2 w}{\partial \theta^2} \right)
\]

\[
\gamma_{x\theta} = \frac{1}{R} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x} - z \left( \frac{2}{R} \frac{\partial^3 w}{\partial x \partial \theta} - \frac{1}{R} \frac{\partial v}{\partial x} \right)
\]

Now that the strain has been defined in terms of the displacements, unit forces and unit moments can be found by integrating the derived stresses over the thickness of the shell as shown below:
\[
\begin{pmatrix}
V_x \\
V_\theta \\
V_{x\theta} \\
M_x \\
M_\theta \\
M_{x\theta} \\
M_{\theta x}
\end{pmatrix}
= \int_{-h/2}^{+h/2}
\begin{pmatrix}
\sigma_x (1 + \frac{z}{R}) \\
\sigma_\theta \\
\tau_{x\theta} (1 + \frac{z}{R}) \\
\sigma_x (1 + \frac{z}{R}) z \\
\sigma_\theta z \\
\tau_{x\theta} (1 + \frac{z}{R}) z \\
\tau_{x\theta} z
\end{pmatrix} dz^3
\] (2.6)

where the stresses in (2.6) are known by substituting (2.5) into (2.2). The \(1 + \frac{z}{R}\) part in the above equations comes from the change in length as \(z\) changes, for the curved differential areas. With the unit forces known the next step is to derive the equations of motion.

### 2.3 Equations of Motion

The equations of motion will now be derived from equilibrium of the differential element. It must be noted that the elements shown in the Figures below are not a true differential element because of the thickness. To be a true differential element the thickness would be \(dR\), but in this case the thickness shown is \(h\). Therefore instead of applying stresses unit forces are applied across the total thickness, \(h\). Also the bar over the top of a unit force or moment denotes that it

\[^3\text{The unit forces and moments can be found in appendix B1}\]
is the force or moment plus the change in that force or moment across that change in distance i.e. 
\[ M_{\theta 1} = M_{\theta 1} + \frac{\partial M_{\theta 1}}{\partial \theta} d\theta. \] 
The radius R is taken to be the mean or middle radius of the cylinder and the positive directions of the coordinate system are shown in red in the following Figures. The sum of the forces in the x, y, and z directions along with the sum of the moments about the x, y, and z axis are given below:

\[ \sum F_x = 0 \]

\[ (V_x + \frac{\partial V_x}{\partial x} dx - V_x) Rd\theta + (V_{\theta x} + \frac{\partial V_{\theta x}}{\partial \theta} d\theta - V_{\theta x}) dx - \rho h R \ddot{u} dx d\theta = 0 \]

\[ -R \frac{\partial V_x}{\partial x} \frac{\partial V_{\theta x}}{\partial \theta} + \rho h \ddot{u} = 0 \] (2.7)
\[ \sum F_y = 0 \]

\[ (V_\theta + \frac{\partial V_\theta}{\partial \theta} d\theta - V_{\theta})dx \cos(\frac{d\theta}{2}) + (V_{\theta R} + \frac{\partial V_{\theta R}}{\partial \theta} d\theta + V_{\theta R})dx \sin(\frac{d\theta}{2}) \]

\[ + (V_{x \theta} + \frac{\partial V_{x \theta}}{\partial x} dx - V_{x \theta})Rd\theta - \rho hR\dot{v} dx d\theta = 0 \]

\[ -\frac{\partial V_\theta}{\partial \theta} - V_{\theta R} - R \frac{\partial V_{x \theta}}{\partial x} + \rho h \dot{v} = 0 \]

\[ \sum F_z = 0 \]

\[ (V_x R + \frac{\partial V_x R}{\partial x} dx - V_x R)Rd\theta + (V_{\theta R} + \frac{\partial V_{\theta R}}{\partial \theta} d\theta - V_{\theta R})dx \cos(\frac{d\theta}{2}) \]

\[ -(V_\theta + \frac{\partial V_\theta}{\partial \theta} d\theta + V_\theta)dx \sin(\frac{d\theta}{2}) - \rho h R \dot{w} dx d\theta \]

\[ -R \frac{\partial V_x R}{\partial x} - \frac{\partial V_{\theta R}}{\partial \theta} + V_\theta + \rho h R \dot{w} = 0 \]

\[ \sum M_x = 0 \]

\[ -(M_{x \theta} + \frac{\partial M_{x \theta}}{\partial x} dx - M_{x \theta})Rd\theta - (M_{\theta} + \frac{\partial M_{\theta}}{\partial \theta} d\theta - M_{\theta})dx + \]

\[ (V_{\theta R} + \frac{\partial V_{\theta R}}{\partial \theta} d\theta + V_{\theta R})dx \cos(\frac{d\theta}{2}) R \sin(\frac{d\theta}{2}) - (V_\theta + \frac{\partial V_\theta}{\partial \theta} d\theta - V_\theta) \sin(\frac{d\theta}{2}) (R \frac{d\theta}{2}) = 0 \]

\[ -R \frac{\partial M_{x \theta}}{\partial x} - \frac{\partial M_{\theta}}{\partial \theta} + RV_{\theta R} = 0 \]

\[ \sum M_y = 0 \]

\[ (M_{x \theta} + \frac{\partial M_{x \theta}}{\partial x} d\theta - M_{x \theta})dx + (M_x + \frac{\partial M_x}{\partial x} dx - M_x)Rd\theta \]

\[ -(V_{x R} + \frac{\partial V_{x R}}{\partial x} dx + V_{x R})Rd\theta(\frac{dx}{2}) = 0 \]

\[ \frac{\partial M_{x \theta}}{\partial \theta} + R \frac{\partial M_x}{\partial x} - RV_{x R} = 0 \]

\[ \sum M_z = 0 \]

\[ (V_{x \theta} + \frac{\partial V_{x \theta}}{\partial x} dx + V_{x \theta})Rd\theta(\frac{dx}{2}) - (V_{\theta x} + \frac{\partial V_{\theta x}}{\partial \theta} d\theta + V_{\theta x})dx R \sin(\frac{d\theta}{2}) = 0 \]

\[ V_{\theta x} = V_{x \theta} \]
The moment equations (2.10 and 2.11) can be solved for $V_x R$ and $V_\theta R$ and then substituted back into equations (2.8 and 2.9) to obtain the three partial differential equations of motion in terms of the unit forces and moments. These equations are shown below:

$$-R \frac{\partial V_x}{\partial x} - \frac{\partial V_\theta}{\partial \theta} + \rho R \ddot{u} = 0$$
$$- \frac{\partial V_\theta}{\partial \theta} - \frac{\partial M_x}{\partial x} - \frac{1}{R} \frac{\partial M_\theta}{\partial \theta} - R \frac{\partial V_x}{\partial x} + \rho R \ddot{v} = 0$$
$$- \frac{\partial^2 M_\theta}{\partial x \partial \theta} - R \frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 M_x}{\partial \theta^2} - \frac{1}{R} \frac{\partial^2 M_\theta}{\partial \theta^2} + V_\theta + \rho R \ddot{w} = 0$$

(2.13)

The unit forces and moments, equations (2.6)$^4$, can now be substituted into equations (2.13), to obtain the three equations of motion in terms of the three displacements. Neglecting terms in the unit forces containing $h^3 / R^3$ the equations become:

$$-R \frac{\partial^2 u}{\partial x^2} - \frac{1}{2} \frac{\partial^2 u}{\partial \theta^2} - \frac{1}{2} \frac{\partial^2 v}{\partial x \partial \theta} - \frac{1}{2} \frac{\partial^2 v}{\partial x^2} + \rho R \ddot{u} = 0$$
$$- \frac{1}{2} \frac{\partial^2 u}{\partial \theta^2} - \frac{1}{2} \frac{\partial^2 v}{\partial \theta^2} - \frac{1}{2} \frac{\partial^2 v}{\partial x \partial \theta} - \frac{1}{2} \frac{\partial^2 v}{\partial x^2} + \rho R \ddot{v} = 0$$
$$-R \frac{\partial u}{\partial \theta} + \frac{1}{12} \frac{\partial v}{\partial \theta} + \frac{R h^2}{12} \frac{\partial^4 w}{\partial x^4} + \frac{w}{12 R} h^2 \frac{\partial^2 w}{\partial x^2} + \frac{h^2}{6 R} \frac{\partial^4 w}{\partial x^2 \partial \theta^2} + \rho R \ddot{w} = 0$$

(2.14)

In equations (2.14) $K = \frac{Eh}{1 - v^2}$. Three coupled partial differential equations in terms of $u$, $v$, and $w$ are now established.
2.4 Assumed Mode Method

The equations of motion in terms of the displacements are now known and one method of solving them is the assumed mode method. For a simply supported cylinder the assumed modes are:

\[ u = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} U_{mn} e^{i\omega t} \cos(\lambda x) \cos(n\theta) \]

\[ v = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} V_{mn} e^{i\omega t} \sin(\lambda x) \sin(n\theta) \quad , \quad \lambda = \frac{m\pi}{L} \quad (2.15) \]

\[ w = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} W_{mn} e^{i\omega t} \sin(\lambda x) \cos(n\theta) \]

where \( m \) and \( n \) are the wave numbers. Equations 2.15 are the same as the assumed modes discussed in section 1.2, and therefore the same boundary conditions apply.

\[ ^4 \text{Expanded unit forces and unit moment can be found in Appendix B} \]
Substituting the assumed modes shapes, (2.15), into the partial differential equations of motion (2.14) one obtains the discretized mass and stiffness matrices for free vibration, which are shown below:

\[
[M](\ddot{\xi}) + [K](\dot{\xi}) = 0
\]

where

\[
[M] = \frac{\rho h R}{K} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

\[
[K] = \begin{pmatrix}
R \lambda^2 + n^2 \frac{1-\nu}{2R} & -\frac{1+\nu}{2} n \lambda & -\nu \lambda \\
-\frac{1+\nu}{2} n \lambda & \frac{1}{R} n^2 + \frac{1-\nu}{2} R \lambda^2 & \frac{1}{R} n \\
-\nu \lambda & \frac{1}{R} n & \frac{R h^2}{12} \lambda^4 + \frac{1}{R} \frac{h^2 \nu}{12R} \lambda^2 + \frac{h^2}{6R} n^2 \lambda^2
\end{pmatrix}
\]

\[
(\xi) = \begin{pmatrix}
U_{mn} \\
V_{mn} \\
W_{mn}
\end{pmatrix}
\]

To find the natural frequencies of the cylinder the dynamic matrix, \(\Lambda\), will be found for the system, and is defined as the inverse of the mass matrix multiplied by the stiffness matrix. The natural frequencies are then found by taking the square root of the eigenvalues of the dynamic matrix \(\Lambda\). The coefficients, \(U_{mn}\), \(V_{mn}\), and \(W_{mn}\) will be real numbers because damping is ignored in this particular model.
2.5 Results

The natural frequencies and modes will be found and checked with results from ANSYS and published papers. The cylinder is made of aluminum where the properties of the cylinder are:

\[ L=0.35 \text{ m}, \ h=0.002 \text{ m}, \ R=0.1 \text{ m}, \ E=70 \text{ Gpa}, \ \rho = 2710 \text{Kg/m}^3, \ \nu = 0.3 \]

The mathematical software package Mathematica was used to compute the natural frequencies and mode shapes by determining the eigenvalues of the dynamic matrix.

Table 2.1 shows the different values of the natural frequencies for various valued of m and n or modes including natural frequencies for \( n=0 \) to \( n=4 \) and \( m=1 \) to \( m=3 \).

<table>
<thead>
<tr>
<th></th>
<th>n=0</th>
<th>n=1</th>
<th>n=2</th>
<th>n=3</th>
<th>n=4</th>
</tr>
</thead>
<tbody>
<tr>
<td>m=1</td>
<td>4502.77</td>
<td>2605.74</td>
<td>1216.97</td>
<td>775.278</td>
<td>881.759</td>
</tr>
<tr>
<td></td>
<td>6658.26</td>
<td>8020.82</td>
<td>11497.3</td>
<td>15919.3</td>
<td>20667.7</td>
</tr>
<tr>
<td></td>
<td>9246.58</td>
<td>13268.5</td>
<td>20118.4</td>
<td>27746.9</td>
<td>35719.6</td>
</tr>
<tr>
<td>m=2</td>
<td>7939.95</td>
<td>5430.71</td>
<td>3294.46</td>
<td>2095.53</td>
<td>1598.15</td>
</tr>
<tr>
<td></td>
<td>9005.54</td>
<td>11365.6</td>
<td>14337.9</td>
<td>18050.6</td>
<td>22293.9</td>
</tr>
<tr>
<td></td>
<td>15511.4</td>
<td>17976.6</td>
<td>23594.1</td>
<td>30470.4</td>
<td>37935.1</td>
</tr>
<tr>
<td>m=3</td>
<td>8040.9</td>
<td>6833.1</td>
<td>4961.9</td>
<td>3538.7</td>
<td>2686.31</td>
</tr>
<tr>
<td></td>
<td>13508.3</td>
<td>14867.9</td>
<td>17530.5</td>
<td>20820.</td>
<td>24619.4</td>
</tr>
<tr>
<td></td>
<td>22994.4</td>
<td>24601.4</td>
<td>28850.8</td>
<td>34698.2</td>
<td>41433.6</td>
</tr>
</tbody>
</table>

Table 2.1: Natural Frequencies in Hz

A graphical representation of Table 2.4 is shows in Figure 2.4, where it can easily be seen that for any given \( m \) and \( n \) value there are three mode shapes.
Each of the three curves in Figure 2.4 belongs to a certain type of mode shape. As explained earlier in section 1.2 for each m and n value a longitudinal, circumferential, and radial mode exists. Therefore in Figure 2.4 the blue, black, and red curves belong to the circumferential, longitudinal, and radial modes respectively.

From Table 2.1 and Figure 2.4 it can be seen that the lowest natural frequency belongs to the radial 1, 3 (m=1, n=3) mode shape (section 2.6 explains mode determination). It must be noted that this is not always the case, and is special to the dimensions, boundary conditions, and properties of the materials in which the cylinder is made of. For an example, a simply supported aluminum shaft with a length of 60 inches, diameter of 3 inches, and thickness of
.125 inches has the lowest natural frequency around 90 Hz which corresponds to the first bending mode.

2.6 Corresponding Mode Shapes

In Table 2.1 it is shown that there are a set of three natural frequencies for any given m and n value. These three natural frequencies represent the radial, circumferential, and longitudinal modes (in no particular order). For instance, when n=3 and m=1, the natural frequencies are 775.3, 15919, and 27746 Hz; note that the natural frequencies are in radians per second in the equations therefore \( f \) must be multiplied by \( 2\pi \) before substitution. Once the natural frequencies are known they can be back substituted into the dynamic matrix to find the corresponding mode shapes. This will result in three equations in terms of \( U, V, \) and \( W \) which all equal zero, but two of the equations will be the same because the determinant of the matrix is zero meaning that the matrix is singular. Knowing this, any two of the three equations are used to solve for the ratio of one displacement with respect to the other. For example, lets pick the natural frequencies when n=3 and m=1 (listed above). When \( f = 775.3 \) Hz is substituted back into the matrix it results in:

\[
\begin{pmatrix}
39.47 & -17.50 & -2.69 \\
-17.50 & 92.73 & 30. \\
-2.69 & 30. & 9.94 \\
\end{pmatrix}
\begin{pmatrix}
U \\
V \\
W \\
\end{pmatrix}
= 0
\]

The top two equations give:

\[
39.5U - 17.5V - 2.7W = 0 \quad (2.16)
\]

\[
-17.5U + 92.7V + 30W = 0 \quad (2.17)
\]

28
Solving the second equation (2.17) for $U$ and then for $V$ results in:

\[
U = \frac{92.7V + 30W}{17.5} \\
V = \frac{17.5U - 30W}{92.7}
\]

Subbing in the above relationships to the first equation (2.16) one at a time, ratios of $W$ to $U$ and $W$ to $V$ can be found. For 775.3 Hz the ratios turn out to be:

\[
39.5\left(\frac{92.7V + 30W}{17.5}\right) - 17.5V - 2.7W = 0 \rightarrow \frac{W}{V} = -\frac{191.4}{65.014} = -2.949 \\
39.5U - 17.5\left(\frac{17.5U - 30W}{92.7}\right) - 2.7W = 0 \rightarrow \frac{W}{U} = -\frac{36.2}{2.963} = -12.21
\]

Therefore from the above ratios, it can be seen that $W$ is the dominant displacement i.e. if $V=1$, $W=-2.9$ and if $U=1$, $W=-12.2$, therefore it can be said that 775.3 Hz is associated with the third radial mode shape where $m=1$ and $n=3$.

This procedure can be repeated to find the mode associated with any natural frequency. For $f = 27746$ Hz, when $m=1$ and $n=3$ the ratio of $W/V=-.3$ and $W/U=-1.15$, therefore $V$ is the largest displacement, therefore 27746 Hz relates to the third circumferential mode shape.

To verify and gain confidence in the analytical model the resulting natural frequencies from the assumed mode method are compared to some published results and with results from ANSYS for two different cylindrical shells. Table 2.2 compares results from ANSYS to the assumed mode results for a cylindrical shell with the following properties:
L = 35 m, h = 0.02 m, R = 0.1 m, E = 70 Gpa, \( \rho = 2710 \text{Kg} / \text{m}^3 \), \( \nu = 0.3 \)

<table>
<thead>
<tr>
<th>Mode Shape</th>
<th>Natural Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>m \ n</td>
<td>ANSYS Assumed mode method</td>
</tr>
<tr>
<td>1 \ 3</td>
<td>748 \ 775</td>
</tr>
<tr>
<td>1 \ 0</td>
<td>4489 (Circumferential) \ 4502</td>
</tr>
<tr>
<td>1 \ 0</td>
<td>6677 (Breathing) \ 6658</td>
</tr>
<tr>
<td>1 \ 1</td>
<td>2599 (Bending) \ 2605</td>
</tr>
<tr>
<td>2 \ 3</td>
<td>2101 \ 2095</td>
</tr>
</tbody>
</table>

Table 2.2: Result comparison for cylindrical shell

Kim and Bert [26] published some results comparing their calculated natural frequencies with experimental results. To gain more confidence in the present model the results are compared to the published results from Kim and Bert in Table 2.3 where the cylinder properties are:

L = 11.74 in, h = 0.02 in, R = 5.836 in, E = 29.5 Mpsi, \( \rho = 0.007345 \text{lbf} \text{s}^2/\text{in}^4 \), \( \nu = 0.285 \)

<table>
<thead>
<tr>
<th>Mode Shape</th>
<th>Natural Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>m \ n</td>
<td>Assumed mode method Kim and Bert (analytical) Kim and Bert (experimental)</td>
</tr>
<tr>
<td>1 \ 1</td>
<td>3269 \ 3270 \ -</td>
</tr>
<tr>
<td>1 \ 2</td>
<td>1861 \ 1862 \ -</td>
</tr>
<tr>
<td>1 \ 3</td>
<td>1101 \ 1108 \ 1001</td>
</tr>
<tr>
<td>1 \ 4</td>
<td>705 \ 705 \ 715</td>
</tr>
<tr>
<td>1 \ 5</td>
<td>497 \ 497 \ 534</td>
</tr>
<tr>
<td>1 \ 6</td>
<td>400 \ 400 \ 410</td>
</tr>
<tr>
<td>1 \ 7</td>
<td>380 \ 380 \ 393</td>
</tr>
<tr>
<td>1 \ 8</td>
<td>416 \ 416 \ 426</td>
</tr>
<tr>
<td>1 \ 9</td>
<td>488 \ 488 \ 495</td>
</tr>
</tbody>
</table>

Table 2.3: Result comparison

The above comparisons show that the present model works well and agrees with a finite element program and published values for two different cylindrical shells.
2.7 Galerkin Method

One of the many ways to find approximate numerical solutions to partial differential equations is known as the Galerkin method. The Galerkin method is a special case of the weighted-residual methods where the weight function, $\Psi_B$, is chosen to be equal to the approximation function $\phi_A$. A general form of an equation to be solved by a weighted-residual method can be of the form:

$$L(u) = F$$  \hspace{1cm} (2.18)

where $L$ is an operator acting on $u$ and $F$ is a known function. A general form of the approximated solution is:

$$u_N = \sum_{A=1}^{N} d_A \phi_A + \phi_0$$  \hspace{1cm} (2.19)

Because $u_N$ is an approximation of the exact solution, when it is substituted into the operator equation 2.19 a residual will exist, meaning that the equation is no longer equal to $F$, but actually equal to $F+Q$, where $Q$ is called the residual. In all weighted-residual methods, not just the Galerkin method, the coefficients $d_A$ are determined by enforcing the residual to become zero in a weighted-integral form, which produces $N$ equations in terms of $N$ $d_A$ coefficients.

The Galerkin method will now be applied to the three equations of motion, for the single-layer cylindrical shell previously derived, to find the natural frequencies and mode shapes.
For a simply supported cylinder, the approximation function in the x, θ, and z coordinates are specified as:

\[
\phi_x^A = \cos\left(\frac{m\pi}{L} x\right)\cos(n\theta) \\
\phi_\theta^A = \sin\left(\frac{m\pi}{L} x\right)\sin(n\theta) \\
\phi_z^A = \sin\left(\frac{m\pi}{L} x\right)\cos(n\theta)
\] (2.20)

where the superscript letter of \(\phi_A\) denotes the corresponding coordinate for that approximation function. If using the Galerkin method, where the weight functions must be equal to the approximation functions, the weight functions must be:

\[
\Psi_x^B = \cos\left(\frac{i\pi}{L} x\right)\cos(j\theta) \\
\Psi_\theta^B = \sin\left(\frac{i\pi}{L} x\right)\sin(j\theta) \\
\Psi_z^B = \sin\left(\frac{i\pi}{L} x\right)\cos(j\theta)
\] (2.21)

Again where the superscript letter of \(\Psi_B\) denotes the corresponding coordinate for the weight function.

Note that A indicates a set of values for the m and n wave numbers and B indicates a set of values for i, and j. For simplified example let A and B take on values of 1 and 2, and let the following Table keep track of the corresponding values of m, n, i, and j.

---

5 Examples of other weighted-residual methods are Petrov-Galerkin, Least-Squares, and Collocation method [27]
Table 2.4: Assigned values of m, n, i, and j with respect to A and B

![Table 2.4](image)

The values of A control the vibrational mode in which the solutions will represent, i.e. in this example the solution will consist of a set of 2 natural frequencies for each degree of freedom, \(u, v, \) and \(w\). For instance the solution will produce the natural frequency for the \(m=1, n=1\) mode in the \(w\) direction, which is usually referred to as bending, along with the \(m=1, n=2\) mode in the \(w\) direction, in which the original circular cross section of the cylinder distorts into an ellipse. Therefore, if a specific mode or modes are wanted just adjust the values of \(n, m, i, j\). Of course A can take on an infinite number, but one must realize that for every additional value of A the number of equations and unknowns will increase by three. For the current example it can be shown that the final matrix is six by six and if A and B each took on a third value, say \(n=3, m=1\) for \(A=3\), and \(i=3, j=1\) for \(B=3\), the resulting matrix would be nine by nine.

Continuing on with applying the Galerkin method to the equations of motion; with the approximation functions defined, the approximated solution for \(u, v, \) and \(w\) (the displacement in the \(x, \theta, \) and \(z\) direction respectively), become:
\[ u_A = \sum_{A=1}^{N} d^x_A \phi^x_A + \phi_0 \]
\[ v_A = \sum_{A=1}^{N} d^\theta_A \phi^\theta_A + \phi_0 \]
\[ w_A = \sum_{A=1}^{N} d^z_A \phi^z_A + \phi_0 \]

where in this case all \( \phi_0 \) are equal to zero\(^6\).

Substituting the approximate solutions into the equations of motion in the x direction results in:

\[
\begin{bmatrix}
\frac{\rho h R}{K} d^x_A + (R \lambda^2 + n^2 \frac{1-v}{2R}) d^x_A - \frac{1+v}{2} n \lambda d^\theta_A - v \lambda d^z_A
\end{bmatrix} \cos\left(\frac{m \pi}{L} x\right) \cos(n \theta) = R
\]

Then this equation is multiplied by its weight-function \( \Psi^x_B \) and integrated over the respective region, which in this case is described by \( 0 \leq x \leq L, \ 0 \leq \theta \leq 2\pi \). The resulting equation is shown below:

\[
\left\{ \int_{0}^{L} \int_{0}^{2\pi} \begin{bmatrix}
\frac{\rho h R}{K} d^x_A + (R \lambda^2 + n^2 \frac{1-v}{2R}) d^x_A - \frac{1+v}{2} n \lambda d^\theta_A - v \lambda d^z_A
\end{bmatrix} \cos\left(\frac{m \pi}{L} x\right) \cos(n \theta) \cos\left(\frac{j \pi}{L} x\right) \cos(j \theta) dx \ d\theta = R
\]

The same procedure will be performed on the remaining two partial differential equations of motion, which will produce a system of equations, in terms of \( d^x_{AB}, \ d^\theta_{AB}, \ d^z_{AB} \).

Arranging the system of equations in to matrix form shown below:

\[
[M]\left(\ddot{\xi}\right) + [K](\xi) = 0
\]

\(^6\) \( \phi_0 \) must satisfy all boundary conditions and for this case zero does
the natural frequencies can easily be found, by again using the properties of the dynamic matrix, $A=M^{-1}K$. The eigenvalues of the dynamic matrix are equal to the square of the natural frequencies and the eigenvectors are the mode shapes.

For the same cylinder as earlier, $L=.35$ m, $h=.002$ m, $R=.1$ m, $E=70$ GPa $\rho = 2710 Kg/m^3$, $\nu = .3$ the Galerkin method produced the following natural frequencies:

<table>
<thead>
<tr>
<th>Mode Shape</th>
<th>Natural Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Assumed mode method</td>
</tr>
<tr>
<td>1 3</td>
<td>775</td>
</tr>
<tr>
<td>1 0</td>
<td>4502</td>
</tr>
<tr>
<td>1 0</td>
<td>6658</td>
</tr>
<tr>
<td>1 1</td>
<td>2605</td>
</tr>
<tr>
<td>2 3</td>
<td>2095</td>
</tr>
</tbody>
</table>

Table 2.5: Comparison of results for Assumed mode and Galerkin method
Chapter 3: Three-Layer Cylindrical Shell

3.1 Introduction

The same process used for determining the natural frequencies and mode shapes for the single-layer cylindrical shell will now be used for determining the natural frequencies and mode shapes of a three-layer cylindrical shell. The only difference between the single-layer and three-layer cylinder is that there are extra shear stresses, which come from the viscoelastic layer (VEM). Throughout this thesis the outermost layer will be called the shell, the middle layer is the VEM, and the innermost layer is referred to as the constraining layer (CL). For simplicity in the mathematical equations the CL will be denoted as the number 1, the VEM as 2 and the Shell as 3.

![Three-Layer Cylinder](image)

Figure 3.1: Three-Layer Cylinder

The same coordinate system as in the single-layer cylinder equations will be used in the three-layer model with the addition of subscripts denoting the corresponding layer.
3.2 Kinetics/Kinematics

Similar to the single-layer cylinder the radial displacement, \( w \), is assumed to be constant across the thickness and therefore same for all layers. Again the displacements in the \( x \) and \( \theta \) (\( u \) and \( v \)) directions of the three layers are assumed to vary linearly in the radial direction, thus the displacement field can be described as:

\[
\begin{align*}
u_1 &= u_1^o + z \beta_{x1} \\
v_1 &= v_1^o + z \beta_{\theta 1} \\
u_2 &= u_2^o + z \beta_{x2} \\
v_2 &= v_2^o + z \beta_{\theta 2} \\
u_3 &= u_3^o + z \beta_{x3} \\
v_3 &= v_3^o + z \beta_{\theta 3}
\end{align*}
\]

where 1, 2, and 3, denote the CL, VEM, and shell layers. The rotation angles of the shell (3) and the constraining layer (1) are defined as:

\[
\begin{align*}
\beta_{x1} &= -\frac{\partial w}{\partial x}, & \beta_{x3} &= -\frac{\partial w}{\partial x} \\
\beta_{\theta 1} &= \frac{v_1}{R_1} - \frac{1}{R_1} \frac{\partial w}{\partial \theta}, & \beta_{\theta 3} &= \frac{v_3}{R_3} - \frac{1}{R_3} \frac{\partial w}{\partial \theta}
\end{align*}
\]

Applying the strain displacement equations (2.3) to the above displacement field, the variation of the strains with respect to the radial, \( z \), location give:
\[ \varepsilon_{xxi} = \frac{du_i}{dx} - \frac{z}{R_i} \frac{\partial^2 w}{\partial x^2} \]
\[ \varepsilon_{\theta \theta i} = \frac{1}{R_i} \frac{dv_i}{d\theta} + \frac{w}{R_i} + z \left( \frac{1}{R_i} \frac{dv_i}{d\theta} - \frac{1}{R_i} \frac{\partial^2 w}{\partial \theta^2} \right) \]
\[ \gamma_{x \theta i} = \frac{1}{R_i} \frac{\partial u_i}{\partial \theta} + \frac{\partial v_i}{\partial x} - z\left( \frac{2}{R_i} \frac{\partial^2 w}{\partial x \partial \theta} - \frac{1}{R_i} \frac{\partial v_i}{\partial x} \right) \] (3.3)

The stress-strain relationships from equations (2.2) still apply to the three-layer cylinder, and will be used.

Kinematics are now needed to find the shear forces created in the VEM. Assuming no slip conditions the equations must satisfy must satisfy:

\[ u_1(x, \theta, \frac{h_1}{2}) = u_2(x, \theta, -\frac{h_2}{2}) \]
\[ v_1(x, \theta, \frac{h_1}{2}) = v_2(x, \theta, -\frac{h_2}{2}) \] (3.4)
\[ u_3(x, \theta, -\frac{h_3}{2}) = u_2(x, \theta, \frac{h_2}{2}) \]
\[ v_3(x, \theta, -\frac{h_3}{2}) = v_2(x, \theta, \frac{h_2}{2}) \]

which leads to:

\[ u_2 = \frac{1}{2} \left[ (u_3 + \frac{h_3}{2} \beta) + (u_1 + \frac{h_1}{2} \beta) \right] \] (3.5)
\[ \beta_{x_2} = \frac{1}{h_2} \left[ (u_3 + \frac{h_3}{2} \beta) - (u_1 + \frac{h_1}{2} \beta) \right] \]
\[
v_2 = \frac{1}{2} \left[ (v_3 - \frac{h_3}{2} \beta_{\theta 3}) + (v_1 + \frac{h_1}{2} \beta_{\theta 1}) \right]
\]
\[
\beta_{\theta 2} = \frac{1}{h_2} \left[ (v_3 - \frac{h_3}{2} \beta_{\theta 3}) - (v_1 + \frac{h_1}{2} \beta_{\theta 1}) \right]
\]

The strains in the VEM are defined below along with the shear stress-shear strain relationship:

\[
\gamma_{xz 2} = \beta_{x 2} + \frac{\partial w}{\partial x}
\]
\[
\gamma_{\theta z 2} = \beta_{\theta 2} - \frac{v_2}{R_2} + \frac{1}{R_2} \frac{\partial w}{\partial \theta}
\]

The unit forces in the VEM are then found by integrating the stress over the thickness.

\[
\begin{pmatrix}
V_{x R 2} \\
V_{\theta R 2}
\end{pmatrix} = \int_{-h_z/2}^{h_z/2} \begin{pmatrix}
G 2 \gamma_{xz 2} (1 + \frac{R_2}{z}) \\
G 2 \gamma_{\theta z 2}
\end{pmatrix} dz
\]

The relationship between \( V_{sR 2} \) and \( \tau_{Rz 2} \) is derived below by taking the sum of the moments about the x and y axis for a differential element in the VEM layer:
Figure 3.2: Equilibrium for VEM element

\[ \sum M_x = 0 \]
\[ (V_\theta R^2 + \frac{\partial V_\theta R^2}{\partial \theta} d\theta + V_\theta R^2) dx \cos \left( \frac{d\theta}{2} \right) R^2 \sin \left( \frac{d\theta}{2} \right) \]
\[ -\tau_\theta R^2 (R^2 + \frac{h^2}{2}) \frac{h^2}{2} dx d\theta - \tau_\theta R^2 (R^2 - \frac{h^2}{2}) \frac{h^2}{2} dx d\theta = 0 \]
\[ V_\theta R^2 = \frac{h^2}{2} \tau_\theta R^2 \]

\[ \sum M_y = 0 \]
\[ -(V_x R^2 + \frac{\partial V_x R^2}{\partial x} dx + V_x R^2) R^2 d\theta \frac{dx}{2} \]
\[ +\tau_{Rx} (R^2 + \frac{h^2}{2}) \frac{h^2}{2} dx d\theta + \tau_{Rx} (R^2 - \frac{h^2}{2}) \frac{h^2}{2} dx d\theta = 0 \]
\[ V_x R^2 = \frac{h^2}{2} \tau_{Rx} \]

It must be noted that \( V_{xR^2} \) does not equal \( \tau_{Rx} \); \( V_{xR^2} \) is a force per unit length and \( \tau_{Rx} \) is a stress.
3.3 Equations of Motion

As before the summation of the unit forces in the x, and θ directions, along with the sum of the moments about the x and θ coordinates will be derived for the shell and constraining layers separately. The sum of the forces in the z direction will be derived for the complete structure (shell, VEM, and CL). The effects of the VEM are already accounted for from the kinetics/kinematics in the previous section. The VEM is assumed to undergo pure shear, consequently, the bending, axial, torsional strain energies are assumed negligible (ie. No \( \sigma_{xx} \), \( M_x \), \( M_{\theta} \), etc.). The equations for the constraining layer will be derived first, the equations for the shell second, and lastly the sum of the forces in the z direction for the complete structure.
(i) Constraining Layer (CL)

**Figure 3.3:** Differential element for the constraining layer.

\[ \sum F_x = 0 \]

\[ (V_{x1} + \frac{\partial V_{x1}}{\partial x} dx - V_{x1}) R_1 d\theta + (V_{\theta x1} + \frac{\partial V_{\theta x1}}{\partial \theta} d\theta - V_{\theta x1}) dx + (R_1 + \frac{\eta}{2}) \tau_{Rx1} dxd\theta - \rho_1 h_1 R_{l1} \ddot{u}_{l1} dxd\theta = 0 \]  \hspace{1cm} (3.11)

\[ -R_1 \frac{\partial V_{x1}}{\partial x} \frac{\partial V_{\theta x1}}{\partial \theta} - (R_1 + \frac{\eta}{2}) \tau_{Rx1} + \rho_1 h_1 R_{l1} \ddot{u}_{l1} = 0 \]

\[ \sum F_y = 0 \]

\[ (V_{\theta 1} + \frac{\partial V_{\theta 1}}{\partial \theta} d\theta - V_{\theta 1}) dx \cos(\frac{d\theta}{2}) + (V_{\theta R1} + \frac{\partial V_{\theta R1}}{\partial \theta} d\theta + V_{\theta R1}) dx \sin(\frac{d\theta}{2}) + (V_{x \theta 1} + \frac{\partial V_{x \theta 1}}{\partial x} dx - V_{x \theta 1}) R_1 d\theta + (R_1 + \frac{\eta}{2}) \tau_{R \theta 1} dxd\theta - \rho_1 h_1 R_{l1} \ddot{v}_{l1} dxd\theta = 0 \]  \hspace{1cm} (3.12)

\[ -\frac{\partial V_{\theta 1}}{\partial \theta} - V_{\theta R1} - R_1 \frac{\partial V_{x \theta 1}}{\partial x} - (R_1 + \frac{\eta}{2}) \tau_{R \theta 1} + \rho_1 h_1 R_{l1} \ddot{v}_{l1} = 0 \]
\[ \sum M_x = 0 \]

\[ -(M_{x\theta 1} + \frac{\partial M_{x\theta 1}}{\partial x} dx - M_{x\theta 1}) R_1 d\theta - (M_{\theta 1} + \frac{\partial M_{\theta 1}}{\partial \theta} d\theta - M_{\theta 1}) dx - \\
(R_1 + \frac{h_1}{2}) \tau R_1 dx d\theta + \frac{h_1}{2} (V_{\theta R 1} + \frac{\partial V_{\theta R 1}}{\partial \theta} d\theta + V_{\theta R 1}) dx \cos(\frac{d\theta}{2}) R_1 \sin(\frac{d\theta}{2}) \]

\[ -(V_{\theta 1} + \frac{\partial V_{\theta 1}}{\partial \theta} - V_{\theta 1}) \sin(\frac{d\theta}{2}) (R_1 \frac{d\theta}{2}) = 0 \]

\[ -R_1 \frac{\partial M_{x\theta 1}}{\partial x} - \frac{\partial M_{\theta 1}}{\partial \theta} - (R_1 + \frac{h_1}{2}) \tau R_1 \theta R_1 + R_1 V_{\theta R 1} = 0 \]

\[ \sum M_y = 0 \]

\[ (M_{\theta x 1} + \frac{\partial M_{\theta x 1}}{\partial \theta} d\theta - M_{\theta x 1}) dx + (M_{x 1} + \frac{\partial M_{x 1}}{\partial x} dx - M_{x 1}) R_1 d\theta \\
+(R_1 + \frac{h_1}{2}) \tau R x_1 d\theta \frac{h_1}{2} (V_{x R 1} + \frac{\partial V_{x R 1}}{\partial x} dx + V_{x R 1}) R_1 d\theta \frac{dx}{2} = 0 \]

\[ \frac{\partial M_{\theta x 1}}{\partial \theta} + R_1 \frac{\partial M_{x 1}}{\partial x} + (R_1 + \frac{h_1}{2}) \tau R x_1 - R_1 V_{x R 1} = 0 \]
(ii) Shell

Figure 3.4: Differential element for the shell layer

\[ \sum F_x = 0 \]

\[ (V_{x3} + \frac{\partial V_{x3}}{\partial x} dx - V_{x3}) R_3 d\theta + (V_{\theta x3} + \frac{\partial V_{\theta x3}}{\partial \theta} d\theta - V_{\theta x3}) dx - (R_3 - \frac{h_3}{2}) \tau_{Rx3} dx d\theta \]

\[ -\rho_3 h_3 R_3 u_3 dx d\theta = 0 \]

\[ \sum F_y = 0 \]

\[ (V_{\theta 3} + \frac{\partial V_{\theta 3}}{\partial \theta} d\theta - V_{\theta 3}) dx \cos(d\theta) + (V_{\theta R3} + \frac{\partial V_{\theta R3}}{\partial \theta} d\theta + V_{\theta R3}) dx \sin(d\theta) + \]

\[ (V_{x\theta 3} + \frac{\partial V_{x\theta 3}}{\partial x} dx - V_{x\theta 3}) R_3 d\theta - (R_3 + \frac{h_3}{2}) \tau_{R\theta 3} dx d\theta - \rho_3 h_3 R_3 u_3 dx d\theta = 0 \]

\[ \frac{\partial V_{\theta 3}}{\partial \theta} - V_{\theta R3} R_3 \frac{\partial V_{x\theta 3}}{\partial x} + (R_3 - \frac{h_3}{2}) \tau_{R\theta 3} + \rho_3 h_3 R_3 v_3 = 0 \]
\[ \sum M_x = 0 \]

\[-(M_{x\theta 3} + \frac{\partial M_{x\theta 3}}{\partial x} dx - M_{x\theta 3}) R_3 d\theta - (M_{\theta 3} + \frac{\partial M_{\theta 3}}{\partial \theta} d\theta - M_{\theta 3}) dx \]

\[-(R_3 - \frac{h_3}{2})_R R_3 d\alpha \frac{h_3}{2} + (V_{R3} + \frac{\partial V_{R3}}{\partial \theta} d\theta + V_{R3}) dx \cos(\frac{d\theta}{2}) R_3 \sin(\frac{d\theta}{2}) = 0 \]  

\[-(V_{R3} + \frac{\partial V_{R3}}{\partial \theta} d\theta - V_{R3}) \sin(\frac{d\theta}{2}) R_3 \frac{d\theta}{2} = 0 \]

\[-R_3 \frac{\partial M_{x\theta 3}}{\partial x} - \frac{\partial M_{\theta 3}}{\partial \theta} - (R_3 - \frac{h_3}{2}) \frac{h_3}{2} \tau R_3 + R_3 V_{R3} = 0 \]  

\[\sum M_y = 0\]

\[(M_{x3} + \frac{\partial M_{x3}}{\partial \theta} d\theta - M_{x3}) dx + (M_{x3} + \frac{\partial M_{x3}}{\partial x} dx - M_{x3}) R_3 d\theta \]

\[+(R_3 - \frac{h_3}{2}) R_3 dx \frac{h_3}{2} - (V_{xR3} + \frac{\partial V_{xR3}}{\partial x} dx + V_{xR3}) R_3 d\theta \frac{dx}{2} = 0 \]  

\[\frac{\partial M_{x3}}{\partial \theta} + R_3 \frac{\partial M_{x3}}{\partial x} + (R_3 - \frac{h_3}{2}) \frac{h_3}{2} \tau R_3 x - R_3 V_{xR3} = 0 \]
(iii) Complete Structure (all Layers) (only showing unit forces in Z direction to reduce clutter)

Figure 3.5: Differential element for the complete structure

\[ \sum F_z = 0 \]

\[ (V_{xR1} + \frac{\partial V_{xR1}}{\partial x} dx - V_{xR1}) R_1 d\theta + (V_{\theta R1} + \frac{\partial V_{\theta R1}}{\partial \theta} dx - V_{\theta R1}) dx \cos \left( \frac{d\theta}{2} \right) \]

\[ - (V_{\theta 1} + \frac{\partial V_{\theta 1}}{\partial \theta} d\theta + V_{\theta 1}) dx \sin \left( \frac{d\theta}{2} \right) - \rho_1 h_1 R_1 \ddot{w} d\theta d\theta \]

\[ + (V_{xR2} + \frac{\partial V_{xR2}}{\partial x} dx - V_{xR2}) R_2 d\theta + (V_{\theta R2} + \frac{\partial V_{\theta R2}}{\partial \theta} d\theta - V_{\theta R2}) dx \cos \left( \frac{d\theta}{2} \right) - \rho_2 h_2 R_2 \ddot{w} d\theta d\theta \]

\[ + (V_{xR3} + \frac{\partial V_{xR3}}{\partial x} dx - V_{xR3}) R_3 d\theta + (V_{\theta R3} + \frac{\partial V_{\theta R3}}{\partial \theta} d\theta - V_{\theta R3}) dx \cos \left( \frac{d\theta}{2} \right) \]

\[ - (V_{\theta 3} + \frac{\partial V_{\theta 3}}{\partial \theta} d\theta + V_{\theta 3}) dx \sin \left( \frac{d\theta}{2} \right) - \rho_3 h_3 R_3 \ddot{w} d\theta d\theta = 0 \]

(3.19)

\[ -R_1 \frac{\partial V_{xR1}}{\partial x} + \frac{\partial V_{\theta R1}}{\partial \theta} + V_{\theta 1} + \rho_1 h_1 R_1 \ddot{w} \]

\[ -R_2 \frac{\partial V_{xR2}}{\partial x} + \frac{\partial V_{\theta R2}}{\partial \theta} + \rho_2 h_2 R_2 \ddot{w} \]

\[ -R_3 \frac{\partial V_{xR3}}{\partial x} + \frac{\partial V_{\theta R3}}{\partial \theta} + V_{\theta 3} + \rho_3 h_3 R_3 \ddot{w} = 0 \]
As before the moment equations will be solved for $V_{x\text{Ri}}$ and $V_{\theta\text{Ri}}$, then substituted into the sum of the forces in the x and $\theta$ equations for the shell and constraining layer, and then into the sum of the forces in the $z$ direction equation for the complete structure. This will produce five partial differential equations in terms of the known unit forces and unit moments. The next step is to substitute the unit forces and moments into these five partial differential equations. This will produce five coupled partial differential equations in terms of the five displacements $U1$, $V1$, $U3$, $V3$, and $W$, again in the form of:

$$[M]\ddot{\xi}+[K]\dot{\xi}=0$$

The above system can be solved for, in the same manner as the previous chapter, to find the natural frequencies and mode shapes of a three-layered cylindrical shell. See Appendix B2 for the five partial differential equations in terms of the unit forces and a listing of the unit forces to determining the equivalent mass and stiffness matrices.

### 3.4 Modeling the VEM

Viscoelastic materials have a special property known as a loss factor which gives the VEM the ability to internally dissipate applied mechanical energy by converting it to heat. For additional information on loss factor refer to Sun [28]. Experimental results show that the shear modulus and loss factor for a VEM varies with frequency but for a free vibration response the shear modulus must be assumed constant, otherwise the equations are not solvable. The loss factor and shear modulus of the VEM are assumed to be constant in the present model which is acceptable because the damping effects are studied at a specific frequency. Therefore, this
assumption of constant properties is applicable. If a frequency response function is to be found the change in properties of the VEM must be accounted for, and could be modeled as in Douglas and Yang [29].

To introduce damping to the system the complex stiffness approach will be used where additional information can be found in Thomson [30]. The complex stiffness approach treats the stiffness of the VEM as a spring with damping capabilities. The shear modulus, $G$, for the VEM will now be complex in the form of $G(1 + \eta i)$, where $\eta$ is called the material loss factor and is specified by the manufacturer of the viscoelastic material.

### 3.5 Results

For the three-layered cylindrical shell the Galerkin method is applied for finding the equivalent mass and stiffness matrices so that the natural frequencies could be found. The resulting natural frequencies are compared to the single-layer cylindrical shell while letting the thicknesses of the shell and VEM or CL and VEM approach zero. The thicknesses could not be set to zero because of singularities, but taking the limit produced natural frequencies very close to the single-layer cylinder. The compassion of the natural frequencies for a cylinder of the following dimensions are shown in Table 3.1.

$L = 0.35 \, m$, $h = h_3 = 0.002 \, m$, $R = R_1 = 0.1 \, m$, $E = E_3 = 70 \, Gpa$, $\rho = \rho_3 = 2710 \, Kg/m^3$, $\nu = \nu_3 = 0.3$
Mode Shape | Natural Frequency (Hz)  
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>m</td>
<td>n</td>
<td>Untreated Cylindrical shell</td>
<td>Three-layered Cylindrical shell</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>750</td>
<td>750</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>4452</td>
<td>4452</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>6588</td>
<td>6586</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2597</td>
<td>2595</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>2082</td>
<td>2081</td>
</tr>
</tbody>
</table>

Table 3.1: Comparison of single-layer cylindrical shell to three-layered with the thickness of the CL and VEM equal to .00001 inch

The results in Table 3.1 show that the three-layered shell equations converge to the single-layer shell solutions.

A second verification of the present approach was conducted by using a finite element software package called ANSYS. It can be seen in Table 3.2 that the comparison between the present approach and results from an equivalent finite element model created in ANSYS are in close agreement when calculating the natural frequency for the first bending mode for different CL and VEM thicknesses.

<table>
<thead>
<tr>
<th>Thicknesses of CL and VEM</th>
<th>Natural Frequency (Hz)</th>
<th>ANSYS</th>
<th>Present Analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>h₁ (in)</td>
<td>h₂ (in)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.1</td>
<td>.1</td>
<td>152</td>
<td>161</td>
</tr>
<tr>
<td>.05</td>
<td>.05</td>
<td>164</td>
<td>166</td>
</tr>
<tr>
<td>.01</td>
<td>.01</td>
<td>173</td>
<td>172</td>
</tr>
</tbody>
</table>

Table 3.2: Comparison among the present analysis and ANSYS for first bending mode (m=1, n=1). Material properties: $E₁ = E₃ = 10E^6$ psi, $v₁ = v₃ = .33$, $ρ₁ = ρ₃ = 2.55E^4$ lb $s^2$/in$^4$, $h₃ = .1$ in, $R_o = 3$ in, $E₂ = 49600$ psi, $v₂ = .24$, $ρ₂ = 1.4E^4$ lb $s^2$/in$^4$.
Table 3.3 shows the damping ratios for the 1, 1 (m=1, n=1) mode. With damping in the system the natural frequencies are now complex and the damping factor in the present approach is defined as the imaginary part of the squared resonant frequency divided by the real part of the squared natural frequency, where the damping factor according to results from ANSYS is defined as twice the real part divided by the imaginary part. The difference in definition of the damping factor comes from the different damping approaches used. In the present approach the complex stiffness approach was used to incorporate the damping, where viscous damping was used in ANSYS.

<table>
<thead>
<tr>
<th>Mode Shape</th>
<th>ANSYS</th>
<th>Present Approach</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Natural Frequency (Hz)</td>
<td>Damping Factor (%)</td>
</tr>
<tr>
<td>1 1</td>
<td>-0.8+159i</td>
<td>1%</td>
</tr>
</tbody>
</table>

Table 3.3: Comparison among the present analysis and ANSYS for first bending mode (m=1, n=1).
Material properties: $E_1 = E_3 = 10E^6$ psi, $v_1 = v_3 = .33$, $\rho_1 = \rho_3 = 2.55E^{-4}$ lb/s$^2$/in$^4$, $h_1 = .1$ in, $h_2 = .05$ in, $E_2 = 49600$ psi, $v_2 = .24$, $\rho_2 = 1.4E^{-4}$ lb/s$^2$/in$^4$, $h_2 = .05$ in, $\eta = 1$. Shell Dimensions: L = 60 in, $R_o = 3$ in, $h_3 = .1$ in.

It is seen in Table 3.3 that the results from ANSYS contain higher damping factors than the results from this thesis. The conclusion was somewhat expected due to the different approaches for handling the damping mathematically, and that the ANSYS model included damping associated with extension and bending of the VEM layer, where the present approach only accounted for damping capabilities due to shear.
Chapter 4: Three-Layer Cylindrical Partially Treated

4.1 Introduction

A partially treated cylinder will now be analyzed to investigate the effects on the natural frequency and damping factor with respect to the coverage length, thicknesses, and material combinations. The partially treated cylinder combines the single-layer equations from chapter 2 with the three-layer equations in chapter 3 in such a way to represent a partially treated cylindrical shell as shown in Figure 4.1.

![Diagram of partially constrained shell]

Figure 4.1: Cross sectional view of partially constrained shell

For the single-layer regions \((0 \leq x \leq A, B \leq x \leq L)\) the single-layer equations are used and for the three-layer region \((A \leq x \leq B)\) the three-layer equations are used.
For this procedure the Galerkin method is applied and the limits of integration in the x direction are modified to account for the change in the shell type. A simplified form of this is shown below:

\[
[M] \{\ddot{\xi}_s\} + [K] \{\dot{\xi}_s\} = 0
\]

where

\[
[M] = \int_0^{A} \int_0^{2\pi} [M_{SingleLayer}] d\theta dx + \int_0^{B} \int_0^{2\pi} [M_{ThreeLayer}] d\theta dx + \int_0^{L} \int_0^{2\pi} [M_{SingleLayer}] d\theta dx,
\]

\[
[K] = \int_0^{A} \int_0^{2\pi} [K_{SingleLayer}] d\theta dx + \int_0^{B} \int_0^{2\pi} [K_{ThreeLayer}] d\theta dx + \int_0^{L} \int_0^{2\pi} [K_{SingleLayer}] d\theta dx
\]

where the \(M_{SingleLayer}\) and \(K_{SingleLayer}\) are from the single-layer equations and are shown in appendix B3.

4.2 Results

To confirm the partially treated cylinder equations the coverage length was set equal to the shell length making the partially treated shell equations represent a fully constrained cylindrical shell. Table 4.1 compares the natural frequencies from the fully treated approach and the partially treated approach by letting \(X=A=0\) and \(X=B=L\).
<table>
<thead>
<tr>
<th>Thicknesses of CL and VEM</th>
<th>Natural Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>h₁ (in)</td>
<td>h₂ (in)</td>
</tr>
<tr>
<td>.1</td>
<td>.1</td>
</tr>
<tr>
<td>.05</td>
<td>.05</td>
</tr>
<tr>
<td>.01</td>
<td>.01</td>
</tr>
</tbody>
</table>

Table 4.1: Comparison among the fully treated analysis and the partially treated analysis, when A=0, B=L, for the first bending mode (m=1, n=1). Material properties: E₁ = E₃ = 10E⁶ psi, ν₁ = ν₃ = .33, ρ₁ = ρ₃ = 2.55E⁻⁴ lbf s²/in⁴, E₂ = 49600 psi, ν₂ = .24, ρ₂ = 1.4E⁻⁴ lbf s²/in⁴, η = 1. Shell Dimensions: L = 60 in, R₀ = 3 in, h₃ = .1.

The partially treated results are also compared with results from ANSYS for various CL and VEM thicknesses. The model in ANSYS was created using solid brick elements (type 185) with 20 elements around the circumference and 30 elements along the length. The batch file used for creating the ANSYS model is included in Appendix D. The results from this model and the present approach are shown in Table 4.2, Figures 4.2 and Figure 4.3. It should be noted that a twice as fine mesh did not change the results by more than .5 percent.
Table 4.2: Comparison between ANSYS and present approach for the first bending mode (m=1, n=1).

Material properties: \( E_1 = E_3 = 10 \times 10^6 \text{ psi}, \ \nu_1 = \nu_3 = 0.33, \ \rho_1 = \rho_3 = 2.55 \times 10^4 \text{ lb s}^2/\text{in}^4, \ \h_3 = 0.1 \text{ in}, \ E_2 = 49600 \text{ psi}, \ \nu_2 = 0.24, \ \rho_2 = 1.4 \times 10^4 \text{ lb s}^2/\text{in}^4, \ \eta = 1. \) Shell Dimensions: L = 60 in, \( R_0 = 3 \text{ in}. \)

![Figure 4.2: Comparison between ANSYS and thesis results for natural frequency vs. coverage length (values from Table 4.2 for h1=h2=0.05)"

It is apparent that an offset exists when comparing the resulting natural frequencies from ANSYS and this thesis from Figure 4.2 and may be due to nodal location used in ANSYS.

\( f_n \) = Natural Frequency

\( \text{DF} \) = Damping Factor
ANSYS may have used the radius entered as the mid-plane radius rather than the outermost radius, which would make the shell slightly larger than what was used in this thesis and would offset the results in a similar manner to what is shown in Figure 4.2

![Damping Factor vs Coverage Length](image)

Figure 4.3: Comparison between ANSYS and thesis results for damping factor vs. coverage length (values from Table 4.2 h1=h2=.05")

It is seen in Figure 4.3 that results from ANSYS contain higher damping factors than the results from this thesis. As stated in Chapter 3 ANSYS includes damping capabilities from extension and contraction of the viscoelastic material, where the present thesis does not.

Another possibility for the differences in the results between ANSYS and the present approach, for both the natural frequency and damping factor, could be due to the assumed shape functions. The three-layer portion of the shaft will obviously reinforce the shell layer and therefore there may be a slight “kink” in the deformed shaft if the CL and VEM are stiff enough, due to material type and/or thicknesses. This “kinking” phenomenon is shown in Figure 4.4, but was not apparent in the ANSYS model.
To gain insight on how the natural frequencies and damping factors varied as some of the key parameters for the system differed; ratios for four key parameters were created and are shown in Table 4.3.

<table>
<thead>
<tr>
<th>Ratio</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>h1/h3</td>
<td>ratio of CL thickness to shell thickness</td>
</tr>
<tr>
<td>h2/h3</td>
<td>ratio of VEM thickness to shell thickness</td>
</tr>
<tr>
<td>E1/E3</td>
<td>ratio of CL Young's modulus to shells Young's modulus</td>
</tr>
<tr>
<td>G2/E3</td>
<td>ratio of VEM shear modulus to shells Young's modulus</td>
</tr>
</tbody>
</table>

Table 4.3: Ratio's of four key parameters

Figures 4.5 through 4.12 show the effects of the non-dimensionalized parameters on the natural frequency and damping factor as the coverage length varies, for m = 1 n = 1 mode, due to the variation of parameters in Table 4.3. The base material and geometrical properties used in the following figures are given below:

\[ E_1 = E_3 = 10E^6 \text{ psi}, \ E_2 = 49600 \text{ psi}, \ h_1 = .05 \text{ in}, \ h_2 = .05 \text{ in}, \ \nu_1 = \nu_3 = .33, \ \nu_2 = .24, \ \rho_1 = \rho_3 = 2.55E^4 \text{ lbf s}^2/\text{in}^4, \ \rho_2 = 1.4E^4 \text{ lbf s}^2/\text{in}^4, \ \eta = 1, \ L = 60 \text{ in}, \ R_o = 3 \text{ in}, \ h_3 = .1 \text{ in} \]
Figure 4.5: Natural Frequency variation with h1/h3

Figure 4.6: Damping Factor variation with h1/h3
Figure 4.7: Natural Frequency variation with h2/h3

Figure 4.8: Damping Factor variation with h2/h3
Figure 4.9: Natural Frequency variation with E1/E3

Figure 4.10: Damping Factor variation with E1/E3
Figure 4.11: Natural Frequency variation with G2/E3

Figure 4.12: Damping factor variation with G2/E3
Figures 4.5 and 4.6 both show that the natural frequency and damping factor increase as the thickness of the CL increases. Figure 4.6 also shows that there is an optimal coverage length for achieving maximum damping factor, which depends on CL thickness.

Figure 4.7 shows that the natural frequency decreases as the VEM thickness increases, where Figure 4.8 shows that thicker VEM produces better damping capability and that an optimal coverage length exists.

Figures 4.9 and 4.10, similar to the effects of increasing h1/h3, show that both the natural frequency and damping factor increase with increasing Young’s modulus for the CL.

Figure 4.11 shows that the natural frequency slightly increases as the shear moduli of the VEM increases. It is shown in Figure 4.12 that an optimal VEM shear moduli exists to achieve the highest damping factor. The red line is the softest shear moduli shown, where the black line is the stiffest moduli shown, and the peak damping factor is achieved between these two values.

Overall Figures 4.5 through 4.12 shown that an optimal coverage length exists, along with an optimal VEM shear moduli.

Results are shown in Figures 4.13 through 4.20 for a shell with thickness of .3 inches with the same variations as in Figures 4.5 through 4.12.
Figure 4.13: Natural Frequency variation with h1/h3

Figure 4.14: Damping Factor variation with h1/h3
Figure 4.15: Natural Frequency variation with h2/h3

Figure 4.16: Damping Factor variation with h2/h3
Figure 4.17: Natural Frequency variation with E1/E3

Figure 4.18: Damping Factor variation with E1/E3
Figure 4.19: Natural Frequency variation with G2/E3

Figure 4.20: Damping Factor variation with G2/E3
Overall Figures 4.13 through 4.20 show similar results to Figures 4.5 through 4.12 except larger damping factors were achieved. The natural frequency actually decreased in most cases, for the case of a thicker shell, which at first may not make sense, but one must remember that the VEM has also gotten thicker because the ratio of h2 to h3 were kept constant, which adds mass to the system without adding much stiffness. Figure 4.20 shows that the optimal ratio of shear moduli of the VEM to the shells Young’s modulus has changed from .0002 to .002 for the thicker shell.

4.3 Conclusion

Overall this thesis proposes an adequate analytical model of a untreated cylindrical shaft along with a partially treated shaft with constrained layer damping. The single-layer results were verified by comparisons with published values along with a well known finite element program; ANSYS. The natural frequencies for the three-layered cylindrical shaft compared well with results from ANSYS and were close to some published values for a fully treated shell. The results in section 4.2 show that no significant damping is created for a cylindrical shell partially treated with cylindrical constrained layer damping for the first bending mode. It must be noted that for other modes, damping capabilities may differ significantly but for the shaft analyzed in this thesis the lowest natural frequency was the first bending mode, and was of importance. An important finding from this thesis is that an optimal thickness, coverage length, and shear modulus for the VEM along with an optimal CL thickness exist for developing the maximum damping factor, which agree with some publicized conclusions.
4.4 Future Considerations

There are some areas where this thesis topic could be expanded and/or

- Introduce new shape functions which allow the “kinking” phenomenon. Refer to [31].

- Incorporate the use of composite materials, where the layer orientations could be analyzed.

- Include the energy dissipation due to the expansion and contraction of the VEM in the longitudinal and circumferential directions\(^9\).

- Conduct experiments to gain confidence in damping factor and natural frequencies for the partially treated shell.

\(^9\) Sylwan [32] includes compressional damping
REFERENCES


Appendix A: Mode Shapes

RADIAL MODE SHAPES

Longitudinal wave number = \( m \)

Circumferential wave number = \( n \)

\( m=1, n=0 \) (First breathing mode)

\( m=1, n=1 \) (First bending Mode)
m=1, n=2

Change in time

m=1, n=3

Change in time

m=2, n=1

Change in time
The programs which were used are given below. The first program code is used in Mathematica to plot the radial mode animations. The second and third programs are m files used in MATLAB to animate longitudinal and circumferential modes.

(*-----------------------------RadialModes.nb*-----------------------------*)

<< Graphics'Animation`
<< Graphics'SurfaceOfRevolution`
m = 2; n = 2;
w = 100; R = .1; L = .35; λ = \(\frac{m \pi}{L}\);
\[ \text{Modeshape} = \]
\[
\text{Table} \left[ \text{SurfaceOfRevolution} \left[ \right. \right.
\begin{align*}
& R + R/3 \sin[m \pi / L x] \cos[n \theta] \cos[w t], \{x, 0, L\}, \\
& \{\theta, 0, 2 \pi\}, \text{Axes} \rightarrow \text{False}, \text{Boxed} \rightarrow \text{False}, \\
& \text{RevolutionAxis} \rightarrow \{1, 0, 0\}, \text{PlotPoints} \rightarrow 50, \\
& \text{PlotRange} \rightarrow \{\{0, L\}, \{-1.5 R, 1.5 R\}, \{-1.5 R, 1.5 R\}\}, \\
& \text{ViewPoint} \rightarrow \{2.216, -1.963, 0.871\}, \\
& \{t, 0, 2 \pi / w, 2 \pi / w / 8\} \right] \right. \]
\[
% ++++++ MATLAB PROGRAM #1 ++++++
%
% Longitudinal Mode shape Animations
%
% MATLAB m file

n=0;

m=2;

L=5;

x00=0;

x11=1;

x22=2;

x33=3;

x44=4; x55=5;

ti=1;

i=1;

dtheta=2*pi/50;

dtime=2*pi/20;

for time=0:dtime:2*pi-.0001
for theta=0: dtheta: 2* pi: 0.0001

    time;

    x0(l,i)=x00+.1*cos(m*pi/L*x00)*cos(n*theta)*cos(time);
    y0(l,i)=sin(theta);
    z0(l,i)=cos(theta);

    x1(l,i)=x11+.1*cos(m*pi/L*x11)*cos(n*theta)*cos(time);
    y1(l,i)=sin(theta);
    z1(l,i)=cos(theta);

    x2(l,i)=x22+.1*cos(m*pi/L*x22)*cos(n*theta)*cos(time);
    y2(l,i)=sin(theta);
    z2(l,i)=cos(theta);

    x3(l,i)=x33+.1*cos(m*pi/L*x33)*cos(n*theta)*cos(time);
    y3(l,i)=sin(theta);
    z3(l,i)=cos(theta);

    x4(l,i)=x44+.1*cos(m*pi/L*x44)*cos(n*theta)*cos(time);
    y4(l,i)=sin(theta);
    z4(l,i)=cos(theta);

    x5(l,i)=x55+.1*cos(m*pi/L*x55)*cos(n*theta)*cos(time);
    y5(l,i)=sin(theta);
    z5(l,i)=cos(theta);

    i=i+1;

end
view([-15 10])

Figure(1);

set(1,'Position',[150,150,700,500]);

set(1,'Visible','off');

axis([-1 L+1 -2 2 -2 2]);

axis manual

hold on

p1=plot3(x0,y0,z0,'r', 'LineStyle','none','Marker','.','EraseMode','xor');

p2=plot3(x1,y1,z1,'b', 'LineStyle','none','Marker','.','EraseMode','xor');

p3=plot3(x2,y2,z2,'g', 'LineStyle','none','Marker','.','EraseMode','xor');

p4=plot3(x3,y3,z3,'r', 'LineStyle','none','Marker','.','EraseMode','xor');

p5=plot3(x4,y4,z4,'b', 'LineStyle','none','Marker','.','EraseMode','xor');

p6=plot3(x5,y5,z5,'g', 'LineStyle','none','Marker','.','EraseMode','xor');

grid on

hold off

F(:,ti)= getframe;

hold off

ti=ti+1;

i=1;

delete(p1,p2,p3,p4,p5,p6);

end

close(1))
pause(2)
view([-15 10])

Figure(1);

set(1,'Position',[150,150,700,500]);
axis([-1 L+1 -2 -2 2 2]);
axis manual
xlabel=('x');

movie(F, 5,50);

%+++++++++++++++++++++++++++++++++++++++ MATLAB PROGRAM #1 ++++++++%

%Circumferential mode animations

n=1;
m=1;
dtheta=2*pi/50;
dtime=2*pi/20;
L=5;
x00=0;
x11=1;
x22=2;
x33=3;
x44=4;
x55=5;
ti=1;
i=1;
for time=0:dtime:2*pi-.0001
for theta=0:dtheta:2*pi-.0001
x0(1,i)=x00;
y0(1,i)=sin(.1*sin(m*pi/L*x00)*sin(n*theta)*cos(time)+theta);
z0(1,i)=cos(.1*sin(m*pi/L*x00)*sin(n*theta)*cos(time)+theta);
x1(1,i)=x11;
y1(1,i)=sin(.1*sin(m*pi/L*x11)*sin(n*theta)*cos(time)+theta);
z1(1,i)=cos(.1*sin(m*pi/L*x11)*sin(n*theta)*cos(time)+theta);
x2(1,i)=x22;
y2(1,i)=sin(.1*sin(m*pi/L*x22)*sin(n*theta)*cos(time)+theta);
z2(1,i)=cos(.1*sin(m*pi/L*x22)*sin(n*theta)*cos(time)+theta);
x3(1,i)=x33;
y3(1,i)=sin(.1*sin(m*pi/L*x33)*sin(n*theta)*cos(time)+theta);
z3(1,i)=cos(.1*sin(m*pi/L*x33)*sin(n*theta)*cos(time)+theta);
x4(1,i)=x44;
y4(1,i)=sin(.1*sin(m*pi/L*x44)*sin(n*theta)*cos(time)+theta);
z4(1,i)=cos(.1*sin(m*pi/L*x44)*sin(n*theta)*cos(time)+theta);
x5(1,i)=x55;
y5(1,i)=sin(.1*sin(m*pi/L*x55)*sin(n*theta)*cos(time)+theta);
z5(1,i)=cos(.1*sin(m*pi/L*x55)*sin(n*theta)*cos(time)+theta);
i=i+1;
end

view([-15 10])

Figure(1);

set(1,'Position',[150,150,700,500]);

set(1,'Visible','off');

axis([-1 L+1 -2 2 -2 2]);

axis manual

hold on

p1=plot3(x0,y0,z0,'r', 'LineStyle','none','Marker','.','EraseMode','xor');

p2=plot3(x1,y1,z1,'b', 'LineStyle','none','Marker','.','EraseMode','xor');

p3=plot3(x2,y2,z2,'g', 'LineStyle','none','Marker','.','EraseMode','xor');

p4=plot3(x3,y3,z3,'r', 'LineStyle','none','Marker','.','EraseMode','xor');

p5=plot3(x4,y4,z4,'b', 'LineStyle','none','Marker','.','EraseMode','xor');

p6=plot3(x5,y5,z5,'g', 'LineStyle','none','Marker','.','EraseMode','xor');

grid on

hold off

F(:,ti)=getframe(1,[91,55,700,500]);

ti=ti+1;

i=1;

delete(p1,p2,p3,p4,p5,p6);

end

close((1))
pause(2)
view([-15 10])

Figure(1);

set(1,'Position',[150,150,700,500]);

axis([-1 L+1 -2 2 -2 2]);

axis manual

xlabel=('x');

movie(F, 10,50);

%clear all
Appendix B: Unit Forces

B.1 Single-Layer Shell

Equations of Motion for Single Layer Cylindrical Shell:

\[-R \frac{\partial V}{\partial x} - \frac{\partial V}{\partial \theta} + \rho h R \ddot{u} = 0\]
\[-\frac{\partial V}{\partial \theta} - \frac{\partial M_{x\theta}}{\partial x} - \frac{1}{R} \frac{\partial M_{\theta}}{\partial \theta} - R \frac{\partial V}{\partial x} + \rho h R \ddot{v} = 0\]
\[-\frac{\partial^2 M_{x\theta}}{\partial x \partial \theta} - R \frac{\partial^2 M_x}{\partial x^2} - \frac{\partial^2 M_{x\theta}}{\partial x \partial \theta} - \frac{1}{R} \frac{\partial^2 M_{\theta}}{\partial \theta^2} + V_{x} + \rho h R \ddot{w} = 0\]

Unit Forces For Single Cylindrical Shell:

\[V_x = K \left( \frac{\partial U}{\partial x} - \frac{h^2}{12R} \frac{\partial^2 W}{\partial x^2} - \frac{h^2}{12R^3} \frac{\partial \nu}{\partial x} - \frac{h^2}{12R^3} \frac{\partial V}{\partial \theta} - \nu \frac{\partial W}{\partial R} + \frac{\nu}{R} \frac{\partial U}{\partial \theta} \right)\]
\[V_{x\theta} = V_{\theta x} = K(1-v) \left( \frac{1}{2} \frac{\partial V}{\partial x} + \frac{h^2}{24R^2} \frac{\partial^2 V}{\partial x \partial \theta} - \frac{h^2}{12R^2} \frac{\partial^2 W}{\partial x \partial \theta} + \frac{1}{2} \frac{\partial U}{\partial \theta} \right)\]
\[V_{\theta} = \frac{K}{R} \left( \frac{\partial U}{\partial \theta} + W + \nu R \frac{\partial U}{\partial x} \right)\]
\[M_{x\theta} = K(1-v) \left( \frac{h^2}{24R^2} \frac{\partial U}{\partial x} + \frac{h^2}{12R} \frac{\partial V}{\partial \theta} - \frac{h^2}{12R} \frac{\partial^2 W}{\partial x \partial \theta} \right)\]
\[M_{\theta x} = K(1-v) \left( \frac{h^2}{24R} \frac{\partial U}{\partial x} - \frac{h^2}{12R} \frac{\partial^2 W}{\partial x \partial \theta} \right)\]
\[M_{\theta} = K \left( -\frac{h^2}{12R^2} \frac{\partial V}{\partial \theta} - \frac{h^2}{12R^2} \frac{\partial^2 W}{\partial \theta^2} - \nu h^2 \frac{\partial^2 W}{\partial x^2} \right)\]
\[M_x = K \left( -\frac{h^2}{12} \frac{\partial^2 W}{\partial x^2} + \frac{h^2}{12R} \frac{\partial U}{\partial x} - \nu h^2 \frac{\partial^2 W}{\partial x \partial \theta} + \nu h^2 \frac{\partial^2 W}{\partial \theta^2} \right)\]
\[K = \frac{Eh}{1-v^2}\]
B.2 Three-Layer Shell

Equation of motion for Three-Layer Cylindrical Shell:

\[-R_1 \frac{\partial V_{x1}}{\partial x} - \frac{\partial V_{\theta1}}{\partial \theta} - \left( R_1 + \frac{h_1}{2} \right) \tau_{R11} + \rho_1 h_1 R_1 \ddot{U}_1 = 0 \]

\[-\frac{\partial V_{\theta1}}{\partial \theta} - \frac{\partial M_{x\theta1}}{\partial x} - \frac{1}{R_1} \frac{\partial M_{\theta1}}{\partial \theta} \left( \frac{h_1}{2} + \frac{h_1^2}{4R_1} + R_1 \right) \tau_{R\theta1} - R_1 \frac{\partial V_{x\theta1}}{\partial x} + \rho_1 h_1 R_1 \ddot{V}_1 = 0 \]

\[-R_3 \frac{\partial V_{x3}}{\partial x} - \frac{\partial V_{\theta3}}{\partial \theta} + \left( R_3 - \frac{h_3}{2} \right) \tau_{R33} + \rho_3 h_3 R_3 \ddot{U}_3 = 0 \]

\[-\frac{\partial V_{\theta3}}{\partial \theta} - \frac{\partial M_{x\theta3}}{\partial x} - \frac{1}{R_3} \frac{\partial M_{\theta3}}{\partial \theta} \left( \frac{h_3}{2} - \frac{h_3^2}{4R_3} - R_3 \right) \tau_{R\theta3} - R_3 \frac{\partial V_{x\theta3}}{\partial x} + \rho_3 h_3 R_3 \ddot{V}_3 = 0 \]

\[-\frac{\partial^2 M_{x\theta1}}{\partial x \partial \theta} - R_1 \frac{\partial^2 M_{x1}}{\partial x^2} - \left( R_1 + \frac{h_1}{2} \right) \frac{h_1}{2} \frac{\partial \tau_{R\theta1}}{\partial x} - \frac{\partial^2 M_{x\theta1}}{\partial x \partial \theta} - \frac{1}{R_1} \frac{\partial^2 M_{\theta1}}{\partial \theta^2} \]

\[-\left( R_1 + \frac{h_1}{2} \right) \frac{h_1}{2} \frac{\partial \tau_{R\theta1}}{\partial x} + V_{\theta1} \]

\[-R_2 \frac{\partial V_{x2}}{\partial x} - \frac{\partial \tau_{Rx2}}{\partial \theta} \]

\[-\frac{\partial^2 M_{x\theta2}}{\partial x \partial \theta} - R_3 \frac{\partial^2 M_{x2}}{\partial x^2} + \left( R_3 - \frac{h_3}{2} \right) \frac{h_3}{2} \frac{\partial \tau_{Rx2}}{\partial x} - \frac{\partial^2 M_{x\theta2}}{\partial x \partial \theta} - \frac{1}{R_3} \frac{\partial^2 M_{\theta2}}{\partial \theta^2} \]

\[-\left( R_3 - \frac{h_3}{2} \right) \frac{h_3}{2} \frac{\partial \tau_{Rx3}}{\partial x} + V_{\theta3} \]

\[+ \left( \rho_1 R_1 h_1 + \rho_2 R_2 h_2 + \rho_3 R_3 h_3 \right) \ddot{W} = 0 \]
Unit Forces for Three Layer Cylindrical Shell:

\[ V_{xi} = K_i \left( \frac{\partial U_i}{\partial x} - \frac{h_i}{12R_i} \frac{\partial^3 W}{\partial x^3} - \frac{h_i^2}{12R_i^2} \frac{\partial U_i}{\partial \theta^2} + \frac{\nu_i \partial V_i}{R_i \frac{\partial \theta^2}{\partial x}} + \frac{\nu_i \partial V_i}{R_i \frac{\partial x}{\partial \theta}} \right) \]

\[ V_{\theta i} = V_{\theta x i} = K_i (1 - \nu_i) \left( \frac{1}{2} \frac{\partial V_i}{\partial x} + \frac{h_i}{24R_i^2} \frac{\partial V_i}{\partial \theta} - \frac{h_i^2}{12R_i^2} \frac{\partial^3 W}{\partial \theta \partial x^2} + \frac{1}{2R_i} \frac{\partial U_i}{\partial \theta} \right) \]

\[ V_\phi = \frac{K_i}{R_i} \left( \frac{\partial V_i}{\partial \theta} + W + \nu_i R_i \frac{\partial U_i}{\partial x} \right) \]

\[ M_{x i} = K_i (1 - \nu_i) \left( \frac{h_i}{24R_i} \frac{\partial U_i}{\partial x} + \frac{h_i}{12R_i} \frac{\partial V_i}{\partial \theta} - \frac{h_i^2}{12R_i} \frac{\partial^3 W}{\partial \theta \partial x^2} \right) \]

\[ M_{\theta i} = K_i (1 - \nu_i) \left( \frac{h_i}{24R_i} \frac{\partial V_i}{\partial x} - \frac{h_i}{12R_i} \frac{\partial^3 W}{\partial \theta \partial x^2} \right) \]

\[ M_{\phi i} = K_i \left( - \frac{h_i^2}{12R_i^2} \frac{\partial^2 W}{\partial \theta \partial x^2} - \frac{h_i}{12R_i^2} \frac{\partial^2 U_i}{\partial x^2} + \frac{\nu_i h_i^2}{12} \frac{\partial^2 W}{\partial x^2} \right) \]

\[ M_{xi} = K_i \left( - \frac{h_i^2}{12} \frac{\partial^2 W}{\partial x^2} + \frac{h_i^2}{12R_i} \frac{\partial U_i}{\partial x} - \frac{\nu_i h_i^2}{12R_i} \frac{\partial^2 W}{\partial \theta^2} + \frac{\nu_i h_i^2}{12R_i} \frac{\partial^2 W}{\partial \theta^2} \right) \]

\[ K_i = \frac{E_i h_i}{1 - \nu_i^2} \]

where \( i = 1 \) or \( 3 \) representing the constraining layer or shell.
B.3 Three-Layer Partially Treated Shell Mass and Stiffness Matrices

\[
[M] = \begin{bmatrix}
M_{11} & 0 & 0 & 0 & 0 \\
0 & M_{22} & 0 & 0 & 0 \\
0 & 0 & M_{33} & 0 & 0 \\
0 & 0 & 0 & M_{44} & 0 \\
0 & 0 & 0 & 0 & M_{55}
\end{bmatrix}, \quad
[K] = \begin{bmatrix}
K_{11} & K_{12} & K_{13} & K_{14} & K_{15} \\
K_{21} & K_{22} & K_{23} & K_{24} & K_{25} \\
K_{31} & K_{32} & K_{33} & K_{34} & K_{35} \\
K_{41} & K_{42} & K_{43} & K_{44} & K_{45} \\
K_{51} & K_{52} & K_{53} & K_{54} & K_{55}
\end{bmatrix},
\]

where,

\[
M_{11} = \int_A^B \int_0^{2\pi} M_{ulul} R_l d\theta d\chi;
\]

\[
K_{11} = \int_A^B \int_0^{2\pi} K_{ulul} R_l d\theta d\chi;
\]

\[
K_{12} = \int_A^B \int_0^{2\pi} K_{ulvl} R_l d\theta d\chi;
\]

\[
K_{13} = \int_A^B \int_0^{2\pi} K_{ulul3} R_l d\theta d\chi;
\]

\[
K_{14} = \int_A^B \int_0^{2\pi} K_{ulvl3} R_l d\theta d\chi;
\]

\[
K_{15} = \int_A^B \int_0^{2\pi} K_{ululw} R_l d\theta d\chi;
\]

\[
M_{22} = \int_A^B \int_0^{2\pi} M_{vlvl} R_l d\theta d\chi;
\]

\[
K_{21} = \int_A^B \int_0^{2\pi} K_{vlul} R_l d\theta d\chi;
\]

\[
K_{22} = \int_A^B \int_0^{2\pi} K_{vlvl} R_l d\theta d\chi;
\]

\[
K_{213} = \int_A^B \int_0^{2\pi} K_{vlul3} R_l d\theta d\chi;
\]

\[
K_{24} = \int_A^B \int_0^{2\pi} K_{vlvl3} R_l d\theta d\chi;
\]

\[
K_{25} = \int_A^B \int_0^{2\pi} K_{vlvlw} R_l d\theta d\chi;
\]
\[
\begin{align*}
M_{33} &= \int_{A}^{B} \int_{0}^{2\pi} \mu_{3u3single} R_3 d\theta \, dx + \int_{A}^{B} \int_{0}^{2\pi} \mu_{3u3} R_3 d\theta \, dx + \int_{B}^{L} \int_{0}^{2\pi} \mu_{3u3single} R_3 d\theta \, dx; \\
K_{31} &= \int_{A}^{B} \int_{0}^{2\pi} \kappa_{3u1} R_3 d\theta \, dx; \\
K_{32} &= \int_{A}^{B} \int_{0}^{2\pi} \kappa_{3v1} R_3 d\theta \, dx; \\
K_{33} &= \int_{A}^{B} \int_{0}^{2\pi} \kappa_{3u3single} R_3 d\theta \, dx + \int_{A}^{B} \int_{0}^{2\pi} \kappa_{3u3} R_3 d\theta \, dx + \int_{B}^{L} \int_{0}^{2\pi} \kappa_{3u3single} R_3 d\theta \, dx; \\
K_{34} &= \int_{A}^{B} \int_{0}^{2\pi} \kappa_{3v3single} R_3 d\theta \, dx + \int_{A}^{B} \int_{0}^{2\pi} \kappa_{3v3} R_3 d\theta \, dx + \int_{B}^{L} \int_{0}^{2\pi} \kappa_{3v3single} R_3 d\theta \, dx; \\
K_{35} &= \int_{A}^{B} \int_{0}^{2\pi} \kappa_{3wsingle} R_3 d\theta \, dx + \int_{A}^{B} \int_{0}^{2\pi} \kappa_{3w} R_3 d\theta \, dx + \int_{B}^{L} \int_{0}^{2\pi} \kappa_{3wsingle} R_3 d\theta \, dx; \\
M_{44} &= \int_{A}^{B} \int_{0}^{2\pi} \mu_{3v3single} R_3 d\theta \, dx + \int_{A}^{B} \int_{0}^{2\pi} \mu_{3v3} R_3 d\theta \, dx + \int_{B}^{L} \int_{0}^{2\pi} \mu_{3v3single} R_3 d\theta \, dx; \\
K_{41} &= \int_{A}^{B} \int_{0}^{2\pi} \kappa_{3v1} R_3 d\theta \, dx; \\
K_{42} &= \int_{A}^{B} \int_{0}^{2\pi} \kappa_{3v3} R_3 d\theta \, dx; \\
K_{43} &= \int_{A}^{B} \int_{0}^{2\pi} \kappa_{3u3single} R_3 d\theta \, dx + \int_{A}^{B} \int_{0}^{2\pi} \kappa_{3u3} R_3 d\theta \, dx + \int_{B}^{L} \int_{0}^{2\pi} \kappa_{3u3single} R_3 d\theta \, dx; \\
K_{44} &= \int_{A}^{B} \int_{0}^{2\pi} \kappa_{3v3single} R_3 d\theta \, dx + \int_{A}^{B} \int_{0}^{2\pi} \kappa_{3v3} R_3 d\theta \, dx + \int_{B}^{L} \int_{0}^{2\pi} \kappa_{3v3single} R_3 d\theta \, dx; \\
K_{45} &= \int_{A}^{B} \int_{0}^{2\pi} \kappa_{3wsingle} R_3 d\theta \, dx + \int_{A}^{B} \int_{0}^{2\pi} \kappa_{3w} R_3 d\theta \, dx + \int_{B}^{L} \int_{0}^{2\pi} \kappa_{3wsingle} R_3 d\theta \, dx; \\
M_{55} &= \int_{A}^{B} \int_{0}^{2\pi} \mu_{3w1} R_1 d\theta \, dx + \int_{A}^{B} \int_{0}^{2\pi} \mu_{2w1} R_2 d\theta \, dx + \int_{A}^{B} \int_{0}^{2\pi} \mu_{3w1single} R_3 d\theta \, dx \\
&\hspace{1cm}+ \int_{A}^{B} \int_{0}^{2\pi} \mu_{3w1} R_3 d\theta \, dx + \int_{B}^{L} \int_{0}^{2\pi} \mu_{3w1single} R_3 d\theta \, dx; \\
K_{51} &= \int_{A}^{B} \int_{0}^{2\pi} \kappa_{1u1} R_1 d\theta \, dx + \int_{A}^{B} \int_{0}^{2\pi} \kappa_{2u1} R_2 d\theta \, dx + \int_{A}^{B} \int_{0}^{2\pi} \kappa_{3u1} R_3 d\theta \, dx; \\
K_{52} &= \int_{A}^{B} \int_{0}^{2\pi} \kappa_{1v1} R_1 d\theta \, dx + \int_{A}^{B} \int_{0}^{2\pi} \kappa_{2v1} R_2 d\theta \, dx + \int_{A}^{B} \int_{0}^{2\pi} \kappa_{3v1} R_3 d\theta \, dx; \\
K_{53} &= \int_{A}^{B} \int_{0}^{2\pi} \kappa_{1u3} R_1 d\theta \, dx + \int_{A}^{B} \int_{0}^{2\pi} \kappa_{2u3} R_2 d\theta \, dx + \int_{A}^{B} \int_{0}^{2\pi} \kappa_{3u3} R_3 d\theta \, dx \\
&\hspace{1cm}+ \int_{A}^{B} \int_{0}^{2\pi} \kappa_{3u3} R_3 d\theta \, dx + \int_{B}^{L} \int_{0}^{2\pi} \kappa_{3u3} R_3 d\theta \, dx; \\
K_{54} &= \int_{A}^{B} \int_{0}^{2\pi} \kappa_{1v3} R_1 d\theta \, dx + \int_{A}^{B} \int_{0}^{2\pi} \kappa_{2v3} R_2 d\theta \, dx + \int_{A}^{B} \int_{0}^{2\pi} \kappa_{3v3} R_3 d\theta \, dx \\
&\hspace{1cm}+ \int_{A}^{B} \int_{0}^{2\pi} \kappa_{3v3} R_3 d\theta \, dx + \int_{B}^{L} \int_{0}^{2\pi} \kappa_{3v3} R_3 d\theta \, dx; \\
K_{55} &= \int_{A}^{B} \int_{0}^{2\pi} \kappa_{1w1} R_1 d\theta \, dx + \int_{A}^{B} \int_{0}^{2\pi} \kappa_{2w2} R_2 d\theta \, dx + \int_{A}^{B} \int_{0}^{2\pi} \kappa_{3w1} R_3 d\theta \, dx \\
&\hspace{1cm}+ \int_{A}^{B} \int_{0}^{2\pi} \kappa_{3w3} R_3 d\theta \, dx + \int_{B}^{L} \int_{0}^{2\pi} \kappa_{3w3} R_3 d\theta \, dx; \\
\end{align*}
\]
and

\[ M_{u11} = h_1 R_1 \rho_1 \cos[n \theta]^2 \cos[\frac{m \pi x}{L}]^2 \]

\[ K_{u11} = \cos[n \theta]^2 \cos[\frac{m \pi x}{L}]^2 \left( \frac{G_2 \left( \frac{h_1}{2} + R_1 \right)}{h_2} + \frac{E_1 h_1 n^2}{2 R_1 (1 + \nu_1)} + \frac{E_1 h_1 m^2 \pi^2 R_1}{L^2 (1 - \nu_1^2)} \right) \]

\[ K_{u1v1} = \cos[n \theta]^2 \cos[\frac{m \pi x}{L}]^2 \left( \frac{-E_1 h_1 m n \pi}{2 L (1 + \nu_1)} - \frac{E_1 h_1^3 m n \pi}{24 L R_1^2 (1 + \nu_1)} \right. \]

\[ \left. - \frac{E_1 h_1 m \pi \nu_1}{L (1 - \nu_1^2)} - \frac{E_1 h_1^3 m \pi \nu_1}{12 L R_1^2 (1 - \nu_1^2)} \right) \]

\[ K_{u1u3} = -\cos[n \theta]^2 \cos[\frac{m \pi x}{L}]^2 \left( \frac{G_2 \left( \frac{h_1}{2} + R_1 \right)}{h_2} \right) \]

\[ K_{u1v3} = 0 \]

\[ K_{u1w} = \cos[n \theta]^2 \cos[\frac{m \pi x}{L}]^2 \left( -\frac{G_2 m \pi \left( \frac{h_1}{2} + R_1 \right)}{L} - \frac{G_2 h_1 m \pi \left( \frac{h_1}{2} + R_1 \right)}{2 h_2 L} \right. \]

\[ \left. - \frac{G_2 h_2 m \pi \left( \frac{h_1}{2} + R_1 \right)}{2 h_2 L} - \frac{E_1 h_1^3 m^2 \pi}{12 L R_1^2 (1 + \nu_1)} - \frac{E_1 h_1^3 m^3 \pi}{12 L^3 (1 - \nu_1^2)} \right. \]

\[ \left. - \frac{E_1 h_1^3 m \pi \nu_1}{L (1 - \nu_1^2)} - \frac{E_1 h_1^3 m \pi \nu_1}{12 L R_1^2 (1 - \nu_1^2)} \right) \]

\[ M_{v11} = h_1 R_1 \rho_1 \sin[\frac{m \pi x}{L}]^2 \sin[n \theta]^2 \]

\[ K_{v11} = \sin[\frac{m \pi x}{L}]^2 \sin[n \theta]^2 \left( \frac{-E_1 h_1 m n \pi}{2 L (1 + \nu_1)} - \frac{E_1 h_1^3 m n \pi S}{24 L R_1^2 (1 + \nu_1)} - \frac{E_1 h_1 m \pi \nu_1}{L (1 - \nu_1^2)} \right) \]

\[ K_{v1v1} = \sin[\frac{m \pi x}{L}]^2 \sin[n \theta]^2 \left( \frac{G_2 \left( \frac{h_1}{2} + R_1 \right)}{2 R_2} + \frac{G_2 h_1^2 \left( \frac{h_1}{2} + R_1 \right)}{8 R_2 R_2} + \frac{G_2 h_2 \left( \frac{h_1}{2} + R_1 \right) S}{2 R_2 R_2} \right. \]

\[ \left. + \frac{E_1 h_1^3 m^2 \pi^2}{8 L^2 R_1 (1 + \nu_1)} + \frac{E_1 h_1^3 m^2 \pi^2 R_1}{2 L^2 (1 + \nu_1)} + \frac{E_1 h_1^3 n^2}{12 R_1^3 (1 - \nu_1^2)} + \frac{E_1 h_1 n^2}{R_1 (1 - \nu_1^2)} \right) \]

\[ K_{v1u3} = 0 \]
$K_{v1v3} = \sin\left[\frac{m \pi x}{L}\right]^2 \sin[n \phi]^2 \left\{ -\frac{G_2 \left(\frac{h_1}{2} + R_1\right)}{h_2} \right\} + \frac{G_2 h_1 \left(\frac{h_1}{2} + R_1\right)}{2 h_2 R_1} + \frac{G_2 \left(\frac{h_1}{2} + R_1\right)}{2 R_2} + \frac{G_2 h_1 \left(\frac{h_1}{2} + R_1\right)}{4 R_1 R_2} + \frac{G_2 h_3 \left(\frac{h_1}{2} + R_1\right)}{2 h_2 R_3} + \frac{G_2 h_1 h_3 \left(\frac{h_1}{2} + R_1\right)}{4 h_2 R_1 R_3} + \frac{G_2 h_3 \left(\frac{h_1}{2} + R_1\right)}{4 R_2 R_3} + \frac{G_2 h_1 h_3 \left(\frac{h_1}{2} + R_1\right)}{8 R_1 R_2 R_3} \right\}$

$K_{vlvl} = \sin\left[\frac{m \pi x}{L}\right]^2 \sin[n \phi]^2 \left\{ \frac{8 G_2 h_1^2 n \left(\frac{h_1}{2} + R_1\right)}{4 h_2 R_1^2} + \frac{3 G_2 h_1 n \left(\frac{h_1}{2} + R_1\right)}{4 R_1 R_2} + \frac{G_2 h_3 n \left(\frac{h_1}{2} + R_1\right)}{2 h_2 R_3} + \frac{G_2 h_1 h_3 n \left(\frac{h_1}{2} + R_1\right)}{4 h_2 R_1 R_3} + \frac{G_2 h_3 n \left(\frac{h_1}{2} + R_1\right)}{4 R_2 R_3} + \frac{G_2 h_1 h_3 n \left(\frac{h_1}{2} + R_1\right)}{8 R_1 R_2 R_3} \right\}$

$M_{u3u3} = h_3 R_3 \rho 3 \cos\left[\frac{m \pi x}{L}\right]^2 \cos[n \phi]^2$

$K_{u3u1} = -\cos\left[\frac{m \pi x}{L}\right]^2 \cos[n \phi]^2 \frac{G_2 \left(-\frac{h_3}{2} + R_3\right)}{h_2}$

$K_{u3v1} = 0$

$K_{u3u3} = \cos\left[\frac{m \pi x}{L}\right]^2 \cos[n \phi]^2 \left\{ \frac{E_3 h_3^2 n^2}{2 R_3 \left(1 + \nu_3\right)} + \frac{E_3 h_3^2 \pi^2 R_3}{L^2 \left(1 - \nu_3^2\right)} \right\}$

$K_{u3v3} = \cos\left[\frac{m \pi x}{L}\right]^2 \cos[n \phi]^2 \left\{ -\frac{E_3 h_3 m n \pi}{2 L \left(1 + \nu_3\right)} - \frac{E_3 h_3^3 m n \pi}{24 L R_3^2 \left(1 + \nu_3\right)} - \frac{E_3 h_3^3 m n \pi \nu_3}{L \left(1 - \nu_3^2\right)} - \frac{E_3 h_3^3 m n \pi \nu_3}{12 L R_3^2 \left(1 - \nu_3^2\right)} \right\}$
\[ K_{v3w} = \cos\left[ \frac{m \pi x}{L} \right]^2 \cos[n \theta]^2 \left( \frac{G_2 m \pi \left( -\frac{h_3}{2} + R_3 \right)}{L} + \frac{G_2 h_1 m \pi \left( -\frac{h_3}{2} + R_3 \right)}{2 h_2 L} \right) \]

\[ + \frac{G_2 h_3 m \pi \left( -\frac{h_3}{2} + R_3 \right)}{2 h_2 L} - \frac{E_3 h_3^3 m n^2 \pi}{12 L R_3^2 \left( 1 + \nu_3 \right)} - \frac{E_3 h_3^3 m^3 \pi^3}{12 L^3 \left( 1 - \nu_3^2 \right)} \]

\[ M_{v3v3} = h_3 R_3 \rho_3 \sin\left[ \frac{m \pi x}{L} \right]^2 \sin[n \theta]^2 \]

\[ K_{v3w} = 0 \]

\[ K_{v3w} = \sin\left[ \frac{m \pi x}{L} \right]^2 \sin[n \theta]^2 \left( \frac{G_2 \left( -\frac{h_3}{2} + R_3 \right)}{h_2} - \frac{G_2 h_1 \left( -\frac{h_3}{2} + R_3 \right)}{2 h_2 R_1} \right) \]

\[ + \frac{G_2 h_3 \left( -\frac{h_3}{2} + R_3 \right)}{2 R_2} - \frac{G_2 h_1 h_3 \left( -\frac{h_3}{2} + R_3 \right)}{4 R_1 R_2} + \frac{G_2 h_3 \left( -\frac{h_3}{2} + R_3 \right)}{2 h_2 R_3} \]

\[ + \frac{G_2 h_3 \left( -\frac{h_3}{2} + R_3 \right)}{4 h_2 R_1 R_3} + \frac{G_2 h_3 \left( -\frac{h_3}{2} + R_3 \right)}{4 R_2 R_3} + \frac{G_2 h_1 h_3 \left( -\frac{h_3}{2} + R_3 \right)}{8 R_1 R_2 R_3} \]

\[ K_{v3w} = \sin\left[ \frac{m \pi x}{L} \right]^2 \sin[n \theta]^2 \left( \frac{E_3 h_3 m \pi}{2 L^2 \left( 1 + \nu_3 \right)} \right) \]

\[ - \frac{E_3 h_3^3 m \pi}{24 L^3 \left( 1 + \nu_3 \right)} - \frac{E_3 h_3^3 m \pi \nu_3}{L \left( 1 - \nu_3^2 \right)} \]

\[ K_{v3w} = \sin\left[ \frac{m \pi x}{L} \right]^2 \sin[n \theta]^2 \left( \frac{G_2 \left( -\frac{h_3}{2} + R_3 \right)}{h_2} - \frac{G_2 \left( -\frac{h_3}{2} + R_3 \right)}{2 R_2} \right) \]

\[ + \frac{G_2 h_3^2 \left( -\frac{h_3}{2} + R_3 \right)}{4 h_2 R_3^2} - \frac{G_2 h_3^2 \left( -\frac{h_3}{2} + R_3 \right)}{8 R_2 R_3^2} - \frac{G_2 h_3 \left( -\frac{h_3}{2} + R_3 \right)}{h_2 R_3} \]

\[ + \frac{G_2 h_3 \left( -\frac{h_3}{2} + R_3 \right)}{2 R_2 R_3} + \frac{E_3 h_3^3 m^2 \pi^2}{8 L^2 R_3 \left( 1 + \nu_3 \right)} + \frac{E_3 h_3 m^2 \pi^2 R_3}{2 L^2 \left( 1 + \nu_3 \right)} \]

\[ + \frac{E_3 h_3^3 n^2}{12 R_3^2 \left( 1 - \nu_3^2 \right)} + \frac{E_3 h_3 n^2}{R_3 \left( 1 - \nu_3^2 \right)} \]
$K_{w3w} = \sin\left[ \frac{m \pi x}{L} \right]^2 \sin[n \theta]^2 \left\{ \begin{array}{c}
\frac{G2 h1 n \left( -\frac{h3}{2} + R3 \right)}{2 h2 R1} - \frac{G2 n \left( -\frac{h3}{2} + R3 \right)}{R2} - \frac{G2 h1 n \left( -\frac{h3}{2} + R3 \right)}{4 R1 R2} \\
\frac{G2 h2^2 n \left( -\frac{h3}{2} + R3 \right)}{4 h2 R3^2} - \frac{G2 h3^2 n \left( -\frac{h3}{2} + R3 \right)}{8 R2 R3^2} - \frac{G2 h3 n \left( -\frac{h3}{2} + R3 \right)}{2 h2 R3} \\
\frac{G2 h1 h3 n \left( -\frac{h3}{2} + R3 \right)}{4 h2 R1 R3} + \frac{3 G2 h3 n \left( -\frac{h3}{2} + R3 \right)}{4 R2 R3} + \frac{G2 h1 h3 n \left( -\frac{h3}{2} + R3 \right)}{8 R1 R2 R3} \\
+ \frac{E3 h3^3 m^2 n \pi^2}{6 L^2 R3 (1 + \nu_3)} + \frac{E3 h3^3 n^3}{12 R3^3 (1 - \nu_3^2)} + \frac{E3 h3 n}{R3 (1 - \nu_3^2)} + \frac{E3 h3^3 m^2 n \pi^2 v_3}{12 L^2 R3 (1 - \nu_3^2)} \end{array} \right\}$

$M_{w1w} = h1 R1 \rho 1 \cos[n \theta]^2 \sin\left[ \frac{m \pi x}{L} \right]^2$

$K_{w1u1} = \cos[n \theta]^2 \sin\left[ \frac{m \pi x}{L} \right]^2 \left\{ \begin{array}{c}
\frac{G2 h1 m \pi \left( \frac{h1}{2} + R1 \right)}{2 h2 L} - \frac{E1 h1^3 m n^2 \pi}{24 L R1^2 (1 + \nu_1)} \\
\frac{E1 h1^3 m^3 \pi^3}{12 L^3 (1 - \nu_1^2)} - \frac{E1 h1 m \pi \nu_1}{L (1 - \nu_1^2)} \end{array} \right\}$

$K_{w1v1} = \cos[n \theta]^2 \sin\left[ \frac{m \pi x}{L} \right]^2 \left\{ \begin{array}{c}
\frac{G2 h1^2 n \left( \frac{h1}{2} + R1 \right)}{8 R1^2 R2} + \frac{G2 h1 n \left( \frac{h1}{2} + R1 \right)}{4 R1 R2} + \frac{E1 h1^3 m^2 n \pi^2}{8 L^2 R1 (1 + \nu_1)} \\
\frac{E1 h1^3 n^3}{12 R1^3 (1 - \nu_1^2)} + \frac{E1 h1 n}{R1 (1 - \nu_1^2)} + \frac{E1 h1^3 m^2 n \pi^2 v_1}{6 L^2 R1 (1 - \nu_1^2)} \end{array} \right\}$

$K_{w1u3} = \cos[n \theta]^2 \sin\left[ \frac{m \pi x}{L} \right]^2 \frac{G2 h1 m \pi \left( \frac{h1}{2} + R1 \right)}{2 h2 L}$

$K_{w1v3} = \cos[n \theta]^2 \sin\left[ \frac{m \pi x}{L} \right]^2 \left\{ \begin{array}{c}
\frac{G2 h1 n \left( \frac{h1}{2} + R1 \right)}{2 h2 R1} + \frac{G2 h1 n \left( \frac{h1}{2} + R1 \right)}{4 R1 R2} \\
+ \frac{G2 h1 h3 n \left( \frac{h1}{2} + R1 \right)}{8 R1 R2 R3} - \frac{G2 h1 h3 n \left( \frac{h1}{2} + R1 \right)}{8 R1 R2 R3} \end{array} \right\}$
\[ K_{\omega_1} = \cos[\pi \theta]^2 \sin\left(\frac{m \pi x}{L}\right)^2 \left( \frac{G_2 h_1 m^2 \pi^2}{2 L^2} \left( \frac{h_1}{2} + R_1 \right) \right. \]
\[ + \frac{G_2 h_1 h_3 m^2 \pi^2}{4 h_2 L^2} \left( \frac{h_1}{2} + R_1 \right) \left. \right) + \frac{G_2 h_1^2 m^2 \pi^2}{8 R_1^2 R_2} \left( \frac{h_1}{2} + R_1 \right) \]

\[ M_{\omega_1} = h_2 R_2 \rho_2 \cos[\pi \theta]^2 \sin\left(\frac{m \pi x}{L}\right)^2 \]

\[ K_{\omega_2} = -\cos[\pi \theta]^2 \sin\left(\frac{m \pi x}{L}\right)^2 \left( \frac{G_2 m \pi R_2}{L} \right) \]

\[ K_{\omega_3} = \cos[\pi \theta]^2 \sin\left(\frac{m \pi x}{L}\right)^2 \left( \frac{G_2 h_1 n}{2 R_1} + \frac{G_2 h_2 n}{2 R_2} + \frac{G_2 h_1 h_2 n}{4 R_1 R_2} \right) \]

\[ K_{\omega_4} = \cos[\pi \theta]^2 \sin\left(\frac{m \pi x}{L}\right)^2 \left( \frac{G_2 h_1 n^2}{2 L^2} \left( \frac{h_1^2}{2} + R_1 \right) \right. \]
\[ + \frac{G_2 h_3 n^2}{2 R_3} \left( \frac{h_3^2}{2} + R_3 \right) \left. \right) \]

\[ M_{\omega_3} = h_3 R_3 \rho_3 \cos[\pi \theta]^2 \sin\left(\frac{m \pi x}{L}\right)^2 \]

\[ K_{\omega_4} = -\cos[\pi \theta]^2 \sin\left(\frac{m \pi x}{L}\right)^2 \left( \frac{G_2 h_3 m \pi}{2 h_2 L} \left( \frac{h_3}{2} + R_3 \right) \right. \]
\[ + \frac{G_2 h_3 n \left( \frac{h_3}{2} + R_3 \right)}{4 R_2 R_3} + \frac{G_2 h_1 h_3 n \left( \frac{h_3}{2} + R_3 \right)}{8 R_1 R_2 R_3} \]
\[ K_{\text{w}3} = \cos[n \theta]^2 \sin \left[ \frac{m \pi x}{L} \right]^2 \left( \frac{G2 h3 m \pi (-h3/2 + R3)}{2 h2 L} - \frac{E3 h3^3 m^2 \pi}{24 L R3^2 (1 + v3)} \right) \]

\[ K_{\text{w}3v3} = \cos[n \theta]^2 \sin \left[ \frac{m \pi x}{L} \right]^2 \left( \frac{G2 h3 n (-h3/2 + R3)}{2 h2 R3} + \frac{G2 h3 n (-h3/2 + R3)}{4 R2 R3} + \frac{E3 h3^3 m^2 n \pi^2}{8 L^2 R3 (1 + v3)} \right) \]

\[ K_{\text{w}3v3w} = \cos[n \theta]^2 \sin \left[ \frac{m \pi x}{L} \right]^2 \left( \frac{G2 h3^2 m^2 \pi^2 (-h3/2 + R3)}{2 I^2} + \frac{G2 h3^2 n^2 (-h3/2 + R3)}{4 h2 R3} - \frac{G2 h3^2 n^2 (-h3/2 + R3)}{8 R2 R3^2} \right) \]

\[ M_{\text{u}3u3\text{single}} = h3 R3 \rho3 \cos \left[ \frac{m \pi x}{L} \right]^2 \cos[n \theta]^2 \]

\[ K_{\text{u}3u3\text{single}} = \cos \left[ \frac{m \pi x}{L} \right]^2 \cos[n \theta]^2 \left( \frac{E3 h3 n^2}{2 R3 (1 + v3)} + \frac{E3 h3 m^2 \pi^2 R3}{L^2 (1 - v3)} \right) \]

\[ K_{\text{u}3v3\text{single}} = \cos \left[ \frac{m \pi x}{L} \right]^2 \cos[n \theta]^2 \left( \frac{E3 h3 m n \pi}{2 L (1 + v3)} - \frac{E3 h3^3 m n \pi}{24 L R3^2 (1 + v3)} \right) \]

\[ K_{\text{u}3w\text{single}} = \cos \left[ \frac{m \pi x}{L} \right]^2 \cos[n \theta]^2 \left( \frac{E3 h3^3 m^3 \pi^3}{12 L R3^2 (1 + v3)} - \frac{E3 h3^3 m^3 \pi^3}{12 L^3 (1 - v3)} \right) \]
\[ M_{v3\text{single}} = h_3^3 R_3^3 \rho_3 \sin\left(\frac{m \pi x}{L}\right)^2 \sin[n \theta]^2 \]

\[ K_{v3\text{single}} = \sin\left(\frac{m \pi x}{L}\right)^2 \sin[n \theta]^2 \left( -\frac{E_3 h_3^3 m n \pi}{2 L (1 + \nu_3)} - \frac{E_3 h_3^3 m n \pi}{24 L R_3^2 (1 + \nu_3)} - \frac{E_3 h_3^3 m n \pi}{L (1 - \nu_3^2)} \right) \]

\[ K_{v3\text{single}} = \sin\left(\frac{m \pi x}{L}\right)^2 \sin[n \theta]^2 \left( -\frac{E_3 h_3^3 m^2 n^2 \pi}{8 L^2 R_3 (1 + \nu_3)} + \frac{E_3 h_3^3 m^2 n^2 \pi}{2 L^2 (1 + \nu_3)} \right) \]

\[ K_{v3\text{single}} = \sin\left(\frac{m \pi x}{L}\right)^2 \sin[n \theta]^2 \left( -\frac{E_3 h_3^3 n^2}{12 R_3^3 (1 - \nu_3^2)} + \frac{E_3 h_3^3 n^2}{R_3 (1 - \nu_3^2)} \right) \]

\[ M_{w3\text{single}} = h_3^3 R_3^3 \rho_3 \cos[n \theta]^2 \sin\left(\frac{m \pi x}{L}\right)^2 \]

\[ K_{w3\text{single}} = \cos[n \theta]^2 \sin\left(\frac{m \pi x}{L}\right)^2 \left( -\frac{E_3 h_3^3 m n^2 \pi}{24 L R_3^2 (1 + \nu_3)} - \frac{E_3 h_3^3 m^3 \pi^3}{12 L^3 (1 - \nu_3^2)} - \frac{E_3 h_3^3 m \pi \nu_3}{L (1 - \nu_3^2)} \right) \]

\[ K_{w3\text{single}} = \cos[n \theta]^2 \sin\left(\frac{m \pi x}{L}\right)^2 \left( -\frac{E_3 h_3^3 m^2 n^2 \pi}{8 L^2 R_3 (1 + \nu_3)} + \frac{E_3 h_3^3 n^3}{12 R_3^3 (1 - \nu_3^2)} + \frac{E_3 h_3 n}{R_3 (1 - \nu_3^2)} \right) \]

\[ K_{w3\text{single}} = \cos[n \theta]^2 \sin\left(\frac{m \pi x}{L}\right)^2 \left( -\frac{E_3 h_3^3 n^2}{6 L^2 R_3 (1 - \nu_3^2)} \right) \]

\[ K_{w3\text{single}} = \cos[n \theta]^2 \sin\left(\frac{m \pi x}{L}\right)^2 \left( \frac{E_3 h_3^3 m^4 \pi^4 R_3}{12 L^4 (1 - \nu_3^2)} + \frac{E_3 h_3^3 m^2 \nu_3}{12 L^2 R_3 (1 - \nu_3^2)} + \frac{E_3 h_3^3 m^2 n^2 \pi^2 \nu_3}{6 L^2 R_3 (1 - \nu_3^2)} \right) \]
Appendix C: Mathematica Program

Mathematica Program for Three-Layer Cylindrical Shaft

Three-layer partially treated cylindrical shell program

Overview:

The lines below are used in Mathematica version 4 to obtain the results presented in this paper for the three-layer cylinder. The program can produce results for a partially treated shell along with a fully treated shell by letting the coverage length be equal to the total shell length. The program was built in a modular form where the “master” file loads sub-files by file name to obtain results. Each file is listed below starting with the file name with the “master” file first and then the sub-files in order of the way they are loaded.

The way the program works: The first portion of the program begins with assumed mode shapes, then describes the strain-displacement, stress-strain relationships, and then it defines the unit forces. With the unit forces now defined in terms of the assumed displacements, the equations of motion in terms of the unit forces are then loaded. This results in the equations of motion in terms of the displacements which are then arranged into matrix form,

\[ [\mathbf{M}]\{\ddot{\mathbf{z}}\} + [\mathbf{K}]\{\mathbf{z}\}, \]

and then the natural frequencies and damping factors are calculated by finding the eigenvalues of the dynamic matrix resulting from the applied Galerkin method.

(*-----------------------------Master.nb-----------------------------*)

\[ \text{\text{\textless\textless d:\textbackslash \textquoteright\textbackslash \textit{\textbackslash thisis\textbackslash results\textbackslash producer\textbackslash UF.m};} \]

\[ \text{\text{\textless\textless d:\textbackslash \textbackslash \textbackslash thisis\textbackslash results\textbackslash producer\textbackslash EQ.m};} \]
\[ h_1 = .05; \]
\[ h_2 = .05; \]
\[ h_3 = .1; \]
\[ \text{RmShaft} = 3; \]
\[ R_1 = \text{RmShaft} - h_3 - h_2 - h_1 / 2; \]
\[ R_2 = \text{RmShaft} - h_3 - h_2 / 2; \]
\[ R_3 = \text{RmShaft} - h_3 / 2; \]
\[ L = 60; \]
\[ E_{m1} = 10 \times 10^6; \]
\[ E_{m3} = 10 \times 10^6; \]
\[ \rho_1 = .000255 (\text{lb}_f \text{s}^2/\text{in}^4); \]
\[ \rho_3 = .000255 (\text{lb}_f \text{s}^2/\text{in}^4); \]
\[ \rho_2 = .00014 (\text{lb}_f \text{s}^2/\text{in}^4); \]
\[ v_1 = .33; \]
\[ v_3 = .33; \]
\[ G_{2x} = G_{2\theta} = 20000 (1 + 1 \text{i}); \]
\[ b_1 = 36 (*\text{coverage Length *}); \]
\[ (*\text{mode*}) m = 1; n = 1; \]
\[ \text{TotalA} = 1; \]
\[ \text{TotalB} = 1; \]
\[ \text{NumbEQ} = 5; \]
\[ m_{\text{Start}} = m; \quad m_{\text{Stop}} = m; \]
\[ n_{\text{Start}} = n; \quad n_{\text{Stop}} = n; \]
\[ i_{\text{Start}} = m; \quad i_{\text{Stop}} = m; \]
\[ j_{\text{Start}} = n; \quad j_{\text{Stop}} = n; \]
\[ wn = \sqrt{\text{Eigenvalues}[\text{Ans}]}; \]
\[ \text{Nat} = \frac{wn}{2\pi}; \]
\[ Wn = \text{Re}[\text{Nat}[[5]]]; \]
\[ \text{squaredNat} = \text{Nat}[[5]]^2; \]
\[ \text{DF} = \text{Im}[\text{squaredNat}] / \text{Re}[\text{squaredNat}] * 100; \]

Natural Frequency (Hz)
\[ Wn \]
168.557

Damping Factor (%)

This file generates the unit forces.
This file was generated automatically by the Mathematica front end.
It contains Initialization cells from a Notebook file, which typically will have the same name as this file except ending in ".nb" instead of ".m".

This file is intended to be loaded into the Mathematica kernel using the package loading commands Get or Needs. Doing so is equivalent to using the Evaluate Initialization Cells menu command in the front end.

DO NOT EDIT THIS FILE. This entire file is regenerated automatically each time the parent Notebook file is saved in the Mathematica front end.
Any changes you make to this file will be overwritten.

\[ u = U_1 \cos[(m \pi)/VL \times] \cos[n \theta]; \]
\[ v = V_1 \sin[(m \pi)/VL \times] \sin[n \theta]; \]
\[ u_3 = U_3 \cos[(m \pi)/VL \times] \cos[n \theta]; \]
\[ v_3 = V_3 \sin[(m \pi)/VL \times] \sin[n \theta]; \]
\[ w = W \sin[(m \pi)/VL \times] \cos[n \theta]; \]
\[ G_1 = E ml A((2(1 + \nu_l));) \]
\[ G_3 = E m_3 A((2(1 + \nu_3));) \]

\[
\begin{align*}
\epsilon_{x x} & = \partial_x u - z \partial_x (v_1 + w); \\
\gamma_{x \theta} & = \gamma_{x \theta} \left[ (1 VR_1) \right] \left( \frac{(1 VR_1)}{x} \right) \left( \frac{1 VR_1}{\partial_x \theta} \right) - \left( \frac{1 VR_1}{\partial_x v_1} \right); \\
\beta_{xx} & = -\partial_x w; \\
\beta_{x \theta} & = \beta_{x \theta} \left[ \left( \frac{(1 VR_1)}{\partial_x \theta} \right) \right];
\end{align*}
\]
\[\frac{\text{vl}}{R_l} - \frac{lVR_l}{\partial \Theta} \]

\[\sigma_{xl} = E_{ml}(l - \nu_l^2)(\epsilon_{xl} + \nu_l \epsilon_{\Theta l});\]

\[\sigma_{\Theta l} = E_{ml}(l - \nu_l^2)(\epsilon_{\Theta l} + \nu_J \epsilon_{xl});\]

\[\tau_{x\Theta 1} = \tau_{\Theta xl} = -G_{l} \Gamma_{x\Theta l};\]

\[\epsilon_{x3} = \frac{\partial u_3}{\partial x} - z \frac{\partial w}{\partial x};\]

\[\epsilon_{\Theta 3} = \frac{1}{lVR_3} \frac{\partial v_3}{\partial \Theta} + wVR_3 + z (\frac{1}{lVR_3} \frac{\partial \Theta}{\partial \Theta} - \frac{1}{lVR_3} \frac{\partial \Theta}{\partial \Theta}) \frac{\partial v_3}{\partial \Theta};\]

\[\Gamma_{x\Theta 3} = \frac{1}{lVR_3} \frac{\partial u_3}{\partial \Theta} - \frac{\partial v_3}{\partial \Theta} \frac{\partial w}{\partial \Theta};\]
\[(2VR3) \partial_{\theta} w - (1VR3) \partial_{x} w = 0,\]

\\[(\text{Beta}_x) - (-\partial_x w);\]

\\ \text{\texttt{IndentingNewLine}}

\\[(\text{Beta}_\theta) = \frac{v_3}{R_3} - (1VR3) \partial_{\theta} w;\]

\\ \text{\texttt{IndentingNewLine}}

\\[(\Sigma_x = (E_m v_3 - h_3 V_2)) \partial_{x} w;\]

\\ \text{\texttt{IndentingNewLine}}

\\[(\text{Tau}_x) = G_3 \partial_{\theta} w;\]

\\ \text{\texttt{IndentingNewLine}}

\\[(u_2 = (1V2) \partial_{x} w;\]

\\ \text{\texttt{IndentingNewLine}}

\\[(\text{Beta}_x) - (1Vh2) \partial_{x} w;\]

\\ \text{\texttt{IndentingNewLine}}

96
\( v_2 = (1V2) \cdot ((v_3 - (h3V2) [\text{Beta}] \cdot (([\text{Theta}]_3) - (h))) (1 V2) + (v_1 + (h1V2) [\text{Beta}] \cdot (([\text{Theta}]_1)) (1 V2)) ) \)

\[ \text{IndentingNewLine} \]
\( [\text{Beta}] \cdot ([\text{Theta}]_2) = (1 V2) \cdot ((v_3 - (h3V2) [\text{Beta}] \cdot ([\text{Theta}]_3)) - (v_1 + (h1V2) [\text{Beta}] \cdot ([\text{Theta}]_1)) ) \)

\[ \text{IndentingNewLine} \]
\( [\text{Gamma}] \cdot [\text{Rx}2] = ([\text{Beta}] \cdot x_2 + ([\text{PartialD}] \cdot x \cdot w) ) \)

\[ \text{IndentingNewLine} \]
\( [\text{Gamma}] \cdot [\text{R}][\text{Theta}2] = ([\text{Beta}] \cdot [\text{Theta}]_2 - v_2 VR2 + (1 VR2) ([\text{PartialD}] \cdot [\text{Theta}] \cdot w) ) \)

\[ \text{IndentingNewLine} \]
\( [\text{Gamma}] \cdot [\text{Rx}]_2 = ([\text{Integral}] \cdot ((-h2)/2) \% (h2/2)) G2x \cdot ([\text{Gamma}] \cdot [\text{Rx}2]) \)

\[ \text{IndentingNewLine} \]
\( [\text{Gamma}] \cdot [\text{R}][\text{Theta}]_2 = ([\text{Integral}] \cdot ((-h2)/2)) \% (h2/2)) G2[\text{Theta}] \)

\[ \text{IndentingNewLine} \]
\( [\text{Gamma}] \cdot [\text{R}][\text{Theta}]_2 = ([\text{Integral}] \cdot ((-h2)/2)) \% (h2/2)) G2 [\text{Theta}] \)

\[ \text{IndentingNewLine} \]
\( \tau_{Rx1} = V \cdot x_2 / h_2 ; \)

\[ \text{IndentingNewLine} \]
\( \tau_{Rx3} = V \cdot x_2 / h_2 ; \)

\[ \text{IndentingNewLine} \]
\( \tau_{Rx2} = V \cdot x_2 / h_2 ; \)

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\( \tau_{Rx3} = V \cdot x_2 / h_2 ; \)

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\( \tau_{Rx3} = V \cdot x_2 / h_2 ; \)
\[ V_{x3} = \int_{-h3/2}^{h3/2} \sigma_{x3} \left(1 + \frac{z}{R3}\right) \, dz; \]
\[ M, V_{\theta x3} = \int_{-h3/2}^{h3/2} \tau_{x3} \left(1 - \frac{z}{R3}\right) \, dz; \]
\[ M_{x3} = \int_{-h3/2}^{h3/2} \sigma_{x3} \left(1 + \frac{z}{R3}\right) \, dz; \]
\[ M_{x\theta3} = \int_{-h3/2}^{h3/2} \tau_{x3} \left(1 + \frac{z}{R3}\right) \, dz; \]
\[ M_{\theta x3} = \int_{-h3/2}^{h3/2} \sigma_{\theta3} \, dz; \]
\[ M_{\theta3} = \int_{-h3/2}^{h3/2} \tau_{\theta3} \, dz; \]

This file loads the equations of motion. This file was generated automatically by the Mathematica front end. It contains Initialization cells from a Notebook file, which typically will have the same name as this file except ending in "nb" instead of "m".
using the Evaluate Initialization Cells menu command in the front end.

DO NOT EDIT THIS FILE. This entire file is regenerated automatically each time the parent Notebook file is saved in the Mathematica front end. Any changes you make to this file will be overwritten.

*************************************************************

StyleBox[(V_\[Theta]R3 = (1VR3) (R3 \[PartialD]_x M_\_x \[Theta]3 + \[PartialD]_\[Theta] M_\[Theta]3))],
FontSize->9]

(V_\_xR3 = (1VR3) ((\[PartialD]_\[Theta] M_\[Theta]x3 + R3 \[PartialD]_x M_\_x x3)));

(Eq3\_single = (-R3) \[PartialD]_x V_\_x3 - \[PartialD]_\[Theta]' V_\[Theta]3 - R3 \[PartialD]_x V_\_x[Theta]3 + \[Rho]3\_3 h3\_u3tt;)[IndentingNewLine]

(EQ3\_single = Collect[Eq3\_single, {Ul, V1, U3, V3, W} (*\((,)(,)(Simplify\))* )];)


(EQ3[\[Theta]single = Collect[Eq3[\[Theta]single, {Ul, V1, U3, V3, W} (*\((,)(,)(Simplify\))* ) ];)

(Eq3\_wsingle = (-R3) \[PartialD]_x V_\_xR3 - \[PartialD]_\[Theta]' V_\[Theta]R3 + V_\[Theta]3 + \[Rho]3\_3 R3\_h3\_w1tt;)[IndentingNewLine]

(EQ3\_wsingle = Collect[Eq3\_wsingle, {Ul, V1, U3, V3, W} (*\((,)(,)(Simplify\))* ) ];)

RowBox[
 RowBox[{"TermsSingle", ",="},
 RowBox[{"", GridBox[
 {\\{\{EQ3\_single\[\((2)\)\]/U3\}, \{EQ3\_single\[\((3)\)\]/V3\}, \{EQ3\_single\[\((4)\)\]/W\}},
 {\\{\{EQ3[\[Theta]single\[\((2)\)\]/U3\}, \{EQ3[\[Theta]single\[\((3)\)\]/V3\}, \{EQ3[\[Theta]single\[\((4)\)\]/W\}},
 {\\{\{EQ3\_wsingle\[\((2)\)\]/U3\}, \{EQ3\_wsingle\[\((3)\)\]/V3\}, \{EQ3\_wsingle\[\((4)\)\]/W\}}
 ]}, "]"}]], ",;"}])
\[M_{u3u3}\text{single} = \text{EQ3xsingle}[[1]]*u3x((U3*u3\text{tt}))/\text{Shapeu3};]\]
\[K_{u3u3}\text{single} = \text{TermsSingle}[[1, 1]]*\text{Shapeu3};]\]
\[K_{u3v3}\text{single} = \text{TermsSingle}[[1, 2]]*\text{Shapeu3};]\]
\[K_{u3w3}\text{single} = \text{TermsSingle}[[1, 3]]*\text{Shapeu3};]\]

\[M_{v3v3}\text{single} = \text{EQ3v3single}[[1]]*v3x((V3*v3\text{tt}))/\text{Shapev3};]\]
\[K_{v3u3}\text{single} = \text{TermsSingle}[[2, 1]]*\text{Shapev3};]\]
\[K_{v3v3}\text{single} = \text{TermsSingle}[[2, 2]]*\text{Shapev3};]\]
\[K_{v3w3}\text{single} = \text{TermsSingle}[[2, 3]]*\text{Shapev3};]\]

\[M_{w3w3}\text{single} = \text{EQ3w3single}[[1]]*w((W*w\text{tt}))/\text{Shapew};]\]
\[K_{w3u3}\text{single} = \text{TermsSingle}[[3, 1]]*\text{Shapew};]\]
\[K_{w3v3}\text{single} = \text{TermsSingle}[[3, 2]]*\text{Shapew};]\]
\[K_{w3w3}\text{single} = \text{TermsSingle}[[3, 3]]*\text{Shapew};]\]

TermsSingle//MatrixForm;

\[V_{\Theta R}\text{Rl} = (lVRl)(Rl'[\Phi x Mx\Theta l + Rl'[\Phi x Mx\Theta l + (R1 + h1/2)]) (h1/2) (Tau)_R[\Theta l)]; =\]

\[Eqlx = -Rl'[\Phi x Vx]\Theta l - Vx\Theta l - Rl'[\Phi x Mx\Theta l + (R1 + h1/2)]) (Tau)_R[\Theta l]; =\]

\[EQlx = \text{Collect}[Eqlx, \{U1, V1, U3, V3, W\} (* \text{Simplify} *) ]; =\]

\[Eql\Theta l = -Vx\Theta l - Vx\Theta l - (R1 + h1/2)]) (Tau)_R[\Theta l]; =\]

\[EQl\Theta l = \text{Collect}[Eql\Theta l, \{U1, V1, U3, V3, W\} (* \text{Simplify} *) ]; =\]
\[
\begin{align*}
V_{xR3} &= (1VR3) \left( \frac{\partial}{\partial \Theta} M_{x3} + (R3 - h3V2) h3V2 \tau_{Rx3} \right); \\
Eq3x &= -R3 \frac{\partial}{\partial x} V_{x3} - \frac{\partial}{\partial \Theta} V_{\Theta x3} + (R3 - h3V2) \tau_{Rx3} + \rho_{3} R_{3} h_{3} v_{3tt}; \\
Eq3\Theta &= - \frac{\partial}{\partial \Theta} V_{\Theta 3} - V_{\Theta R3} - R3 \frac{\partial}{\partial x} V_{x\Theta 3} + (R3 - h3V2) \tau_{R\Theta 3} + \rho_{3} R_{3} h_{3} v_{3tt}; \\
Eqlw &= -Rl \frac{\partial}{\partial x} V_{xRl} - \frac{\partial}{\partial \Theta} V_{\Theta Rl} + V_{\Theta 1} + \rho_{l} R_{l} h_{l} w_{ltt}; \\
Eq2w &= -R2 \frac{\partial}{\partial x} V_{xR2} - \frac{\partial}{\partial \Theta} V_{\Theta R2} + \rho_{2} R_{2} h_{2} w_{1tt}; \\
Eq3w &= -R3 \frac{\partial}{\partial x} V_{xR3} - \frac{\partial}{\partial \Theta} V_{\Theta R3} + V_{\Theta 3} + \rho_{3} R_{3} h_{3} w_{1tt}; \\
\text{Collect} & \quad \text{Collect} & \quad \text{Collect}.
\end{align*}
\]
"0", \(\{\text{EQ1}_x[\{(5)\}]/W\}\),
\(\{\text{EQ1}_x[\{(4)\}]/U1\}, \{\text{EQ1}_x[\{(3)\}]/V1\},
"0", \(\{\text{EQ1}_x[\{(2)\}]/V3\}, \{\text{EQ1}_x[\{(5)\}]/W\}\),
\(\{\text{EQ3}_x[\{(2)\}]/U1\},
"0", \(\{\text{EQ3}_x[\{(3)\}]/U3\}, \{\text{EQ3}_x[\{(4)\}]/ V3\}, \{\text{EQ3}_x[\{(5)\}]/W\}\),
\{ 
"0", \(\{\text{EQ3}_x[\{(2)\}]/V1\}, \{\text{EQ3}_x[\{(4)\}]/
U3\}, \{\text{EQ3}_x[\{(3)\}]/V3\}, \{\text{EQ3}_x[\{(5)\}]/
W\}\),
\(\{\text{EQ1}_w[\{(4)\}]/U1\}, \{\text{EQ1}_w[\{(3)\}]/V1\}, \{\text{EQ1}_w[\{(2)\}]/
U3\}, \{\text{EQ1}_w[\{(6)\}]/W\}\),
\(\{\text{EQ2}_w[\{(2)\}]/U1\}, \{\text{EQ2}_w[\{(4)\}]/V1\}, \{\text{EQ2}_w[\{(3)\}]/
U3\}, \{\text{EQ2}_w[\{(6)\}]/W\}\),
\(\{\text{EQ3}_w[\{(2)\}]/U1\}, \{\text{EQ3}_w[\{(3)\}]/V1\}, \{\text{EQ3}_w[\{(4)\}]/
U3\}, \{\text{EQ3}_w[\{(6)\}]/W\}\}
\})

\(!/\!(\text{Shape1} = \\
\text{Cos}[(i [\Pi]/VL \times] \text{Cos}[j \text{[Theta]}]);)[\text{IndentingNewLine}]
\text{Shape1} = \text{Sin}[(i [\Pi]/VL \times] \text{Sin}[j \text{[Theta]}]);][n]
\text{Shape1} = \text{Cos}[(i [\Pi]/VL \times] \text{Cos}[j \text{[Theta]}]);)[\text{IndentingNewLine}]
\text{Shape1} = \text{Sin}[(i [\Pi]/VL \times] \text{Sin}[j \text{[Theta]}]);][n]
\text{Shape1} = \text{Cos}[(i [\Pi]/VL \times] \text{Cos}[j \text{[Theta]}]);)[\text{IndentingNewLine}]
\text{Shape1} = \text{Sin}[(i [\Pi]/VL \times] \text{Sin}[j \text{[Theta]}]);][n]
\text{Shape1} = \text{Cos}[(i [\Pi]/VL \times] \text{Cos}[j \text{[Theta]}]);)[\text{IndentingNewLine}]
\text{Shape1} = \text{Sin}[(i [\Pi]/VL \times] \text{Sin}[j \text{[Theta]}]);][n]
\text{Shape1} = \text{Cos}[(i [\Pi]/VL \times] \text{Cos}[j \text{[Theta]}]);)[\text{IndentingNewLine}]
\text{Shape1} = \text{Sin}[(i [\Pi]/VL \times] \text{Sin}[j \text{[Theta]}]);][n]
\text{Shape1} = \text{Cos}[(i [\Pi]/VL \times] \text{Cos}[j \text{[Theta]}]);)[\text{IndentingNewLine}]
\text{Shape1} = \text{Sin}[(i [\Pi]/VL \times] \text{Sin}[j \text{[Theta]}]);][n]
\text{Shape1} = \text{Cos}[(i [\Pi]/VL \times] \text{Cos}[j \text{[Theta]}]);)[\text{IndentingNewLine}]
\text{Shape1} = \text{Sin}[(i [\Pi]/VL \times] \text{Sin}[j \text{[Theta]}]);][n]
\text{Shape1} = \text{Cos}[(i [\Pi]/VL \times] \text{Cos}[j \text{[Theta]}]);)[\text{IndentingNewLine}]
\text{Shape1} = \text{Sin}[(i [\Pi]/VL \times] \text{Sin}[j \text{[Theta]}]);][n]
\text{Shape1} = \text{Cos}[(i [\Pi]/VL \times] \text{Cos}[j \text{[Theta]}]);)[\text{IndentingNewLine}]
\text{Shape1} = \text{Sin}[(i [\Pi]/VL \times] \text{Sin}[j \text{[Theta]}]);][n]
\text{Shape1} = \text{Cos}[(i [\Pi]/VL \times] \text{Cos}[j \text{[Theta]}]);)[\text{IndentingNewLine}]
\text{Shape1} = \text{Sin}[(i [\Pi]/VL \times] \text{Sin}[j \text{[Theta]}]);][n]
\text{Shape1} = \text{Cos}[(i [\Pi]/VL \times] \text{Cos}[j \text{[Theta]}]);)[\text{IndentingNewLine}]
\text{Shape1} = \text{Sin}[(i [\Pi]/VL \times] \text{Sin}[j \text{[Theta]}]);][n]
\text{Shape1} = \text{Cos}[(i [\Pi]/VL \times] \text{Cos}[j \text{[Theta]}]);)[\text{IndentingNewLine}]
\text{Shape1} = \text{Sin}[(i [\Pi]/VL \times] \text{Sin}[j \text{[Theta]}]);][n]
\text{Shape1} = \text{Cos}[(i [\Pi]/VL \times] \text{Cos}[j \text{[Theta]}]);)[\text{IndentingNewLine}]
\text{Shape1} = \text{Sin}[(i [\Pi]/VL \times] \text{Sin}[j \text{[Theta]}]);][n]
\text{Shape1} = \text{Cos}[(i [\Pi]/VL \times] \text{Cos}[j \text{[Theta]}]);)[\text{IndentingNewLine}]
\text{Shape1} = \text{Sin}[(i [\Pi]/VL \times] \text{Sin}[j \text{[Theta]}]);][n]
\text{Shape1} = \text{Cos}[(i [\Pi]/VL \times] \text{Cos}[j \text{[Theta]}]);)[\text{IndentingNewLine}]
\text{Shape1} = \text{Sin}[(i [\Pi]/VL \times] \text{Sin}[j \text{[Theta]}]);][n]
\( (K_{u3v3} = \text{Terms}[\{[3, 4]\}] \ast \text{Shapeu3;}])[(\text{IndentingNewLine})
\( (K_{u3w} = \text{Terms}[\{[3, 5]\}] \ast \text{Shapeu3;})]
\]

\( (M_{v3v3} = \text{EQ3}\{[\Theta]\} \ast (V3*V3tt)) \ast \text{Shaperv3;}])[(\text{IndentingNewLine})
\( (K_{v3u1} = \text{Terms}[\{[4, 1]\}] \ast \text{Shaperv3;}])[(\text{IndentingNewLine})
\( (K_{v3v1} = \text{Terms}[\{[4, 2]\}] \ast \text{Shaperv3;}])[(\text{IndentingNewLine})
\( (K_{v3u3} = \text{Terms}[\{[4, 3]\}] \ast \text{Shaperv3;}])[(\text{IndentingNewLine})
\( (K_{v3v3} = \text{Terms}[\{[4, 4]\}] \ast \text{Shaperv3;}])[(\text{IndentingNewLine})
\( (K_{v3w} = \text{Terms}[\{[4, 5]\}] \ast \text{Shaperv3;}])
\]

\( (M_{w1w1} = \text{EQ1w}\{[\{1\}]\}] \ast (W*Wttt)) \ast \text{Shapew;}])[(\text{IndentingNewLine})
\( (K_{w1u1} = \text{Terms}[\{[5, 1]\}] \ast \text{Shapew;}])[(\text{IndentingNewLine})
\( (K_{w1v1} = \text{Terms}[\{[5, 2]\}] \ast \text{Shapew;}])[(\text{IndentingNewLine})
\( (K_{w1u3} = \text{Terms}[\{[5, 3]\}] \ast \text{Shapew;}])[(\text{IndentingNewLine})
\( (K_{w1v3} = \text{Terms}[\{[5, 4]\}] \ast \text{Shapew;}])[(\text{IndentingNewLine})
\( (K_{w1w} = \text{Terms}[\{[5, 5]\}] \ast \text{Shapew;}])
\]

\( (M_{w2w1} = \text{EQ2w}\{[\{1\}]\}] \ast (W*Wttt)) \ast \text{Shapew;}])[(\text{IndentingNewLine})
\( (K_{w2u1} = \text{Terms}[\{[6, 1]\}] \ast \text{Shapew;}])[(\text{IndentingNewLine})
\( (K_{w2v1} = \text{Terms}[\{[6, 2]\}] \ast \text{Shapew;}])[(\text{IndentingNewLine})
\( (K_{w2u3} = \text{Terms}[\{[6, 3]\}] \ast \text{Shapew;}])[(\text{IndentingNewLine})
\( (K_{w2v3} = \text{Terms}[\{[6, 4]\}] \ast \text{Shapew;}])[(\text{IndentingNewLine})
\( (K_{w2w} = \text{Terms}[\{[6, 5]\}] \ast \text{Shapew;}])
\]

\( (M_{w3w1} = \text{EQ3w}\{[\{1\}]\}] \ast (W*Wttt)) \ast \text{Shapew;}])[(\text{IndentingNewLine})
\( (K_{w3u1} = \text{Terms}[\{[7, 1]\}] \ast \text{Shapew;}])[(\text{IndentingNewLine})
\( (K_{w3v1} = \text{Terms}[\{[7, 2]\}] \ast \text{Shapew;}])[(\text{IndentingNewLine})
\( (K_{w3u3} = \text{Terms}[\{[7, 3]\}] \ast \text{Shapew;}])[(\text{IndentingNewLine})
\( (K_{w3v3} = \text{Terms}[\{[7, 4]\}] \ast \text{Shapew;}])[(\text{IndentingNewLine})
\( (K_{w3w} = \text{Terms}[\{[7, 5]\}] \ast \text{Shapew;}])
\]

(* ---------------------------------------------Galerkin.m------------------------------------------------*)

This file was generated automatically by the Mathematica front end.
It contains Initialization cells from a Notebook file, which typically
will have the same name as this file except ending in ".nb" instead of
".m".

This file is intended to be loaded into the Mathematica kernel using
the package loading commands Get or Needs. Doing so is equivalent to
using the Evaluate Initialization Cells menu command in the front end.
DO NOT EDIT THIS FILE. This entire file is regenerated automatically each time the parent Notebook file is saved in the Mathematica front end. Any changes you make to this file will be overwritten.

\[ A = \frac{L}{2} - \frac{bl}{2}; \]
\[ B = \frac{L}{2} + \frac{bl}{2}; \]

\[ A = \frac{L}{2} - \frac{bl}{2}; \]
\[ B = \frac{L}{2} + \frac{bl}{2}; \]

\[
A = \frac{L}{2} - \frac{bl}{2}; \\
B = \frac{L}{2} + \frac{bl}{2};
\]

\[
A = \frac{L}{2} - \frac{bl}{2}; \\
B = \frac{L}{2} + \frac{bl}{2};
\]
\[ \text{Stiff}_{l + a, 1 + b} = \int_{A}^{B} \int_{0}^{2\pi} (K_{vl}vl) R_l \, d\Theta \, dx \]

\[ \text{StiffT}_{l + a, 1 + TotalB + b} = \int_{A}^{B} \int_{0}^{2\pi} (K_{vl}vl) R_l \, d\Theta \, dx \]

\[ \text{Stiff}_{l + a, 1 + b, 2 + TotalB} = \int_{A}^{B} \int_{0}^{2\pi} (K_{vl}vl) R_l \, d\Theta \, dx \]

\[ \text{Mass}_{l + a, 1 + b} = \int_{A}^{B} \int_{0}^{2\pi} (M_{ul}ul) R_l \, d\Theta \, dx \]
\[ \int_{0}^{\pi} (K_u u_3) \, \mathrm{d} \theta \]

\[ \frac{\partial}{\partial x} \]

\[ R_1 \int_{A}^{B} (K_u u_3) \, \mathrm{d} \theta \]

\[ \text{for} \ i = 1 \]
\[
\text{Mass} \left[ l + \text{TotalA} + a, \\
1 + \text{TotalB} + \\
b \right] = \\
\int \text{A\%B} \left( \int 0 \% (2 \pi / \text{M\_v1v1}) \\
\right) R1 \left( \text{DifferentialD}\right) \left[ \Theta \right] \\
\text{DifferentialD} [x]),
\]

\[
\text{Stiff} \left[ l + \text{TotalA} + a, \\
1 + \text{TotalB} + \\
b \right] = \\
\int \text{A\%B} \left( \int 0 \% (2 \pi / \text{K\_vlu1}) \\
\right) R1 \left( \text{DifferentialD}\right) \left[ \Theta \right] \\
\text{DifferentialD} [x]),
\]

\[
\text{Stiff} \left[ l + \text{TotalA} + a, \\
1 + \text{TotalB} + \\
b \right] = \\
\int \text{A\%B} \left( \int 0 \% (2 \pi / \text{K\_vlu3}) \\
\right) R1 \left( \text{DifferentialD}\right) \left[ \Theta \right] \\
\text{DifferentialD} [x]),
\]
\[\text{Stiff}[(1 + \text{TotalA} + a,} \\
1 + \text{TotalB} + b +} \\
2 \text{TotalB})] = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (K_{vlv3})} \\
R1 \int_{0}^{\pi} (K_{vlw})} \\
(\text{DifferentialD}[\text{Theta}]} \\
(\text{DifferentialD}[\text{x}])],} \\
\text{Stiff}[(1 + \text{TotalA} + a,} \\
1 + 2 \text{TotalB} + b +} \\
2 \text{TotalB})] = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (M_{u3u3single})} \\
R1 \int_{0}^{\pi} (M_{u3u3single})} \\
(\text{DifferentialD}[\text{Theta}]} \\
(\text{DifferentialD}[\text{x}]),} \\
\text{Stiff}[(1 + \text{TotalA} + a,} \\
1 + 2 \text{TotalB} + b)]],} \\
\text{Equation3} \text{Sum} \text{Force} \text{in} \text{x} \text{for} 3,} \\
\text{Mass}[(1 + 2 \text{TotalA} + a,} \\
1 + 2 \text{TotalB} + b)]],} \\
\text{Sum} \text{Force} \text{in} \text{r} \text{for} 3,} \\
\text{Mass}[(1 + 2 \text{TotalA} + a,} \\
1 + 2 \text{TotalB} + b)]],} \\
\text{Sum} \text{Force} \text{in} \text{z} \text{for} 3,} \\
\text{Mass}[(1 + 2 \text{TotalA} + a,} \\
1 + 2 \text{TotalB} + b)]],} \\
\text{Sum} \text{Force} \text{in} \text{r} \text{for} 3,} \\
\text{Mass}[(1 + 2 \text{TotalA} + a,} \\
1 + 2 \text{TotalB} + b)]],} \\
\text{Sum} \text{Force} \text{in} \text{z} \text{for} 3,} \\
\text{Mass}[(1 + 2 \text{TotalA} + a,} \\
1 + 2 \text{TotalB} + b)]],} \\
\text{Sum} \text{Force} \text{in} \text{r} \text{for} 3,} \\
\text{Mass}[(1 + 2 \text{TotalA} + a,} \\
1 + 2 \text{TotalB} + b)]],} \\
\text{Sum} \text{Force} \text{in} \text{z} \text{for} 3,} \\
\text{Mass}[(1 + 2 \text{TotalA} + a,} \\
1 + 2 \text{TotalB} + b)]],} \\
\text{Sum} \text{Force} \text{in} \text{r} \text{for} 3,} \\
\text{Mass}[(1 + 2 \text{TotalA} + a,} \\
1 + 2 \text{TotalB} + b)]],} \\
\text{Sum} \text{Force} \text{in} \text{z} \text{for} 3,} \\
\text{Mass}[(1 + 2 \text{TotalA} + a,} \\
1 + 2 \text{TotalB} + b)]],}
\[\text{Stiff}[l + 2 \text{ TotalA } + a, 1 + b]] = \int_A^B (K_u3v1) R3 \text{ \{DifferentialD\} } \theta \text{ \{DifferentialD\} } x)\]

StyleBox["';", FontColor->RGBColor[0, 0, 1]]

RowBox[{\{Stiff\}[(1 + 2 \text{ TotalA } + a, 1 + \text{ TotalB } + b\})] = \int_A^B (K_u3v1) R3 \text{ \{DifferentialD\} } \theta \text{ \{DifferentialD\} } x)\]

StyleBox["';", FontColor->RGBColor[0, 0, 1]]

RowBox[{\{Stiff\}[(1 + 2 \text{ TotalA } + a, 1 + b + 2 \text{ TotalB})\}], "]="

RowBox[{\{\int_A^B (K_u3v1) R3 \text{ \{DifferentialD\} } \theta \text{ \{DifferentialD\} } x)\}], "+",

StyleBox["';", FontColor->RGBColor[0, 0, 1]]

RowBox[{\{\int_A^B (K_u3v1) R3 \text{ \{DifferentialD\} } \theta \text{ \{DifferentialD\} } x)\}], "+",

StyleBox["';", FontColor->RGBColor[0, 0, 1]]

RowBox[{\{\int_A^B (K_u3v1) R3 \text{ \{DifferentialD\} } \theta \text{ \{DifferentialD\} } x)\}], "+",

StyleBox["';", FontColor->RGBColor[0, 0, 1]]

RowBox[{\{\int_A^B (K_u3v1) R3 \text{ \{DifferentialD\} } \theta \text{ \{DifferentialD\} } x)\}], "+",

StyleBox["';", FontColor->RGBColor[0, 0, 1]]
\[
\frac{\text{d}x}{\text{d}t} = \frac{1}{2} \left( \int_0^{\pi} \int_0^{\pi} (K_{u3v3single} R3) \text{d}\Theta \text{d}x \right) + \frac{1}{2} \left( \int_0^{\pi} \int_0^{\pi} (K_{u3w} R3) \text{d}\Theta \text{d}x \right) + \frac{1}{2} \left( \int_0^{\pi} \int_0^{\pi} (K_{u3wsingle} R3) \text{d}\Theta \text{d}x \right)
\]

\[
\text{Mass} = 110
\]
\begin{align*}
&\text{RowBox}[\{\text{\(\int_0^A\int_0^{2\pi} (M_{v3v3single}) R^3 \text{\(d\)}\Theta \text{\(d\)}x\)}\}, "+", \\
&\text{StyleBox}[\{\text{\(\int_A^B\int_0^{2\pi} (M_{v3v3}) R^3 \text{\(d\)}\Theta \text{\(d\)}x\)}\}, "+", \{\text{\(\int_{\int_0^{A\int_0^{2\pi} (K_{v3ul}) R^3 \text{\(d\)}\Theta \text{\(d\)}x\)}\},

&\text{StyleBox}[\{\{\text{\(\int_{\int_0^{2\pi}} (K_{v3single}) R^3 \text{\(d\)}\Theta \text{\(d\)}x\)}\}, "+", \{\text{\(\int_{\int_0^{2\pi}} (K_{v3ul}) R^3 \text{\(d\)}\Theta \text{\(d\)}x\)}\},"]
\end{align*}
FontColor->RGBColor[0, 0, 1],
"+", \(\int_0^{2\pi} K v_3 u_3 \text{single} R^3 \text{DifferentialD}[\Theta] \text{DifferentialD}[x]) \), ";", ";\"
RowBox[\{(\text{Stiff})[(1 + 3) TotalA a, 1 + TotalB b + 2 TotalB])\}, ",="
RowBox[\{(\int_0^A \int_0^{2\pi} K v_3 \text{single} R^3 \text{DifferentialD}[\Theta] \text{DifferentialD}[x]) \}
\\text{R3} \text{DifferentialD}[\Theta] \text{DifferentialD}[x])\}, ",+", ";\"
RowBox[\{(\int_0^A \int_0^{2\pi} K v_3 \text{single} R^3 \text{DifferentialD}[\Theta] \text{DifferentialD}[x]) \}
\\text{R3} \text{DifferentialD}[\Theta] \text{DifferentialD}[x])\}, ";+", ";\"
RowBox[\{(\int_0^A \int_0^{2\pi} K v_3 \text{single} R^3 \text{DifferentialD}[\Theta] \text{DifferentialD}[x]) \}
\\text{R3} \text{DifferentialD}[\Theta] \text{DifferentialD}[x])\}, ";+", ";\"
RowBox[\{(\int_0^A \int_0^{2\pi} K v_3 \text{single} R^3 \text{DifferentialD}[\Theta] \text{DifferentialD}[x]) \}
\\text{R3} \text{DifferentialD}[\Theta] \text{DifferentialD}[x])\}, ";+", ";\"
RowBox[\{(\int_0^A \int_0^{2\pi} K v_3 \text{single} R^3 \text{DifferentialD}[\Theta] \text{DifferentialD}[x]) \}
\\text{R3} \text{DifferentialD}[\Theta] \text{DifferentialD}[x])\}, ";+", ";\"
RowBox[\{(\int_0^A \int_0^{2\pi} K v_3 \text{single} R^3 \text{DifferentialD}[\Theta] \text{DifferentialD}[x]) \}
\\text{R3} \text{DifferentialD}[\Theta] \text{DifferentialD}[x])\}, ";+", ";\"
RowBox[\{(\int_0^A \int_0^{2\pi} K v_3 \text{single} R^3 \text{DifferentialD}[\Theta] \text{DifferentialD}[x]) \}
\\text{R3} \text{DifferentialD}[\Theta] \text{DifferentialD}[x])\}, ";+", ";\"
RowBox[\{(\int_0^A \int_0^{2\pi} K v_3 \text{single} R^3 \text{DifferentialD}[\Theta] \text{DifferentialD}[x]) \}
\\text{R3} \text{DifferentialD}[\Theta] \text{DifferentialD}[x])\}, ";+", ";\"
RowBox[\{(*Equation 5 Sum Force in w) for 1 + 2 + 3*) \}
\\text{FontColor->RGBColor[1, 0, 0]}\]
\[ Mass[I + 4 TotalA + a, 1 + 4 TotalB + b] = \]
\[\int_A^B \int_0^{2\pi} M_{wl} R_1 \, d\theta \, dx \]
\[+ \int_A^B \int_0^{2\pi} M_{w2} R_2 \, d\theta \, dx \]
\[+ \int_A^B \int_0^{2\pi} M_{w3} R_3 \, d\theta \, dx \]
\[\int_A^B \int_0^{2\pi} K_{wl} R_1 \, d\theta \, dx \]
\[+ \int_A^B \int_0^{2\pi} K_{w2} R_2 \, d\theta \, dx \]
\[+ \int_B^L \int_0^{2\pi} K_{w3} R_3 \, d\theta \, dx \]
\[ \begin{align*} 
&\int_A^B \left( \int_0^{2\pi} (K_{wlvl}) R_l \, d\theta \, dx \right) + \int_A^B \left( \int_0^{2\pi} (K_{w2vl}) R_2 \, d\theta \, dx \right) \\
&\quad + \int_A^B \left( \int_0^{2\pi} (K_{w3vl}) R_3 \, d\theta \, dx \right), \\
&\quad \text{Stiff} = \left( 1 + 4 \right) \text{TotalA} + a, \\
&\quad 1 + b + 2\text{TotalB}, \\
&\quad \text{Stiff} = \left( 1 + \right) \text{TotalA} + a, \\
&\quad 1 + \text{TotalB} + b + 2\text{TotalB}. 
\end{align*} \]
\[
\begin{align*}
&\text{StyleBox}\left[\left(\int_{A}^{B} \left(\int_{0}^{2\pi} (K_{\text{w2v3}} R_{2} \, d\Theta \, dx) \right) d\Theta \, dx\right) + \left(\int_{0}^{A} \left(\int_{0}^{2\pi} (K_{\text{w3v3single}} R_{3} \, d\Theta \, dx) \right) dt \, dt \right) + \left(\int_{B}^{L} \left(\int_{0}^{2\pi} (K_{\text{w3v3single}} R_{3} \, d\Theta \, dx) \right) \right) \right]\right]
\end{align*}
\]

\[
\begin{align*}
&\text{RowBox}\left[\left\{\text{Stiff}[((1 + 4a) \, \text{TotalA} + a, 1 + 2b \, \text{TotalA} + a, 1 + 2b \, \text{TotalB}, b, b)]\right\}\right]
\end{align*}
\]

\[
\begin{align*}
&\text{StyleBox}\left[\left(\int_{A}^{B} \left(\int_{0}^{2\pi} (K_{\text{w1w}} R_{1} \, d\Theta \, dx) \right) d\Theta \, dx\right) + \left(\int_{0}^{A} \left(\int_{0}^{2\pi} (K_{\text{w2w}} R_{2} \, d\Theta \, dx) \right) \right) + \left(\int_{B}^{L} \left(\int_{0}^{2\pi} (K_{\text{w3wsingle}} R_{3} \, d\Theta \, dx) \right) \right) \right]
\end{align*}
\]

\[
\begin{align*}
&\text{RowBox}\left[\left\{\text{Stiff}[((1 + 4a) \, \text{TotalA} + a, 1 + 2b \, \text{TotalB}, b)\right\}\right]
\end{align*}
\]
Appendix D: ANSYS BATCH File

ANSYS BATCH File

The following lines of code can be copied/typed into a text file and read into ANSYS version 6.1 to create a partially treated three-layer cylinder which the complex natural frequencies for the damped solution are calculated. This was the file used to create the results presented in this thesis.

Code Starts below this line

/ BATCH

!!!!!!!File CREATED BY NATE SMITH 5/5/04 THIS FILE SETS UP
!!!!!!!AND SOLVES FOR A 3 LAYERD CYLINDER WITH DAMPING
! /COM,ANSYS RELEASE 6.1 UP20020321 16:19:08 05/05/2004
/input,start61,ans,'C:\PROGRAM FILES\ANSYS INC\ANSYS61\docu\1,1,1,1,1
!*  
/ PREP7 

!!!!!!!!!!!!List of all Variables

! Shell Length 
* set,L,60 

! Coverage Length 
* set,COV,36 

! outer radius of shell 
* set,Ros,3 

! thickness of shell 
* set,ts,1 

! thickness of VEM 
* set,tv,.05 

! Thickness of CL 
* set,tc,.05 
* set,za,(L-COV)/2 
* set,zb,(L-COV)/2+cov 
* set,ris,ros-ts 
* set,rov,ris 
* set,riv,ris-tv 
* set,roc,riv 
* set,ric,riv-tc 

!!!!!!!!!!!!!!!!!!!!!!!!ELEMENT TYPE AND MATERIAL PROPERTIES 
ET,1,SOLID185 
!*
!*  
MPTEMP,.......
MPTEMP,1,0
MPDATA,EX,1,,10e6
MPDATA,PRXY,1,,33
MPTEMP,.......
MPTEMP,1,0
MPDATA,DENS,1,,000255
MPTEMP,.......
MPTEMP,1,0
MPDATA,EX,2,,49600
MPDATA,PRXY,2,,24
MPTEMP,.......
MPTEMP,1,0
MPDATA,DENS,2,,00014
MPTEMP,.......
MPTEMP,1,0
MPDATA,DAMP,2,,0061

!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!CREATIONG OF VOLUMES
! MPLIST,ALL,,EVLT
! /REPLET,RESIZE
CYLIND,roc,ric,za,zb,0,360,
CYLIND,rov,riv,za,zb,0,360,
CYLIND,ros,ris,0,L,0,360,
! /VIEW, 1,1,1,1
! /ANG, 1
! /REP,FAST
FLST,5,39,4,ORDE,9
FITEM,5,1
FITEM,5,-14
FITEM,5,16
FITEM,5,21
FITEM,5,-36
FITEM,5,45
FITEM,5,-48
FITEM,5,53
FITEM,5,-56
!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!LOCAL LENGTHS
FLST,5,12,4,ORDE,6
FITEM,5,17
FITEM,5,-20
FITEM,5,37
FITEM,5,-40
FITEM,5,57
FITEM,5,-60
CM,_Y,LINE
LSEL,,P51X
CM,_Y1,LINE
CMSEL,,Y
!*
LESIZE,,Y1,2,,,,,,1
!*

FLST,5,48,4,ORDE,6
FITEM,5,1
FITEM,5,-16
FITEM,5,21
FITEM,5,-36
FITEM,5,41
FITEM,5,-56
CM,_Y,LINE
LSEL,,P51X
CM,_Y1,LINE
CMSEL,,Y
!*
LESIZE,,Y1,,5,,1
!*
!!!!!!!!!!!!!!!!!!!!!!!!!!!!Mesh
CM,_Y,VOLU
VSEL,,2
CM,_Y1,VOLU
CMSEL,,S,Y
!*
CMSEL,,S,Y1
VATT,2,1,0
CMSEL,,S,Y
CMDELE,,Y
CMDELE,,Y1
!*
CM,_Y,VOLU
VSEL,,1
CM,_Y1,VOLU
CHKMSH,'VOLU'
CMSEL,,S,Y
!*
VSWEEP,,Y1
!*

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CMDELE,_Y
CMDELE,_Y1
CMDELE,_Y2
!* CM,_Y,VOLU VSEL,,,, 2 CM,_Y1,VOLU CHKMSH,'VOLU' CMSEL,S,_Y
!* VSWEEP,_Y1 
!* CMDELE,_Y CMDELE,_Y1 CMDELE,_Y2
!* CM,_Y,VOLU VSEL,,,, 3 CM,_Y1,VOLU CHKMSH,'VOLU' CMSEL,S,_Y
!* VSWEEP,_Y1 
!* CMDELE,_Y CMDELE,_Y1 CMDELE,_Y2
!* NUMMRG,ALL,, ,LOW 
!!!!!!!!!BoundaryCondition 
!!!!!!!!!!BOUNDARY CONDITIONS 
FLST,2,16,4,ORDE,2 FITEM,2,41 
FITEM,2,-56 
!* /GO DL,P51X,,UX, 
FLST,2,16,4,ORDE,2 FITEM,2,41 
FITEM,2,-56 
!* /GO DL,P51X,,UY, /PREP7
!!!!!!!!!!!!!!!!!!!!!!!!!!!SOLUTION SETUP

!FINISH
!/SOLU
!*  
!ANTYPE,2
!*  
!MODOPT,LANB,4
!EQSLV,SPAR
!MXPAND,4,0
!LUMP,0
!PSTRES,0
!*  
!MODOPT,LANB,1,100,500,OFF
FINISH
!/SOLU
!*  
!ANTYPE,2
!*  
!MODOPT,damp,4
!EQSLV,FRT
!MXPAND,4,0
!LUMP,0
!PSTRES,0
!*  
!MODOPT,damp,5,100,300,OFF
!!!!!!!!!!!!!!!!!!!!!!!!!!!SOLVE
!/STATUS,SOLU
  SOLVE