Modeling problems in mucus viscoelasticity and mucociliary clearance

Michael M. Norton

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Modeling Problems in Mucus Viscoelasticity and Mucociliary Clearance

Michael M. Norton

A Thesis Submitted in Partial Fulfillment of the Requirement
for the Master of Science in Mechanical Engineering

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July 2009
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Modeling Problems in Mucus Viscoelasticity and Mucociliary Clearance

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Modeling Problems in Mucus Viscoelasticity and Mucociliary Clearance

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I’d like to thank, foremost, Dr. Risa Robinson for courageously letting me tackle this fascinating subject and supporting me throughout the research.

Much of the supporting mathematics would not have been doable in such a short period of time if it were not for Dr. Weinstein, who always had time to teach a new technique. We can only hope that the future will test the strength of his constitution with actual “mucus.”

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Finally, I thank all of my friends and family whom must have thought I had dropped off the face of the Earth for weeks at a time, but who always gave their support. There are, of course, those who knew exactly where I was: Sylvan Hemmingway and cubicle-mates, thank you.

To my parents Sharon and Robert, and Annie Reeds I owe a special thanks for going above and beyond by reading some of my first drafts and listening to me as I’d try to make sense of what I was learning.
Abstract

From the common cold and allergies to severe chronic and acute respiratory impairments, the function of the body’s mucociliary clearance mechanism plays a primary defense role. Mucus demonstrates numerous non-Newtonian behaviors which set it apart from viscous fluids. Among them: Bingham plastic behavior, shear-thinning, and elasticity on short time scales due to relaxation time. Experimental evidence suggests that certain rheologies promote effective transport. In an effort to reveal the mechanisms controlling transport, models are developed.

Firstly, a steady state model which idealizes the mucus as a rigid body is created in order to bring together disparate bodies of experimental work from the literature. The force balance reveals that the force cilia are capable of exerting cannot be related, simply, to the velocity of mucus. That is, only a fraction of the force cilia are capable of exerting is required to steadily transport mucus at the velocities observed experimentally. Likewise, the velocities estimated by this model when cilia force is the input are overestimated by one to two orders of magnitude. This incongruity formally motivates the inclusion of one of mucus’s rheological behaviors, stress relaxation.

The first viscoelastic problem considered is the response of the linear Maxwell fluid to an oscillating plate. Though a problem commonly discussed in textbooks on theoretical viscoelasticity, the complete analytical solutions are not provided. Here, solutions are found and graphed in terms of the phase and amplitude of the velocity field resultant from the oscillations of the plate; all derivations are shown in their entirety. The effects of stress relaxation (sometimes referred to as memory) and inertia on phase and amplitude are shown to have frequency dependence. Furthermore, it is shown that oscillatory shear perturbations to a viscoelastic Maxwell fluid always travel further and faster away from the source as Deborah number (a dimensionless parameter governing the importance of viscoelastic forces, De=0 corresponds to a Newtonian fluid) is increased. The limitation of the linear Maxwell fluid is illustrated by attempting to apply the constitutive equation to a thin film flow problem. It is found that the stress field of the solution only differs from the viscous case if the boundary conditions are transient; that is, the constitutive equation cannot account for the changes in stress that occur over space. The time derivative must be replaced by a Convected Derivative to achieve the
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Three problems were completed using the Upper Convected Maxwell model for viscoelasticity. The first considers a purely unidirectional shear flow which, unlike a viscous fluid, possesses tensile stresses along streamlines. The model posits that these additional stresses are essential for transport by allowing regions which are actively sheared, to hold up inactive regions. A novel relationship between applied stress and relaxation time is developed; the model shows that increasing the relaxation time of mucus decreases the amount of stress that must be imparted by cilia. In the second two problems, the UCM equations are simplified via a perturbation series expansion for small Weissenberg number (also a dimensionless group governing the importance of viscoelastic forces). This technique allows the analytically solvable viscous (also referred to as the unperturbed or order one) solutions to be used to estimate the impact of small amounts of stress memory. It is found that elasticity increases the developing region of a viscous flow; all stress components are convected downstream due to flow memory. Likewise, in the sinusoidally varying stress case, the velocity field is always shifted further away from the phase of the applied stress as viscoelastic forces are increased. It is also found that the departure from the viscous solution is dramatically reduced if the stress distribution is moving at the same velocity as the mucal flow. This shows the benefit of an antiplectic wave speed (the physiologically relevant case in which the phase of the cilial beat is moving opposite to transport) as there is no danger that these two can be in phase with one another.

Model restrictions prevent quantitative gauges of transport efficiency as a function of metachronal wave parameters and relaxation time to be made. Several additional problems are proposed to address unanswered modeling questions and experimental solutions for the lack of rheological data on tracheal mucus are suggested.
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<th>Description</th>
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<tbody>
<tr>
<td>CBF</td>
<td>Ciliary Beat Frequency</td>
</tr>
<tr>
<td>ASL</td>
<td>Airway Surface Liquid</td>
</tr>
<tr>
<td>PCL</td>
<td>Periciliary Layer</td>
</tr>
<tr>
<td>MW</td>
<td>Metachronal Wave</td>
</tr>
<tr>
<td>ODE</td>
<td>Ordinary Differential Equation</td>
</tr>
<tr>
<td>PDE</td>
<td>Partial Differential Equation</td>
</tr>
<tr>
<td>UCM</td>
<td>Upper-Convected Maxwell</td>
</tr>
<tr>
<td>$G^*(\omega)$</td>
<td>Complex dynamic modulus</td>
</tr>
<tr>
<td>$G'(\omega)$</td>
<td>real part of dynamic modulus $G^*$, Storage Modulus</td>
</tr>
<tr>
<td>$G''(\omega)$</td>
<td>imaginary part of dynamic modulus $G^*$, Loss Modulus</td>
</tr>
<tr>
<td>$G$</td>
<td>Modulus of rigidity</td>
</tr>
<tr>
<td>$\eta^*(\omega)$</td>
<td>Complex viscosity</td>
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<tr>
<td>$\eta'(\omega)$</td>
<td>real part of complex viscosity $\eta^*$</td>
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<tr>
<td>$\eta''(\omega)$</td>
<td>imaginary part of complex viscosity $\eta^*$</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Newtonian Viscosity</td>
</tr>
<tr>
<td>$\eta_0$</td>
<td>Zero Shear Rate Viscosity</td>
</tr>
<tr>
<td>$\phi$</td>
<td>Volume fraction or phase depending on context</td>
</tr>
<tr>
<td>$\Theta$</td>
<td>Phase shift</td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>Amplitude</td>
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<tr>
<td>$\psi$</td>
<td>Stream function</td>
</tr>
<tr>
<td>$c$</td>
<td>phase/wave speed</td>
</tr>
<tr>
<td>De</td>
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<td>Density</td>
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<tr>
<td>kDa</td>
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1 Introduction

1.1 Motivation for Research and Research Goals

From the common cold and allergies to severe chronic and acute respiratory impairments, the function of the body’s mucociliary clearance mechanism plays a primary defense role by removing pathogens from the respiratory tract. The mucociliary system is dynamic and able to respond to the specific clearance needs of an individual. As a result, gaining a physical understanding of the dynamics at work by experimental methods has proven problematic. Acquiring knowledge about the rheological properties of mucus and cilial kinematics and dynamics, and their interaction is challenging because the procedures required are complex, which limit sample groups and hinder reproducibility.

Characterizing the viscoelastic rheology of mucus is a particularly convoluted process. Unlike Newtonian viscous fluids which possess a straightforward connection between strain rate and force, the response of mucus is dependent on the spatial scale, time scale and magnitude of an applied quantity (be it strain, strain rate, or stress). This means that even the geometry of the rheological device used to characterize mucus samples is a variable that must be considered in experimentation. Mucus is thus non-Newtonian in the broadest sense; it possesses the readily observable characteristic of stress relaxation, yielding, and shear-thinning. Furthermore, mucus is a composite of chemical species. The structure of mucus’s polymer network is highly sensitive to the concentration of its primary components, the glycoproteins mucins. Accordingly the rheological properties are highly sensitive. The introduction of additional constituents, such as other biological macromolecules and dissolved ions or pH variability, also impact mucus rheology in health and disease.

The body strives to maintain an optimal mucus rheology. In order to garner an understanding of why one rheology is preferred over another, a variety of model problems are completed with the aim of revealing the role of one of mucus’s non-Newtonian features: relaxation time. First, we take a pedagogical approach; known parameters from the literature are used in simple steady state models. Linear
viscoelasticity is then explored through the classic oscillatory plate problem and a thin-film flow problem reminiscent of tracheal mucus transport. The solution to the oscillatory shear problem, though readily attainable by analytical means, is not provided in texts. It is therefore derived in its entirety here. The results will inform some general insights about viscoelastic media; shear disturbances travel faster and further than they do in Newtonian media.

However the linear Maxwell model does not translate well to flow situations in which large displacements occur. This becomes apparent when the model is applied to the thin film flow problem; viscoelastic effects only alter the solution to stress if the applied boundary conditions are transient despite the fact that fluid traveling in the flow experiences changing conditions which, intuitively, should draw out the effects of relaxation time. A more robust model utilizing the Upper Convected time derivative in the Maxwell constitutive equation which allows stress and strain states to convect with fluid flow is thus utilized. Three additional model problems which probe different aspects of mucociliary transport are created using the improved Maxwell equation. One relates the shear that must be applied by cilia to relaxation time. Another illustrates the effects of relaxation time on the developing region of a viscous flow. Finally, a third shows the effect of boundary condition transience (intended to model the naturally occurring wave-like coordination between cilia). Limitations found in this second viscoelastic model and the methods used to solve the problems are also discussed.

This thesis concludes with a proposal for the next stage of research. Improvements on the models developed in this thesis are one avenue of investigation. However, it is suggested that the next iteration of this research be experimental in nature. Pointed empirical studies are crucial to elevate models from mathematical exercises to quantitative predictors.

1.2 Epithelial Anatomy

Epithelial membranes are present wherever there is an internal or external surface and are thus present throughout the body of multi-cellular organisms, serving a variety of functions. Of these epithelia, cilia bound epithelium in the respiratory tract is a special incarnation which transports mucus and whatever else may lie upon or within it by ciliary
action. In the human body, ciliated epithelia is present in the nasal cavity, paranasal sinuses, eustachian tubes, middle ear, pharynx, trachea, oviducts and cervix in females, ductuli efferentes in males, and the ependymal lining of the brain.\(^1\) In the respiratory context, the load transported by the mucus often comprises environmental particulates, bacteria, or cellular waste. The load and mucus are transported from finer generations of the lung up to the esophagus where it is removed from the respiratory system. The ciliary escalator, as well as airway peristalsis and cough, maintain open breathing passages and prevent infection. Malfunction of any component of the mucociliary clearance mechanism (as is found in Asthma, Chronic Obstructive Pulmonary Disease, Primary Ciliary Dyskinesia, Bronchiectasis, Chronic Bronchitis, and Cystic Fibrosis) reduces the respiratory system’s ability to discard unwanted matter.

The components of the functioning mucociliary mechanism can, essentially, be categorized into two factions: the cells, and the secretions of those cells. Without cilia, it is easy to imagine evacuation of unwanted respiratory particulates would be dramatically reduced, limited to the other methods of clearance mentioned. Less obvious, there is evidence to suggest that omitting mucus from the system would also diminish the evacuation of pathogens even if cilia function were maintained. Though highly efficient movers, cilia cannot move a load without properly being coupled to it via the viscoelastic gel, mucus.\(^2\)

1.2.1 Cilia

The living part of the mucociliary mechanism is the epithelial cells, which can be subdivided further into ciliated cells and mucus secreting cells. The mucus appears to rest on a bed of cilia 5-7\( \mu \)m long in the trachea, 2-3 \( \mu \)m in the intrapulmonary bronchi (much shorter than those found in water propelled organisms).\(^3\) Cilia are 0.1\( \mu \)m\(^4\) in radius, and spaced 0.3\( \mu \)m\(^4\) to 0.05\( \mu \)m\(^5\) from each other at the base. Arranged in a dense mat, 200-300 per cell, cilia are thin cytoplasmic extrusions from the substrata cells.

Viewing the cross section of a cilium would reveal a nine-fold symmetry in the cytoskeleton, or axoneme. These protein filaments that give the protrusion its structure run the length of the cilium, terminating at the apical cap. This is drawn schematically in
Figure 1.1 and pictured in Figure 1.2; longitudinal and axial views are provided. Nine microtubule doublets (comprising an A subfiber and a B subfiber) arranged radially surround two microtubule singlets. The radial fibers connect to each other via nexin links and to the center structure by radial spokes. During the beat cycle, the radial doublets are moved relative to each other via a pair of attached mechanochemical motors called dyneins. These “walking proteins” are part of a larger class of chemicals called ATPases which use the chemical energy of adenosine triphosphate (ATP) to perform work. As the fibers slide relative to each other bending occurs. A study by Dentler 1982\(^6\) on rabbit and chicken tracheal cilia finds through examination of the capping structure during bending, that all 9 fibers as well as the central pair of singlets remain fixed in the cap in most cases. This supports the observation of ciliary twisting during bending; a cilium must twist if the distal ends of the microtubules are to remain planar (fixed in the cap) during sliding and bending. A more detailed mechanical account of the internal ciliary motor can be found in Lindeman 1994.\(^7\)

Cilia are classified by their internal structure. In addition to the 9+2 variety described, cilia are also found in the typically immotile 9+0 configuration, a more detailed overview can be found in Satir and Christensen 2007.\(^8\) In this thesis, cilia will always refer to the former as they are the variety present in the respiratory tract. In recent years 9+0 cilia have received attention for their key role in mammalian embryo development.\(^9,10\)
Figure 1.1 (left): Schematic drawing representing key structural features of a cilium, a) longitudinal; and b-e) cross-sectional views of the structure of a cilium from the respiratory tract. The cross section of a cilium varies; different structures are present where the cilia attach to the cell (basal body), and at its end (ciliary cap). The features of the cross section mentioned can be seen in e): (A) and (B) subfibers, (r) radial spokes, (n) nexin links, (o) and (i) outer and inner dynein arms, (h) dilated heads of radial spokes for attachment to (p) central microtubules.3

Figure 1.2 (right): Upper Left Inset: cross-sectional view of cilium structure.8 Upper-Right Inset: longitudinal view of ciliary cap and crown. The cilia in the image are approximately 5.1µm in length.11

An individual cilium beats in a cyclic manner. The entire cycle is composed of an effective stroke (approximately one quarter of the cycle) in which the cilia straightens itself and moves toward the intended direction of propulsion. It is only during this portion of the stroke in which contact with the mucus is made. The effective stroke is followed by the recovery stroke (the remaining three quarters of the beat cycle) in which the cilium bends and curls to the side, avoiding the mucus layer to prevent retrograde mucus flow, and returns to its original position.8,12 Fluid flow analysis discussed in the modeling history section discusses this advantage in greater detail.

Only the tips of cilium ever engage the mucus, Figure 1.3. Contact between the cilia and mucus is achieved with the help of hair-like ‘claws’ embedded in the apical cap.
which enable the cilia to drag the semi-solid towards its destination, Figure 1.4. These structures are not present in water propelling cilia and flagella. This supports the hypothesis that they play a role exclusive to mucus transport. The additional grip of the hooking mechanism is made possible by a cluster of three to seven thin projections of approximately 30nm in length and 10nm in diameter. Examination of the cilia surface chemistry in frog palate studies indicates a negative charge over its length, which further affects how cilia adhere to the mucus layer during the effective stroke. Both of these cilial features may affect mucus clearance; however, experiments have not been conducted that specifically address these features.

A group of cilia beat together in a semi-organized fashion. Each cilium beats slightly out of phase with adjacent cilia. The wave of activity, known as the metachronal wave (MW), is said to travel essentially opposite relative to the direction of propulsion or in an antiplectic fashion. The wave also has a slight laoplectic component and therefore travels back and to the right of the moving mucus. If one could visualize this motion in the trachea, the patches of activity would appear to corkscrew down the airway. It should be noted, however, that the recent experimental work of Ryser and colleagues 2007 have found contrary results regarding the direction of wave propagation. The
length of a complete wave is debated somewhat in the literature, figures as low as 10µm and as high as 100µm have been observed by Lucas and Douglas 1934\textsuperscript{17} and Ryser,\textsuperscript{16} respectively. The wave travels between 100 and 400 µm/sec.\textsuperscript{16} There continues to be debate as to whether ciliary coordination exists solely as the result of hydrodynamic coupling, or a form of chemical communication, or both.

**Figure 1.5 (left):** Cilia Beat: Positions 1-9 represent the recovery stroke, 10-12 the effective stroke. The arrow indicates the direction of propulsion (2D motion assumption).

**Figure 1.6 (right):** Rabbit trachea epithelium: (e) represents the direction of the effective stroke and (m) the metachronal wave.\textsuperscript{14}

**Figure 1.7 (left):** Diagram illustrating the four principle metachronal wave types. In the case of mammalian tracheal cilia, the wave would be directed to the upper left, mostly antiplectic but partially laeoplectic.

**Figure 1.8 (right):** Image showing the contrast between an antiplectic metachronal wave and a symplectic wave.\textsuperscript{18}
1.2.2 Mucus

The inert component of the mucociliary mechanism is composed of a watery periciliary fluid in which cilia swim, and the much thicker mucus which rides on top of the periciliary layer and ciliary mat. In this case, “thick” and “thin” describe the apparent viscosity of the two fluids, not their physical dimension. The forces that keep the mucus layer aloft are twofold. Motivated by capillary forces, periciliary fluid (similar in composition to interstitial fluid) rises between the cilia up to their tips. Additionally, the mucus is restricted from diffusing into the space between cilia due to its large molecular size. In a healthy individual, 2% of the mucus is made of the macromolecular glycoprotein mucin. Interstitial fluid and an array of other biological chemicals with a variety of functions contribute to the remainder of mucus’s total weight. Beyond having properties that permit adherence to unwanted bacteria and other airway debris, mucus is also tuned to work with the cilial beat frequency due to its timescale dependent behavior. In a general sense, mucus possesses characteristics of both a solid and a liquid; it has the ability to store energy (via elasticity) and dissipate it (via viscosity). However, time scale, strain and strain-rate determine which rheological feature, solid or fluid, will dominate.

The mucin polymers which account for mucus’s complex rheology are secreted by both goblet cells as well as mucus cells found in glands. Mucins are categorized as either membrane bound or of a secreted gel-forming variety. In the human respiratory tract, there are four gel-forming mucins which are secreted to form the mucus film: MUCs 2, 5B, 5AC, and 19. The varying lengths of mucins make the physical character of mucus highly dependent on the proportions of these mucins and the presence of any additional constituents. While it is known that mucus composition changes due to illness and can be inconsistent amongst healthy individuals as well, a direct understanding of what each particular mucin contributes rheologically is not known. As a result, there is not a standard, widely accepted quantitative description of mucus. Aggravating this variability in its properties, illustrated in Figure 1.9, is the difficulty in fully characterizing a given sample in the first place. Mucus demonstrates a variety of non-Newtonian effects which require separate tests. Among them: Bingham plastic behavior.
in which a yield stress must be reached before a fluid like behavior is exhibited; shear-thinning in which viscosity decreases as strain-rate increases; and a slow dissipation of elastic potential (relaxation time). There is some debate in related literature as to whether shear-thinning is dependent on the duration of time of the applied strain-rate, known as thixotropy. Though it will be made clear in later sections, this thesis focuses exclusively on the elasticity of mucus. This is a modeling constraint; in actual polymeric solutions, a multitude of complex phenomena are often exhibited.

The majority of what is known about the mechanical properties of mucus has been found by macro-rheological studies by way of creep, oscillatory, and stress sweep modes of commercial rheometers (either cone and plate or parallel plate) of "composite" mucus samples. Composite, in this case, refers to the mixture found in-vivo which contains all mucin glycol-proteins as well as numerous other chemical species. These are the traditional tools of the engineer and are particularly useful when large volumes of a test specimen can be acquired.

Shearing oscillatory tests of mucus are quite prevalent in the literature. Less common (but no less important) are creep and stress sweep tests. Characterization of mucus’s spinnability has been performed by some researchers. Shear-free tests of extensional viscosity are, to the best of our knowledge, non-existent for human mucus though micro-rheological techniques have been employed on viscoelastic excretions of other species.

Material properties of oscillatory tests are typically reported in terms of either the real \( G'(\omega) \) and imaginary \( G''(\omega) \) parts of the dynamic modulus \( G^*(\omega) = G'(\omega) + iG''(\omega) \), storage and loss moduli, respectively, or \( |G^*| \) and the loss tangent \( G''/G' \). The dynamic modulus can be thought of as a frequency dependent rigidity. For comparison, ideal elastic solids respond immediately to imposed strains, while purely viscous fluids, which experience forces proportional to strain rate, respond precisely with a 90-degree phase lag. The stronger the elastic response of a material at a given angular frequency \( \omega \) the more of the response will be in phase with the input and the larger the storage modulus \( G' \). It is important to note that \( G' \) and \( G'' \) are coupled to each other even in simplified viscoelastic materials. For example, one cannot change the viscous component of a material without changing the frequency dependence of both Storage and
Loss moduli. These concepts will be discussed further in Chapter 3, when elementary viscoelastic constitutive equations are introduced.

To date, the most complete characterization of human mucus was completed by Davis and Dippy 1969, who performed three tests on the sputum extracted from patients with acute bronchitis. And while other studies have been completed in recent years, multiple quantization methods are not often used.

![Figure 1.9: Experimentally gathered dynamic moduli, the ratio of stress to strain, as a function of angular frequency. Davis and Dippy 1969 examined sputum, Albertini-Yagi 2004: induced sputum and directly collected samples, Trindade 2008: cleft palate vs. non-cleft palate, King 1979: cystic fibrosis, Gelman and Meyer 1979: dynamic moduli at resonance for five samples - untreated, heated (30 min at 50 °C), and 1,10, and 50 µl treatments of Glutaraldehyde which induced crosslinking (listed in order of increasing dynamic modulus), and Lorenzi 1992: the effect of saline treatments on collected rat mucus.](image-url)
The response of mucus samples to imposed stresses, strains, and strain rates is quite complex; the diversity of responses to small amplitude oscillatory tests is shown in Figure 1.9. The complexity of glycol-protein chains when dispersed with other chemical species allows numerous ways for the interaction to be modified, both positively and negatively, by environmental factors, disease, medical treatment, and even the physical handling of the samples prior to analysis. The basis of mucolytic therapy is the disruption of any one of the modes of entanglement that mucin polymers may have with each other. These modes include the covalent S-S bonds between polymer subunits, ionic bonds between charged regions of the glycoproteins, hydrogen bonds linking oligosaccharide side-chains, van der Waals' forces, physical entanglement, and complications due to the addition of DNA and/or Actin filaments in diseased states. The chemical and timescale dependency of mucus's mechanical properties is clear from the variability of data collected from mammalian mucus samples in Figure 1.9.

Correlating viscoelastic properties with transport rate has been investigated numerous times. Notably, Gelman and Meyer conclude that altering the dissipative component of bovine cervical mucus has little effect on its transference rate on an excised frog palate. Increasing the elastic character, however, increases transport to a point, and decrease it thereafter. This is a compelling result and motivates the discussion of future experimental works discussed in Chapter 5. Currently, there is no experimental work that has pinpointed the precise chemical and micro structural elements which maximize transportability. Gelman and Meyer made note of the fact that the viscosity and elasticity of mucus should not be taken as entirely independent. The work of King (1980) concludes, through the study of guaran (a polymer extracted from guar beans) on a frog palate, that transference rates always decrease as elasticity increases. Yates performed experiments on rat and chicken trachea; the results provide insights into transference rates but are less rigorous from a rheological stance.

A slightly different approach is investigated by Puchelle who finds that mucus with varying elasticities and viscosities are capable of being transported with efficiency as long as the sample possesses a certain degree of spinnability, a measure of a liquid's ability to be drawn into fiber without rupturing. Spinnability, like the dynamic modulus, is a bulk property which has a molecular origin that depends on both viscous
and elastic capabilities. This characteristic is also related to the extensional viscosity of a material. To date, there is not a theoretical framework for smoothly connecting various bodies of rheological data on a viscoelastic material such as mucus into a cohesive model. One test, unfortunately, cannot uniquely predict the results of the other. For example, it would be quite convenient to generate extensional viscosity or spinnability information directly from the dynamic moduli of Figure 1.9.

The rheology of mucus and the action of cilia come together to produce a net mucal flow; some experimental values for its average velocity are reported in Table 1. The general impression of the interaction between cilia and mucus is that cilia beat on a time scale faster than that of mucus’s ability to relax. As a result, the cilia interact with the mucus as if it were a solid instead of a highly viscous fluid. However, it has been shown through experiment that there is an optimal coupling of the two factors. The idea that a solid interaction is all that is necessary for transport may be an oversimplification; this is evidenced by the spinnability correlation.

<table>
<thead>
<tr>
<th>Velocity [µm/sec]</th>
<th>Species</th>
<th>Technique</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>167</td>
<td>Human (Male)</td>
<td>Non-invasive estimate</td>
<td>Miller 1968\textsuperscript{30}</td>
</tr>
<tr>
<td>225</td>
<td>Rat</td>
<td>Non-invasive estimate</td>
<td>Miller 1968</td>
</tr>
<tr>
<td>250</td>
<td>Cat</td>
<td>Non-invasive estimate</td>
<td>Miller 1968</td>
</tr>
<tr>
<td>39.7</td>
<td>Human</td>
<td>Cultured Cells - Human</td>
<td>Matsui et al. 1998\textsuperscript{31}</td>
</tr>
<tr>
<td>900 (maximum)</td>
<td>Excised Bullfrog Palate</td>
<td>Fluorescent Tracers</td>
<td>Winet et al. 1982\textsuperscript{32}</td>
</tr>
<tr>
<td>93-262</td>
<td>Excised Chicken Trachea</td>
<td>-</td>
<td>Yates 1980\textsuperscript{27}</td>
</tr>
</tbody>
</table>

\textbf{Table 1.1:} Mean (unless otherwise stated) transports rates of mucus in a variety of experimental setups.
1.3 Mucociliary Model Review

The modeling of ciliary and flagellar dynamics has its roots in the study of microorganism propulsion. As the field developed, many researchers extrapolated their understanding of mobile organisms to respiratory cilia applications. Because the anatomy and movement of cilia have already been discussed and not all models developed for flagella are valid for cilia (the converse is also true), this review will neglect flagella (which differ from a dynamics aspect) from the modeling history, unless applicable.

Blake performed one of the first rigorous mathematical models developed to explain the propulsion created by cilia in 1971. The development of the model was motivated by a curiosity in the movement of aquatic microorganisms (the multicellular Opalina in this case). Cilia, as they are often found in a dense mat, are modeled as a continuous undulating surface by use of small perturbation techniques. The MW is modeled as a traveling wave around a hemispherical body shown in Figure 1.10.

![Figure 1.10: Illustration of the spherical envelope model by Blake.](image)

Blake expanded this work to consider organisms with larger radii of curvature and those which were essentially planar in 1971. His results show that for ciliated organisms, the elongated torpedo shape was most favorable for Stokes flow propulsion, as more cilia would be ‘rowing’ in a beneficial direction. The model is only valid for symplectic metachronal activity (effective stroke is in the same direction of the wave), as is present in the Opalina. In the antiplectic case cilia motion can create significant gaps in the ciliary mat, invalidating the assumptions of the model. Also, the model is limited; it can only account for only velocities up to one half of the wave speed. The position of the surface approximates the position of the propelling portion of the cilia, their tips. This is
not necessarily an accurate representation of the cilia as it couples the wave speed with the speed of the cilia, this limitation was discussed by Blake and Sleigh 1974.\footnote{Blake examines the details of ciliary propulsion by modeling individual cilium as flexing cylindrical bodies anchored at one point. Both effective and recovery strokes are examined; the advantages of a recovery stroke become apparent. As a cilium bends to the side, it finds itself much closer to the cell surface where, due to the no slip condition, the flow is considerably slower than the flow developed near the tips of the cilia and easier to move against. Thus, a cilium finds less resistance when recoiling (moving against the flow) to ready itself for another effective stroke. Furthermore, cilia during the recovery stroke move less rigidly than during the effective stroke. Coiling back to their original positions by moving as parallel as possible to the imposing flow minimizes its resistance and backflow. The beat pattern used was gathered from experimental observation. Hand sketches were converted to models by a Fourier-series least-squares approach; however, the data was only planar so the model is limited as such. An infinite plane array of cilia was considered in which substantial averaging was employed to simplify calculations. The end result was a set of velocity profiles which were a function of the distance from the cell surface only (time averaged out) as well as plots for the bending moments and forces experienced by cilia in the three micro-organisms, \textit{Opalina}, \textit{Paramecium}, and \textit{Pleurobrachia}. This method overcomes the aforementioned limitations of the envelope model; it is known as the cilia sub-layer model because it considers the profile developed within the ciliary region, not just beyond it.

Utilizing this sub-layer technique, Blake creates a theoretical mucus clearance model by adding a dual layer nature to the fluid in which the cilia inhabit. The lower serous layer is given a comparatively lesser viscosity than the layer which rides atop it. Again, force singularities are distributed along the cilia centerline, which beats rhythmically in a bio-mimetic fashion to cause flow. Blake’s model permits significant averaging to take place because the oscillatory component of the flow (that is, the portion of the profile which shows both positive and retrograde movement) is said to be comparatively small. Air pathway orientation (gravity effects) and airway flow reversal from respiration are also considered, these flows are superimposed on those developed by the cilia. The model displays a heavy significance on the characteristics of the periciliary
fluid, its depth and viscosity, on mucus removal rather than the characteristics of mucus itself.  

Liron and Mochon developed a discrete cilia time-dependent model, a continuation of Blake’s sub-layer model, which gives a more detailed account of the flow in the near wall regions where oscillations (forward and reverse flow) take place. Again, an infinite plane array of cilia is considered and averaging the flow perpendicular to the net flow direction is employed. Though the cilial array is planar, their motion is one dimensional - coordinated such that the wave travels either opposite or in the direction of net flow (antiplectic and symplectic, respectively). That is, every cilia perpendicular to the flow is in phase. As with Blake’s approach, the forces the cilia generate are calculated iteratively from their velocity relative to the velocity of the flow field. With the added complexity of transience, Liron and Mochon assume a ‘one-to-one correspondence between the flow (and forces) and the configuration of the cilium (or cilia).” In other words, because the forces are unknown, they can be restricted to the same periodicity and phase shift of the ciliary movement itself. The result is a two-dimensional, unsteady flow which accounts for cilia interaction. The findings are an expanded version of Blake’s model. Mathematically, the result can be reduced directly to Blake’s model.

In order to overcome the computational difficulties of the cilia sublayer model, Keller et al. developed a model, termed the traction-layer model, in which the discrete cilia are replaced by a continuous, unsteady body force field in the periciliary region. Following an approach developed by Gray and Hancock for spermatozoa propulsion, the viscous forces on a slender body (an elongated cylinder) are found by utilizing two coefficients, tangential and normal. The final flow field is honed in on by an iterative approach; the body force distribution and flow field are solved for at each step. The results predict an oscillatory component that is on the same order of magnitude with the mean flow; compared to Blake, stronger backflow in the lower regions of the flow is predicted.

Liron and Rozenson took into account the non-Newtonian character of mucus (a linearized Oldroyd model was used) in a model and considered two cases: one where cilia remained (for their entire cycle) beneath the mucus, and another where the tips entered
the layer during the effective stroke. The cilia were modeled as an infinite array of pulsing forces, the magnitude of the pulse equal to the force developed by a cilium during it effective stroke. The metachronal-wave was simulated by adding a phase shift to the frequency of the pulses. The results support the claim that ciliary contact with the mucus layer is necessary for transport; however the model only serves one purpose, as the prediction of flows in the serous layer is not as realistic as previous modeling attempts.\textsuperscript{42}

Transport increases due to impingement are also predicted by Blake and Winet. Their model represents a cilium as a rigidly pivoting rod. The difference between the effective and recovery strokes is simulated by changing the length of the rod. That is, the rod is longer during its effective stroke than its recovery stroke. The model predicts dramatic improvements in the transport of mucus when the mucus layer is within the range of the effective stroke. However, as with Liron and Rozenson,\textsuperscript{42} the validity of the model beyond this conclusion is questionable due to the unrealistic flow in the serous region.\textsuperscript{43}

Building upon their work of integrating slender body theory into mucociliary transport which neglected tip penetration,\textsuperscript{44} Blake and Fulford consider ciliary interactions with the mucus/serous interface. The deformation of the interface due to the movement of an impinging body is calculated along with the force distribution on that body for a range of fluid viscosities. It is found that a higher viscosity in the upper layer results in a depression of that layer in the direction of the impinging body’s motion. That is, mucus may be pulled down in front of a cilium slightly during the effective stroke. It is difficult to confirm this with experimental evidence.\textsuperscript{45} A later study investigates the effect of penetration depth. Mucus velocity is found to increase as ciliary incursion furthers, but only to a point; after which velocity decreases. The model finds that this impingement is only crucial when few cilia are present (or active). Dense mats of cilia can, theoretically, propel mucus effectively by shearing the periciliary layer with no direct mucus contact.\textsuperscript{4}

Advances in computational solving power have allowed more involved cilia models to be actualized. Dillon\textsuperscript{2006}\textsuperscript{46} started from the ground up by modeling the internal structure of a cilium in a 2D sense. The microtubule doublets and nexin links are accounted for by elastic connections. The cilia’s motors, the dynein arms, are represented
as moveable links – their connections to the microtubules can be changed during the course of the simulation to induce sliding between the microtubules in relation to one another causing bending. The movements of cilia are coupled to the fluid portion of the model and are represented by body force (as opposed to a no-slip condition at its surface). The model represents a significant gain in the field. Ciliary beats are not fixed; as a result, insights can be gained as to the effects of the fluid(s) on the behavior of the cilia\textsuperscript{47}. Work in this area has also been performed by Gueron & Liron\textsuperscript{48,49} and Gueron & Gurevich\textsuperscript{50,51}.

As with Dillon, the internal structures of the cilia were modeled, three-dimensional in this case; individual cilium autonomously tailored their effective strokes to the viscosity of the fluid present. Furthermore, metachronal coordination also developed due to hydrodynamic coupling between cilia. The benefit of the third dimension is the ability to observe twisting of cilium\textsuperscript{50}. Though it is also observed by Lenz and Ryksin that coordination can develop by hydrodynamic interaction, they caution that the existence of hydrodynamic coupling does not imply the lack of coordination by other means, such as chemical communication\textsuperscript{52}. The most elaborate of this class of simulations was performed by Mitran\textsuperscript{53} who also included non-linear viscoelasticity via the Upper Convected Maxwell model (a model explored in Chapter 4).

What is seen as lacking in these models is not detail or complexity, but the development of theses regarding which mechanics of the mucus and cilia dominate mucociliary transport. For a material as complex as mucus, it is crucial to understand which rheological characteristics benefit transport and which do not. Which properties did the body intend to impart to mucus and which are necessary to achieve this end? In this thesis, mucus was modeled with a relaxation time, not simply because it has been thoroughly shown to exhibit this characteristic, but to create illustrative models that show how relaxation time can positively affect transport.
1.4 Scope of Research

- **Chapter 2:** Developed a steady state transport model in order to relate stress applied by cilia to average mucus transport velocity. The model includes two layers, one the Newtonian periciliary region and the other a rigid body representing the mucus. Variables in the models are extracted from relevant experimental works or theoretical models found in the literature. The model shows the discrepancy between multiple bodies of work and the necessity of including viscoelasticity in any transport model.

- **Chapter 3:** Linear viscoelasticity was investigated through the linear Maxwell fluid model which idealizes the viscoelastic response as fluid like on long time scales and solid-like or elastic on short time scales. The Maxwell model is implemented in an oscillatory unidirectional flow as well as a thin-film flow situation.
  - Results of the oscillating plate or “Ferry Shear Wave” problem are often discussed in texts with analytical solutions omitted. All solutions in this thesis are completed with full derivations in order to make the solutions more malleable. For example, the shear stress or the amplitude and phase of the velocity field as a function of position are readily attainable if the analytical solution is known.
  - The linear Maxwell model is applied to a thin-film flow situation. The solution is found to be unrealistic in that viscoelastic effects are only observable when boundary conditions are transient. This is due to the Lagrangian reference frame from which the linear Maxwell model is written. This problem exposes the limitations of the model and motivates the used of the Upper Convected derivative in Chapter 4.
• **Chapter 4.** The inability of the Linear Maxwell model to accurately predict stress in flows with large deformations is addressed by introducing the Upper Convected Derivative into the constitutive equation. Three modeling problems are completed using the Upper Convected Maxwell or UCM model:

  • Unidirectional shear flow assumptions are applied to the UCM fluid equations revealing the fact that normal or tensile stresses arise which are proportional to applied shear, the square of the shear rate, and relaxation time. The presence of tensile forces motivates the construction of a model designed to estimate the relaxation time and shear stress needed to generate tensile forces sufficient to “hold up” regions of the mucal flow which are not actively sheared by the cilia.

  • By expanding variables in the UCM model in terms of powers of small Weissenberg number (small viscoelastic stress assumption), the effects of relaxation time on shear and normal stresses in the transitional region between shear and shear-free boundary conditions are estimated.

  • Utilizing the same Weissenberg number expansion of the stick-slip problem, the effects the wavelength and phase speed of a sinusoidally distributed stress have on the velocity field are estimated.

• **Chapter 5.** Microrheological techniques utilizing Confocal Microscopy are proposed as a way of improving the current understanding of mucus’ fine structure and bulk rheological behavior as a function of chemical variables.
2 Steady State Model

The initial and most fundamental mucus transport model represents the mucus film as a rigid body; the purpose of this model is to establish an order of magnitude estimate of the relationship between the forces exerted by cilia on the mucus and the flow velocity of the mucus. Though experimental works have been completed which quantify various aspects of the mucociliary mechanism, a work which measures all relevant parameters of mucal flow and cilial dynamics in single experimental setup does not exist. Because a force balance must always hold, this model is a fundamental step in moving towards a more detailed model. In this section, the development of the steady state model will be introduced and the implementation of parameters taken from the literature, discussed.

2.1 Problem setup

Under the rigid body assumption, neither finite nor continuous deformation under shear of the mucus occurs; velocity is continuous throughout the mucus film. The rigid body limit can conceptually be reached by either an infinitely large elastic modulus in a Hookean model or an infinitely large viscosity in a Newtonian model. This is an acceptable naïve assumption, since under steady state conditions mucus appears to have a very high viscosity (a tendency to continuously strain albeit slowly given a steadily applied stress). Additionally, the flow profiles observed via tracers implanted in the mucus in works by Winet 1980-1987\textsuperscript{32,54,55,56} and Matsui 1998\textsuperscript{31} show velocity to be essentially constant across normal mucus film thicknesses.

In the steady state model, shown in figure 2.1, there are two layers shown: periciliary layer (PCL) and mucus layer, light and dark gray respectively. If mucus slab is to maintain constant velocity $V$, then a force must be applied to balance the viscous drag induced by the sliding of the mucus slab across the PCL and the weight of the slab itself.
Figure 2.1: Schematic of rigid body problem. Material properties and dimensions are shown in black, and velocities are shown in white. The matching condition between the two free bodies is velocity $V_M$.

The flow profile of the periciliary region is found by solving Navier-Stokes equation under the unidirectional flow assumption. No-slip (velocity continuity) is the only matching condition between the PCL and mucus regions. The stress applied by cilia, though denoted by $\tau$, does not represent shear applied to the interface per se. The force required to maintain velocity of the mucus slab is averaged over area and thus given a symbol typical of stress. This ambiguity must be dealt with formerly when mucus is modeled as a viscous fluid instead of a rigid body. This problem is included in Appendix B.2; however, it will not be a part of the discussion.

2.2 Parameters from the Literature and their Limitations

Reasonable values for the parameters for this model must be extracted from various experimental works from the literature. The depth of the periciliary layer is taken to be the typical length of cilia in the trachea, $7\mu m$. The rabbit tracheal slices of Sanderson and Sleigh\textsuperscript{14} report thicknesses between 1-10$\mu m$; the mucus layer is thus given a thickness of 10$\mu m$. Both the periciliary layer and mucus are assigned a density equal to that of water. In this representation the parameters which have the greatest impact on the outcome are the viscosity of the periciliary layer and the duration of the interaction between the cilia and mucus; they are also the most malleable of the parameters. Each
parameter that is extracted from the literature and used in the model will be discussed in terms of its validity before results are presented.

### 2.2.1 Effective Viscosity

While the interstitial fluid itself is composed of essentially salt water, the presence of the cilia introduces a solid phase thereby increasing the effective viscosity of the layer as whole to much beyond that of water. The viscosity of the PCL has a large effect on the velocity and stress estimates made by the steady state model. Therefore, having a method by which to estimate the degree to which cilia add resistance to the mucus slab as it slides along the PCL is important to estimating relevant stresses and velocities.

Correlations relating the concentration or volume fraction of a solid phase in dilute and concentrated dispersions to the effective viscosity can be useful in understanding how cilia introduce substantial flow resistance in the periciliary region. For cilia of radius $r$ and separated by distance $d_{cilia}$ packed on a square grid, the volume fraction $\phi$ is given by:

$$\phi = \frac{\pi r^2}{(2r + d_{cilia})^2} \tag{2.1}$$

However, typical correlations between the volume fraction of a dispersed phase and effective viscosity (presented in texts) cannot be justly used due to the deformability, anchoring, active forcing and spatially varying concentrating of the cilia (during the beat cycle the spacing between cilia varies greatly as can be seen in the micrographs of Sleigh shown earlier) as these correlations often consider rigid particles which are completely mobile, dispersed evenly, and not their own source of propulsion. None the less, an order of magnitude effective viscosity is surmised using the Krieger-Dougherty equation, eqn. 2.2 which calculates a factor to amplify the solvent

* Rectangular packing is used here in order to coincide with geometric assumptions made by Teff whose data will also be used in the steady state model; this assumption only affects the maximum volume fraction estimate.
viscosity $\eta$, based on the ratio of volume fraction to the maximum volume fraction (both are arrived at by geometric considerations) and a shape correction factor $[\eta]$. 

$$\eta_{\text{effective}} = \eta_s \left(1 - \frac{\phi}{\phi_m}\right)^{[\eta]_m} \tag{2.2}$$

The shape correction factor or intrinsic viscosity for a rod like geometry was derived by Barnes$^{57}$ and accounts for the rod-like geometry of the cilia, eqn. 2.3. The axial ratio is defined as the length, $7\mu$m, of the cilia divided by their diameter, $0.2\mu$m.

$$[\eta]_{\text{rods}} = \frac{7(\text{axial ratio})^5}{100} \tag{2.3}$$

In lieu of an experimentally acquired maximum volume fraction $\phi_m$, the geometric limit of eqn. 2.1 ($d_{\text{cilia}}=0$), 0.79, is used. A plot of the factor multiplying the solvent viscosity in eqn. 2.2 is plotted as a function of volume fraction below in Figure 2.2. Using the above relationships it is found that the effective viscosity can, at maximum, be one order of magnitude greater than the solvent viscosity. The effect of increasing the viscosity of the PCL on predicted stress and velocities will be discussed in Section 2.3.

**Figure 2.2:** Viscosity amplification due to a solid phase as a function of volume fraction. The curve asymptotes at $\phi=\phi_m=0.79$. This model does not predict an order of magnitude increase greater than 10.
2.2.2 Force Distribution

The method for experimentally gathering the force exerted by cilia can currently only be performed ex-vivo; this has been accomplished for a number of cilia equipped species or eukaryotic flagella by a variety of methods including optical tweezers and atomic force microscopy, a review of the measurements taken can be found in Teff 2007. The results of Teff are performed on cultured human ciliated cells so are of primary interest (though it should be noted that all measurements were taken on a cilial mat devoid of mucus). The agreement to the molecular origin of force generation as it is presented in their work is compelling; however, the results are not immediately applicable to the force balance problem posed here. That is, calculating a time averaged stress from the data presented is not a straightforward process.

In the study, ATP is introduced to the specimen to induce a spike in cilia beat frequency in order to coordinate optical and atomic force microscopy equipment. A following study by the same group shows that the frequency linearly affects the force exerted by cilium during the effective stroke due to geometrically consistent beating. However, the force is not gathered in the latter nor is the frequency explicitly given in the former.

Furthermore, while the mechanical/chemical response of ciliated cells to applied loads is not fully understood, it has been shown that variation of the load on cilia increase ATP concentrations in the epithelial tissue. The same has also been proven for the eukaryotic flagella of the sea-urchin spermatozoa. It is therefore reasonable to suspect that the application of mucus combined with a vertical orientation will also induce a change in the system; this must be considered when trying to mesh various sorts of experimental data and physical results with the model problems presented here.

In this particular case, the force estimated in one experimental geometry may not be applicable to another. The cultures in Teff are beating in the absence of mucus and oriented horizontally; they may therefore change their behavior if given a substance to propel against gravity. This is not intended as a critique; the goal of their work was to gather data of unperturbed cilia.

In an attempt to extrapolate a reasonable time averaged shear it is assumed that the force remains constant as the cilium passes through the mucus. Based on the ideal
cilia beat illustrated in Figure 1.5 the duration of cilia-mucus contact or “contact fraction” is estimated to comprise one tenth of the entire cycle. Unfortunately, the average stress calculation is extremely sensitive to this parameter, a variable which lacks direct experimental quantization. Active research in this area is being completed by a research group at the University of North Carolina\textsuperscript{62}; however at the time of this work contact between live mucus and cilia has not been observed.

The stress estimated by the work of Teff\textsuperscript{58} is shown below in Table 2.1 The geometric considerations are consistent with those of Section 2.2.1. Starting with the maximum force of one cilium, 210pN, an average stress of 69 Pa is estimated. The stress estimated here will be employed in the steady state model to estimate mucus transport velocity in Section 2.3.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Units</th>
<th>Teff et al. 2007\textsuperscript{58}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cilium spacing</td>
<td>[m]</td>
<td>3.50E-07</td>
</tr>
<tr>
<td>Cilium radius</td>
<td>[m]</td>
<td>1.00E-07</td>
</tr>
<tr>
<td>Cilia fraction</td>
<td>[%]</td>
<td>10.39</td>
</tr>
<tr>
<td>Area</td>
<td>[m^2]</td>
<td>3.025E-13</td>
</tr>
<tr>
<td>Contact fraction</td>
<td>[-]</td>
<td>0.1</td>
</tr>
<tr>
<td>Maximum force ~1 cilium</td>
<td>[N]</td>
<td>2.10E-10</td>
</tr>
<tr>
<td>Per period average</td>
<td>[N]</td>
<td>2.10E-11</td>
</tr>
<tr>
<td>Average Stress</td>
<td>[Pa]</td>
<td>69.42</td>
</tr>
</tbody>
</table>

**Table 2.1**: Average stress generated by an active mat of cilia. The assumed parameter which could not be grounded in experimental works from the literature is the “Contact Fraction” which determines how long cilia engage the mucus during their beat cycle.

### 2.2.3 Velocity

There are several in-vivo and ex-vivo experimental works which provide mucus velocities that can be used in this model; a short list of these values has already been provided in Table 1.1. Additionally, by using knowledge of respiratory tract geometry mucus flow velocity has been predicted by researchers. The velocities of tracheal transport from Asgharian 2001\textsuperscript{63} (plotted below in Figure 2.3) and ex-vivo transport of
mucus by bullfrog palate by Winet 1982\textsuperscript{32} will be used in eqn. 2.5. These two works were chosen so that a theoretical prediction of the in-vivo situation could be compared to an experimental ex-vivo work; both are on the same order of magnitude, hundreds of microns per second.

![Mucus Velocity as a function of branch generation](image)

**Figure 2.3:** Theoretical mucus velocity as a function of pulmonary branch generation by Asgharian.\textsuperscript{63}

### 2.3 Results and Conclusions

The steady state force balance, when solved for velocity or stress, yield equations 2.4 and 2.5, respectively. Below, $\rho$, $\eta$, $\tau$, $h$, $g$, and $V$ stand for density, Newtonian viscosity, stress applied by cilia, film thickness, acceleration due to gravity, and velocity; subscripts M and PCL refer to mucus and periciliary layer fluid, respectively. Details of the formulation can be found in Appendix B.1.

\[
V_M = -\frac{\rho g h_{PCL}}{\eta_{PCL}} \left( \frac{h_{PCL}}{2} + h_M \right) + \frac{\tau_{cilia} h_{PCL}}{\eta_{PCL}} \tag{2.4}
\]

\[
\tau_{cilia} = \frac{\eta_{PCL} V_M}{h_{PCL}} + \rho g \left( \frac{h_{PCL}}{2} + h_M \right) \tag{2.5}
\]
The parameters, as they are derived from the aforementioned experimental and theoretical works, when implemented in the steady state model through 2.4 or 2.5 yield the results show below in Table 2.2. In works where average velocities are given, stress is the result; when stress is given, velocity is the result. The model shows that there is at least a two order of magnitude difference between the experimental values used in one equation with the theoretical value predicted by the other. For example, when one begins with the stress imparted by cilia (first results column in Table 2.2) the mucus velocity predicted overshoots actually experimental results. Likewise, starting with mucus transport velocities underestimates the required applied stress.

<table>
<thead>
<tr>
<th>参数</th>
<th>单位</th>
<th>Teff et al. (^{58})</th>
<th>Winet (^{32})</th>
<th>Asgharian (^{63})</th>
</tr>
</thead>
<tbody>
<tr>
<td>mucus thickness, (h_M)</td>
<td>[m]</td>
<td>1.00E-05</td>
<td>1.00E-05</td>
<td>1.00E-05</td>
</tr>
<tr>
<td>PCL thickness, (h_{PCL})</td>
<td>[m]</td>
<td>5.00E-06</td>
<td>5.00E-06</td>
<td>5.00E-06</td>
</tr>
<tr>
<td>Viscosity of PCL, (\eta_{PCL})</td>
<td>[Pa*sec]</td>
<td>1.00E-02</td>
<td>1.00E-02</td>
<td>1.00E-02</td>
</tr>
<tr>
<td>Density, (\rho)</td>
<td>[kg * m^-3]</td>
<td>1.00E+03</td>
<td>1.00E+03</td>
<td>1.00E+03</td>
</tr>
<tr>
<td>Applied Stress, (\tau_{cilia})</td>
<td>[Pa]</td>
<td>6.94E+01</td>
<td>1.02E+00</td>
<td>3.06E-01</td>
</tr>
<tr>
<td>mean mucus velocity, (V_M)</td>
<td>[m*s^-1]</td>
<td>3.46E-02</td>
<td>4.50E-04</td>
<td>9.17E-05</td>
</tr>
<tr>
<td></td>
<td>[um*s^-1]</td>
<td>3.46E+04</td>
<td>4.50E+02</td>
<td>9.17E+01</td>
</tr>
<tr>
<td></td>
<td>[mm*min^-1]</td>
<td>2.08E+03</td>
<td>2.70E+01</td>
<td>5.50E+00</td>
</tr>
</tbody>
</table>

**Table 2.2:** Results from rigid body force balance using both stress and velocity as inputs. Bolded quantities mark inputs from the respective experimental work or model. Cells shaded light gray indicate the resultant quantity calculated from 2.4 (first column of results) or 2.5 (second and third columns).

Increasing the viscosity according to Section 2.2.1 partly alleviates this disparity, though only by an order of magnitude at best. Altering the contact fraction from Section 2.2.2 is another possible way to bring the two sets of data into closer agreement. However, a quick calculation quickly shows that the brevity of mucociliary contact would need to be increased two orders of magnitude to a thousandth of the beat cycle. Taken at face value, however, one can conclude from this model that all of the force the cilia are capable of imparting is not achieved in-vivo. Further work on this model is hindered by only having indirect access to the necessary data.
3 Mucus as a Linear Maxwell Fluid

The discrepancy between the two bodies of available data in the previous chapter, the force applied by the cilia and the mucus velocity, in the rigid slab force balance has a few possible causes. The errors introduced by attempting to mesh different bodies of experimental work have been discussed; however, they are but one source. The unaddressed matter is the complex material properties of mucus; mucus’s response to mechanical perturbations has already been shown to be quite different than both ideal Hookean solids and Newtonian fluids in the literature review of Section 1.2.2; this must be considered explicitly. Chapter 2 revealed that, perhaps, all of the momentum from the cilia was not imparted productively into mucus transport; an explanation may rest with the complex mechanical characteristics of mucus. The rheological properties of mucus impact how cilia initially engage mucus and impart momentum, but also how it flows in response. The problems considered henceforth will focus on the latter by considering different boundary conditions and constitutive equations and expressing, mathematically, the role of these properties in transport.

In this chapter, the fundamental concept of relaxation time will be developed by contrasting Newtonian, Hookean, and Maxwellian media in Section 3.1. The response of the Maxwell fluid constitutive equation to suddenly applied stress, strain, and strain-rate will be examined in Section 3.2. Next, the flow profiles of Newtonian and Maxwellian fluids will be contrasted over a range of timescales in a shear-driven unidirectional flow scenario (Section 3.3). Rheological measurements from the literature will be considered briefly for the purposes of extracting parameters useable in theoretical models (3.4). Lastly in this chapter, the linear Maxwell model will be applied to a flow situation reminiscent of the thin film transport which takes places in the trachea (3.5). These results motivate the use of a more complex constitutive equation, the Upper Convected Maxwell model, which will be used in problems in Chapter 4.

3.1 Relaxation Time

Mucus’s most dominant rheological property is its tendency to behave more like an elastic solid on short time scales. A “catch-all” constitutive equation for viscoelastic
media does not exist. Therefore, models must be built up to what is required for accurate simulation rather than whittled down to what is needed from a general law. Introducing a model which accounts for stress relaxation is the most fundamental building block. Mucus is capable of additional behaviors beyond timescale dependence whose implications will be discussed in later sections.

When mucus is impinged upon abruptly, the relationship between stress and strain tends towards that of a solid. A force which is suddenly applied to mucus by cilia, for instance, results in a strain that is roughly in phase with the force. In other words, the force is proportional to and in phase with the strain, not the rate at which the strain is applied. Alternatively, the force applied to a Newtonian fluid will always be proportional to and in phase with the strain rate, but (90 degrees) out of phase with the strain.

The simplest viscoelastic fluid model which allows for elastic behavior on short time scales is the Maxwell constitutive equation for stress and strain rate. The Maxwell model should be thought of as the addition of elasticity to an ideal Newtonian fluid; it can be represented schematically as a viscous dashpot of viscosity $\eta_0$ (commonly referred to as the zero-shear rate viscosity) and a Hookean spring of rigidity $G$ (often defined in terms of the relaxation time and zero-shear rate viscosity) in series as shown below in Figure 3.1.

![Figure 3.1: Schematic of the Maxwell fluid element. The rigidity of the elastic element $G$ is defined in terms of the relaxation time $\lambda$ and zero-shear rate viscosity $\eta_0$. This notation is common in texts.](image)

Just as every link in a chain must transmit the same amount of force, the schematic shows that stress is constant throughout the element. That is, both the elastic and viscous elements are in the same state of stress at all times. On the other hand, the strain or strain-rate of the Maxwell element as a whole receives contributions from both elements so that the total strain or strain-rate is the sum of the Hookean and Newtonian elements. The
differential equation describing the relationship between stress and strain-rate is shown below in eqn. 3.1 where \( \lambda, \tau, \dot{\gamma}, \) and \( \eta_0 \) represent relaxation time, stress, strain-rate and zero-shear rate viscosity, respectively. The time constant \( \lambda \) arises naturally when deriving the differential equation from the schematic. It is the ratio between the values of the viscosity and rigidity of the element. In the next section, it will be shown to govern the exponential decay or build-up of stress within the element when a strain-rate is imposed.

\[
\lambda \frac{\partial \tau}{\partial t} + \tau = \eta_0 \dot{\gamma}
\]  

(3.1)

The application of a constant force to a Maxwellian fluid results in continuous deformation just like a Newtonian fluid; however, it is also capable of storing energy elastically like a solid. In the Maxwell model the relaxation time dictates the rate at which elastic potential energy is dissipated by viscosity. In other words, a strained infinitesimal Maxwell fluid element will become less stressed over time; the duration of which the material maintains elastic stress before dissipating is governed by the time constant. Each of the ideal elements in the model can be accessed in two hypothetical limits, this will be explored mathematically in the next section. For now it suffices to say that the response of a Maxwell fluid to a step input is entirely solid like (strain is entirely proportional to stress) at the instant of the step. The solid like response wanes as time goes on; eventually an entirely Newtonian relationship between stress and strain-rate is approached.

This is evidenced by eqn. 3.1; the importance of the relaxation time is proportional to the time rate of change of the stress. The infinitesimally short time scale of the step function allows the left most term to dominate. In this limit, the constitutive equation reduces to Hooke’s Law (recalling that the relaxation time can be defined as the zero-shear rate viscosity divided by the elastic or rigidity modulus). Alternatively, a force applied over a long time scale causes the left most term to become negligible, ushering a purely Newtonian behavior (strain-rate is entirely proportional to stress). It will be common in later sections to non-dimensionalize this constitutive equation and ones like it with appropriate scales and show the relative importance of the terms formally.
The Maxwell model has limitations in its ability to match empirical behavior of viscoelastic media. Just as the Hookean model for relating stress and strain is only valid for infinitesimal displacement gradients, so too is the elastic element in the Maxwell model. While the strain contributions of the viscous and elastic elements can be extracted from the model and this gradient assessed, the fact that these elements cannot be physically attributed to a particular physical element because of the assumed homogeneity of the material poses a conceptual problem. Thus, the model as it stands now is typically only capable of describing the behavior of a real viscoelastic material in a narrow range of circumstances. This fact will be demonstrated in sub-sections to follow after the ramifications of relaxation time are illustrated.

### 3.2 Response of the Maxwell Fluid Model to Sudden Perturbations

What does rapid application of a strain rate do, mechanically, that a slow application does not? In this section the constitutive equation of the linear Maxwell fluid will be examined in order to answer this question. The question is being posed: if an infinitesimal Maxwell fluid element during its flow experiences an unsteady strain, strain-rate or stress, how will it respond? A sudden increase in strain rate (such as that from a step-input) allows the stress to build in the elastic portion of the mucus before it can be dissipated. The viscous element behaves rigidly and allows the applied to strain to communicate with the fluid’s elastic element. It is in this way that viscosity and elasticity play off of each other in the Maxwell model. A weak viscous component actually hinders the accessibility of elasticity; the elastic dumbbells that are being represented in the model glide over each other instead of being strained. Derivations of the response of the Maxwell model (eqn. 3.1) can be found in Appendix C. The resulting equations are shown on the next page in Table 3.1:
Forcing Function Strain \( t > 0 \): Strain Rate \( t > 0 \): Stress \( t > 0 \):

<table>
<thead>
<tr>
<th>Forcing Function</th>
<th>Strain ( t &gt; 0 ):</th>
<th>Strain Rate ( t &gt; 0 ):</th>
<th>Stress ( t &gt; 0 ):</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step Strain</td>
<td>( \gamma_H = e^{-\frac{t}{\lambda}} )</td>
<td>( \dot{\gamma}_H = -\frac{\Gamma}{\lambda} e^{-\frac{t}{\lambda}} )</td>
<td>( \tau = \eta_0 \Gamma e^{-\frac{t}{\lambda}} )</td>
</tr>
<tr>
<td></td>
<td>( \gamma_N = \Gamma \left[ 1 - e^{-\frac{t}{\lambda}} \right] )</td>
<td>( \dot{\gamma}_N = \Gamma e^{-\frac{t}{\lambda}} )</td>
<td></td>
</tr>
<tr>
<td>Step Strain Rate</td>
<td>( \gamma_H = \lambda \Gamma \left( 1 - e^{-\frac{t}{\lambda}} \right) )</td>
<td>( \dot{\gamma}_H = \Gamma e^{-\frac{t}{\lambda}} )</td>
<td>( \tau = \eta_0 \dot{\Gamma} \left( 1 - e^{-\frac{t}{\lambda}} \right) )</td>
</tr>
<tr>
<td></td>
<td>( \gamma_N = \dot{\Gamma} \left( t - \lambda + \lambda e^{-\frac{t}{\lambda}} \right) )</td>
<td>( \dot{\gamma}_N = \dot{\Gamma} \left( 1 - e^{-\frac{t}{\lambda}} \right) )</td>
<td></td>
</tr>
<tr>
<td>Step Stress</td>
<td>( \gamma_H = \frac{T}{G} )</td>
<td>( \dot{\gamma}_H = \frac{T \lambda}{\eta_0} )</td>
<td>( \tau = T )</td>
</tr>
<tr>
<td></td>
<td>( \gamma_N = \frac{T}{\eta_0} t )</td>
<td>( \dot{\gamma}_N = \frac{T}{\eta_0} )</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.1: Response of Maxwell fluid constitutive equation to three suddenly applied quantities: strain, strain rate, and stress. Subscripts denote the Hookean and Newtonian elements of the Maxwell model, H and N, respectively. Quantities represented by capital Greek letters \( \Gamma, \Gamma' \) and \( T \) correspond to the applied quantities: strain, strain-rate, and stress, respectively. Maintaining their definitions from previous sections: \( G, \eta_0, \) and \( \lambda \) correspond to rigidity, zero-shear rate viscosity, and relaxation time, respectively. All strains, strain rates, and stresses are for \( t > 0 \).

The shift in strain-rate contribution between the elastic element and viscous element as well as the gradual rise in stress is plotted below in Figure 3.2. In the context of mucociliary interaction; the sudden application of a strain-rate by cilia as they first contact the mucus does not immediately result in a stress in the fluid – it slowly builds to that of the Newtonian stress. In contrast, a Newtonian fluid feels a stress immediately and a Hookean solid shows stress only once a strain has resulted from the strain rate. The elastic component is solely responsible for stress at \( t=0^+ \); in other words, the Maxwellian fluid is effectively an ideal solid at this instant. However, there is no stress in the material at the instance of application because though the strain rate is non-zero, no strain has been achieved.
After the initial application $t>0$, the elastic and Newtonian components interact with each other as shown in Figure 3.2. The elastic component continues to strain, but the rate at which it does so exponentially decreases as the Newtonian element catches up and continues to take up more and more of the imposed strain rate. At equilibrium ($t \to \infty$), the elastic element has strained a finite amount which is proportional to the relaxation time and the imposed strain rate. The magnitude of this strain is not a function of how the strain rate was applied (step-function as was described or otherwise); however, the time it takes to achieve that strain is sensitive to the transience of the strain-rate.

![Figure 3.2](image_url)

**Figure 3.2:** Left, strain rate as a function of time for the viscous and elastic elements, and the Maxwell fluid element as a whole in response to a suddenly applied strain rate. Right, stress as a function of time for the same suddenly applied strain rate. Analytical expressions for these plots can be found in the third and fourth rows of Table 3.1.

The described lag in stress build-up may or may not be beneficial. On one hand, the cilia encounter less resistance at first as evidenced by Figure 3.2, this clearly makes the passage of cilia through the mucus and the completion of its effective stroke easier. On the other, compared to a Newtonian fluid, the maximum shear developed is less. As was established in the two simple model problems (mucus as a rigid body and mucus as a Newtonian fluid), there is a minimum shear required to maintain positive transport; this lag appears to hinder the cilia's ability to achieve their full potential; this is the apparent trade-off. The benefit, however, is intriguing and appears when the cilia disengage the mucus. A strain rate is no longer imposed on the mucus, yet the stress that has been built during the cilia's effective stroke lingers, declining exponentially. This stress, while it remains may continue to drive fluid flow in a beneficial way.
This suggests a connection between the wavelength (and possibly the direction) of the metachronal wave and the relaxation time. This idea will be explored somewhat in Section 3.5 and explicitly in Chapter 4. The relaxation time must be sufficiently long or the fluid velocity sufficiently rapid over inactive portions of the ciliary mat. If the relaxation time in conjunction with flow velocity is insufficient to maintain stress and flow, gravity can potentially take over causing reversal.

3.3 Response of the Maxwell Fluid Model Due to an Oscillating Plate

In the previous section, the transient response of a Maxwell fluid to short time scale perturbations was considered. This section will demonstrate the Maxwellian viscoelastic response over a range of time scales by varying the frequency of an oscillating boundary condition. The geometry, shown below in Figure 3.3, is applicable to parallel plate rheometers. Flow profile frequency dependence will be discussed first for Newtonian fluids and then for Maxwellian fluids.

\[
\begin{align*}
1. & \quad u(t, y = 1) = 0 \\
2. & \quad \tau_{xy}(t, y = 1) = 0 \\
3. & \quad u(t, y \to \infty) = 0 \\
& \quad u(t, y = 0) = \cos(t)
\end{align*}
\]

Figure 3.3: Shear-wave problem diagram. A sinusoidally varying velocity is applied at the bottom boundary. Solutions to no-slip, slip, and semi-infinite domain conditions were found; see Appendix D.

There are two equations governing this problem: conservation of momentum and the Maxwell fluid constitutive equation. After simplifying the equations for a unidirectional flow, assuming a purely oscillatory solution, and non-dimensionalizing the variables, the problem is reduced to a single ordinary differential equation as seen in eqn.
3.2. The variables $Wo$, $De$, $\bar{u}^0$, $\bar{y}$ represent Womersley number, Deborah number, and dimensionless velocity amplitude and y position, respectively.

\[
Wo^2 \bar{u}^0 (i - De) = \frac{\partial^2 \bar{u}^0}{\partial \bar{y}^2}
\] (3.2)

There are two dimensionless quantities which arise naturally when non-dimensionalizing variables of the Maxwell constitutive equation and the conservation of momentum equations present in eqn. 3.2; the Womersley number and the Deborah number. The non-dimensionalizing procedure for this problem is shown in Appendix Section D.1. The Womersley number, eqn. 3.3, like the Reynolds number, is the ratio of inertial to viscous forces in the problem in which the timescale used is the angular frequency of the oscillating plate. The other parameters: $\rho, \eta_0$, and $h$, maintain their definitions from previous sections representing density, zero-shear rate viscosity and film thickness or plate gap.

\[
Wo = h \sqrt{\frac{\rho \omega}{\eta_0}}
\] (3.3)

The Deborah number, eqn. 3.4, describes the relative importance of elasticity in the problem and is the product of the angular frequency $\omega$ and relaxation time $\lambda$. Increasing the relaxation time or decreasing the time scale of the problem (increasing the angular frequency) both serve to increase the importance of elasticity in the solution.

\[
De = \lambda \omega
\] (3.4)

The solution for eqn. 3.2 is found for three sets of boundary conditions. In one case, the opposing wall is represented by a no-slip, zero flow velocity boundary condition. Problems were also completed for shear-free opposing wall and semi-infinite cases. The results for shear-free and semi-infinite case will be discussed in this section. However, solutions for all three boundary conditions can be found in Appendix D. The no-slip case solution will be discussed in greatest detail as it is the most reminiscent of
the in-vivo scenario. The semi-infinite case will also be discussed as the real and imaginary parts of the Eigen values take on special physical meaning.

In order to illustrate the effect of elasticity on a unidirectional shear flow, the amplitude $\Lambda$ and phase shift $\Theta$ for a Newtonian fluid (De=0) are shown first in Figure 3.4. As the Womersley number is increased the propagation of the disturbance (the oscillating wall) attenuates more dramatically. In other words, the faster the plate oscillates the smaller the region of influence the plate has. This is similar to pulling a rug out from underneath a piece of furniture. Pulling slowly on the rug drags the furniture along with it (the rug and furniture are in phase). Friction (or in our case, viscosity) connects the two objects. Alternatively, pulling the rug quickly leaves the furniture only slightly displaced. The furniture felt much less of the imposed velocity and thus moved out of phase with the rug. This is the high Womersley number case; amplitude attenuates quicker and the difference in phase between layers of fluid is increased.

![Figure 3.4: Amplitude $\Lambda$ and Phase Shift $\Theta$ plots as functions of position and Womersley number for a Newtonian fluid (De=0) subject to a sinusoidally varying wall velocity ($y=0$) and shear-free wall ($y=1$). Phase shift is measured in radians away from the input, $\cos(t)$. All quantities have been non-dimensionalized.](image-url)
Figure 3.5: Amplitude $\Lambda$ and Phase Shift $\Theta$ plots as functions of position and Deborah number for a fixed Womersley number of 1.2 subject to a sinusoidally varying wall velocity ($y=0$) and shear-free wall ($y=1$). Phase shift is measured in radians away from the input, $\cos(t)$. All quantities have been non-dimensionalized.

Ideally, it would be beneficial to probe the effects of elasticity in a scenario devoid of inertia. However, for the infinite plate problem at hand this is not possible. In order to investigate the effect of elasticity alone, it would be necessary to consider a less ideal geometry (in terms of mathematic simplicity) such as a finite object, an oscillating sphere for example. In the geometry used, as the Womersley number goes to zero, the flow profile becomes linear, regardless of Deborah number. It is for this reason that plotted in Figure 3.5 are cases for varying Deborah with a non-zero Womersley number (Wo=1.2) chosen only for plot clarity. It is clear that as Deborah number is increased the amplitude imposed by the wall does not diminish exponentially as it did for the Newtonian fluid, instead there is a gain as one moves further from the wall. The dip in De=6 case is due to interference from shear waves reflected off of the opposing wall.

The effect Deborah number on the attenuation and phase of the propagating disturbance can perhaps be understood more simply by examining the solution to the semi-infinite case in which there is no opposing wall, see Appendix Section for a full derivation of the solution D.5. Quantities alpha and beta are plotted below in Figure 3.6 and correspond to the decay and wave speed of the disturbance. (The wave speed is not equivalent to beta for the bounded cases.) The plot shows that as Deborah number increases, perturbations travel faster and decay less.
Figure 3.6: Attenuation (Alpha) and Phase Speed (Beta) of velocity plotted as a function of Deborah number for the Oscillating wall problem on a semi-infinite domain. All work for this problem can be found in Appendix Section D.5. The Womersley number is equal to one.

Actual mucus flow in the trachea falls in the low Womersley number, high Deborah number regime; Table 3.2 below shows these values. That is, inertial effects are negligible while elastic effects heavily influence the flow. Choosing the time scale in this problem is straightforward; however, for a viscoelastic flow which experiences spatial as well as temporal perturbations, other time scales may prove more relevant. Furthermore, it will be shown in the next section that extracting a representative relaxation time from a real viscoelastic material is not straightforward as most viscoelastic media possess a continuous spectrum of relaxation times. The Deborah numbers calculated below use a time scale typical of tracheal cilia; two of the relaxation times were taken from the data of Davis and Dippy 1969.20. One comes directly from creep study data while the other is extracted as the longest relaxation time from oscillatory data. The process for the latter is detailed in the next section. The third relaxation time used, 3 seconds, is taken from Mitran 200753 who used a relaxation time of 3 seconds in a mucociliary model; an experimental source was not cited.
length scale \( h \) [m] & 1.00E-06 & 1.00E-05 & 1.00E-04 \\
\hline
time scale \( \omega \) [sec\(^{-1}\)] & 1.26E+02 & 1.26E+02 & 1.26E+02 \\
\hline
Density \( \rho \) [kg*m\(^{-3}\)] & 1.00E+03 & 1.00E+03 & 1.00E+03 \\
\hline
zero shear rate viscosity \( \eta_0 \) [Pa*sec] & 2.43E+03 & 3.52E+03 & 2.43E+03 \\
\hline
\( Wo \): & 7.19E-06 & 5.97E-05 & 7.19E-04 \\
\hline
time scale \( \omega \) [sec\(^{-1}\)] & 1.26E+02 & 1.26E+02 & 1.26E+02 \\
relaxation time \( \lambda \) [sec] & *2.41E+04 & **8.30E+02 & 3.00E+00 \\
\hline
\( De \): & 3.02E+06 & 1.04E+05 & 3.78E+02 \\
\hline

Table 3.2: Womersley and Deborah numbers calculated from physical parameters. From left to right, the length scales are 1, 10, and 100 microns which are roughly the upper and lower bounds of mucus film thicknesses and the wavelength of the metachronal wave. *Relaxation time from creep study. **Longest relaxation time from discrete relaxation spectrum. The zero-shear rate viscosities are extracted from data on the lowest frequency or the creep study, 3.52E3 and 2.43E4 Pa*sec, respectively.

The insight that one takes away from this example is that the presence of elasticity in an otherwise viscous fluid allows an oscillating disturbance to perturb a larger region of that media. The addition of elasticity results in a fluid which is a better communicator. If one assumes that mucus can be modeled sufficiently using the Maxwell constitutive equation, then some of the benefits of elasticity can be anticipated should this model be employed in a flow situation.

### 3.4 Relaxation Spectrum of Mucus

The previous section described the behavior of a Maxwellian viscoelastic material in a geometry like that of a commercial rheometer. However, typically only inputs and outputs are recorded (though material properties can in some cases be backed out from analyzing experimental flow profiles in a rheometer\(^{64}\)). Materials which exhibit complex frequency dependent properties (such as mucus) are quantified in terms of either the complex dynamic modulus or complex viscosity. The choice is a matter of preference; the dynamic modulus can be thought of as quantifying how elastic the media in question...
is (and thus has units of force per unit area) while the complex viscosity (possessing units of force per area times time) assesses with how fluid-like the material is.

The Maxwellian fluid possesses an analytical form for both the dynamic modulus and complex viscosity which are derived in Appendix D.5 and shown below in Table 3.3. The Real and Imaginary parts are plotted along the magnitude of the moduli as a function of Deborah number as well to illustrate the shift of a Maxwellian fluid from viscous behavior at low frequencies to elastic behavior at high frequencies.

### Complex Viscosity

<table>
<thead>
<tr>
<th>Definition</th>
<th>$\eta^* = \frac{\tau^0}{\gamma^0} = \eta^0 + \eta^i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real Part</td>
<td>$\eta' = \frac{\eta_0}{1 + De^2}$</td>
</tr>
<tr>
<td>Imaginary Part</td>
<td>$\eta'' = \frac{-De \eta_0}{1 + De^2}$</td>
</tr>
<tr>
<td>Magnitude</td>
<td>$</td>
</tr>
</tbody>
</table>

### Dynamic Modulus

<table>
<thead>
<tr>
<th>Definition</th>
<th>$G^* = \frac{\tau^0}{\gamma^0} = G^0 + G^i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real Part</td>
<td>$G' = \frac{De \eta_0}{1 + De^2}$</td>
</tr>
<tr>
<td>Imaginary Part</td>
<td>$G'' = \frac{\omega \eta_0}{1 + De^2}$</td>
</tr>
<tr>
<td>Magnitude</td>
<td>$</td>
</tr>
</tbody>
</table>

**Table 3.3:** Complex Viscosity and Complex Dynamic Modulus are defined along with their Real and Imaginary parts.

**Figure 3.7:** Complex Dynamic Modulus (left) and Complex Viscosity (right) with their respective real and imaginary parts for a Maxwell fluid are plotted as a function of Deborah number. The modulii and viscosities have been non-dimensionalized with respect to the model’s elastic/shear modulus and zero shear-rate viscosity, respectively.
Mucus, in actuality, possesses a continuous spectrum of relaxation times. Thus, using the equations in Table 3.3 can only describe the frequency dependent rheological properties of mucus over a narrow range of frequencies. A discrete estimation of this spectrum can be estimated using dynamic material data collected across a range of frequencies. The equations for the Loss and Storage moduli for a Maxwellian fluid with a discrete relaxation spectrum are shown below in eqn. 3.5. Indices $k$ and $j$ count relaxation times and the frequencies at which data was gathered, respectively. In the other words, the response of the material at the $j^{th}$ frequency is due to the ensemble response of $N$ Maxwellian elements.

$$G'(\omega_j) = \sum_{k=1}^{N} \frac{\eta_{0k} \lambda_k \omega_j^2}{1 + (\omega_j \lambda_k)^2}$$
$$G''(\omega_j) = \sum_{k=1}^{N} \frac{\eta_{0k} \omega_j}{1 + (\omega_j \lambda_k)^2}$$

(3.5)

One method which can be used to generate the model parameters ($\eta_{0k}$ and $\lambda_k$) is to make a reasonable guess for the relaxation time and run an iterative numerical scheme to guess positive values of the zero shear rate viscosity such that the quantity below in eqn. 3.6, the square of the error between experimental data and the fit, is minimized. A slight modification of this procedure is used to generate the data in Table 3.4. The free parameter $m$ is used with the angular frequency to aid in the process of guessing a relaxation time such that: $1/\omega^n = \lambda$ (the units do not need to be consistent as this relationship is only being used to aid the iterative solver). Additionally, the assumption that the viscosity parameters always decrease with increasing angular frequency is imposed.

$$\sum_{j=1}^{N} \left\{ \left[ \frac{G'(\omega_j)}{G'_{j}} - 1 \right]^2 + \left[ \frac{G''(\omega_j)}{G''_{j}} - 1 \right]^2 \right\}$$

(3.6)
The discrete relaxation spectrum is fit reasonably well to the storage modulus if outliers are omitted (the anomalies were due to a common but often unavoidable phenomenon in shear rheological measurements, sample fracture). Achieving a good fit was difficult; in part, this is due to the method used to minimize eqn. 3.6, Microsoft’s Excel solver. An iterative solver was chosen for two reasons: a linear regression type fitting scheme is not appropriate as it does not allow the relaxation time to be a free parameter. Additionally, the result often yields negative values for the viscosity even when relaxation times were assumed a priori. Physically, the discrepancy can be in large part attributed to the shear-rate dependence of mucus’s response; a mechanism which is not explicitly accounted for by the Maxwell model. This issue has been addressed in the literature, in particular by Papanastasiou and colleagues\textsuperscript{65}, in which the constitutive equation is divided into a linear frequency dependent portion and a non-linear shear-rate dependent portion. The procedure is applicable to any viscoelastic material which exhibits these two dependences and could be used on the data of Davis and Dippy to potentially extract information on the shear-thinning behavior of mucus.

Table 3.4: Discrete Relaxation Spectrum parameters acquired by processing oscillatory rheological data from Davis and Dippy.

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$m$</th>
<th>$\lambda$</th>
<th>$\eta_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00791</td>
<td>0.00224</td>
<td>0.00830</td>
<td>0.00352</td>
</tr>
<tr>
<td>0.00250</td>
<td>0.00377</td>
<td>0.00183</td>
<td>0.00636</td>
</tr>
<tr>
<td>0.00791</td>
<td>0.00327</td>
<td>0.00177</td>
<td>0.00382</td>
</tr>
<tr>
<td>0.00250</td>
<td>0.00126</td>
<td>0.00089</td>
<td>0.00382</td>
</tr>
<tr>
<td>0.00158</td>
<td>0.00164</td>
<td>0.00166</td>
<td>0.00382</td>
</tr>
<tr>
<td>0.00199</td>
<td>0.00470</td>
<td>0.00142</td>
<td>0.00382</td>
</tr>
<tr>
<td>0.00250</td>
<td>0.00100</td>
<td>0.00064</td>
<td>0.00382</td>
</tr>
</tbody>
</table>
Figure 3.8: Storage Modulus (Real part of Complex Dynamic Modulus) of the Relaxation Spectrum of Mucus. The thick line indicates the sum of multiple relaxation times, and thin lines, the response of individual relaxation times. Data points from which the Relaxation Spectrum was generated are labeled as white points; excluded data as solid points.

Figure 3.9: Loss Modulus (Imaginary part of Complex Dynamic Modulus) of the Relaxation Spectrum of Mucus. The thick line indicates the sum of multiple relaxation
times, and thin lines, the response of individual relaxation times. Data points from which the Relaxation Spectrum was generated are labeled as white points; excluded data as solid points.

### 3.5 Thin Film Flow of a Linear Maxwell Fluid

The previous sections have established the role of elasticity and relaxation in affecting the stress/strain relationship of a material in simple flow scenarios and have also shown, by assuming a spectrum of Maxwellian relaxation times, that experimental data can be fitted reasonably well to a simple model—especially if the relaxation time near the time-scale of interest is known. The foundation is sufficient to apply the Maxwell model to a flow situation like that found in the trachea. The boundary condition chosen to emulate the periodic and wave-like coordination of the cilia is a simple sinusoidal traveling wave. The governing equations and boundary conditions have non-dimensionalized with respect to angular frequency, velocity amplitude of sinusoidally varying component of boundary condition, wavelength and film thickness; the lower boundary condition is shown in non-dimensional form by eqn. 3.7 where \( c \) is \( \pm 1 \) to all the direction of the wave to change.

\[
\begin{align*}
 u(x, y = 0, t) &= F(x, t) = 1 + \cos(2\pi x + ct) \\
\end{align*}
\]  

(3.7)

An analytical solution is made possible due to two assumptions: purely viscous flow and small aspect ratio or thin domain. The latter is also called the thin-film or lubrication approximation; the derivation of which is shown in appendix section Appendix F. These are both called assumptions because no flow can be devoid of inertia or be “infinitely” thin but by non-dimensionalizing the governing equations it can readily be shown which terms dominate the system. All of the steps to arrive at the solution for the velocity components can also be found in the Appendix. In the derivation, a low Womersley number is assumed per the calculations from Section 3.3 while the effect of a small aspect ratio on the governing equations is shown more formally. The solutions for the \( x \)-component \( u \) and \( y \)-component \( v \) of the velocity field are below in eqns. 3.8 and 3.9 and plotted in Figure 3.10. The solution is left in terms of the boundary condition \( F(x,t) \) where convenient.
\[ u = -3 \left[ \left( 1 + \frac{\sin(2\pi + ct)}{2\pi} - \frac{\sin(ct)}{2\pi} \right) - F(x, t) \right] \left( \frac{y^2}{2} - y \right) + F(x, t) \]  
\[ (3.8) \]

\[ v = -\frac{\partial F}{\partial x} \left[ 3 \left( \frac{y^3}{6} - \frac{y^2}{2} \right) + y \right] \]  
\[ (3.9) \]

**Figure 3.10:** Vector field quiver plot (left) and velocity magnitude contour (right) of the solution to the thin film approximation problem.

The solution to the velocity field does not readily yield insights into the role of viscoelasticity on the flow; there is no Deborah number dependence. The shear stress, however, needs to be found by solving eqn. 3.10 which is simply the dimensionless version of one of the Maxwell constitutive equations for unidirectional flow (see Appendix F.1). Its solution will comprise sinusoids whose amplitude will depend on Deborah number.

\[ \text{De} \frac{\partial \tau_{xy}}{\partial t} + \tau_{xy} = \frac{\partial u}{\partial y} \]  
\[ (3.10) \]

These sinusoids, on average, do not add an additional shear stress and therefore do not help address the incongruity between the aforementioned set of experimental data. Additionally, the ordinary differential equation only allows for functionality in time to influence the shear stress field. However, because the flow has a net velocity an element of fluid may experience a strain and then be convected downstream. Under this
constitutive equation, viscoelastic effects can only be seen if the boundary condition has a time varying character. This is an unreasonable constraint as it does not allow cilia to have an effect on the down-stream cilia. Addressing this limitation will be the focus of Chapter 4.

### 3.6 Alternative Scaling of Linear Maxwell Model

The asymptotics used to arrive at the thin film flow equations of the previous section assumed a certain flow regime which was devoid of inertia. The scaling used was traditional for a Newtonian fluid and presumes that the only forces at work are either viscous or inertial. However, the introduction of relaxation time introduces additional physics. In this section, an alternative manipulation of the Maxwell constitutive and unidirectional conservation of momentum equation will be discussed in order to elucidate how inertia and relaxation time governed the flow profiles in the solution of the oscillatory plate example.

Three scaled equations governing a viscoelastic shear flow different that the one used in Section 3.3 are presented in eqns. 3.11-3.13. Appropriate derivatives have been taken in order to combine the equations of momentum and Maxwellian viscoelasticity. All variables are dimensionless, and all parameters maintain their meaning from previous sections. Full derivations of the below equations are shown in Appendix G. The reason there are three scales presented is because there are three terms present and it is unclear which terms should balance one another. It is no surprise, however, that the dimensionless quantity that emerges is the same regardless of this choice. What sets these equations apart from the ones used in prior sections is that no time scale was imposed; instead, it was left as a variable and found by allowing different pairs of terms to be of equal order.

\[
\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \left( \frac{\lambda \eta_0}{\rho h^2} \right) \frac{\partial^2 u}{\partial y^2} \tag{3.11}
\]
One of the benefits of scaling the governing equations over simply plotting results for various parameters is the ability to examine certain limits. For example, in eqn. 3.11 which represents a case where inertial forces are always on par with viscoelastic ones shows that if inertia is small, then the RHS dominates and the unidirectional Viscous flow equation results (regardless of relaxation time). Alternatively, if this same limit is applied to eqn. 3.12 where viscous and viscoelastic forces were assumed to be the same order of magnitude, the result is the wave equation, a hyperbolic PDE. Finally, a case where inertial and viscous forces are matched in magnitude yields an ODE devoid of spatial dependence when the limit is taken. In other words, different behaviors can be drawn out depending on which assumptions are made during the scaling of the equations.

The scaled equations here did not yield the Womersley and Deborah numbers of Section 3.3 because a time scale was not imposed a priori (in previous problems the angular frequency was the time scale). Instead, three different time scales arose as a result of different scaling choices; they are shown below (eqns. 3.14-3.16) in the same order as their corresponding scaled equations.

\[ t = \tilde{t} \lambda \]  
(3.14)

\[ t = \tilde{t} \eta_0 \sqrt{\frac{\lambda \rho}{\eta_0}} \]  
(3.15)

\[ t = \tilde{t} \rho h^2 \frac{1}{\eta_0} \]  
(3.16)
The three paths that led to the equations are shown step by step in Appendix G. It is by examining the corresponding the values of the time scales associated with the three scaled equations that one is able to choose which is the most appropriate for a given situation. For example, if the reciprocal of the angular frequency of the oscillating plate from is like the relaxation time, then the proper equation to solve is 3.11. On the other hand if the time scale is like that of eqn. 3.15 then perhaps eqn. 3.12 is a better representation of the dominate terms in the equation and so on. What is being revealed is that under different circumstances, different terms in governing equation have a chance to dominate. For the oscillating plate problem, we see that there were in fact three flow regimes. If at first the results of problems such as those in Section 3.3 seem nonsensical, then examining the terms that dominate in the scaled equations can add meaning to what is being observed.
4 Upper Convected Maxwell Model

In Section 3.1 the concept of relaxation time was introduced. However, after attempting to employ it in a flow situation reminiscent of mucus transport in the trachea in Section 3.5 it was found that the linear model was incapable of allowing the stress state of the fluid to be convected. To solve this shortcoming, the upper-convected time derivative is introduced to accomplish the proper Lagrangian to Eulerian reference frame transformation for the tensorial information. It is defined below in eqn. 4.1:

\[ \frac{\partial}{\partial t} \tau_{ij} = \frac{\partial \tau_{ij}}{\partial t} + u \cdot \nabla \tau - \{ \tau \cdot \nabla v \}^T - \{ \tau \cdot \nabla v \} \]

This time derivative is not unique in allowing the convection of the stress tensor components; however, it is the most prevalent in the literature and will be used in this work. The derivative is shown expanded in Cartesian coordinates in Appendix A.2. In this section, the improved Maxwell model will be used to attempt a theoretical connection between the wavelength and wave speed of the metachronal wave and the relaxation time of mucus. First, the potential role of tensile forces is explored in a purely unidirectional case. Then, the effect of relaxation time on the length of the developing region between a simple shear flow and a shear-free flow is illustrated by performing a perturbation series for small Weissenberg number (weak elasticity or short strain memory assumption) about the Newtonian case. In this section, the Weissenberg number serves the same role as the Deborah number but is more appropriate for flows in which large deformations are possible. Finally, the same perturbation method will be used to examine the effects of elasticity on a flow which is subject to a transient sinusoidally varying wall stress.
4.1 Tension in a Unidirectional Shear Flow

Before the convected derivative is implemented in a complex flow problem one of its effects, the emergence of tension along streamlines, will be illustrated in a simple unidirectional flow and used to propose a connection between the force exerted by cilia, relaxation time, and metachronal wavelength. Examples of the effect of tensile forces on real viscoelastic flows are die exit swell and the rod climbing ability of polymeric solutions. In the first example, the sudden change in shear allows the flow to recoil forcing it bulge. In the second, polymer solutions can be shown to climb a spinning rod; the circulating shear flow creates tension along the streamlines which draws the fluid towards the rod.

In this problem, as with the two examples, tension will be created through shear in portions of the flow and allowed to relax in others in an alternating fashion. The tension developed where shear is applied must be sufficient to support the mass of the inactive portion of the flow; a schematic is shown below in Figure 4.1. The major assumptions in this calculation are that the transitions that occur between the active and inactive regions are instantaneous, and that the inactive regions have the ability to transmit tensile forces despite being a liquid. There is evidence in the literature to suggest that mucus not only possesses a high viscosity at rest, but also possesses a yield stress which must be surpassed in order for it to flow. Deformations which occur below this stress are recoverable. This phenomenon is not a feature of either the linear Maxwell model or UCM model and is beyond the scope of this thesis but does help to justify this simple model.

![Figure 4.1: Problem schematic for a unidirectional flow of a UCM fluid. Gray regions are actively sheared by stress T while...](image-url)
white regions are in shear-free flow. In order to estimate the tension necessary to hold up the inactive region, a force balance around box A is performed.

The derivation for the dimensionless group of parameters in eqn. 4.2 is shown in Appendix G. Briefly, the problem is solved by first finding the necessary shear to support the active (gray) part of the film. The total tension in the flow is found by simplifying the UCM equations for a unidirectional flow and integrating across the thickness of the film. An additional force balance is performed on the inactive region marked by ‘A’ Figure 4.1. Grouping all of the parameters together ($T, \gamma, \rho, \eta_0, \lambda$, and $g$ representing applied shear, metachronal wavelength, density, zero shear-rate viscosity, relaxation time, and acceleration due to gravity, respectively) results in a dimensionless group, eqn. 4.2, which must be greater than 0.75 in order for the tension developed in the active region to be large enough to support the inactive region.

$$\frac{T^2 \lambda}{\eta_0 \rho g} \geq \frac{3}{4}$$

In addition to the steady-state rigid body model discussed earlier, this group of parameters provides an avenue by which the stress can be estimated from known physical parameters. The most note-worthy aspect of this result is the derived relationship between stress and relaxation time. Stress is found to be inversely proportional the square root of the relaxation time; thus, as relaxation time increases the demand on cilia decreases. This is due directly to the fact that the tensile forces are proportional to the square of the shear-rate and the relaxation time.

Figure 4.2 below shows this relationship for three different wavelengths of the metachronal wave (100µm being most indicative of the physiological case). If the extremely long relaxation times used in earlier sections are used in the model, a vanishingly low stress is predicted. A relaxation time of 3 seconds, representing the healthy case, predicts stresses on the order of tens of Pascals. This is a reasonable stress estimate as it is roughly in line with the stresses calculated in Chapter 2.
However, the purely monotonic relationship between required stress and relaxation time and the other parameters in the problem reveals the shortcomings of this model (and perhaps the UCM model in general). The dimensionless group predicts only the positive effects of elasticity and only the negative effects of a large wavelength. This is inconsistent with the experimental observation that an optimal mucus rheology exists and also implies that there is no tuning between the rheology of the mucus and the parameters which govern the coordination of the cilia. Perhaps if the shear-thinning character of mucus were taken into account, diminishing returns for large applied forces could be predicted. This model, while simple, provides a way in which the importance of viscoelastic effects can be estimated without considering complex flow phenomena.

![Figure 4.2: Minimum applied stress that must be applied by cilia as a function of relaxation time and metachronal wavelength. 1,10 and 100 micron wavelength curves are plotted in varying shades of blue.](image)
4.2 Stick Slip Transition of a UCM fluid

In order to directly address the shortcomings of the linear viscoelastic model; the UCM is used to anticipate the effects of relaxation time on the stick-slip flow transition. The stick-stick boundary conditions are chosen to be reminiscent of the leading or trailing edge of the active part of the metachronal wave and are represented in this problem as a step function at the origin. In the previous section an instantaneous transition between regions of applied shear and slip regions was assumed; here, the validity of that assumption will be tested for a viscous fluid. This problem is of interest because the interaction of viscous and elastic effects can be examined without considering inertia, a feat that was not possible in the unidirectional flows considered in Chapter 3. The inherent non-linearity of the UCM prohibits simple tractable solutions by analytical methods. A perturbation solution for small Weissenberg number was attempted. A forcing function for the order Weissenberg number PDE could not be generated; however, order Weissenberg number solutions for the stress components are readily obtained. These stress components allow a $Wi^2$ correction to the stream function to exist. However, the assessment of the behavior of the $O(Wi)$ stress terms is enough to ascertain the behavior of the $O(Wi^2)$ stream function and thus illustrates the role of elasticity. A similar problem which shows stream function deviations at $O(Wi)$ is discussed in Section 4.3.

4.2.1 Problem Setup

The stream function is employed to combine the two second order PDEs of the Navier Stokes’ equation into one 4th order PDE, the biharmonic equation. The benefit of this approach is in reducing the number of equations that must be solved simultaneously. The cost of this approach is that the order of the differential equation is increased, thus increasing the number of boundary conditions that must be imposed. Furthermore, there are restrictions on the order of the boundary conditions when using separation of variables; Eigen functions can only be found that satisfy boundary conditions that two orders apart. For example, if the stream function is fixed at the top and bottom boundaries
of the domain, then the problem must also be subject to fixed second derivatives with respect to y (if the basis is found with respect to y). This is the case for the problem at hand as seen in Figure 4.3; for this application of the biharmonic equation, the second derivative with respect to y at the boundaries corresponds to an applied shear stress so long as there is no flow through the boundary.

\[
\Psi(x, y = 1) = \frac{1}{3} \left. \frac{\partial^2 \psi}{\partial y^2} \right|_{x, y = 1} = 0
\]

\[
\Psi(x, y = 0) = 0 \left. \frac{\partial^2 \psi}{\partial y^2} \right|_{x, y = 0} = f(x, t)
\]

**Figure 4.3:** Problem schematic for Stick-Slip and Gaussian Stress (introduced in the next section) problems. Boundary conditions are only applied at the top and bottom of the domain as the disturbances for both of these problems are near the origin and decay exponentially in both directions.

In order to find a solution the boundary conditions are homogenized using the polynomial \( \phi \), below. Separation of variables is used and since there are no boundary conditions applied at the left and right boundaries, only the Eigen functions which go to zero for large \( x \) in either direction are permitted as solutions. The problem is inevitably split into left and right halves; requirements for a continuous solution with continuous derivatives allow the coefficients to the general solution to be found. The entire solution procedure can be found in Appendix I.

The solution form used to develop the order one and order Weissenberg number governing equations is shown below in eqn. 4.3 for the stream function; the same power expansion is applied to all variables.

\[
\psi = \psi_0 + We \psi_1
\] (4.3)
The perturbation method assumes that the expansion parameter (Weissenberg number) is small so that the next order term is only a small alteration about the first order solution. Thus the solutions found in this section and the next cannot be extended to situations involving the extremely high physiologically relevant Weissenberg numbers that result from some of the relaxation times estimates found in Chapter 3.

The equations governing the order one or \( O(1) \) (viscous) and \( O(Wi) \) behavior are shown below in eqns. 4.4 and 4.5 behavior For this particular problem it was found that an \( O(Wi) \) solution for the stream function did not exist. Higher order corrections were found to exist, but were not solved for as the \( O(Wi) \) solutions for the stresses were just as informative. It can be seen by examining eqns. 4.5 that if \( \psi_1 \) is zero, then the stress components can be found simply by taking derivatives of the viscous solution.

**Order 1** \hspace{1cm} (4.4)

\[
\begin{align*}
\tau_{x0} &= 2\psi_0^{1,0,0} \\
\tau_{y0} &= \psi_0^{0,2,0} - \psi_0^{2,0,0} \\
\tau_{y0} &= -2\psi_0^{1,1,0}
\end{align*}
\]

**Order Wi** \hspace{1cm} (4.5)

\[
\begin{align*}
\tau_{x1} &= 2\psi_1^{1,1,0} + 2\left[ -\psi_0^{1,1,1} + (\psi_0^{0,2,0})^2 + 2(\psi_0^{1,1,0})^2 + \psi_0^{1,0,0,0} \psi_0^{1,2,0} - \psi_0^{2,0,0} \psi_0^{2,0,0} - \psi_0^{0,1,0} \psi_0^{2,1,0} \right] \\
\tau_{y1} &= (\psi_1^{0,2,0} - \psi_1^{2,0,0}) + (\psi_0^{2,0,1} - \psi_0^{0,2,1}) - (\psi_0^{1,0,0} \psi_0^{2,1,0} + \psi_0^{1,1,0} \psi_0^{1,2,0} - 2\psi_0^{1,1,0} (\psi_0^{0,2,0} + \psi_0^{2,0,0}) \ldots + (\psi_0^{0,1,0} \psi_0^{3,0,0} + \psi_0^{1,0,0} \psi_0^{3,0}) \ldots) \\
\tau_{y1} &= -2\psi_1^{1,1,0} + 2\left[ -\psi_0^{1,1,1} + (\psi_0^{2,0,0})^2 + 2(\psi_0^{1,1,0})^2 - \psi_0^{1,0,0} \psi_0^{1,2,0} - \psi_0^{2,0,0} \psi_0^{2,1,0} + \psi_0^{0,1,0} \psi_0^{2,1,0} \right]
\end{align*}
\]

### 4.2.2 Calculation of Weissenberg Number

The parameter which governs the importance of viscoelastic terms in the non-dimensionalized governing equations is the Weissenberg number. Unlike the Deborah number used in the oscillating plate problem, the proper time scale for a viscoelastic flow
is often a characteristic strain rate. This strain characterizes how fast fluid elements are deforming through either shear, elongation, or both. Unlike a Newtonian fluid, viscoelastic fluids resist deformational changes due to fluid memory. This tendency to remain un-deformed begins to dominate flow dynamics if either the strain rate or relaxation time is large. In the stick slip problem, deformation is due largely to shear. Accordingly, the characteristic strain rate is calculated from the linear shear profile imposed as the far upstream condition as shown below in eqn. 4.6. The shear applied by the wall and the zero-shear rate viscosity are given by $T$ and $\eta_0$, respectively (in strongly elongational flows, other strain rates may be used).

$$Wi = \lambda \frac{T}{\eta_0} \quad (4.6)$$

The Weissenberg numbers using this strain rate are shown below in Table 4.1. Two relaxation time/viscosity pairs are used from previous sections. The applied shear is the value estimated from $T_{eff}$. The results indicate that the viscoelastic response is more pronounced in mucus with the longer relaxation time associated with the purulent sputum collected.

<table>
<thead>
<tr>
<th></th>
<th>Healthy</th>
<th>Unhealthy (Bronchitis)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Applied Shear</td>
<td>7.00E+01</td>
<td>7.00E+01</td>
</tr>
<tr>
<td>Zero Shear Rate Viscosity</td>
<td>2.43E+03</td>
<td>3.52E+03*</td>
</tr>
<tr>
<td>Time Scale</td>
<td>2.88E-02</td>
<td>1.99E-02</td>
</tr>
<tr>
<td>Relaxation Time</td>
<td>3.00E+00</td>
<td>8.30E+02*</td>
</tr>
<tr>
<td>Wi</td>
<td>1.04E+02</td>
<td>4.17E+04</td>
</tr>
</tbody>
</table>

Table 4.1: Weissenberg number estimates based on parameters of the metachronal wave. The negative wave speed indicates antiplectic propagation. *This is the longest relaxation time (with corresponding zero-shear rate viscosity) extracted from the dynamic moduli data of Davis and Dippy 1969.
4.2.3 Results and Conclusion

The final unperturbed solution for the stream function is shown below in eqn. 4.7 and plotted in Figure 4.4. The solution is written in terms of left and right halves. Symbols $\phi$, $\phi_+$, $u(x)$, and $\gamma$ represent the polynomial used to homogenize the boundary conditions, $\phi$ directly left and right of the origin after being integrated with the basis vector $\sin(\gamma x)$, the Heaviside step function, and Eigen values, respectively.

$$\psi = \psi_h + \phi$$

$$\phi = u(x) \left( \frac{y^3}{6} - \frac{y^2}{2} + \frac{y}{3} \right) + \left( -\frac{y}{3} \right)$$

$$\psi_h = \sum_{n=1}^{\infty} \sin(n\gamma) \left[ \left( -\frac{\phi_+ - \phi_-}{2} \right) e^{\gamma x} + \left( \frac{\gamma^2 (\phi_+ - \phi_-)}{2(\gamma + 1)} \right) x e^{\gamma x}, x \leq 0 \right. \right.$$ 

$$\left. + \left( \frac{\phi_+ - \phi_-}{2} \right) e^{-\gamma x} + \left( \frac{\phi_+ - \phi_-}{2(\gamma + 1)} + \frac{\gamma (\phi_+ - \phi_-)}{2} - \frac{\phi_+ - \phi_-}{2} \right) x e^{-\gamma x}, x \geq 0 \right.$$ 

$$\phi_+ = \frac{\langle \sin(n\gamma), \left( \frac{1}{6} y^3 - \frac{1}{2} y^2 \right) \rangle}{\langle \sin(n\gamma), \sin(n\gamma) \rangle}$$

$$\phi_- = \frac{\langle \sin(n\gamma), \left( -\frac{1}{3} y \right) \rangle}{\langle \sin(n\gamma), \sin(n\gamma) \rangle}$$

$$\gamma = n\pi$$

To quantify the effect of the stress discontinuity on the flow, the norm with respect to the far field flow profile (up and down stream) is found and plotted as a function of $x$ on a semi-log scale in Figure 4.5. Choosing the point at which the flow has
stabilized up and down stream is up to the discretion of the observer. In this case it will be said that the effect of the stress discontinuity has dissipated (that is the flow resembles a simple shear flow to the right or a plug flow to the left) by three film thicknesses up and downstream of the origin because the norm has decreased approximately three orders of magnitude by these distances. The results show that a flow despite being devoid of inertia requires distance along the flow path to adjust to new boundary conditions. Because cilia detach from the mucus, there is reason to believe that much of the efficacy of mucociliary transport hinges on the ability of active cilia to act at a distance through the mucus. For this to be possible, one would expect the effect of the stress discontinuity to have a larger region of influence on a viscoelastic flow compared to a purely viscous flow.

![Figure 4.4: Stream function contour plot of flow with an applied stress-discontinuity. The bottom boundary to the right comprises a "stick" or applied stress condition; the bottom-left, a stress-free condition.](image)
Figure 4.5: The norm with respect to the far-field solutions is plotted as a function of x-position. Far away from the origin in both the negative and positive direction on the x axis the flow profile is unperturbed; shear-free and linear in shear, respectively. The norm is intended to quantify the spatial region of influence of the shear discontinuity (the smaller the norm, the more the flow has approached the far-field solution); from the figure it can be seen that the flow has nearly stabilized by two or three film heights (the norm has decreased 3 orders of magnitude by that distance both up and downstream).

Using eqns. 4.4 and 4.5 the viscous and \( O(Wi) \) solutions for shear and stream wise normal stress are found for the flow downstream of the stress discontinuity (the flow is moving to the left and there is no shear applied at the bottom boundary). Solutions are shown in eqns. 4.8 and 4.9 and plotted in Figure 4.6 and Figure 4.7. The ‘-’ symbols indicate that the solutions shown are only for the left half of the domain. Symbols \( \phi \), and \( \gamma \) maintain their definitions. Additional constants \( C \), \( D \) and \( y_l \) are defined in eqn. 4.10.

\[
\begin{align*}
\tau_{xy} &= \tau_{xy0} + Wi \tau_{xy1} \\
\tau_{xy0} &= \sum_{n=1}^{\infty} \left\{ e^{\pi} \sin(p) \left\{ - \gamma^2 C - Dx - 2D\gamma - C\gamma^2 - D\gamma^2 x \right\} + \frac{1}{3} y, \gamma^2 \sin(p) \right\} 
\end{align*}
\]
\[ \tau_{xy1-} = \sum_{n=1}^{\infty} \left\{ e^{2\gamma x} \sin(\gamma y) \cos(\gamma y) \left[ -6D^2 \gamma^2 - 4CD \gamma^3 - 4D^2 \gamma^3 x - \frac{5}{3} D \right] \right\} \\
+ e^{2\gamma x} \sin(\gamma y) \cos(\gamma y) \left[ -\frac{5}{3} D \gamma^3 y_1 - C \gamma^3 y_1 - D \gamma^4 y_1 x \right] \right\} \]

\[ \tau_{xx-} = \tau_{xx0-} + Wi \tau_{xx1-} \]  \hspace{1cm} (4.9)

\[ \tau_{xx0-} = \sum_{n=1}^{\infty} e^{2\gamma x} \cos(\gamma y)[2\gamma D + 2\gamma C + D \gamma x] \]

\[ \tau_{xx1-} = \sum_{n=1}^{\infty} \left\{ e^{2\gamma x} \cos(\gamma y)^2 \left[ 4D^2 \gamma^2 + 4CD \gamma^3 + 2C^2 \gamma^4 + 4D^2 \gamma^3 x + 2D^2 \gamma^4 x^2 \right] \right\} \\
+ e^{2\gamma x} \cos(\gamma y)^2 \left[ \frac{4}{3} D \gamma^3 y_1 + \frac{2}{3} C \gamma^3 y_1 + \frac{2}{3} D \gamma^4 x y_1 \right] \right\} \\
+ e^{2\gamma x} \sin(\gamma y)^2 \left[ -2D^2 \gamma^2 + 2C^2 \gamma^4 + 4CD \gamma^4 x + 2D^2 \gamma^4 x^2 \right] \right\} \\
+ e^{2\gamma x} \sin(\gamma y)^2 \left[ -\frac{4}{3} D \gamma^3 y_1 - 2C \gamma^3 y_1 - 2D \gamma^4 x y_1 + \frac{2}{9} \gamma^4 y_1^2 \right] \right\} \]

\[ D = \left( \frac{\phi_- - \phi_+}{2(\gamma + 1)} \right) + \frac{\gamma(\phi_- - \phi_+)}{2} - \frac{(\phi_- - \phi_+)}{2} \]

\[ C = \left( -\frac{(\phi_- - \phi_+)}{2} \right) \]  \hspace{1cm} (4.10)

\[ y_1 = \frac{\langle \sin(\gamma y), y \rangle}{\langle \sin(\gamma y), \sin(\gamma y) \rangle} \]

The solution is plotted for a Weissenberg number of 0.1, this keeps the size of the viscoelastic perturbation to within ten percent of the size of the viscous solution. The plots of both the shear and normal flow stresses illustrate the convection of stresses; the contours are essentially shifted normal to the direction of fluid flow. For small Weissenberg numbers fluid memory only allows convection of stress states to be convected by only small fractions of the film height past the viscous cases. However, the effects of much larger Weissenberg numbers can be extrapolated, albeit qualitatively, from these results. One can expect to see the transition length continue to grow as viscoelastic stresses dominate over the viscous stresses. Introducing transience into this
problem is possible by considering a moving stress discontinuity; however, this will be explored in the next section using a sinusoidally varying wall stress.

**Figure 4.6:** Shear stress $\tau_{xy}$ contour plot showing viscous stress (black line), $Wi=0.1$ (cyan). The stream function is plotted in the background (light gray). The solution is plotted only for the region downstream of the change in boundary conditions.
Figure 4.7: Stream wise normal stress $\tau_{xx}$ contour plot showing viscous stress (black line), $Wi=0.1$ (cyan). The stream function is plotted in the background (light gray). The solution is plotted only for the region downstream of the change in boundary conditions.
4.3 UCM with Sinusoidally Varying Wall Stress

Building upon the work of the previous section, a stress forcing function was chosen which would be comparatively straightforward to deal with mathematically and for which an \( O(Wi) \) correction existed. The function, a sinusoidally varying stress, was chosen to possess a time dependent phase which could explore the effects of a moving disturbance in a UCM fluid and reveal the role, if any, that the metachronal wave plays in transport. There were many options when it came to choosing a function with which to explore the dynamics of a UCM fluid. The effects of elasticity on a flow will again be examined by quantifying the shift away from the Newtonian solution, in this case, because the stress varies regularly.

4.3.1 Problem Setup

The geometry of the problem is identical to that of the previous problem, only the applied stress (labeled as \( f(x,t) \) in Figure 4.3) has been changed to eqn. 4.11 (or eqn. 4.13 in terms of the wave speed) where \( a \) is the wave number, \( b \) is the angular frequency, and \( c \) is the wave speed. The definition of phase speed or wave speed is shown in eqn. 4.12, where \( \varphi \) represents the phase of the sinusoid which is both a function of \( x \) position and time.

\[
\frac{\partial^2 \psi}{\partial y^2} \bigg|_{x,y=0,t} = f(x,t) = 1 + \sin(ax + bt) \tag{4.11}
\]

\[
\varphi = ax + bt
\]

\[
\frac{d\varphi}{dt} = \frac{d\varphi}{dx} \frac{dx}{dt} = \frac{d\varphi}{dx} \frac{dx}{dt} = b/a = -c \tag{4.12}
\]
\frac{\partial^2 \psi}{\partial y^2} \bigg|_{x,y=0,t} = 1 + \sin[a(x + (b/a)\tau)] 
\frac{\partial^2 \psi}{\partial y^2} \bigg|_{x,y=0,t} = 1 + \sin[a(x - ct)] \tag{4.13}

In addition to the constraints on Weissenberg number size that applied to Section 4.2, the other free parameters in the problem will have restricted values as well such that $O(Wi)$ terms remain small compared to the unperturbed result. Nonetheless, some of elasticity’s influence on a flow with transient boundary conditions can be estimated by this approach. The flow field that results from the imposed sinusoidal stress is not the primary result of interest. The choice to use the stream function to simplify the process of finding a solution limits the choice of boundary conditions as was discussed in the previous section. Therefore, regardless of the nature of the stress applied in this problem, the flow rate remains a part of the prescribed boundary conditions. In other words, it is not possible to apply a stress and look at the effect on flow rate as say, a function of Weissenberg number. Instead, the phase shift $\Theta$ (eqn. 4.16) away from the Newtonian solution as a function of Weissenberg number, wavelength, and wave speed is found.

\[
\psi = \Lambda \sin(ax + \Theta) \\
= \Lambda \left[ \sin(ax)\cos(\Theta) + \cos(ax)\sin(\Theta) \right] \\
= \Pi_1 \sin(ax) + \Pi_2 \cos(ax) \tag{4.14}
\]

\[
\Pi_1 = \Lambda \cos(\Theta) \\
\Pi_2 = \Lambda \sin(\Theta) \tag{4.15}
\]

\[
\Theta = \text{ArcTan} \left( \frac{\Pi_2}{\Pi_1} \right) \tag{4.16}
\]
4.3.2 Calculation of Weissenberg Number

In addition to the time scale used to calculate the Weissenberg number in the stick-slip problem (characteristic strain rate), the sinusoidally varying stress problem introduces a relevant time scale. In order to arrive at this time scale, one must consider how often a mucus fluid element cycles between maximum and minimum stress applied by the boundary condition. This frequency will be a function of the average mucus velocity $V_m$, and the parameters of the metachronal wave: its phase speed $c$ and wavelength $\gamma$ as shown below in eqn. 4.17.

$$ Wi = \lambda \frac{|V_m - c|}{\gamma} \quad (4.17) $$

The Weissenberg numbers calculated for this alternate time scale are shown below in Table 4.2. Again, two relaxation times are used and indicate that the viscoelastic response is more pronounced in mucus with longer relaxation times and typically associated with some ailment.

<table>
<thead>
<tr>
<th></th>
<th>Healthy</th>
<th>Unhealthy (Bronchitis)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wavelength $\gamma$</td>
<td>1.00E-04</td>
<td>1.00E-04</td>
</tr>
<tr>
<td>Wave Speed $c$</td>
<td>-4.00E-04</td>
<td>-4.00E-04</td>
</tr>
<tr>
<td>Mucus Velocity $V_m$</td>
<td>1.00E-04</td>
<td>1.00E-04</td>
</tr>
<tr>
<td>Time Scale</td>
<td>2.00E-01</td>
<td>2.00E-01</td>
</tr>
<tr>
<td>Relaxation Time $\lambda$</td>
<td>3.00E+05</td>
<td>8.30E+02*</td>
</tr>
<tr>
<td>$Wi$</td>
<td>1.50E+01</td>
<td>4.15E+03</td>
</tr>
</tbody>
</table>

Table 4.2: Weissenberg number estimates based on parameters of the metachronal wave. The negative wave speed indicates antiplectic propagation. *This is the longest relaxation time extracted from the dynamic moduli data of Davis and Dippy 1969.20
4.3.3 Results and Conclusion

Separation of variables and Fourier Transforms are used to find both order one and order Weissenberg number parts of the solution, eqns. 4.18 and 4.19. All of the variables in the solutions below maintain their definitions from previous sections.

\[ \psi_0 = \psi_h + \phi \]
\[ \psi_h = \sin(ax + bt) \sum_{n=1}^{\infty} \frac{(a^2 + 2\gamma^2 a^2)}{2(\gamma^2 + \gamma^2)} \sin(n\gamma) \]  
\[ \phi = f(x, t) \left( -\frac{y^3}{6} + \frac{y^2}{2} - \frac{y}{3} \right) + \frac{1}{3} y \]
\[ \gamma = n\pi \]

\[ \psi_1(x, y, 0) = \sum_{n=1}^{\infty} \frac{a^2 \cos(ax) \sin(n\gamma)}{48\gamma^5(4a^4 + 5a^2\gamma^2 + \gamma^4)^2} \left\{ \begin{array}{c} -36a^2b\gamma^2 - 72b\gamma^4 \\ + 9a^3(-53 + 4\gamma^2) \\ + a(-282\gamma^2 + 8\gamma^4) \\ - 366\gamma^4 + 64\gamma^6 \\ + a\gamma^2 \sin(ax) \\ + a^4(-438 + 64\gamma^2) \\ + a^2\gamma^2(-867 + 128\gamma^2) \end{array} \right\} \]  

Interpreting the results by attributing the theoretical phase shift phenomena to physics of the relaxation time is somewhat convoluted by the fact that the flow is never allowed to fully develop. The flow is constantly perturbed by the varying shear stress and never becomes shear-free. From previous sections exploring linear viscoelastic effects (Sections 3.3 and 3.6 in particular) it was found that elasticity in a fluid increases the distance that disturbances propagate and the speed at which they do so. Section 4.2 illustrated how stress states can be convected. However, the effects of a constantly changing stress have not been illustrated. First, the effects of wavelength will be addressed and then those associated with the wave speed.

In all of the wave speed cases (Figure 4.8, Figure 4.9, and Figure 4.10), increasing the wavelength of the applied stress worked to decrease the magnitude of the phase shift.
at all Weissenberg numbers. Recalling the lessons learned about linear viscoelasticity in Chapter 3, it is understood that the more abrupt the change a Maxwellian fluid element experiences, the more its elastic character will be surface and thus induce a phase shift. In the extreme case of a fluid exiting a channel or pipe, the change in applied stresses is abrupt; if the fluid is viscoelastic one can expect the magnitude of its response (recoil and swell due to dormant tensile forces) to be large. If it were somehow possible to gradually back off the stress the observed exit-swell would be smaller. This is also the case for the sinusoidally varying stress. While the interface is constrained to be flat, the effect of relaxation time manifests itself in the flow’s memory of the upstream boundary conditions and removing the stress allows the fluid to attempt some strain recovery. Increasing the abruptness of the change (decreasing the wavelength) or increasing the Weissenberg both make the viscoelastic effects more pronounced by reducing the flow’s time to relax/adjust to new boundary conditions or increasing the fluid's memory of upstream stresses, respectively.

![Figure 4.8: Phase $\Theta$ (in radians) contour plot at $\psi(0,0.5,0)$ as a function of Weissenberg number and wavelength for a non dimensional wave speed of 1 (traveling in the direction of fluid flow to the right). Increasing the wavelength for a non-zero Weissenberg number shifts the flow upstream. Points below the dashed white line indicate that $Wi\psi/\psi_0$ has exceeded 0.1 and cannot be considered valid.](image-url)
Figure 4.9: Phase $\Theta$ (in radians) contour plot at $\psi(0,0.5,0)$ as a function of Weissenberg number and wavelength for no wave speed. Increasing the Weissenberg number amplifies the effect that changing the wavelength will have on the shift of the flow. Increasing the wavelength for a non-zero Weissenberg number shifts the flow downstream; in the limit of large wavelengths the flow approaches the Newtonian solution (zero phase shift). Points below the dashed white line indicate that $Wi\psi_1/\psi_0$ has exceeded 0.1 and cannot be considered valid.

Figure 4.10: Phase $\Theta$ (in radians) contour plot at $\psi(0,0.5,0)$ as a function of Weissenberg number and wavelength for a wave speed of -1 (traveling to the left, opposing the direction of fluid flow). Increasing the wavelength for a non-zero Weissenberg number shifts the flow downstream. Points below the dashed white line indicate that $Wi\psi_1/\psi_0$ has exceeded 0.1 and cannot be considered valid.
The zero Weissenberg number case (Stoke’s flow) is reversible and thus entirely ambivalent to any transience in the boundary conditions. However, the addition of elasticity makes the flow sensitive to temporal shifts in the applied stress distribution; the wave speed. Figure 4.11 below shows how allowing the stress distribution to move to the right induces an upstream phase shift while an antiplectic (leftward) moving distribution causes the flow to shift downstream. Equations 4.12 and 4.13 show the definition of wave speed for a sinusoid and apply it to the boundary condition used in this problem. These relationships are consistent with those discussed between wavelength and phase shift and can be understood by considering the diagram in Figure 4.13. The figure illustrates a UCM fluid element as it is convected to the right. A negative or antiplectic wave speed imposes more rapid changes from which the fluid takes longer to relax. A symplectic wave imposes less rapid changes and thus tends towards zero phase shift up to a point (c~0.29, this speed varies weakly with Weissenberg number) after which the flow shifts upstream, out of phase with the applied stress. The phase shifts are of opposite sign because the wave speeds are also of opposite sense. If one were to observe the transience flows created by the moving stress distributions simultaneously they would find that as the stress of one was increasing, the stress of the other would be decreasing.
Figure 4.11: Phase (in radians) contour plot at $\psi(0,0.5,0)$ as a function of Weissenberg number and non-dimensional wave speed. The non-dimensional wavelength is fixed at 10. Stress waves traveling symplectically (with the flow) induce an upstream phase shift while antiplectic waves induce a downstream phase shift. A standing stress distribution causes a slight downstream shift in the stream function for non-zero Weissenberg numbers.
Figure 4.12: The region of validity for phase shift at $\psi(0.0.5,0)$ as a function of Weissenberg number and non-dimensional wave speed. The non-dimensional wavelength is fixed at 10. Gray indicates $Wi\psi_1/\psi_0 > 0.1$. White, $Wi\psi_1/\psi_0 < 0.1$.

Figure 4.13: Schematic illustrating how a UCM fluid element (maroon) experiences different stresses as it is convected to the right (orange). The curves represent the phase of the imposed stress. From left to right, the waves are traveling upstream, downstream, and downstream at the same velocity as the fluid velocity of the element. In the first case, the fluid element is exposed to more rapidly changing conditions. As the wave speed approaches that of the fluid element, changing becomes less pronounced (middle) until they disappear completely (right).

This problem establishes how elasticity is sensitive to the abruptness of a change in boundary conditions both spatially and temporally; the results are summarized below
in Table 4.3. The results, essentially, indicate how often a fluid element is contacted by cilia; the more often the interaction, the higher the magnitude of the phase shifts. However, the role of elasticity in mucociliary transport is not entirely elucidated by this problem. The phase shift appears to be beneficial in that the flow can retain stress while no stress is being applied; however, the problem is not ideal for correlating stress with flow rate to truly gauge efficiency.

<table>
<thead>
<tr>
<th></th>
<th>Viscous Fluid</th>
<th>UCM Fluid</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Wavelength</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Increase</td>
<td>No change</td>
<td>Upstream</td>
</tr>
<tr>
<td>Decrease</td>
<td>No change</td>
<td>Downstream</td>
</tr>
<tr>
<td><strong>Wave Speed</strong></td>
<td></td>
<td></td>
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<tr>
<td>Symplectic</td>
<td>No change</td>
<td>Upstream</td>
</tr>
<tr>
<td>Antiplectic</td>
<td>No change</td>
<td>Downstream</td>
</tr>
<tr>
<td>Zero</td>
<td>No change</td>
<td>Downstream</td>
</tr>
</tbody>
</table>

*Table 4.3* A summary of the functionality between parameters as found by examining Figures 4.4-4.7. **Positive** value for the phase shift indicates a shift to the *left* (upstream). A **negative** value for the phase shift indicates a shift to the *right* (downstream).
5 Conclusion and Discussion

5.1 Summary of Results

This thesis has considered numerous problems intended to investigate various aspects of mucociliary transport. A steady state model developed by completing a straightforward force balance on the mucus was completed. Inputting empirical data illustrated that a more complex constitutive equation for mucus had to be considered; experimental values for stress and velocity disagreed with theoretical values by at least one order of magnitude.

The initial literature review revealed a heavy focus on the elasticity of mucus on short time scales as the dominant rheologically complex behavior of mucus. This mindset seemed a promising lead for understanding mucus, and considerable time was spent developing the concept of relaxation time through the linear Maxwell model, a viscoelastic fluid model in which the media is Newtonian at rest but possesses elasticity on short time scales. It was shown through a complete derivation of the classic oscillating plate problem (a flow problem described but un-derived in texts) that unsteady shear disturbances propagate further and faster when the media is a Maxwell fluid.

In order to investigate how this time scale dependent elastic behavior might manifest itself in a flow situation, both linear and non-linear Maxwell models were examined in a flow which possessed a sinusoidally varying, unsteady velocity or stress distribution. The UCM provided the most realistic results of the two in that it allowed stress information to be convected in the fluid, a feature which linear viscoelasticity does not include. Additionally, the UCM equations are able to predict tensile forces along streamlines, a phenomenon which can be observed experimentally. A model is constructed which relied on the tensile stresses developed through shear to hold up regions of the mucal flow which did not directly engage the cilia; stress applied by cilia was found to be inversely proportional to the square root of the relaxation time.

The UCM model is further explored by considering two additional problems: a flow with a stress discontinuity (often referred to as the “stick-slip” problem) and another with a transient, sinusoidally varying stress. In order to find analytical solutions, all
variables were expanded in powers of small Weissenberg number. The equations developed governing the perturbation solution allow the Newtonian (\(Wi=0\)) case to be found first. The effect of small amounts of elasticity on the velocity and stress fields could then be estimated analytically using the Newtonian solution. The solutions for the perturbed stress fields confirmed intuition; all stress components are convected downstream from their positions in the Newtonian flow. This problem demonstrated that the larger developing region of mucus may be part of its functionality in the respiratory tract.

Similar conclusions can be arrived at by observing the effect of Weissenberg number, wavelength, and wave speed on the second UCM problem. Additionally, a potential benefit of antiplectic cilial coordination became apparent; symplectic coordination (cilia phase velocity has the same sense mucus velocity) can potentially mollify viscoelastic effects if the phase speed is identical to the mucus velocity. In order for relaxation time to have an effect on the flow, it must changes in boundary conditions. However, if these changes are in phase with the bulk mucal flow, very little variation is experienced by mucus. It seems from these results that an antiplectic metachronal wave has the advantage of ensuring that regardless of mucus velocity, the wave of cilial activity will never travel at the same velocity.

An understanding of how elasticity allows a flow to maintain stress and thus promote stress continuity was gained. However, new questions and goals arose. Firstly, due to the analytical methods used in Chapter 4, a physiologically relevant flow regime could not be modeled. This does not invalidate the insights gained in Chapter 4; it simply means that there is more work to be done if quantitative results of the mucal flow field and stresses are sought.

Secondly, the problems considered here strictly deal with how a viscoelastic flow might respond to various stress or velocity boundary conditions. For a viscoelastic fluid, as was discussed in Chapter 3, stresses due to an imposed strain or strain-rate take time to develop. This means that the beat frequency in concert with the relaxation time directly affect the stress that develops at the PCL-mucus interface. In other words, limiting boundary conditions by considering more explicitly the dynamics of the mucociliary interaction may allow the detriments of large Weissenberg number flows to be
understood and a stronger correlation with flow velocity to be made. The results of this thesis, by contrast, show only the benefits of stress relaxation.

Finally, because mucus possesses rheological features other than those considered by the UCM, it may be that mechanisms required for transport were left out a priori. Mucus, unlike a Maxwell fluid, can possess structure at rest which would be capable of transmitting stresses effectively across regions where cilia are unengaged. A more complete exploration of yielding and shear-thinning would need to be completed in order to understand whether this effect dominates over relaxation time.

In the course of understanding how mucus is moved by the body, questions arose regarding what mucus’s primary function was according the human body. If the purpose of mucus is simply to trap varied pathogens, a complex chemistry would be required and it seems that a complex rheology might exist as secondary feature. And though it seems likely that mucus serves equally well multiple roles, the question is posed to illustrate the fact that, currently, the chemical roles and mechanical roles of mucus are difficult to separate. This connection, like that between form and function, is ubiquitous in nature and fuels research in multiple fields. To further understand mucus, experimental work focused directly on the rheology of mucus at the physiologically relevant time and length scales is essential. Information gained from pointed experimental efforts can be fed back into the modeling effort.

5.2 Future Work

Several of the issues that arose when solving problems in Chapter 4 could be alleviated by utilizing numerical techniques. However, one might consider, first, attempting analytical solutions for the large (as opposed to small) Weissenberg number case which was shown to be more representative of the physiologically relevant regime. Though the analytical methods to handle the large stress gradients of boundary layers become complex, large Weissenberg number asymptotics have received some attention by theoreticians of viscoelastic flows.\(^\text{67}\)

Numerical methods are suggested because they would have the advantage of being free of restrictions imposed by separation of variables, Fourier transforms, and Perturbation theory. For example, in order to perform separation of variables it was
necessary that the boundary conditions differ by two derivatives. This restriction could be
relaxed. Additionally, Fourier transforms, while a very powerful technique, are limited to
certain classes of functions. A numerical solution would allow a custom function to be
generated, perhaps one that cannot be expressed simple using fundamental functions. One
can move away from idealized sinusoids, for example. Implementation of periodic
boundary conditions may also be desirable. Highly viscoelastic flows, like high Reynolds
number flow, will still have to be treated carefully, however.

While numerous improvements can be made to the methods used to find solutions
in Chapter 4; further investigation should consider other viscoelastic models. The major
limitation of the Maxwell model in terms of its applicability to mucus is the fact that an
undisturbed Maxwellian fluid is Newtonian. Mucus, in most cases, possesses structure at
rest and while this structure may be highly altered by shear forces, its ability to transmit
forces like a solid may be a crucial feature for transport. Tensile forces in the UCM only
arise in the presence of velocity gradients. Reviewing the literature available on fiber
spinning of polymeric solutions and elongational or extensional flows is recommended as
a starting point for developing a new set of governing equations to use. Furthermore,
there is substantial research available on the molecular theory of viscoelastic fluids which
may allow integration of shear-thinning; finitely extensible nonlinear elastic or FENE
models are one such example. A literature search on the mechanical characterization of
biological fluids similar to tracheal mucus may provide insights that are applicable to
understanding mucus as well.

On a more fundamental note, all work with linear viscoelasticity has not been
exhausted. Just as the early flagella and cilia models discussed in Chapter 1 worked
extensively with Stokeslets and Slender Body Theory, the same type of models may be
extendable to the linear Maxwell Model. Considering, for example, how the disturbances
created by small oscillating sphere propagate would be an excellent addition to the
oscillating wall problem. A literature search on this topic has not been completed; a
reader interested in these topics may consider looking up Green’s functions for
Maxwellian media.

These suggested models, though they may be good exercises, suffer due to lack of
experimental data on mucus rheology and muco-ciliary interaction. Ultimately, a more
complete empirical characterization of dominant mucal mechanics will need to be achieved so that modeling efforts can become more informed. A connection between the chemical composition of mucus, its microstructure, and thus its transportability should be a high priority goal.
Appendix A Viscoelastic Models: Scaling and Perturbation Equations

A.1 Linear Maxwell Model

Constitutive Equation for relating stress and strain rate:
\[
\lambda \frac{\partial \tau}{\partial t} + \tau = \eta_0 \dot{\gamma}
\] (A.1)

Stress is constant throughout the element (Hookean and Newtonian elements have the same stress)
\[
\tau_H = \tau_N = \tau
\] (A.2)

Stress in Hookean elastic element:
\[
\tau_H = G \gamma_H
\] (A.3)

Stress in Newtonian viscous element:
\[
\tau_N = \eta_0 \dot{\gamma}_N
\] (A.4)

The strain and strain-rate of the Maxwell element is the sum of the strains and strain rates from the Hookean and Newtonian elements.
\[
\gamma = \gamma_H + \gamma_N \quad \text{(A.5)}
\]
\[
\dot{\gamma} = \dot{\gamma}_H + \dot{\gamma}_N \quad \text{(A.6)}
\]

A.2 Upper-Convected Maxwell Model

\[
\lambda \frac{\mathbf{D} \tau}{\mathbf{D}t} + \tau = \eta_0 \dot{\gamma}
\] (A.7)

Definition of Upper-Convected Derivative
\[
\frac{\mathbf{D} \tau}{\mathbf{D}t} = \frac{\partial \tau}{\partial t} + \mathbf{u} \cdot \nabla \tau - \{\tau \cdot \nabla \mathbf{v}\}^T - \{\tau \cdot \nabla \mathbf{v}\}
\] (A.8)

Expand and simplify for the Cartesian, z-independent case
\[
\lambda \left[ \frac{\partial \tau_{xx}}{\partial t} + u_x \frac{\partial \tau_{xx}}{\partial x} + u_y \frac{\partial \tau_{xx}}{\partial y} - 2 \left( \tau_{xx} \frac{\partial u_x}{\partial x} + \tau_{xy} \frac{\partial u_x}{\partial y} \right) \right] + \tau_{xx} = 2\eta_0 \frac{\partial u_x}{\partial x} \\
\lambda \left[ \frac{\partial \tau_{yx}}{\partial t} + u_x \frac{\partial \tau_{yx}}{\partial x} + u_y \frac{\partial \tau_{yx}}{\partial y} - \tau_{yx} \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) - \tau_{yy} \frac{\partial u_x}{\partial y} - \tau_{xx} \frac{\partial u_y}{\partial x} \right] + \tau_{yx} = \eta_0 \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \\
\lambda \left[ \frac{\partial \tau_{yy}}{\partial t} + u_x \frac{\partial \tau_{yy}}{\partial x} + u_y \frac{\partial \tau_{yy}}{\partial y} - \tau_{yy} \left( \frac{\partial u_y}{\partial x} + \frac{\partial u_y}{\partial y} \right) \right] + \tau_{yy} = 2\eta_0 \frac{\partial u_y}{\partial y}
\]

(A.9)

**A.3 UCM Scaling and Perturbation in $$\varepsilon$$ and $$D_e$$**

**Governing Equations:**
UCM (see above section)

**Momentum**

\[
\rho \left[ \frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} \right] = - \frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \\
\rho \left[ \frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} \right] = - \frac{\partial P}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}
\]

(A.10)

**Continuity**

\[
\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = 0
\]

(A.11)

**Boundary Conditions:**

At $$y=0$$ for all $$x$$:

- $$u=B\cos(\omega t+kx)$$ or $$u=B\cos[k(Vt+x)]$$, and $$v=0$$

  - $$B$$ is the amplitude of the varying component of the wall velocity
  - $$k$$ is the wave number ($$2\pi/\gamma$$)
  - $$\gamma$$ wavelength
  - $$V$$ is the wave speed
  - $$\omega$$ is the angular frequency

At $$y=h$$:

- $$u=0$$ and $$v=0$$

**Scaling Choices:**

Angular frequency is a natural time scale to choose because of the boundary condition

- $$y$$ scale is the height of the film/gap – $$h$$
- $$x$$ scale is the wavelength as it appears in the boundary condition - $$\gamma$$
\( x = \bar{x} \gamma \)
\( y = \bar{y} h \)
\( t = \frac{\tilde{t}}{\omega} \) \hspace{1cm} (A.12)

For velocity scales there are options:
1. The amplitude of the boundary condition
2. The wave speed
3. A scale based on time and length scales already found in the problem

Going with option 3

Reasons:
1. In some problems, velocity may not be prescribed in the boundary condition
2. In the CM model, the wave speed will surely impact the flow; however the degree to which is not understood. We don’t want the problem to fall apart if there is not a wave speed.
3. This leaves option 3; frequency will always be a parameter in cilia problems.

\( u_x = \bar{u}_x \omega \gamma \)
\( u_y = \bar{u}_y A \) \hspace{1cm} (A.13)

Using continuity, arrive at a scale for \( u_y \) – find \( A \)

\( \frac{A}{h} \frac{\partial \bar{u}_y}{\partial y} + \frac{\omega \gamma}{\gamma} \frac{\partial \bar{u}_x}{\partial x} = 0 \)

\( \frac{A}{h} \sim \frac{\omega \gamma}{\gamma} \rightarrow A \sim \omega h \) \hspace{1cm} (A.14)

\( u_x = \bar{u}_x \omega \gamma \)
\( u_y = \bar{u}_y \omega h \) \hspace{1cm} (A.15)
Begin with the 2nd Maxwell equation with unknown scales for the stress tensor components:

\[ y = \bar{y} h \]
\[ x = \bar{x} \gamma \]
\[ u_x = \bar{u}_x \omega \gamma \]
\[ u_y = \bar{u}_y \omega \h \]
\[ \tau_{xx} = \bar{\tau}_{xx} A \]
\[ \tau_{xy} = \bar{\tau}_{xy} B \]
\[ \tau_{yy} = \bar{\tau}_{yy} C \]

\[
\begin{align*}
\lambda \frac{\partial \tau_{xy}}{\partial t} + \lambda u_x \frac{\partial \tau_{xy}}{\partial x} + \lambda u_y \frac{\partial \tau_{xy}}{\partial y} - \lambda \tau_{xy} \frac{\partial u_x}{\partial x} - \lambda \tau_{xy} \frac{\partial u_y}{\partial y} - \lambda \tau_{xx} \frac{\partial u_x}{\partial x} - \lambda \tau_{yy} \frac{\partial u_y}{\partial y} + \tau_{xy} &= \eta_0 \frac{\partial u_x}{\partial y} + \eta_0 \frac{\partial u_y}{\partial x} \\
(\lambda \omega B) \frac{\partial \tau_{xy}}{\partial t} + \frac{\lambda \omega B}{\gamma} u_x \frac{\partial \tau_{xy}}{\partial x} + \frac{\lambda \omega B}{h} u_y \frac{\partial \tau_{xy}}{\partial y} - \frac{\lambda \omega B}{\gamma} \tau_{xx} \frac{\partial u_y}{\partial x} &= \tau_{xy} \\
... - \frac{\lambda \omega B}{h} \tau_{yy} \frac{\partial u_x}{\partial y} - \frac{\lambda \omega B}{\gamma} \tau_{xy} \frac{\partial u_y}{\partial x} &= \tau_{xy} + B \tau_{xy} = \frac{\eta_0 \omega y}{h} \frac{\partial u_x}{\partial y} + \frac{\eta_0 \omega h}{\gamma} \frac{\partial u_y}{\partial x} \\
(\lambda \omega B) \frac{\partial \tau_{xy}}{\partial t} + (\lambda \omega B) u_x \frac{\partial \tau_{xy}}{\partial x} + (\lambda \omega B) u_y \frac{\partial \tau_{xy}}{\partial y} - (\lambda \omega B) \tau_{xy} \frac{\partial u_x}{\partial x} &= \tau_{xy} \\
... - (\lambda \omega B) \tau_{yy} \frac{\partial u_x}{\partial y} - (\lambda \omega B) \tau_{xy} \frac{\partial u_y}{\partial x} &= \tau_{xy} + B \tau_{xy} = \frac{\eta_0 \omega }{\epsilon} \frac{\partial u_x}{\partial y} + \frac{\eta_0 \omega e}{\gamma} \frac{\partial u_y}{\partial x} \\
\end{align*}
\]

Balance \[ \tau_{xy} \] with dominant shear stress term

\[ B = \frac{\eta_0 \omega}{\epsilon} \]  

(A.19)

Substitute back into 2nd Maxwell Equation

\[
\begin{align*}
\left( \frac{\lambda \omega^2 \eta_0}{\epsilon} \right) \frac{\partial \tau_{xx}}{\partial t} + \left( \frac{\lambda \omega^2 \eta_0}{\epsilon} \right) u_x \frac{\partial \tau_{xx}}{\partial x} + \left( \frac{\lambda \omega^2 \eta_0}{\epsilon} \right) u_y \frac{\partial \tau_{xy}}{\partial y} - \left( \frac{\lambda \omega^2 \eta_0}{\epsilon} \right) \tau_{xx} \frac{\partial u_x}{\partial x} &= \tau_{xx} \\
... - \left( \frac{\lambda \omega^2 \eta_0}{\epsilon} \right) \tau_{yy} \frac{\partial u_x}{\partial y} - \left( \frac{\lambda \omega^2 \eta_0}{\epsilon} \right) \tau_{xy} \frac{\partial u_y}{\partial x} &= \tau_{xy} + B \tau_{xy} = \left( \frac{\eta_0 \omega e}{\gamma} \right) \frac{\partial u_x}{\partial y} + \left( \frac{\eta_0 \omega e}{\gamma} \right) \frac{\partial u_y}{\partial x} \\
\end{align*}
\]

(A.20)

Motivated by the fact that normal stresses in the x direction arise from shear stress in viscoelastic unidirectional flow, let their respective terms be of the same order in the x-momentum equation.
\[
\rho \left[ \frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} \right] = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \\
(\rho \omega^2 \gamma) \frac{\partial u_x}{\partial t} + \left( \frac{\rho \omega^2 \gamma}{\gamma} \right) u_x \frac{\partial u_x}{\partial x} + \left( \frac{\rho \omega^2 \gamma}{h} \right) u_y \frac{\partial u_x}{\partial y} = -\left( \frac{G}{\gamma} \right) \frac{\partial P}{\partial x} + \frac{A}{\gamma} \frac{\partial \tau_{xx}}{\partial x} + \left( \frac{\eta_0 \omega \gamma}{hh} \right) \frac{\partial \tau_{xy}}{\partial y}
\]

For \( \tau_{st} \) to be on the same order as \( \tau_{xy} \) is follows that:

\[
A = \frac{\eta_0 \omega \gamma}{hh} = \frac{\eta_0 \omega}{\varepsilon^2}
\]

In the case where viscoelastic and inertial effects are neglected, pressure must balance the viscous shear term such that:

\[
G = \frac{\eta_0 \omega}{\varepsilon^2}
\]

Using the 2\(^{nd}\) Maxwell Equation, find a scale for \( \tau_{xy} \) such that as \( \varepsilon \to 0 \), \( \tau_{xy} \) does not gain importance in the equation. There is no basis \( \tau_{xy} \) becoming important in this limit as it was found to be non-influential in the unidirectional case.

\[
\left( \frac{\lambda \omega^2 \eta_0}{\varepsilon} \right) \frac{\partial \tau_{xx}}{\partial t} + \left( \frac{\lambda \omega^2 \eta_0}{\varepsilon} \right) u_x \frac{\partial \tau_{xx}}{\partial x} + \left( \frac{\lambda \omega^2 \eta_0}{\varepsilon} \right) u_y \frac{\partial \tau_{xx}}{\partial y} - \left( \frac{\lambda \omega^2 \eta_0}{\varepsilon} \right) \tau_{xy} \frac{\partial u_x}{\partial x} \\
... - \left( \frac{\lambda \omega^2 \eta_0}{\varepsilon} \right) \tau_{xy} \frac{\partial u_y}{\partial y} - \left( \frac{\lambda \omega \eta_0 \omega}{\varepsilon^2} \right) \tau_{xy} \frac{\partial u_x}{\partial x} - \left( \frac{\lambda \omega \eta_0 \omega \varepsilon}{\varepsilon^2} \right) \tau_{xy} \frac{\partial u_y}{\partial x} = \left( \frac{\eta_0 \omega}{\varepsilon} \right) \frac{\partial u_x}{\partial y} \left( \eta_0 \omega \varepsilon \right) \frac{\partial u_y}{\partial x}
\]

\[
\lambda \omega \left[ \frac{\partial \tau_{xy}}{\partial t} + u_x \frac{\partial \tau_{xy}}{\partial x} + u_y \frac{\partial \tau_{xy}}{\partial y} - \tau_{xy} \frac{\partial u_x}{\partial y} - \tau_{xy} \frac{\partial u_y}{\partial x} \right] + \tau_{xy} = \frac{\partial u_x}{\partial y} + \left( \varepsilon^2 \right) \frac{\partial u_y}{\partial x}
\]

\[
C = \eta_0 \omega
\]

\[
D \left[ \frac{\partial \tau_{xy}}{\partial t} + u_x \frac{\partial \tau_{xy}}{\partial x} + u_y \frac{\partial \tau_{xy}}{\partial y} - \tau_{xy} \frac{\partial u_x}{\partial y} - \tau_{xy} \frac{\partial u_y}{\partial x} - \tau_{yy} \frac{\partial u_x}{\partial y} - \tau_{xx} \frac{\partial u_y}{\partial x} \right] + \tau_{xy} = \frac{\partial u_x}{\partial y} + \left( \varepsilon^2 \right) \frac{\partial u_y}{\partial x}
\]
All of the scales have been found, implement:

**Y-Momentum**

\[
\rho \left[ (\omega^2) \frac{\partial u_y}{\partial t} + \left( \frac{\omega^2 h}{\gamma} \right) u_x \frac{\partial u_y}{\partial x} + \left( \frac{\omega^2 h}{h} \right) u_y \frac{\partial u_y}{\partial y} \right] = -\left( \frac{\eta_0 \omega}{\epsilon^2 h} \right) \frac{\partial P}{\partial y} + \left( \frac{\eta_0 \omega}{\epsilon h} \right) \frac{\partial \tau_{xy}}{\partial x} + \left( \frac{\eta_0 \omega}{h} \right) \frac{\partial \tau_{yy}}{\partial y}
\]

\[
e^{-2} \frac{\rho \omega h^2}{\eta_0} \left[ \frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} \right] = -\frac{\partial P}{\partial y} + (\epsilon) \frac{\partial \tau_{xy}}{\partial x} + (\epsilon^2) \frac{\partial \tau_{yy}}{\partial y}
\]

1st Maxwell

\[
\lambda \left[ \left( \frac{\eta_0 \omega^2}{\epsilon^2} \right) \frac{\partial \tau_{xx}}{\partial t} + \left( \frac{\eta_0 \omega^2 \gamma}{\epsilon^2 \gamma} \right) u_x \frac{\partial \tau_{xx}}{\partial x} + \left( \frac{\eta_0 \omega^2 h}{\epsilon^2 h} \right) u_y \frac{\partial \tau_{xx}}{\partial y} - 2 \left( \frac{\eta_0 \omega^2 \gamma}{\epsilon^2 \gamma} \right) \tau_{xx} \frac{\partial u_x}{\partial x} + \left( \frac{\eta_0 \omega^2 \gamma}{\epsilon h} \right) \tau_{xx} \frac{\partial u_x}{\partial y} \right]...
\]

\[
+ \left( \frac{\eta_0 \omega}{\gamma} \right) \tau_{xx} = -2 \left( \frac{\eta_0 \omega}{\gamma} \right) \frac{\partial u_x}{\partial x}
\]

3rd Maxwell

\[
\lambda \left[ \frac{\partial \tau_{xy}}{\partial t} + u_x \frac{\partial \tau_{xy}}{\partial x} + u_y \frac{\partial \tau_{xy}}{\partial y} - 2 \left( \tau_{xy} \frac{\partial u_x}{\partial x} + \tau_{yy} \frac{\partial u_y}{\partial y} \right) \right] + \tau_{yy} = 2 \eta_0 \frac{\partial u_y}{\partial y}
\]

\[
\lambda \left[ \left( \frac{\eta_0 \omega^2}{\gamma} \right) \frac{\partial \tau_{yy}}{\partial t} + \left( \frac{\eta_0 \omega^2 \gamma}{\gamma} \right) u_x \frac{\partial \tau_{yy}}{\partial x} + \left( \frac{\eta_0 \omega^2 h}{h} \right) u_y \frac{\partial \tau_{yy}}{\partial y} - 2 \left( \frac{\eta_0 \omega^2 h}{h} \right) \tau_{yy} \frac{\partial u_y}{\partial x} + \left( \frac{\eta_0 \omega^2 h}{\epsilon} \right) \tau_{yy} \frac{\partial u_y}{\partial y} \right]...
\]

\[
+ \left( \frac{\eta_0 \omega h}{h} \right) \tau_{yy} = 2 \left( \frac{\eta_0 \omega h}{h} \right) \frac{\partial u_y}{\partial y}
\]

\[
\text{De} \left[ \frac{\partial \tau_{xx}}{\partial t} + u_x \frac{\partial \tau_{xx}}{\partial x} + u_y \frac{\partial \tau_{xx}}{\partial y} - 2 \left( \tau_{xx} \frac{\partial u_x}{\partial x} + \tau_{xy} \frac{\partial u_x}{\partial y} \right) \right] + \tau_{xy} = 2 \frac{\partial u_x}{\partial x}
\]
Full Set:

\[
De \left[ \frac{\partial \tau_{xx}}{\partial t} + u_x \frac{\partial \tau_{xx}}{\partial x} + u_y \frac{\partial \tau_{xx}}{\partial y} - 2 \left( \tau_{xx} \frac{\partial u_x}{\partial x} + \tau_{xy} \frac{\partial u_y}{\partial y} \right) \right] + \tau_{xx} = 2(\varepsilon^2)\frac{\partial u_x}{\partial x} \\
De \left[ \frac{\partial \tau_{yy}}{\partial t} + u_x \frac{\partial \tau_{yy}}{\partial x} + u_y \frac{\partial \tau_{yy}}{\partial y} - 2 \left( \tau_{yy} \frac{\partial u_y}{\partial y} + \tau_{xy} \frac{\partial u_x}{\partial x} \right) \right] + \tau_{xy} = \frac{\partial u_y}{\partial y} + (\varepsilon^2)\frac{\partial u_x}{\partial x} \\
De \left[ \frac{\partial \tau_{yy}}{\partial t} + u_x \frac{\partial \tau_{yy}}{\partial x} + u_y \frac{\partial \tau_{yy}}{\partial y} - 2 \left( \tau_{xx} \frac{\partial u_y}{\partial y} + \tau_{xy} \frac{\partial u_x}{\partial x} \right) \right] + \tau_{xy} = 2\frac{\partial u_x}{\partial y} \\
Wo^2 \left[ \frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} \right] = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} - g_x \varepsilon \\
\varepsilon^2 Wo^2 \left[ \frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} \right] = -\frac{\partial P}{\partial y} + (\varepsilon)\frac{\partial \tau_{xy}}{\partial x} + (\varepsilon^2)\frac{\partial \tau_{yy}}{\partial y} - g_y \varepsilon^2
\]

\[
De = \lambda \omega \\
Wo = h \left( \frac{\rho \omega}{\eta_0} \right) \sqrt{\frac{\rho \omega}{\eta_0}} \\
\varepsilon = \frac{\rho \omega}{\gamma_0}
\]

\[
\begin{align*}
y &= \bar{y}h \\
x &= \bar{x} \gamma \\
u_x &= \bar{u}_x \omega \gamma \\
u_y &= \bar{u}_y \omega h \\
\tau_{xx} &= \tau_{xx} \frac{\eta_0 \omega}{\varepsilon^2} \\
\tau_{xy} &= \tau_{xy} \frac{\eta_0 \omega}{\varepsilon} \\
\tau_{yy} &= \tau_{yy} \frac{\eta_0 \omega}{\varepsilon}
\end{align*}
\]

\[
\begin{align*}
g_x &= \bar{g} \cos \theta \\
g_y &= \bar{g} \sin \theta \\
\bar{g} &= \frac{\rho gh}{\eta_0 \omega}
\end{align*}
\]
Define a new Deborah number such that:

\[ \overline{De} = \frac{De}{\epsilon} \]

This will allow a perturbation series in both \( \overline{De} \) and \( \epsilon \) to be constructed.

\[
\epsilon \overline{De} \left[ \frac{\partial \tau_{xx}}{\partial t} + u_x \frac{\partial \tau_{xx}}{\partial x} + u_y \frac{\partial \tau_{xx}}{\partial y} - 2 \left( \tau_{xx} \frac{\partial u_x}{\partial x} + \tau_{xy} \frac{\partial u_y}{\partial y} \right) \right] + \tau_{xx} = 2(\epsilon^2) \frac{\partial u_x}{\partial x}
\]

\[
\epsilon \overline{De} \left[ \frac{\partial \tau_{xy}}{\partial t} + u_x \frac{\partial \tau_{xy}}{\partial x} + u_y \frac{\partial \tau_{xy}}{\partial y} - \tau_{xy} \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) - \tau_{yy} \frac{\partial u_x}{\partial y} - \tau_{xy} \frac{\partial u_y}{\partial x} + \tau_{yy} \right] + \tau_{xy} = \frac{\partial u_x}{\partial y} + (\epsilon^2) \frac{\partial u_y}{\partial x}
\]

\[
\epsilon^2 \overline{Wo}^2 \left[ \frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} \right] = - \frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} - \overline{g}_x
\]

\[
\epsilon^4 \overline{Wo}^2 \left[ \frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} \right] = - \frac{\partial P}{\partial y} + (\epsilon) \frac{\partial \tau_{xy}}{\partial x} + (\epsilon^2) \frac{\partial \tau_{yy}}{\partial y} - \overline{g}_y
\]

\[
\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0
\]
Perform Asymptotics:
We look at the solution of the above rescaled equations in the limit as \( \varepsilon \to 0 \), holding \( De \) and \( Wo \) fixed. In this limit, we expand dependent variables as follows:

\[
\begin{align*}
    u_x(x, y; \varepsilon, De, Wo) &\sim u_{x0}(x, y) + \varepsilon u_{x1}(x, y; De, Wo) + O(\varepsilon^2) \\
    u_y(x, y; \varepsilon, De, Wo) &\sim u_{y0}(x, y) + \varepsilon u_{y1}(x, y; De, Wo) + O(\varepsilon^2) \\
    \tau_{xx}(x, y; \varepsilon, De, Wo) &\sim \tau_{x0}(x, y) + \varepsilon \tau_{x1}(x, y; De, Wo) + O(\varepsilon^2) \\
    \tau_{xy}(x, y; \varepsilon, De, Wo) &\sim \tau_{y0}(x, y) + \varepsilon \tau_{y1}(x, y; De, Wo) + O(\varepsilon^2) \\
    \tau_{yy}(x, y; \varepsilon, De, Wo) &\sim \tau_{y0}(x, y) + \varepsilon \tau_{y1}(x, y; De, Wo) + O(\varepsilon^2) \\
    P(x, y; \varepsilon, De, Wo) &\sim P_0(x, y) + \varepsilon P_1(x, y; De, Wo) + O(\varepsilon^2)
\end{align*}
\]

Substituting into the differential equation system, and ordering terms, results in the following:

**O(1):**

\[
\begin{align*}
    \tau_{x0} &= 0 \\
    \tau_{y0} &= \frac{\partial u_{x0}}{\partial y} \\
    \tau_{y0} &= 2 \frac{\partial u_{y0}}{\partial y} \\
    0 &= -\frac{\partial P_0}{\partial x} + \frac{\partial \tau_{x0}}{\partial x} + \frac{\partial \tau_{y0}}{\partial y} \\
    \frac{\partial P_0}{\partial y} &= 0 \\
    \frac{\partial u_{x0}}{\partial x} + \frac{\partial u_{y0}}{\partial y} &= 0
\end{align*}
\]

**O(\varepsilon):**

\[
\begin{align*}
    \tau_{x1} &= -De \left[ \frac{\partial \tau_{x0}}{\partial t} + u_{x0} \frac{\partial \tau_{x0}}{\partial x} + u_{y0} \frac{\partial \tau_{x0}}{\partial y} - 2 \left( \tau_{x0} \frac{\partial u_{x0}}{\partial x} + \tau_{x0} \frac{\partial u_{y0}}{\partial y} \right) \right] \\
    \tau_{y1} &= \frac{\partial u_{x1}}{\partial y} - De \left[ \frac{\partial \tau_{y0}}{\partial t} + u_{x0} \frac{\partial \tau_{y0}}{\partial x} + u_{y0} \frac{\partial \tau_{y0}}{\partial y} - \tau_{y0} \frac{\partial u_{x0}}{\partial x} - \tau_{y0} \frac{\partial u_{y0}}{\partial y} - \tau_{x0} \frac{\partial u_{x0}}{\partial x} - \tau_{x0} \frac{\partial u_{x0}}{\partial y} \right] \\
    \tau_{y1} &= 2 \frac{\partial u_{y1}}{\partial y} - De \left[ \frac{\partial \tau_{y0}}{\partial t} + u_{x0} \frac{\partial \tau_{y0}}{\partial x} + u_{y0} \frac{\partial \tau_{y0}}{\partial y} - 2 \left( \tau_{y0} \frac{\partial u_{x0}}{\partial x} + \tau_{y0} \frac{\partial u_{y0}}{\partial y} \right) \right] \\
    0 &= -\frac{\partial P_1}{\partial x} + \frac{\partial \tau_{x1}}{\partial x} + \frac{\partial \tau_{y1}}{\partial y} \\
    \frac{\partial P_1}{\partial y} &= \frac{\partial \tau_{y0}}{\partial x} \\
    \frac{\partial u_{x1}}{\partial x} + \frac{\partial u_{y1}}{\partial y} &= 0
\end{align*}
\]
A.4 UCM Scaling and Perturbation in Wi using Stream Function

*The Deborah number of previous scalings has been replaced by the Weissenberg number (Wi). The Weissenberg number has the same role as the Deborah number, quantifying the importance of elastic effects in the flow. The difference between the two dimensional groups is the time scale which multiplies the relaxation time, a characteristic strain rate for the Weissenberg number and angular frequency for the Deborah number.

**Subscripts denote the order of the function in the Weissenberg number expansion.

***Superscripts denote the number of derivatives that have been taken with respect to a given variable (x,y,t).

\[ \tau = \frac{\tau}{T} \quad \bar{u} = u \frac{\eta_0}{Th} \quad \bar{v} = v \frac{\eta_0}{Th} \quad \bar{\psi} = \psi \frac{Th^2}{\eta_0} \quad Wi = \lambda \frac{T}{\eta_0} \]

\[ u_y = -\frac{\partial \psi}{\partial x} \quad u_x = \frac{\partial \psi}{\partial y} \]

\[ \text{Wi} \left[ \frac{\partial \tau_{xx}}{\partial t} + \frac{\partial \psi}{\partial y} \frac{\partial \tau_{xx}}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \tau_{xx}}{\partial y} - 2 \left( \tau_{xx} \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} + \tau_{xy} \frac{\partial \psi}{\partial y} \frac{\partial \psi}{\partial y} \right) \right] + \tau_{xx} = 2 \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} \]

\[ \text{Wi} \left[ \frac{\partial \tau_{xy}}{\partial t} + \frac{\partial \psi}{\partial y} \frac{\partial \tau_{xy}}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \tau_{xy}}{\partial y} - \tau_{xy} \frac{\partial \psi}{\partial y} \frac{\partial \psi}{\partial y} + \tau_{xx} \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} \right] + \tau_{xy} = \frac{\partial \psi}{\partial y} \frac{\partial \psi}{\partial x} \]

\[ \text{Wi} \left[ \frac{\partial \tau_{yy}}{\partial t} + \frac{\partial \psi}{\partial y} \frac{\partial \tau_{yy}}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \tau_{yy}}{\partial y} + 2 \left( \tau_{xx} \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} + \tau_{xy} \frac{\partial \psi}{\partial y} \frac{\partial \psi}{\partial y} \right) \right] + \tau_{yy} = -2 \frac{\partial \psi}{\partial y} \frac{\partial \psi}{\partial x} \]

\[ 0 = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xx}}{\partial y} \]

\[ 0 = -\frac{\partial P}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \]

\[ \tau_{xx}(x, y; \text{Wi}) \sim \tau_{xx0} + \text{Wi} \tau_{xx1} + \text{Wi}^2 \tau_{xx2} \]

\[ \tau_{xy}(x, y; \text{Wi}) \sim \tau_{xy0} + \text{Wi} \tau_{xy1} + \text{Wi}^2 \tau_{xy2} \]

\[ \tau_{yy}(x, y; \text{Wi}) \sim \tau_{yy0} + \text{Wi} \tau_{yy1} + \text{Wi}^2 \tau_{yy2} \]

\[ \psi(x, y; \text{Wi}) \sim \psi_0 + \text{Wi} \psi_1 + \text{Wi}^2 \psi_2 \]

Order 1

\[ \tau_{xx0} = 2 \psi_0^{1,1,0} \]

\[ \tau_{xy0} = \psi_0^{0,2,0} - \psi_0^{3,0,0} \]

\[ \tau_{yy0} = -2 \psi_0^{1,1,0} \]
\[ \nabla^4 \psi_\varphi = 0 \]

**Order Wi**

\[
\tau_{\varphi x} = 2 \psi_1^{1,1,0} + 2 \left[ -\psi_0^{1,1,1} + (\psi_0^{0,2,0})^2 + 2(\psi_0^{1,1,0})^2 + \psi_0^{1,0,0} \psi_0^{1,2,0} - \psi_0^{2,0,0} \psi_0^{0,2,0} - \psi_0^{0,1,0} \psi_0^{2,1,0} \right]
\]

\[
\tau_{\varphi y} = (\psi_1^{0,2,0} - \psi_0^{2,0,0}) + (\psi_0^{1,0,1} - \psi_0^{0,2,1}) - (\psi_0^{1,0,0} \psi_0^{2,1,0} + \psi_0^{0,1,0} \psi_0^{1,2,0}) - 2 \psi_0^{1,1,0} (\psi_0^{0,2,0} + \psi_0^{2,0,0})...
\]

\[
\tau_{\varphi y} = -2 \psi_1^{1,1,0} + 2 \left[ -\psi_0^{1,1,1} + (\psi_0^{0,2,0})^2 + 2(\psi_0^{1,1,0})^2 - \psi_0^{1,0,0} \psi_0^{1,2,0} - \psi_0^{2,0,0} \psi_0^{2,0,0} + \psi_0^{0,1,0} \psi_0^{2,1,0} \right]
\]

\[ \nabla^2 \psi_1 = \Omega(x, y, t) \]

\[ \Omega(x, y, t) = (\psi_0^{0,4,1} + \psi_0^{4,0,1}) + (\psi_0^{1,0,0} \psi_0^{5,0,0} - \psi_0^{0,5,0} \psi_0^{1,0,0}) + (\psi_0^{0,1,0} \psi_0^{1,4,0} - \psi_0^{1,0,0} \psi_0^{4,1,0})...
\]

\[ + 2(\psi_0^{2,2,1} - \psi_0^{1,0,0} \psi_0^{2,3,0} + \psi_0^{0,1,0} \psi_0^{3,2,0}) \]
Appendix B Derivations for Chapter 1.4: Steady State Problems

B.1 Mucus as a Rigid Body

Figure B.1: Schematic for steady state model, mucus is modeled as a rigid body.

Flow Profile of periciliary layer:

\[ u_{PCL} = \frac{\rho_{PCL} g}{\eta_{PCL}} y^2 + \left( \frac{V}{h_{PCL}} - \frac{\rho_{PCL} g (h_{PCL})}{2 \eta_{PCL}} \right) y \]  

(B.1)

Shear at the interface between the periciliary layer and mucus slab:

\[ \tau_{PCL} = \rho_{PCL} g y + \left( \frac{\eta_{PCL} V}{h_{PCL}} - \frac{\rho_{PCL} g (h_{PCL})}{2} \right) \]

\[ \tau_{PCL} (y = h_{PCL}) = \left( \frac{\eta_{PCL} V}{h_{PCL}} + \frac{\rho_{PCL} g (h_{PCL})}{2} \right) \]  

(B.2)

Force balance on slab:

\[ \sum_{A} F_y = \sigma_{cilia} - \rho_M g h_M - \left( \frac{\eta_{PCL} V}{h_{PCL}} + \frac{\rho_{PCL} g (h_{PCL})}{2} \right) = 0 \]  

(B.4)

The force per unit area required to maintain steady-state transport of the slab is \( \sigma_{cilia} \).

Letting the densities of the two bodies be equal and solve for mucus velocity:

\[ V = -\frac{\rho g h_{PCL}}{\eta_{PCL}} \left( \frac{h_{PCL}}{2} + h_M \right) + \frac{\sigma_{cilia} h_{PCL}}{\eta_{PCL}} = 0 \]  

(B.5)

And, solve for cilia force per unit area:
\[ \sigma_{cilia} = \frac{\eta_{PCL} V}{h_{PCL}} + \rho g \left( \frac{h_{PCL}}{2} + h_M \right) \]  

(B.6)

### B.2 Mucus as a Viscous Fluid

![Diagram of Mucus as a Viscous Fluid](image)

Figure B.2: Left: Schematic of Mucus as a viscous fluid body problem. Right: Free body diagram of Mucus-PCL Interface

Find velocity profile of mucus layer:

\[
0 = \frac{d^2 u_M}{dy^2} \eta_M - \rho_M g
\]

\[ u_M = \frac{\rho_M g}{2\eta_M} y^2 + Ay + B \]  

(B.7)

Subject to conditions:

The stress free mucus-air interface:

\[ \tau_M \bigg|_{y=h_M} = 0 \]  

(B.8)

Prescribed average flow velocity:

\[
\frac{\int_0^{h_M} u_M \, dy}{h_M} = V_M
\]  

(B.9)
\[ u_M = \frac{\rho_M g}{\eta_M} \left( \frac{y^2}{2} - h_M y + \frac{h_M^2}{3} \right) + V_M \]  
\tag{B.10}

The velocity and stress at the PCL-mucus interface:
\[ u_M (0) = \frac{\rho_M g}{\eta_M} \left( \frac{h_M^2}{3} \right) + V_M \]  
\tag{B.11}
\[ \tau_M (0) = -\rho_M g h_M \]  
\tag{B.12}

Find velocity profile of periciliary layer:
\[ 0 = \frac{d^2 u_{PCL}}{dy_{PCL}^2} \eta_{PCL} - \rho_M g \]  
\tag{B.13}
\[ u_{PCL} = \frac{\rho_M g}{2\eta_{PCL}} y_{PCL}^2 + Ay_{PCL} + B \]

Subject to conditions:

No slip at top and bottom of periciliary fluid (velocity continuity between the two layers)
\[ u_{PCL} (0) = 0 \]  
\tag{B.14}
\[ u_{PCL} (h_{PCL}) = u_M (0) \]  
\tag{B.15}

\[ u_{PCL} = \frac{\rho_{PCL} g}{\eta_{PCL}} \left[ \frac{1}{2} y_{PCL}^2 + \frac{\eta_{PCL} \rho_M}{\eta_M \rho_{PCL}} \left( \frac{h_M^2}{3h_{PCL}} \right) y_{PCL} - \frac{1}{2} h_{PCL} y_{PCL} \right] + \frac{V_M y_{PCL}}{h_{PCL}} \]  
\tag{B.16}

Force balance at PCL-Mucus Interface:
\[ \tau_{PCL} (h_{PCL}) = \rho_{PCL} g h_{PCL} \left[ \frac{1}{2} \eta_{PCL} \rho_M \left( \frac{h_M}{h_{PCL}} \right)^2 + \frac{1}{2} \right] + \frac{\eta_{PCL} V_M}{h_{PCL}} \]  
\tag{B.17}
\[ \tau_M (0) = -\rho_M g h_M \]  
\tag{B.18}

\[ \sum F_y = 0 = \sigma W + WL \left[ -\tau_{PCL} (h_{PCL}) + \tau_M (0) \right] \]  
\tag{B.19}
\[
\frac{\sigma}{L} = [\tau_{p_{CL}}(h_{p_{CL}}) - \tau_{M}(0)] \quad \text{(B.20)}
\]

\[
T = [\tau_{p_{CL}}(h_{p_{CL}}) - \tau_{M}(0)]
\]

*\(\sigma\) has units of force per unit length, \(T\) has units of force per unit area

\[
T = \rho_{p_{CL}}gh_{p_{CL}} \left[ \frac{1}{3} \eta_{p_{CL}} \rho_{M} \left( \frac{h_{M}}{h_{p_{CL}}} \right)^2 + \frac{1}{2} \right] + \frac{\eta_{p_{CL}}V_{M}}{h_{p_{CL}}} + \rho_{M}gh_{M} \quad \text{(B.21)}
\]

\[
\rho_{p_{CL}} \approx \rho_{M} \quad \text{(B.22)}
\]

\[
T = \rho gh_{p_{CL}} \left[ \frac{1}{3} \left( \frac{h_{M}}{h_{p_{CL}}} \right)^2 \left( \frac{\eta_{p_{CL}}}{\eta_{M}} \right) + \left( \frac{h_{M}}{h_{p_{CL}}} \right) + \frac{1}{2} \right] + \frac{\eta_{p_{CL}}V_{M}}{h_{p_{CL}}} \quad \text{(B.23)}
\]

\[
V_{M} = \frac{Th_{p_{CL}}}{\eta_{p_{CL}}} - \frac{\rho gh_{p_{CL}}}{\eta_{p_{CL}}} \left[ \frac{1}{3} \left( \frac{h_{M}}{h_{p_{CL}}} \right)^2 \left( \frac{\eta_{p_{CL}}}{\eta_{M}} \right) + \left( \frac{h_{M}}{h_{p_{CL}}} \right) + \frac{1}{2} \right] \quad \text{(B.24)}
\]
Appendix C Derivations for 3.2: Response of Maxwell Model to Sudden Perturbations

C.1 Step Stress Input

Sudden application of stress $T$

$$\tau = u(t)T$$ (C.1)

$$\frac{d\tau}{dt} = \delta(t)T$$ (C.2)

Apply to the linear Maxwell constitutive equation

$$T[\dot{\lambda} \delta(t) + u(t)] = \eta_0 \dot{\gamma}$$ (C.3)

Where $u(t)$ and $\delta(t)$ are the Heaviside step and Dirac Delta functions, respectively.

Convert to the Laplace domain:

$$\mathcal{L}\{\dot{\gamma}(t)\} = \Gamma(s)$$ (C.4)

$$\mathcal{L}\{\dot{\gamma}(t)\} \Rightarrow T\left[\frac{\dot{\lambda}}{s} + \frac{1}{s}\right] = \eta_0 (s\Gamma - \gamma(0))$$ (C.5)

Solve for $\Gamma$

$$\Gamma = \frac{T}{\eta_0} \left(\frac{\dot{\lambda}}{s} + \frac{1}{s^2}\right)$$ (C.6)

Invert Transform

$$\mathcal{L}^{-1}\{\Gamma(s)\} = \gamma(t)$$ (C.7)

$$\gamma = \frac{T}{\eta_0} (\dot{\lambda}u(t) + tu(t))$$ (C.8)

Following directly from C.2:

$$\dot{\gamma} = \frac{T}{\eta_0} (\dot{\lambda}\delta(t) + u(t))$$ (C.9)
<table>
<thead>
<tr>
<th>Strain Rate (t&gt;0):</th>
<th>Strain(t&gt;0):</th>
<th>Stress(t&gt;0):</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \dot{\gamma}_H = 0 ) (C.10)</td>
<td>( \gamma_H = \frac{T}{G} ) (C.11)</td>
<td>( \tau_H = \tau_N = \tau ) (A.2)</td>
</tr>
<tr>
<td>( \dot{\gamma}_N = \frac{T}{\eta_0} ) (C.12)</td>
<td>( \gamma_N = \frac{T}{\eta_0} - t ) (C.13)</td>
<td>( \tau_H = \tau_N = \tau ) (A.2)</td>
</tr>
<tr>
<td>( \dot{\gamma} = \dot{\gamma}_H + \dot{\gamma}_N ) (A.6)</td>
<td>( \dot{\gamma} = \gamma_H + \gamma_N ) (A.5)</td>
<td>( \tau = Tu(t) ) (C.2)</td>
</tr>
</tbody>
</table>

### C.2 Step Strain Input

Sudden application of strain \( \textit{capital gamma} \) to the linear Maxwell constitutive equation

\[
\lambda \frac{\partial \tau}{\partial t} + \tau = \eta_0 \dot{\gamma}
\]

\[
\lambda \tau + \int_0^t \tau(t') dt' = \eta_0 \dot{\gamma}
\]

Define the strain as:

\[
\gamma = u(t) \Gamma
\]

where \( u(t) \) is the Heaviside step function and convert to the Laplace domain

\[
\mathcal{L}\{\tau(t)\} = T(s)
\]

\[
\mathcal{L}\{\ldots\} \Rightarrow \lambda T + \frac{1}{s} T = \eta_0 \frac{1}{s} \Gamma
\]

Solve for \( T \)

\[
T = \frac{\eta_0 \Gamma}{\lambda} \frac{1}{s + \frac{1}{\lambda}}
\]

Invert Transform

\[
\mathcal{L}^{-1}\{T(s)\} = \tau(t)
\]

\[
\tau = \frac{\eta_0 \Gamma}{\lambda} e^{-\frac{t}{\lambda}}
\]
### C.3 Step Strain-Rate Input

Define the strain rate as:
\[ \dot{\gamma} = u(t) \Gamma \]
where \( u(t) \) is the Heaviside step function and convert to the Laplace domain

\[ \mathcal{L}\{ \tau(t) \} = T(s) \]
\[ \mathcal{L}\{ \tau(t) \} \Rightarrow \alpha(sT - \tau(0)) + T = \eta_0 \frac{1}{s} \dot{\gamma} \]

Solve for \( T \)
\[ T = \eta_0 \frac{1}{s(\alpha s + 1)} \dot{\gamma} \]

Invert Transform
\[ \mathcal{L}^{-1}\{ T(s) \} = \tau(t) \]
\[ \tau = \eta_0 \dot{\gamma} \left( 1 - e^{-\frac{t}{\lambda}} \right) \]

<table>
<thead>
<tr>
<th>Strain Rate ((t &gt; 0)):</th>
<th>Strain ((t &gt; 0)):</th>
<th>Stress ((t &gt; 0)):</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \dot{\gamma}_H = -\frac{\Gamma}{\lambda} e^{\frac{-t}{\lambda}} ) (C.14)</td>
<td>( \gamma_H = \Gamma e^{\frac{-t}{\lambda}} ) (C.15)</td>
<td>( \tau_H = \tau_N = \tau ) (A.2)</td>
</tr>
<tr>
<td>( \dot{\gamma}_N = \Gamma e^{\frac{-t}{\lambda}} ) (C.16)</td>
<td>( \gamma_N = \Gamma \left( 1 - e^{\frac{-t}{\lambda}} \right) ) (C.17)</td>
<td>( \tau_H = \tau_N = \tau ) (A.2)</td>
</tr>
<tr>
<td>( \dot{\gamma} = \dot{\gamma}_H + \dot{\gamma}_N ) (A.6)</td>
<td>( \gamma = \gamma_H + \gamma_N ) (A.5)</td>
<td>( \tau = \eta_0 \frac{\Gamma}{\lambda} e^{\frac{-t}{\lambda}} ) (C.18)</td>
</tr>
</tbody>
</table>
Appendix D Derivations for 3.3: Response of Maxwell Model to Oscillatory Perturbations

D.1 Non-Dimensionalized Governing Equations

Conservation of momentum:
\[
\rho \frac{\partial u}{\partial t} = \frac{\partial \tau_{yx}}{\partial y}
\]  
(D.1)

Linear Maxwell Equation
\[
\lambda \frac{\partial \tau_{yx}}{\partial t} + \tau_{xx} = \eta_0 \frac{\partial u}{\partial y}
\]  
(D.2)

Assume Forms for solution:
\[ u = \text{Re}\{u^0 e^{i\omega t}\} \]  
(D.3)
\[ \tau_{yx} = \text{Re}\{\tau_{yx}^0 e^{i\omega t}\} \]  
(D.4)

Substitute:
\[
\rho \frac{\partial u^0 e^{i\omega t}}{\partial t} = e^{i\omega t} \frac{\partial \tau_{yx}^0}{\partial y}
\]  
(D.5)
\[
\lambda \frac{\partial \tau_{yx}^0 e^{i\omega t}}{\partial t} + \tau_{yx}^0 e^{i\omega t} = \eta_0 \frac{\partial u^0 e^{i\omega t}}{\partial y}
\]  
(D.6)

Simplify:
\[
\rho u^0 i\omega = \frac{\partial \tau_{yx}^0}{\partial y}
\]  
(D.7)
\[
\lambda \tau_{yx}^0 i\omega + \tau_{yx}^0 = \eta_0 \frac{\partial u^0}{\partial y}
\]  
(D.8)

Combine Equations
\[
\rho u^0 i\omega = \frac{\partial \tau_{yx}^0}{\partial y}
\]
\[
\frac{\rho}{\eta_0} u^0 (i - \lambda \omega) = \frac{\partial^2 u^0}{\partial y^2}
\]  
(D.9)
Non-dimensionalized with respect to gap width $h$ and imposed velocity amplitude $U$

\[
\frac{\rho w U}{\eta_0} \bar{u}^0(i - \lambda w) = \frac{U}{h^2} \frac{\partial^2 \bar{u}^0}{\partial y^2}
\]

\[
\frac{\rho w h^2}{\eta_0} \bar{u}^0(i - \lambda w) = \frac{\partial^2 \bar{u}^0}{\partial y^2}
\]  

(D.10)

Two non-dimensional groups are now present: the Womersley number $Wo$ and the Deborah number $De$, which describe the relative importance of inertia and viscosity, and elasticity, respectively. (The over bars indicating non-dimensional variables have been dropped)

\[
Wo^2 u^0(i - De) = \frac{\partial^2 u^0}{\partial y^2}
\]  

(D.11)

The Eigen values for the problem are thus

\[
m^2 = Wo^2 (i - De)
\]  

(D.12)

**D.2 Complex Roots**

\[
m^2 = Wo^2 (-De + i) = -\delta + \gamma
\]  

(D.13)

Such that:

\[
\delta = DeWo^2
\]

\[
\gamma = Wo^2
\]  

(D.14)

Let:

\[
m = re^{i\phi} = \alpha + \beta i
\]  

(D.15)

Substitute [3] back into definition of $m^2$ [1]:

\[
r^2 e^{2i\phi} = -\delta + \gamma
\]  

(D.16)

\[
i = e^{\frac{i\pi}{2 + 2n\pi}}
\]

\[
1 = e^{2n\pi i}
\]

\[
\ldots
\]
apply Euler's Identity:
\[ r^2 e^{2i\phi} = -\delta e^{2n\pi} + \gamma e^{\left(\frac{\pi}{2} + 2n\pi\right)} \]

apply Euler's Identity:
\[ r^2 [\cos(2\phi) + i\sin(2\phi)] = ... \]
\[ ... - \delta [\cos(2n\pi) + i\sin(2n\pi)] + \gamma \left[ \cos\left(\frac{\pi}{2} + 2n\pi\right) + i\sin\left(\frac{\pi}{2} + 2n\pi\right) \right] \] (D.17)
equating respective real and imaginary parts, respectively:
\[ r^2 \cos(2\phi) = -\delta \cos(2n\pi) + \gamma \cos\left(\frac{\pi}{2} + 2n\pi\right) = -\delta \] (D.18)
\[ r^2 \sin(2\phi) = -\delta \sin(2n\pi) + \gamma \sin\left(\frac{\pi}{2} + 2n\pi\right) = \gamma \] (D.19)
dividing [6b] by [6a] and solve for \( \phi \):
\[ \phi = \frac{\pi}{2} (2n + 1) - \frac{1}{2} \arctan\left(\frac{\gamma}{\delta}\right) \quad n = 0, 1, 2, ... \] (D.20)

\[ n = 0... \quad n = 1... \quad n = 2... \phi_0 = \phi_2 \]
\[ \phi = \frac{\pi}{2} - \frac{1}{2} \arctan\left(\frac{\gamma}{\delta}\right) \quad \phi = \frac{3\pi}{2} - \frac{1}{2} \arctan\left(\frac{\gamma}{\delta}\right) \quad \phi = \frac{5\pi}{2} - \frac{1}{2} \arctan\left(\frac{\gamma}{\delta}\right) \]

Taking advantage of the trigonometric identity and the definitions of \( \delta \) and \( \gamma \), add the squares of [6a] and [6b] to find \( r \):
\[ r = \text{Wol}[1 + (De)^2]^{\frac{1}{2}} \] (D.21)

Substitute [8] and [7] back into definition of \( m \) [3] to find \( \alpha \) and \( \beta \):
\[ m = re^{\left[\frac{\pi}{2} (2n + 1) - \frac{1}{2} \arctan\left(\frac{\gamma}{\delta}\right)\right]} \quad n = 0, 1 \] (D.22)

apply Euler's Identity:
\[ m = r \left[ \cos\left(\frac{\pi}{2} (2n + 1) - \frac{1}{2} \arctan\left(\frac{\gamma}{\delta}\right)\right) + i \sin\left(\frac{\pi}{2} (2n + 1) - \frac{1}{2} \arctan\left(\frac{\gamma}{\delta}\right)\right) \right] \quad n = 0, 1 \] (D.23)
In the context of the Ferry Shear wave problem, the pair of Eigen values describe the attenuation and phase shift of waves traveling from the source and those reflected off of the boundary wall (if it is present) traveling towards the source of the disturbance, the oscillating wall.

### D.3 Oscillatory Velocity, No-Slip

**Apply Boundary Conditions to General Solution**

\[ u^0 = C_1 e^{(\alpha+i\beta)y} + C_2 e^{-(\alpha+i\beta)y} \]

Boundary Conditions:

\[ u^0(0) = 1 \]
\[ u^0(1) = 0 \]

\[ C_1 + C_2 = 1 \]
\[ 0 = C_1 e^{(\alpha+i\beta)} + C_2 e^{-(\alpha+i\beta)} \]
\[ 0 = (1 - C_2) e^{(\alpha+i\beta)} + C_2 e^{-(\alpha+i\beta)} \]

\[ C_2 = \frac{-e^{(\alpha+i\beta)}}{e^{-(\alpha+i\beta)} - e^{(\alpha+i\beta)}} \]

Separating \( C_2 \) into Real and Imaginary parts and replacing exponentials with hyperbolic functions:

\[ C_2 = \frac{-e^{\alpha}(\cos(\beta) + i\sin(\beta))}{e^{-\alpha}(\cos(\beta) - i\sin(\beta)) - e^{\alpha}(\cos(\beta) + i\sin(\beta))} \]

\[ C_2 = \frac{-e^{\alpha}(\cos(\beta) + i\sin(\beta))}{e^{-\alpha}\cos(\beta) - ie^{-\alpha}\sin(\beta) - e^{\alpha}\cos(\beta) - ie^{\alpha}\sin(\beta)} \]
\[ C_2 = \frac{-e^{\alpha}(\cos(\beta) + i\sin(\beta))}{\cos(\beta)(e^{-\alpha} - e^{\alpha}) - i\sin(\beta)(ie^{\alpha} + ie^{-\alpha})} \]

\[ C_2 = \frac{e^{\alpha}}{2} \frac{\cos(\beta) + i\sin(\beta)}{\cos(\beta)\sinh(\alpha) + i\sin(\beta)\cosh(\alpha)} \]

\[ C_2 = \frac{e^{\alpha}}{2} \frac{\cos(\beta) + i\sin(\beta)}{\text{Conj}(\text{Denom})} \]

\[ C_2 = \frac{e^{\alpha}}{2} \frac{[\cos(\beta) + i\sin(\beta)](\cos(\beta)\sinh(\alpha) - i\sin(\beta)\cosh(\alpha))}{[\cos^2(\beta)\sinh^2(\alpha) + \sin^2(\beta)\cosh^2(\alpha)]} \]

\[ [\cos(\beta) + i\sin(\beta)](\cos(\beta)\sinh(\alpha) - i\sin(\beta)\cosh(\alpha)) = \]

\[ \begin{align*}
1. \cos(\beta)\cos(\beta)\sinh(\alpha) &= \cos^2(\beta)\sinh(\alpha) \\
2. \cos(\beta)(-i)\sin(\beta)\cosh(\alpha) &= (-i)\cos(\beta)\sin(\beta)\cosh(\alpha) \\
3. i\sin(\beta)\cos(\beta)\sinh(\alpha) &= (i)\sin(\beta)\cos(\beta)\sinh(\alpha) \\
4. i\sin(\beta)(-i)\sin(\beta)\cosh(\alpha) &= \sin^2(\beta)\cosh(\alpha)
\end{align*} \]

\[ C_2 = \Pi_R + \Pi_Z i \]

\[ \Pi_R = \left( \frac{e^{\alpha}}{2} \right) \frac{\sin^2(\beta)\cosh(\alpha) + \cos(\beta)\sinh(\alpha)}{\sinh^2(\beta)\cos^2(\alpha) + \cosh(\beta)\sin^2(\alpha)} \]

\[ \Pi_Z = \left( \frac{e^{\alpha}}{2} \right) \frac{\cos(\beta)\sin(\beta)[\sinh(\alpha) - \cosh(\alpha)]}{\sinh^2(\beta)\cos^2(\alpha) + \cosh(\beta)\sin^2(\alpha)} \]

Convert \( C_2 \) into polar form:

\[ C_2 = \Pi_R + \Pi_Z i = \zeta e^{i\phi} \]

\[ \zeta = \sqrt{(\Pi_R)^2 + (\Pi_Z)^2} \]

\[ \phi = \arctan \left( \frac{\Pi_Z}{\Pi_R} \right) \]

\[ C_1 = 1 - C_2 \]

\[ C_1 = 1 - \zeta e^{i\phi} \]

Plug coefficients back into General Solution
and multiply out to isolate real and imaginary parts:

\[ u^0 = (1 - \zeta e^{i\phi})e^{(\alpha + i\beta)y} + (\zeta e^{i\phi})e^{-(\alpha + i\beta)y} \]

\[ u = \text{Re}\{u^0 e^{it}\} \]
\[ u = (1 - \zeta e^{i\phi}) e^{(\alpha + \beta) t} e^{i\phi} + (\zeta e^{i\phi}) e^{-(\alpha + \beta) t} e^{i\phi} \]
\[ u = e^{\alpha t + i(\alpha + \beta) t} + \zeta e^{-(\alpha + i(\alpha + \phi - \beta))} - e^{\alpha + i(\alpha + \phi - \beta)} \]

1. \( V e^{\alpha t + i(\alpha + \beta) t} = V e^{\alpha t} [\cos(\alpha t + \beta y) + i \sin(\alpha t + \beta y)] \)
2. \( \zeta e^{-\alpha t + i(\alpha + \phi - \beta) t} = \zeta e^{-\alpha t} [\cos(\alpha t + \phi - \beta y) + i \sin(\alpha t + \phi - \beta y)] \)
3. \( -\zeta e^{\alpha t + i(\alpha + \phi - \beta) t} = -\zeta e^{\alpha t} [\cos(\alpha t + \phi + \beta y) + i \sin(\alpha t + \phi + \beta y)] \)

Take the real parts only:

\[ u = e^{\alpha t} \cos(t + \beta y) + \zeta e^{-\alpha t} [\cos(t + \phi - \beta y) - \zeta e^{\alpha t} \cos(t + \phi + \beta y)] \]

Using identity:
\[ \cos(u \pm v) = \cos(u) \cos(v) \mp \sin(u) \sin(v) \]
\[ u = t \]
\[ v = \phi - \beta y \text{ or } \beta y \text{ (for first term)} \]

1. \( \cos(t + \beta y) = \cos(t) \cos(\beta y) - \sin(t) \sin(\beta y) \)
2. \( \cos(t + \phi - \beta y) = \cos(t) \cos(\phi - \beta y) - \sin(t) \sin(\phi - \beta y) \)
3. \( \cos(t + \phi + \beta y) = \cos(t) \cos(\phi + \beta y) - \sin(t) \sin(\phi + \beta y) \)

\[ u = e^{\alpha t} \cos(t) \cos(\beta y) - e^{\alpha t} \sin(t) \sin(\beta y) + \zeta \left[ e^{-\alpha t} \cos(t) \cos(\phi - \beta y) - e^{-\alpha t} \sin(t) \sin(\phi - \beta y) \ldots \right] \]
\[ \ldots - e^{\alpha t} \cos(t) \cos(\phi + \beta y) + e^{\alpha t} \sin(t) \sin(\phi + \beta y) \]

\[ u = \cos(t) \left[ e^{\alpha t} \cos(\beta y) + \zeta e^{-\alpha t} \cos(\phi - \beta y) - \zeta e^{\alpha t} \cos(\phi + \beta y) \right] + \ldots \]
\[ \ldots \sin(t) \left[ - e^{\alpha t} \sin(\beta y) - \zeta e^{-\alpha t} \sin(\phi - \beta y) + \zeta e^{\alpha t} \sin(\phi + \beta y) \right] \]

Assume form for \( u \):

\[ u = \Lambda \cos(t + \Theta) \]
\[ u = \Lambda \cos(t) \cos(\Theta) - \Lambda \sin(t) \sin(\Theta) \]
\[ u = \cos(t) \left[ e^{\alpha t} \cos(\beta y) + \zeta e^{-\alpha t} \cos(\phi - \beta y) - \zeta e^{\alpha t} \cos(\phi + \beta y) \right] + \ldots \]
\[ \ldots \sin(t) \left[ - e^{\alpha t} \sin(\beta y) - \zeta e^{-\alpha t} \sin(\phi - \beta y) + \zeta e^{\alpha t} \sin(\phi + \beta y) \right] \]

\[ \Lambda \cos(\Theta) = e^{\alpha t} \cos(\beta y) + \zeta e^{-\alpha t} \cos(\phi - \beta y) - \zeta e^{\alpha t} \cos(\phi + \beta y) = \Pi_1 \]
\[ -\Lambda \sin(\Theta) = -e^{\alpha t} \sin(\beta y) - \zeta e^{-\alpha t} \sin(\phi - \beta y) + \zeta e^{\alpha t} \sin(\phi + \beta y) = \Pi_2 \]

\[ \Lambda = \sqrt{(\Pi_1)^2 + (\Pi_2)^2} \]
\[ \Theta = \arctan \left( \frac{\Pi_2}{\Pi_1} \right) \]
Solution with all the parts:

\[ u = \Lambda \cos(wt + \Theta) \]

\[ \Lambda = \sqrt{\left(\Pi_1 \right)^2 + \left(\Pi_2 \right)^2} \]

\[ \Theta = \arctan\left(\frac{-\Pi_2}{\Pi_1}\right) \]

\[ \Pi_1 = e^{\alpha y} \cos(\beta y) + \zeta e^{-\alpha y} \cos(\varphi - \beta y) - \zeta e^{\alpha y} \cos(\varphi + \beta y) \]

\[ \Pi_2 = -e^{\alpha y} \sin(\beta y) - \zeta e^{-\alpha y} \sin(\varphi - \beta y) + \zeta e^{\alpha y} \sin(\varphi + \beta y) \]

\[ \zeta = \sqrt{\left(\Pi_R \right)^2 + \left(\Pi_Z \right)^2} \]

\[ \varphi = \arctan\left(\frac{\Pi_Z}{\Pi_R}\right) \]

\[ \Pi_R = \left(\frac{e^{\alpha}}{2}\right) \frac{\sin^2(\beta) \cosh(\alpha) + \cos(\beta) \sinh(\alpha)}{\sinh^2(\beta) \cos^2(\alpha) + \cosh(\beta) \sin^2(\alpha)} \]

\[ \Pi_Z = \left(\frac{e^{\alpha}}{2}\right) \frac{\cos(\beta) \sin(\beta) [\sinh(\alpha) - \cosh(\alpha)]}{\sinh^2(\beta) \cos^2(\alpha) + \cosh(\beta) \sin^2(\alpha)} \]

\[ \alpha = r \cos\left(\frac{\pi}{2} - \frac{1}{2} \arctan\left(\frac{1}{De}\right)\right) \]

\[ \beta = r \sin\left(\frac{\pi}{2} - \frac{1}{2} \arctan\left(\frac{1}{De}\right)\right) \]

\[ r = Wo \left[1 + (De)^2\right]^{\frac{1}{2}} \]
D.4 Oscillatory Velocity, Slip

\[ u^0 = C_1 e^{(\alpha+\beta)y} + C_2 e^{-(\alpha+\beta)y} \]

Boundary Conditions:

\[ u^0(0) = 1 \]
\[ u^0(1) = 0 \]

\[ C_1 + C_2 = 1 \]
\[ 0 = C_1 e^{(\alpha+\beta)} + C_2 e^{-(\alpha+\beta)} \]
\[ 0 = (1-C_2) e^{(\alpha+\beta)} + C_2 e^{-(\alpha+\beta)} \]
\[ C_2 = \frac{-e^{(\alpha+\beta)}}{e^{-(\alpha+\beta)} - e^{(\alpha+\beta)}} \]

Separating \( C_2 \) into Real and Imaginary parts and replacing exponentials with hyperbolic functions:

\[ C_2 = \frac{-e^{(\alpha+\beta)}}{e^{-(\alpha+\beta)} - e^{(\alpha+\beta)}} \]

\[ C_2 = \frac{-e^a (\cos(\beta)+i\sin(\beta))}{e^{-a} (\cos(\beta)-i\sin(\beta)) - e^a (\cos(\beta)+i\sin(\beta))} \]

\[ C_2 = \frac{-e^a (\cos(\beta)+i\sin(\beta))}{e^{-a} \cos(\beta)-ie^{-a} \sin(\beta)-e^a \cos(\beta)-ie^a \sin(\beta)} \]

\[ C_2 = \frac{e^a (\cos(\beta)+i\sin(\beta))}{\cos(\beta)(e^{-a}-e^a)-i\sin(\beta)(ie^a+iie^{-a})} \]

\[ C_2 = \frac{e^a}{2 \cos(\beta)\sinh(\alpha)+i\sin(\beta)\cosh(\alpha)} \]

\[ C_2 = \frac{e^a}{2 \left[ \cos(\beta)\sinh(\alpha)+i\sin(\beta)\cosh(\alpha) \right]} \]

\[ C_2 = \frac{e^a}{2} \left[ \cos(\beta)+i\sin(\beta) \left[ \cos(\beta)\sinh(\alpha)-i\sin(\beta)\cosh(\alpha) \right] \right] \]

\[ \left[ \cos(\beta)+i\sin(\beta) \left[ \cos(\beta)\sinh(\alpha)-i\sin(\beta)\cosh(\alpha) \right] \right] = \]

1. \( \cos(\beta)\cos(\beta)\sinh(\alpha) = \cos^2(\beta)\sinh(\alpha) \)
2. \( \cos(\beta)(-i)\sin(\beta)\cosh(\alpha) = (-i)\cos(\beta)\sin(\beta)\cosh(\alpha) \)
3. \( i\sin(\beta)\cos(\beta\beta)\sinh(\alpha) = (i)\sin(\beta)\cos(\beta)\sinh(\alpha) \)
4. \( i\sin(\beta)(-i)\sin(\beta)\cosh(\alpha) = \sin^2(\beta)\cosh(\alpha) \)

\[ C_2 = \Pi_k + \Pi_z i \]
\[ \Pi_R = \left( \frac{e^{\alpha}}{2} \right) \frac{\sin^2(\beta) \cosh(\alpha) + \cos(\beta) \sinh(\alpha)}{\sinh^2(\beta) \cos^2(\alpha) + \cosh(\beta) \sin^2(\alpha)} \]

\[ \Pi_Z = \left( \frac{e^{\alpha}}{2} \right) \frac{\cos(\beta) \sin(\beta) \{ \sinh(\alpha) - \cosh(\alpha) \}}{\sinh^2(\beta) \cos^2(\alpha) + \cosh(\beta) \sin^2(\alpha)} \]

Convert C2 into polar form:

\[ C_2 = \Pi_R + \Pi_Z i = \zeta e^{i \phi} \]

\[ \zeta = \sqrt{\left( \Pi_R \right)^2 + \left( \Pi_Z \right)^2} \]

\[ \phi = \arctan \left( \frac{\Pi_Z}{\Pi_R} \right) \]

\[ C_1 = V - C_2 \]

\[ C_1 = V - \zeta e^{i \phi} \]

Plug coefficients back into General Solution
and multiply out to isolate real and imaginary parts:

\[ u^0 = \left( 1 - \zeta e^{i \phi} \right) e^{(\alpha + \beta)y} + \left( \zeta e^{i \phi} \right) e^{-(\alpha + \beta)y} \]

\[ u = \text{Re} \{ u^0 e^{-it} \} \]

\[ u = \left( 1 - \zeta e^{i \phi} \right) e^{(\alpha + \beta)y} e^{-it} + \left( \zeta e^{i \phi} \right) e^{-(\alpha + \beta)y} e^{-it} \]

\[ u = e^{\alpha y + i(t + \beta y)} + \zeta \left( e^{-\alpha y + (t + \phi - \beta y)} - e^{\alpha y + (t + \phi + \beta y)} \right) \]

4. \quad e^{\alpha y + i(t + \beta y)} = e^{\alpha y} [\cos(t + \beta y) + i \sin(t + \beta y)]

5. \quad \zeta e^{-\alpha y + (t + \phi - \beta y)} = \zeta e^{-\alpha y} [\cos(t + \phi - \beta y) + i \sin(t + \phi - \beta y)]

6. \quad - \zeta e^{\alpha y + (t + \phi + \beta y)} = -\zeta e^{\alpha y} [\cos(t + \phi + \beta y) + i \sin(t + \phi + \beta y)]

Take the real parts only:

\[ u = e^{\alpha y} [\cos(\alpha x + \beta y)] + \zeta e^{\alpha y} [\cos(t + \phi - \beta y)] - \zeta e^{\alpha y} [\cos(t + \phi + \beta y)] \]

Using identity:

\[ \cos(u \pm v) = \cos(u) \cos(v) \mp \sin(u) \sin(v) \]

\[ u = \alpha x \]

\[ v = \phi - \beta y \text{ or } \beta y \text{ (for first term)} \]

4. \quad \cos(t + \beta y) = \cos(t) \cos(\beta y) - \sin(t) \sin(\beta y)

5. \quad \cos(t + \phi - \beta y) = \cos(t) \cos(\phi - \beta y) - \sin(t) \sin(\phi - \beta y)

6. \quad \cos(t + \phi + \beta y) = \cos(t) \cos(\phi + \beta y) - \sin(t) \sin(\phi + \beta y) \]
Assume form for \( u \):

\[
\begin{align*}
  u &= e^{\alpha y} \cos(t) \cos(\beta y) - e^{\alpha y} \sin(t) \sin(\beta y) + \zeta \\
  &= \cos(t) \left[ e^{\alpha y} \cos(\beta y) + \zeta e^{-\alpha y} \cos(\varphi - \beta y) - \zeta e^{\alpha y} \cos(\varphi + \beta y) \right] + \ldots \\
  &\quad + \sin(t) \left[ -e^{\alpha y} \sin(\beta y) - \zeta e^{-\alpha y} \sin(\varphi - \beta y) + \zeta e^{\alpha y} \sin(\varphi + \beta y) \right]
\end{align*}
\]

Assume form for \( u \):

\[
\begin{align*}
  u &= \Lambda \cos(t + \Theta) \\
  u &= \Lambda \cos(\alpha \tau) \cos(\Theta) - \Lambda \sin(t) \sin(\Theta) \\
  &= \cos(t) \left[ V e^{\alpha y} \cos(\beta y) + \zeta e^{-\alpha y} \cos(\varphi - \beta y) - \zeta e^{\alpha y} \cos(\varphi + \beta y) \right] + \ldots \\
  &\quad + \sin(t) \left[ -V e^{\alpha y} \sin(\beta y) - \zeta e^{-\alpha y} \sin(\varphi - \beta y) + \zeta e^{\alpha y} \sin(\varphi + \beta y) \right]
\end{align*}
\]

\[
\begin{align*}
  \Lambda \cos(\Theta) &= e^{\alpha y} \cos(\beta y) + \zeta e^{-\alpha y} \cos(\varphi - \beta y) - \zeta e^{\alpha y} \cos(\varphi + \beta y) = \Pi_1 \\
  -\Lambda \sin(\Theta) &= -e^{\alpha y} \sin(\beta y) - \zeta e^{-\alpha y} \sin(\varphi - \beta y) + \zeta e^{\alpha y} \sin(\varphi + \beta y) = \Pi_2
\end{align*}
\]

\[
\begin{align*}
  \Lambda &= \sqrt{(\Pi_1)^2 + (\Pi_2)^2} \\
  \Theta &= \arctan \left( \frac{\Pi_2}{\Pi_1} \right)
\end{align*}
\]
Solution with all the parts:

\[ u = \Lambda \cos(wt + \Theta) \]  
\[ \Lambda = \sqrt{(\Pi_1)^2 + (\Pi_2)^2} \]  
\[ \Theta = \arctan \left( \frac{-\Pi_2}{\Pi_1} \right) \]  
\[ \Pi_1 = e^{\alpha y} \cos(\beta y) + \zeta e^{-\alpha y} \cos(\varphi - \beta y) - \zeta e^{\alpha y} \cos(\varphi + \beta y) \]  
\[ \Pi_2 = -e^{\alpha y} \sin(\beta y) - \zeta e^{-\alpha y} \sin(\varphi - \beta y) + \zeta e^{\alpha y} \sin(\varphi + \beta y) \]  
\[ \zeta = \sqrt{(\Pi_R)^2 + (\Pi_Z)^2} \]  
\[ \varphi = \arctan \left( \frac{\Pi_Z}{\Pi_R} \right) \]  
\[ \Pi_R = \left( e^{\alpha} \frac{\sin^2(\beta)}{2} \right) \frac{\cosh(\alpha)}{\sinh^2(\beta)} + \left( -e^{\alpha} \frac{\sin(\beta)}{2} \right) \frac{\sinh^2(\beta)}{\cosh(\alpha)} \right] \]  
\[ \Pi_Z = \left( e^{\alpha} \frac{\sin^2(\beta)}{2} \right) \frac{\cosh(\alpha)}{\sinh^2(\beta)} + \left( -e^{\alpha} \frac{\sin(\beta)}{2} \right) \frac{\sinh^2(\beta)}{\cosh(\alpha)} \right] \]  
\[ \alpha = r \cos \left( \frac{\pi}{2} - \frac{1}{2} \arctan \left( \frac{1}{De} \right) \right) \]  
\[ \beta = r \sin \left( \frac{\pi}{2} - \frac{1}{2} \arctan \left( \frac{1}{De} \right) \right) \]  
\[ r = Wo \left[ 1 + (De)^{\frac{1}{2}} \right] \]
D.5 Oscillatory Velocity, Semi-Infinite Domain

\[ u^0 = C_1 e^{(\alpha + \beta)y} + C_2 e^{-(\alpha + \beta)y} \]  
(D.29)

Boundary Conditions:
\[ u^0(0) = 1 \]
\[ u^0(y \to \infty) = 0 \]

\( C_1 = 0 \)
\( C_2 = 1 \)

Plug coefficients back into General Solution
and multiply out to isolate real and imaginary parts:

\[ u^0 = e^{-(\alpha + \beta)y} \]
\[ u = \text{Re}\{u^0 e^{it}\} \]
\[ u = e^{-(\alpha + \beta)y} e^{it} \]
\[ u = e^{-\alpha t + i(t - \beta y)} = e^{-\alpha t} [\cos(t - \beta y) + i \sin(t - \beta y)] \]

Take the real parts only:

\[ u = e^{-\alpha t} \cos(t - \beta y) \]

Using identity:
\[ \cos(u \pm v) = \cos(u) \cos(v) \mp \sin(u) \sin(v) \]
\[ u = t \]
\[ v = \varphi - \beta y \text{ or } \beta y \text{ (for first term)} \]
\[ \cos(t - \beta y) = \cos(t) \cos(-\beta y) + \sin(t) \sin(-\beta y) \]

\[ u = e^{-\alpha t} [\cos(t) \cos(\beta y) - \sin(t) \sin(\beta y)] \]

Assume form for \( u \):
\[ u = \Lambda \cos(t + \Theta) \]
\[ u = \Lambda \cos(t) \cos(\Theta) - \Lambda \sin(t) \sin(\Theta) \]

\[ \Lambda \cos(\Theta) = e^{-\alpha t} \cos(\beta y) = \Pi_1 \]
\[ -\Lambda \sin(\Theta) = -e^{-\alpha t} \sin(\beta y) = \Pi_2 \]
\[ \Lambda = \sqrt{\left(\Pi_1\right)^2 + \left(\Pi_2\right)^2} \]

\[ \Theta = \arctan \left( \frac{\Pi_2}{\Pi_1} \right) \]

Solution with all the parts:

\[ u = \Lambda \cos(t + \Theta) \] \hspace{1cm} (D.30)

\[ \Lambda = \sqrt{\left(\Pi_1\right)^2 + \left(\Pi_2\right)^2} \] \hspace{1cm} (D.31)

\[ \Theta = \arctan \left( -\frac{\Pi_2}{\Pi_1} \right) \] \hspace{1cm} (D.32)

\[ \Pi_1 = \Lambda \cos(\Theta) = e^{\omega y} \cos(\beta y) \]

\[ \Pi_2 = -\Lambda \sin(\Theta) = -e^{\omega y} \sin(\beta y) \] \hspace{1cm} (D.33)

\[ \alpha = r \cos \left( \frac{\pi}{2} - \frac{1}{2} \arctan \left( \frac{1}{De} \right) \right) \]

\[ \beta = r \sin \left( \frac{\pi}{2} - \frac{1}{2} \arctan \left( \frac{1}{De} \right) \right) \]

\[ r = \omega_0 \left[ 1 + (De)^2 \right] \]

Phase Speed

\[ \phi = t + \Theta \]

\[ \frac{d\phi}{dt} = \frac{d\phi}{dy} \frac{dy}{dt} \]

\[ \frac{dy}{dt} = \frac{d\phi}{dt} / \frac{d\phi}{dy} = b/a = -c \]

\[ \frac{d\phi}{dt} = 1 \]

\[ \frac{d\phi}{dy} = \frac{d\Theta}{dy} = \Theta = \beta \]

\[ c = -\beta \] \hspace{1cm} (D.34)
Appendix E Derivations for 3.4: Relaxation Spectrum of Mucus

E.1 Dynamic Modulus of a Maxwell Fluid

\[ \tau = \text{Re}\{\tau^0 e^{i\omega t}\} \]  
(E.1)

\[ \gamma = \text{Re}\{\gamma^0 e^{i\omega t}\} \]  
(E.2)

\[ \gamma = \frac{\tau}{G} + \frac{1}{\eta_0} \]  
(E.3)

\[ iw\gamma^0 e^{i\omega t} = iw\tau^0 e^{i\omega t} \frac{1}{G} + \tau^0 e^{i\omega t} \frac{1}{\eta_0} \]  
(E.4)

\[ iw\gamma^0 = iw\tau^0 \frac{1}{G} + \tau^0 \frac{1}{\eta_0} \]  
(E.5)

\[ \frac{iw}{(iw\frac{1}{G} + \frac{1}{\eta_0})} = \frac{\tau^0}{\gamma^0} \]  
(E.6)

\[ \frac{\tau^0}{\gamma^0} = \frac{iwG\left(\frac{1}{\lambda} - iw\right)}{iw + \frac{1}{\lambda}\left(\frac{1}{\lambda} - iw\right)} = \frac{\left(\frac{iwG}{\lambda} + w^2G\right)}{1 + \lambda^2w^2} = G' + G''i \]  
(E.7)

\[ G^* = G' + G''i \]  
(E.8)

\[ G'' = \frac{w\eta_0}{1 + \lambda^2w^2} \]  
(E.9)

\[ G' = \frac{w^2\lambda^2G}{1 + \lambda^2w^2} \]  
(E.10)

\[ |G^*| = \sqrt{G'^2 + G''^2} \]  
(E.11)

E.2 Complex Viscosity of a Maxwell Fluid

\[ \tau = \text{Re}\{\tau^0 e^{i\omega t}\} \]  
(E.12)
\[
\dot{\gamma} = \text{Re}\{\dot{\gamma}^0 e^{i\omega t}\}
\]  
(E.13)

Maxwell Fluid Constitutive Equation
\[
\dot{\gamma} = \frac{\tau}{G} + \frac{\tau^0}{\eta_0}
\]  
(E.14)

Substitute and simplify:
\[
\dot{\gamma}^0 e^{i\omega t} = i\omega \tau^0 e^{i\omega t} \frac{1}{G} + \tau^0 e^{i\omega t} \frac{1}{\eta_0}
\]  
(E.15)

\[
\dot{\gamma}^0 = i\omega \tau^0 \frac{1}{G} + \tau^0 \frac{1}{\eta_0}
\]  
(E.16)

\[
\frac{1}{i\omega \frac{1}{G} + \frac{1}{\eta_0}} = \frac{\tau^0}{\gamma^0}
\]  
(E.17)

\[
\frac{\tau^0}{\dot{\gamma}^0} = \frac{\eta_0}{(1 + i\omega \lambda)} = \frac{(\eta_0 - i\omega \lambda \eta_0)}{(1 + \omega^2 \lambda^2)} = \frac{(\eta_0 - i\eta_0 De)}{(1 + De^2)}
\]  
(E.18)

\[
\eta^* = \frac{\tau^0}{\dot{\gamma}^0} = \eta' + \eta'' i
\]  
(E.19)

\[
\eta' = \frac{\eta_0}{(1 + De)}
\]  
(E.20)

\[
\eta'' = -De \eta_0, \quad \frac{-De \eta_0}{(1 + De^2)}
\]  
(E.21)

\[
|\eta^*| = \sqrt{\eta'^2 + \eta''^2}
\]  
(E.22)
Appendix F Derivations for 3.5:
Thin Film Flow of a Linear Maxwell Fluid

Continuity:
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]

Momentum:
\[ 0 = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xx}}{\partial x} \]
\[ 0 = -\frac{\partial P}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} \]

Maxwell Fluid Equations:
\[ \lambda \frac{\partial \tau_{xy}}{\partial t} + \tau_{xy} = \eta_0 \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \]
\[ \lambda \frac{\partial \tau_{xx}}{\partial t} + \tau_{xx} = \eta_0 \left( \frac{\partial u}{\partial x} \right) \]
\[ \lambda \frac{\partial \tau_{yy}}{\partial t} + \tau_{yy} = \eta_0 \left( \frac{\partial v}{\partial y} \right) \]

Boundary Conditions:
\[ u(0) = A + B \cos \left( \frac{2\pi}{\gamma} x + \omega t \right) = F(x, t) \quad v(0) = 0 \]
\[ \tau_{xy}(h) = 0 \quad v(h) = 0 \]

Non Dimensionalize governing equations and boundary conditions:
Known scales:
\[ \tilde{t} = \omega \]
\[ x = \frac{x}{\gamma} \]
\[ y = \frac{y}{h} \]
\[ \tilde{u} = \frac{u}{U} \]
\[ \tilde{v} = \frac{v}{\epsilon U} \]

Apply known scales to stress ODEs:
\[ \lambda \omega \frac{\partial \tau_{xy}}{\partial t} + \tau_{xy} = \eta_0 \left( \frac{U}{h} \frac{\partial u}{\partial y} + \frac{U \epsilon}{\gamma} \frac{\partial v}{\partial x} \right) \]
\[ \lambda \omega \frac{\partial \tau_{xx}}{\partial t} + \tau_{xx} = \eta_0 2 \frac{U}{\gamma} \frac{\partial u}{\partial x} \]
\[ \lambda \omega \frac{\partial \tau_{yy}}{\partial t} + \tau_{yy} = \eta_0 2 \frac{U \epsilon}{h} \frac{\partial v}{\partial y} \]

\[ \left( \frac{\lambda \omega h}{U \eta_0} \frac{\partial \tau_{xy}}{\partial t} + \frac{h \tau_{xy}}{U \eta_0} \right) = \frac{\partial u}{\partial y} + \epsilon^2 \frac{\partial v}{\partial x} \]
\[ ? = \frac{U \eta_0}{h} \]
\[ \text{De} \frac{\partial \tau_{xy}}{\partial t} + \tau_{xy} = \frac{\partial u}{\partial y} + \epsilon^2 \frac{\partial v}{\partial x} \]

\[ \text{De} \frac{\partial \tau_{xx}}{\partial t} + \tau_{xx} = 2 \epsilon \frac{\partial u}{\partial x} \]
\[ \text{De} \frac{\partial \tau_{yy}}{\partial t} + \tau_{yy} = 2 \epsilon \frac{\partial v}{\partial y} \]
0 = \frac{-\partial P}{\partial x} \gamma U \eta_0 + \frac{\partial \tau_{xy}}{\partial y} + \varepsilon \frac{\partial \tau_{xx}}{\partial x}

\varepsilon = \frac{U \eta_0}{\varepsilon h}

0 = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \varepsilon \frac{\partial \tau_{xx}}{\partial x}

0 = -\frac{\partial P}{\partial y} + \varepsilon^2 \frac{\partial \tau_{xy}}{\partial x} + \varepsilon \frac{\partial \tau_{yy}}{\partial y}

Let A and B = U
Therefore, after non-dimensionalizing with respect to U the boundary conditions become:

\begin{align*}
\tau_{xy}(0) &= 1 + \cos[(2\pi x + \xi t)] = F(x, t) \\
\nu(0) &= 0 \\
\tau_{xy}(1) &= 0 \\
\nu(1) &= 0
\end{align*}

Take the limit as epsilon goes to zero (thin film approximation)

\begin{align*}
0 &= -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \\
0 &= -\frac{\partial P}{\partial y}
\end{align*}

\begin{align*}
De \frac{\partial \tau_{xy}}{\partial t} + \tau_{xy} &= \frac{\partial u}{\partial y} \\
De \frac{\partial \tau_{xx}}{\partial t} + \tau_{xx} &= 0 \\
De \frac{\partial \tau_{yy}}{\partial t} + \tau_{yy} &= 0
\end{align*}

(F.1)

Take derivative of shear stress ode with respect to y

\begin{align*}
De \frac{\partial^2 \tau_{xy}}{\partial t \partial y} + \frac{\partial \tau_{xy}}{\partial y} &= \frac{\partial^2 u}{\partial y^2}
\end{align*}

And the NS-X with respect to t

\begin{align*}
0 &= -\frac{\partial^2 P}{\partial x \partial t} + \frac{\partial^2 \tau_{xy}}{\partial y \partial t}
\end{align*}

And substitute back into NS-X
\[ 0 = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \]

\[ 0 = -\left( \frac{\partial P}{\partial x} + D e \frac{\partial^2 P}{\partial x \partial t} \right) + \frac{\partial^2 u}{\partial y^2} \]

\[ \frac{\partial^2 u}{\partial y^2} = \left( \frac{\partial P}{\partial x} + D e \frac{\partial^2 P}{\partial x \partial t} \right) \]

\[ u = \left( \frac{\partial P}{\partial x} + D e \frac{\partial^2 P}{\partial x \partial t} \right) \frac{y^2}{2} + C_y + D \]

Apply Boundary Conditions:

\[ u(0) = D = F(x, t) \]
\[ \left. \frac{\partial u}{\partial y} \right|_{y=1} = \left( \frac{\partial P}{\partial x} + D e \frac{\partial^2 P}{\partial x \partial t} \right) + C = 0 \]

\[ D = F(x, t) \]
\[ C = -\left( \frac{\partial P}{\partial x} + D e \frac{\partial^2 P}{\partial x \partial t} \right) \]

\[ u = \left( \frac{\partial P}{\partial x} + D e \frac{\partial^2 P}{\partial x \partial t} \right) \frac{y^2}{2} - \left( \frac{\partial P}{\partial x} + D e \frac{\partial^2 P}{\partial x \partial t} \right) y + F(x, t) \]

\[ u = \left( \frac{\partial P}{\partial x} + D e \frac{\partial^2 P}{\partial x \partial t} \right) \left( \frac{y^2}{2} - y \right) + F(x, t) \]  \hspace{1cm} (F.2)

Using continuity:

\[ \int_0^1 \frac{\partial u}{\partial x} \, dy = \int_0^1 -\frac{\partial v}{\partial y} \, dy \]
\[ v(0) = 0 \]
\[ v(1) = 0 \]

\[ \frac{\partial}{\partial x} \int_0^1 u \, dy = 0 \]
\[ \int_{0}^{1} u dy = G(t) \]

\[ \int_{0}^{1} u dy = \left( \frac{\partial P}{\partial x} + D e \frac{\partial^2 P}{\partial x \partial t} \right) \left( -\frac{1}{3} \right) + F(x,t) \]

\[ \left( \frac{\partial P}{\partial x} + D e \frac{\partial^2 P}{\partial x \partial t} \right) = -3 \left[ G(t) - F(x,t) \right] \quad (F.3) \]

There can be no driving pressure difference on account that the flow is entirely shear-driven, integrate across a wavelength to eliminate the pressure gradient terms and find G(t)

\[ \int_{0}^{1} \frac{\partial P}{\partial x} dx + D e \frac{\partial}{\partial t} \int_{0}^{1} \frac{\partial P}{\partial x} dx = -3 \int_{0}^{1} \left[ G(t) - F(x,t) \right] dx \]

\[ 0 = -3 \left[ G(t) - \left( 1 + \frac{1}{2\pi} \sin(2\pi + ct) - \frac{1}{2\pi} \sin(ct) \right) \right] \]

\[ G(t) = \left( 1 + \frac{1}{2\pi} \sin(2\pi + ct) - \frac{1}{2\pi} \sin(ct) \right) \quad (F.4) \]

Substitute F.4 into F.3 and F.2

\[ u = -3 \left[ \left( 1 + \frac{\sin(2\pi + ct)}{2\pi} - \frac{\sin(ct)}{2\pi} \right) - F(x,t) \right] \left( \frac{y^2}{2} - y \right) + F(x,t) \quad (F.5) \]

Use continuity equation to find v component

\[ \frac{\partial u}{\partial x} = \frac{\partial F}{\partial x} \left[ 3 \left( \frac{y^2}{2} - y \right) + 1 \right] \]

\[ \frac{\partial v}{\partial y} = -\frac{\partial F}{\partial x} \left[ 3 \left( \frac{y^2}{2} - y \right) + 1 \right] \]

\[ v = -\frac{\partial F}{\partial x} \left[ 3 \left( \frac{y^2}{6} - \frac{y^2}{2} \right) + y \right] \quad (F.6) \]
\[ F(x,t) = 1 + \cos(2\pi x + ct) \]
\[ \frac{\partial F}{\partial t} = -2\pi \sin(2\pi x + ct) \]
Appendix G Derivations for 3.6:
Alternative Scaling of Linear Maxwell Model

Conservation of momentum:
\[ \rho \frac{\partial u}{\partial t} = \frac{\partial \tau_{yx}}{\partial y} \]  
(G.1)

Linear Maxwell Equation
\[ \lambda \frac{\partial \tau_{yx}}{\partial t} + \tau_{yx} = \eta_0 \frac{\partial u}{\partial y} \]  
(G.2)

Take the derivative with respect to t of the momentum equation and with respect to y of the Maxwell equation:
\[ \rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 \tau_{yx}}{\partial t \partial y} \]  
(G.3)
\[ \lambda \frac{\partial^2 \tau_{yx}}{\partial y \partial t} + \frac{\partial \tau_{yx}}{\partial y} = \eta_0 \frac{\partial^2 u}{\partial y^2} \]  
(G.4)

Substitute G.1 and G.3 into G.4:
\[ \lambda \rho \frac{\partial^2 u}{\partial t^2} + \rho \frac{\partial u}{\partial t} = \eta_0 \frac{\partial^2 u}{\partial y^2} \]  
(G.5)

Non-dimensionalize with respect to the following scales:
\[ u = \bar{u} U \]
\[ y = \bar{y} h \]
\[ t = \bar{t} A \]
Where U is flow velocity, h is channel or film height (perpendicular to flow), and A is an undefined time scale for t. Over bars representing non-dimensionalized quantities are omitted in the following equations:
\[ \frac{\lambda \rho U}{A^2} \frac{\partial^2 u}{\partial t^2} + \frac{\rho U}{A} \frac{\partial u}{\partial t} = \frac{U \eta_0}{h^2} \frac{\partial^2 u}{\partial y^2} \]  
(G.8)
\[ \frac{\lambda}{A} \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \frac{A \eta_0}{\rho h^2} \frac{\partial^2 u}{\partial y^2} \]  
(G.9)
There are multiple ways that A can be scaled. The two left terms can be assumed to be of equal magnitude. Alternatively, either of the terms on the left can be balanced with the term on the right. It should be noted that the left most term multiplying the second derivative is associated with viscoelastic effects. All three avenues for a selection of A will be explored.

Path 1:

\[
\left( \frac{\lambda}{A} \right) \frac{\partial^2 u}{\partial t^2} \sim (1) \frac{\partial u}{\partial t} \tag{G.10}
\]

\[
A = \lambda \tag{G.11}
\]

\[
\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \frac{\lambda \eta_0}{\rho h^2} \frac{\partial^2 u}{\partial y^2} \tag{G.12}
\]

Path 2:

\[
\left( \frac{\lambda}{A} \right) \frac{\partial^2 u}{\partial t^2} \sim \left( \frac{A \eta_0}{\rho h^2} \right) \frac{\partial^2 u}{\partial y^2} \tag{G.13}
\]

\[
A = h \sqrt{\frac{\lambda \rho}{\eta_0}} \tag{G.14}
\]

\[
\frac{\partial^2 u}{\partial t^2} + h \sqrt{\frac{\rho}{\lambda \eta_0}} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} \tag{G.15}
\]

Path 3:

\[
\left( \frac{A \eta_0}{\rho h^2} \right) \frac{\partial^2 u}{\partial y^2} \sim (1) \frac{\partial u}{\partial t} \tag{G.16}
\]

\[
A = \frac{\rho h^2}{\eta_0} \tag{G.17}
\]

\[
\frac{\lambda \eta_0}{\rho h^2} \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} \tag{G.18}
\]
Appendix H Derivations for 4.1:
Tension in a Unidirectional Shear Flow

Problem Setup:
Flow has two alternating regions: one which is actively sheared, one with no imposed shear

Assumptions:
1. Developing regions between the two regions are negligible small (rapid transition)
2. The off-regions can be supported by tension as if they comprised a rigid body

The constitutive relations for a flow which is unidirectional in the x direction reduce to:

\[ \tau_{xx} = 2\lambda \tau_{xy} \frac{\partial u_x}{\partial y} \]
\[ \tau_{xy} = \eta_0 \frac{\partial u_x}{\partial y} \]
\[ \tau_{yy} = 0 \]

Apply a linear shear profile to the ‘on’ region

\[ \tau_{xy} = T \left( 1 - \frac{y}{H} \right) = \eta_0 \frac{\partial u_x}{\partial y} \]
\[ \frac{\partial u_x}{\partial y} = \frac{T}{\eta_0} \left( 1 - \frac{y}{H} \right) \]

\[ \tau_{xx} = 2\lambda \tau_{xy} \frac{\partial u_x}{\partial y} \]
\[ \tau_{xx} = 2 \frac{\lambda}{\eta_0} T^2 \left( 1 - \frac{y}{H} \right)^2 \]

In the active portion of the slab (length \( y/2 \)), the applied shear must balance the weight of the slab. The tension is equal on the top and bottom, canceling from the force balance.

\[ \sum F_y = T w \frac{y}{2} - H w \frac{y}{2} \rho g = 0 \]
\[ T = H \rho g \]

In the inactive portion, shear is not present to support the slab, the tension developed in the active region must be strong enough to support this weight.
\[\sum F_y = 2 \frac{\lambda}{\eta_0} w T^2 \int_0^H \left(1 - \frac{y}{H}\right)^2 \, dy - Hw \frac{\gamma}{2} \rho g \geq 0\]

\[2 \frac{\lambda}{\eta_0} T^2 \frac{H}{3} - H \frac{\gamma}{2} \rho g \geq 0\]

\[\frac{4}{3} \geq \frac{\gamma_0 \rho g}{T^2 \lambda}\]

Or

\[\frac{T^2 \lambda}{\gamma_0 \rho g} \geq \frac{3}{4}\]
Appendix I Derivations for 4.2: Stick Slip Transition of a Stokes Flow

Non-Dimensionalized PDE
Where $U$ is the average flow velocity and $h$ is the film height.

$$\nabla^4 \psi = \frac{\partial^4 \psi}{\partial x^4} + \frac{\partial^4 \psi}{\partial y^4} + 2 \frac{\partial^4 \psi}{\partial x^2 \partial y^2} = 0$$

$$\tau = \frac{T}{T} \quad \bar{u} = \frac{\eta_0}{Th} \quad \bar{v} = \frac{\eta_0}{Th} \quad \bar{\psi} = \frac{\psi}{\eta_0} \quad De = \frac{\lambda}{\eta_0}$$ \hspace{1cm} (I.1)

$$u_x = -\frac{\partial \psi}{\partial x} \quad u_y = \frac{\partial \psi}{\partial y}$$ \hspace{1cm} (I.2)

$$\left(\frac{T h^2}{h^2 \eta_0}\right) \frac{\partial^4 \psi}{\partial x^4} + \left(\frac{T h^2}{h^2 \eta_0}\right) \frac{\partial^4 \psi}{\partial y^4} + \left(2 \frac{T h^2}{h^2 \eta_0}\right) \frac{\partial^4 \psi}{\partial x^2 \partial y^2} = 0$$ \hspace{1cm} (I.3)

$$\frac{\partial^4 \bar{\psi}}{\partial \bar{x}^4} + \frac{\partial^4 \bar{\psi}}{\partial \bar{y}^4} + 2 \frac{\partial^4 \bar{\psi}}{\partial \bar{x}^2 \partial \bar{y}^2} = 0$$

*Over-bars omitted henceforth

Non-Dimensionalized Boundary Conditions

<table>
<thead>
<tr>
<th>Top Boundary</th>
<th>Bottom Boundary</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_{xy} = 0$</td>
<td>$\tau_{xy} = f(x,t)$</td>
</tr>
</tbody>
</table>
| $\frac{\partial^2 \psi}{\partial y^2} = 0$ | $\frac{\partial^2 \psi}{\partial y^2} = f(x,t)$ | \hspace{1cm} (I.4)

Where $f(x,t)$ is a function that is zero as ‘$x \to -\infty$’ and 1 as ‘$x \to \infty$’, in this case the Heaviside step function was used.

$$\left.\frac{\partial^2 \psi}{\partial y^2}\right|_{x,y=1} = 0 \quad \left.\frac{\partial^2 \psi}{\partial y^2}\right|_{x,y=0} = f(x,t) = u(x)$$ \hspace{1cm} (I.5)

Constant streamline across upper and lower domain boundaries:

*The value of the stream function at the top boundary is chosen to be equal to the flow present over a plate with a prescribed shear stress, no slip condition over its surface and no stress at the top of the film.

$$\psi(x, y = 1) = \frac{1}{3} \quad \psi(x, y = 0) = 0$$

Homogenized Boundary Conditions
\[
\psi = \psi_h + \phi
\]

\[
\frac{\partial^2 \psi_h}{\partial y^2} \bigg|_{x,y=1} = 0 \quad \frac{\partial^2 \psi_h}{\partial y^2} \bigg|_{x,y=0} = 0 \\
\psi_h(x, y = 1) = 0 \quad \psi_h(x, y = 1) = 0
\]

\[\text{(I.6)}\]

\[
\frac{\partial^2 \phi}{\partial y^2} \bigg|_{x,y=1} = 0 \quad \frac{\partial^2 \phi}{\partial y^2} \bigg|_{x,y=0} = f \\
\phi(x, y = 1) = -\frac{1}{3} \quad \phi(x, y = 0) = 0
\]

Creating homogenizing function of satisfy boundary conditions:
Assume polynomial form (order one higher than highest order boundary condition)

\[
\phi = Ay^3 + By^2 + Cy + D
\]

\[
\frac{\partial^2 \phi}{\partial y^2} = 6Ay + 2B
\]

\[\text{(I.7)}\]

\[
0 = 6A + 2B \quad B = f \quad A + B + C + D = -\frac{1}{3} \quad D = 0
\]

\[
A = -\frac{f}{6} \quad -\frac{f}{6} + \frac{f}{2} + C + 0 = -\frac{1}{3} \\
C = -\frac{1}{3} - \frac{f}{3}
\]

\[
\phi = -\frac{f}{6} y^3 + \frac{f}{2} y^2 + \left(-\frac{1}{3} - \frac{f}{3}\right)y
\]

\[\text{(I.8)}\]

\[
f(x) = -u(x)
\]

\[
\phi_+ = \frac{1}{6} y^3 - \frac{1}{2} y^2 \\
\phi_- = \left(-\frac{1}{3}\right)y
\]

\[\text{(I.9)}\]

Create forcing function:

\[
\nabla^4 \psi = \nabla^4 \psi_h + \nabla^4 \phi = 0
\]

\[\text{(I.10)}\]
\[ \nabla^4 \psi_h = -\nabla^4 \phi \] (I.11)

\[ \frac{\partial^4 \phi}{\partial x^4} = \frac{d^4 f}{dx^4} \left[ \left( \frac{1}{6} \right) y^3 + \left( -\frac{1}{2} \right) y^2 + \left( \frac{1}{3} \right) y \right] \]
\[ \frac{\partial^4 \phi}{\partial y^4} = 0 \]
\[ 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} = \frac{d^2 f}{dx^2} (2y - 2) \]

\[ -\nabla^4 \phi = \frac{d^4 f}{dx^4} \left[ \left( \frac{1}{6} \right) y^3 + \left( -\frac{1}{2} \right) y^2 + \left( \frac{1}{3} \right) y \right] + \frac{d^2 f}{dx^2} (2y - 2) \] (I.12)

Separation of Variables
\[ \psi_h (x, y) = X(x)Y(y) \] (I.13)

\[ Y \frac{d^4 X}{dx^4} + X \frac{d^4 Y}{dy^4} + 2 \frac{d^2 Y}{dy^2} \frac{d^2 X}{dx^2} = -\nabla^4 \phi \] (I.14)

\[ Y \left[ \frac{d^4 X}{dx^4} + X \left( \frac{1}{Y} \frac{d^4 Y}{dy^4} \right) + 2 \frac{d^2 Y}{dy^2} \left( \frac{1}{Y} \frac{d^2 Y}{dy^2} \right) \right] = -\nabla^4 \phi \] (I.15)

Two Eigen Value Problems
\[ \left( \frac{1}{Y} \frac{d^4 Y}{dy^4} \right) = k_1 \]
\[ \left( \frac{1}{Y} \frac{d^2 Y}{dy^2} \right) = k_2 \] (I.16)

2\textsuperscript{nd} Order
\[ \frac{d^2 Y}{dy^2} - k_2 Y = 0 \] (I.17)

Let k be some number gamma for which gamma squared is less than zero
\[ k_2 = -\gamma^2 \] (I.18)
\[ \frac{d^3 Y}{dy^3} + \gamma^2 Y = 0 \] (I.19)

There are two imaginary eigen values for this ODE
\[ r = \pm i \gamma \] (I.20)
\[ Y = C_1 \sin(\gamma y) + C_2 \cos(\gamma y) \] (I.21)
The only permitted solution $\in [0:1]$ given the boundary conditions is \( \sin \) so long as $\gamma = n\pi$.

4\textsuperscript{th} Order

\[
\frac{d^4Y}{dy^4} - k_1 Y = 0 \tag{I.22}
\]

Let \( k \) be some number lambda for which lambda to the 4\textsuperscript{th} is greater than zero

\[ k_2 = \lambda^4 \tag{I.23} \]

\[
\frac{d^4Y}{dy^4} - \lambda^4 Y = 0 \tag{I.24}
\]

\[
\left( r^4 - \lambda^4 \right) = \left( r^2 - \lambda^2 \right)^2 = (r + \lambda)(r - \lambda)(r + \lambda i)(r - \lambda i) \tag{I.25}
\]

\[ Y = C_1 \sin(\lambda y) + C_2 \cos(\lambda y) + C_3 e^{\lambda y} + C_4 e^{-\lambda y} \tag{I.26} \]

Again, however, $\in [0:1]$ the only permitted solution is the first where $\lambda = n\pi$ (which is equal to gamma)

The two problems can be written now as

\[
\left( \frac{1}{Y} \frac{d^4Y}{dy^4} \right) = \gamma^4 \tag{I.27}
\]

\[
\left( \frac{1}{Y} \frac{d^3Y}{dy^3} \right) = -\gamma^2 \tag{I.28}
\]

The original PDE must now be written as a sum (the constants C1 have been absorbed into X which has not been solved for yet):

\[
\sum_{n=1}^{\infty} \sin(n\pi y) \left[ \frac{d^4X}{dx^4} - 2\gamma^2 \frac{d^2X}{dx^2} + \gamma^4 X \right] = -\nabla^4 \phi \tag{I.29}
\]

In order to isolate the ODE in X, find the inner product with respect to the basis function, \( \sin(n\pi y) \), of both sides:

\[
\langle \sin(m\pi y), \sin(n\pi y) \rangle \left[ \frac{d^4X}{dx^4} - 2\gamma^2 \frac{d^2X}{dx^2} + \gamma^4 X \right] = \langle \sin(n\pi y), -\nabla^4 \phi \rangle \tag{I.30}
\]

\[
\left[ \frac{d^4X}{dx^4} - 2\gamma^2 \frac{d^2X}{dx^2} + \gamma^4 X \right] = \frac{\langle \sin(m\pi y), -\nabla^4 \phi \rangle}{\langle \sin(m\pi y), \sin(n\pi y) \rangle} \]

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\[ m = n \]

\[
\langle \sin(n\pi y), \sin(n\pi y) \rangle = \int_0^1 \sin^2(n\pi y)\,dy = \frac{1}{2} \quad (I.31)
\]

\[
\langle \sin(n\pi y), -\nabla^4 \phi \rangle = \int_0^1 \sin(n\pi y) \left[ \frac{d^4 f}{dx^4} \left( \frac{1}{6} y^3 + \left( -\frac{1}{2} \right) y^2 + \left( \frac{1}{3} \right) y \right) + \frac{d^2 f}{dx^2} (2y - 2) \right] dy \quad (I.32)
\]

\[
\int_0^1 \sin(n\pi y) y^3\,dy = \frac{(-1)^n(6 - n^2 \pi^2)}{n^3 \pi^3}
\]

\[
\int_0^1 \sin(n\pi y) y^2\,dy = -2 - 2(-1)^n + (-1)^n n^2 \pi^2
\]

\[
\int_0^1 \sin(n\pi y) y\,dy = \frac{(-1)^n}{n\pi}
\]

\[
\int_0^1 \sin(n\pi y)\,dy = -\frac{1 + (-1)^n}{n\pi}
\]

\[
\langle \sin(n\pi y), -\nabla^4 \phi \rangle = \ldots
\]

\[
= \frac{d^4 f}{dx^4} \left[ \frac{1}{(n\pi)^3} \right] + \frac{d^2 f}{dx^2} \left[ -\frac{2}{n\pi} \right] \quad (I.33)
\]

\[
= \frac{d^4 f}{dx^4} D_{n1} + \frac{d^2 f}{dx^2} D_{2n}
\]

\[
D_{n1} = \left[ \frac{1}{(n\pi)^3} \right]
\]

\[
D_{2n} = \left[ -\frac{2}{n\pi} \right]
\]

At this point in the solution procedure each half of the domain must be considered separately. That is, one cannot differentiate across the step function. This reduces the problem to two homogenous differential equations:

\[
\frac{d^4 X_+}{dx^4} - 2\gamma^2 \frac{d^2 X_+}{dx^2} + \gamma^4 X_+ = 0, x \geq 0 \quad (I.34)
\]

\[
\frac{d^4 X_-}{dx^4} - 2\gamma^2 \frac{d^2 X_-}{dx^2} + \gamma^4 X_- = 0, x \leq 0
\]

The general solutions of which are:
\[ X_+ = Ae^{\gamma x} + Be^{-\gamma x} + Cxe^{\gamma x} + Dxe^{-\gamma x} \]
\[ X_- = Ee^{\gamma x} + Fe^{-\gamma x} + Gxe^{\gamma x} + Hxe^{-\gamma x} \]  
(I.35)

Only some of the above solutions are permitted; we require the solutions go to zero away from the origin.
\[ X_+ = Be^{-\gamma x} + Dxe^{-\gamma x} \]
\[ X_- = Ee^{\gamma x} + Gxe^{\gamma x} \]  
(I.36)

Four conditions are needed to solve for the constants in the equation set. The homogenous solution is allowed to be discontinuous at the origin. However, the final solution must be continuous even at the origin. This will be the first condition. The remaining three will be generated simply by requiring continuity of the derivatives at the origin.

\[ X_+ = Be^{-\gamma x} + Dxe^{-\gamma x} \]
\[ \frac{dX_+}{dx} = -\gamma Be^{-\gamma x} + De^{-\gamma x} - \gamma Dxe^{-\gamma x} \]
\[ \frac{d^2 X_+}{dx^2} = \gamma^2 Be^{-\gamma x} - 2\gamma De^{-\gamma x} + \gamma^2 Dxe^{-\gamma x} \]
\[ \frac{d^3 X_+}{dx^3} = -\gamma^3 Be^{-\gamma x} + 3\gamma^2 De^{-\gamma x} - \gamma^3 Dxe^{-\gamma x} \]
\[ X_- = Ee^{\gamma x} + Gxe^{\gamma x} \]
\[ \frac{dX_-}{dx} = \gamma Ee^{\gamma x} + Ge^{\gamma x} + \gamma Gxe^{\gamma x} \]
\[ \frac{d^2 X_-}{dx^2} = \gamma^2 Ee^{\gamma x} + 2\gamma Ge^{\gamma x} + \gamma^2 Gxe^{\gamma x} \]
\[ \frac{d^3 X_-}{dx^3} = \gamma^3 Ee^{\gamma x} + 3\gamma^2 Ge^{\gamma x} + \gamma^3 Gxe^{\gamma x} \]  
(I.37)

\[ E + \phi_- = B + \phi_+ \]  
(I.38)

\[ \phi_+ = \frac{\begin{pmatrix} \sin(\gamma y) \gamma y^2 - \frac{1}{2} \gamma y^3 \\ \sin(\gamma y) \gamma y \end{pmatrix}}{\begin{pmatrix} \sin(\gamma y) \sin(\gamma y) \end{pmatrix}} \]
\[ \phi_- = \frac{\begin{pmatrix} \sin(\gamma y) - \frac{1}{3} y \\ \sin(\gamma y) \sin(\gamma y) \end{pmatrix}}{\begin{pmatrix} \sin(\gamma y) \sin(\gamma y) \end{pmatrix}} \]

\[-\gamma B + D = \gamma E + G \]
\[\gamma^2 B - 2\gamma D = \gamma^2 E + 2\gamma G \]
\[-\gamma^3 B + 3\gamma^2 D = \gamma^3 E + 3\gamma^2 G \]  
(I.39)
\[
\begin{bmatrix}
1 & 0 & -1 & 0 \\
-\gamma & 1 & -\gamma & -1 \\
\gamma^2 & -2\gamma & -\gamma^2 & -2\gamma \\
-\gamma^3 & 3\gamma^2 & -\gamma^3 & -3\gamma^2
\end{bmatrix}
\begin{bmatrix}
B \\
D \\
E \\
G
\end{bmatrix} = \begin{bmatrix}
\phi_- - \phi_+ \\
0 \\
0 \\
0
\end{bmatrix}
\]

(I.40)

\[
\begin{bmatrix}
\frac{1}{2} & \frac{3}{4\gamma} & 0 & \frac{1}{4\gamma^2} \\
\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
-\frac{1}{2} & -\frac{3}{4\gamma} & 0 & \frac{1}{4\gamma^2} \\
\frac{\gamma^2}{2(\gamma+1)} & \frac{\gamma}{2(\gamma+1)} & -\frac{1}{2(\gamma+1)} & \frac{1}{2(\gamma+1)}
\end{bmatrix}
\begin{bmatrix}
\phi_- - \phi_+ \\
0 \\
0 \\
0
\end{bmatrix}
\]

(I.41)

\[
\begin{bmatrix}
\frac{\phi_- - \phi_+}{2(\gamma+1)} & \frac{\gamma(\phi_- - \phi_+)}{2} - \frac{(\phi_- - \phi_+)}{2} \\
\frac{\phi_- - \phi_+}{2(\gamma+1)} & \frac{\gamma^2(\phi_- - \phi_+)}{2(\gamma+1)}
\end{bmatrix}
\]

(I.42)

Final Solution:

\[
\psi = \psi_h + \phi
\]

(I.43)

\[
\phi = u(x)\left(\frac{\gamma^3}{6} - \frac{\gamma^2}{2} + \frac{\gamma}{3}\right) - \left(\frac{\gamma}{3}\right)
\]

\[
\psi_h = \sum_{n=1}^{\infty} \sin(\gamma y) \begin{cases} Ee^{\gamma x} + Gxe^{\gamma x}, x \leq 0 \\
Be^{-\gamma x} + Dxe^{-\gamma x}, x \geq 0 \end{cases}
\]

\[
B, D, E, G = \begin{bmatrix}
\phi_- - \phi_+ \\
\frac{\phi_- - \phi_+}{2(\gamma+1)} + \frac{\gamma(\phi_- - \phi_+)}{2} - \frac{(\phi_- - \phi_+)}{2} \\
\frac{\gamma^2(\phi_- - \phi_+)}{2(\gamma+1)}
\end{bmatrix}
\]

\[
\phi_+ = \frac{\sin(\gamma y), \frac{\gamma^3}{6} - \frac{\gamma^2}{2} y^2}{\sin(\gamma y), \sin(\gamma y)}
\]
\[ \phi = \begin{pmatrix} \sin(\gamma), \left( -\frac{1}{3} y \right) \\ \langle \sin(\gamma), \sin(\gamma) \rangle \end{pmatrix} \]

\[ \gamma = n\pi \]
Appendix J Derivations for 4.3: UCM with a Sinusoidally Varying Wall Stress

Starting with eqn. 1.33, the forcing function for stress is chosen:

\[ f(x) = 1 + \sin(ax + bt) \]  \hspace{1cm} (J.1)

\[ \frac{df}{dx} = a \cos(ax + bt) \]
\[ \frac{d^2 f}{dx^2} = -a^2 \sin(ax + bt) \]  \hspace{1cm} (J.2)
\[ \frac{d^3 f}{dx^3} = -a^3 \cos(ax + bt) \]
\[ \frac{d^4 f}{dx^4} = a^4 \sin(ax + bt) \]

The ODE in \( x \) with forcing function

\[ \left[ \frac{d^4 X}{dx^4} - 2 \gamma^2 \frac{d^2 X}{dx^2} + \gamma^4 X \right] = \left\langle \sin(j \pi y), -\nabla^4 \phi \right\rangle \]
\[ \left\langle \sin(j \pi y), \sin(n \pi y) \right\rangle \]  \hspace{1cm} (J.3)

Becomes:

\[ \frac{d^4 X}{dx^4} - 2 \gamma^2 \frac{d^2 X}{dx^2} + \gamma^4 X = \sin(ax + bt) \left[ a^4 D_{n1} - a^2 D_{n2} \right] \]  \hspace{1cm} (J.4)

Conversion to frequency domain (Fourier space)

\[ \int_{-\infty}^{\infty} e^{-ax} \left\{ \frac{d^4 X}{dx^4} - 2 \gamma^2 \frac{d^2 X}{dx^2} + \gamma^4 X \right\} dx = \sin(ax + bt) \left[ a^4 D_{n1} - a^2 D_{n2} \right] \]  \hspace{1cm} (J.5)

\[ (\omega^4) Z(\omega) - 2 \gamma^2 (\omega^2) Z(\omega) + \gamma^4 Z(\omega) = \left[ a^4 D_{n1} - a^2 D_{n2} \right] \int_{-\infty}^{\infty} e^{-ax} \sin(ax + bt) dx \]  \hspace{1cm} (J.6)

\[ Z(\omega) \left[ \omega^4 + 2 \gamma^2 \omega^2 + \gamma^4 \right] = \left[ a^4 D_{n1} - a^2 D_{n2} \right] \int_{-\infty}^{\infty} e^{-ax} \sin(ax + bt) dx \]  \hspace{1cm} (J.7)

\[ (\omega^4) Z(\omega) - 2 \gamma^2 (\omega^2) Z(\omega) + \gamma^4 Z(\omega) = \left[ a^4 D_{n1} - a^2 D_{n2} \right] \int_{-\infty}^{\infty} e^{-ax} \sin(ax + bt) dx \]  \hspace{1cm} (J.8)
\[ Z(\omega) \left[ \omega^4 + 2\gamma^2 \omega^2 + \gamma^4 \right] = [a^4 D_{n1} - a^2 D_{n2}] \int_{-\infty}^{\infty} e^{-ax} \sin(ax + bt) dx \tag{J.9} \]

\[ Z(\omega) \left[ \omega^4 + 2\gamma^2 \omega^2 + \gamma^4 \right] = [a^4 D_{n1} - a^2 D_{n2}] \int_{-\infty}^{\infty} \frac{e^{ibt} e^{i\omega x} - e^{-ibt} e^{-i\omega x}}{2i} dx \tag{J.10} \]

\[ Z(\omega) \left[ \omega^4 + 2\gamma^2 \omega^2 + \gamma^4 \right] = [a^4 D_{n1} - a^2 D_{n2}] \sqrt{2\pi} \frac{e^{ibt} \delta(\omega-a) - e^{-ibt} \delta(\omega+a)}{2i} \tag{J.11} \]

\[ Z(\omega) \left[ \omega^4 + 2\gamma^2 \omega^2 + \gamma^4 \right] = [a^4 D_{n1} - a^2 D_{n2}] \sqrt{2\pi} \frac{e^{-ibt} \delta(\omega+a) - e^{ibt} \delta(\omega-a)}{2} \tag{J.12} \]

\[ Z(\omega) = [a^4 D_{n1} - a^2 D_{n2}] \sqrt{2\pi} \frac{e^{-ibt} \delta(\omega+a) - e^{ibt} \delta(\omega-a)}{2(\omega^2 + \gamma^2)^2} \tag{J.13} \]

\[ X(x) = \frac{[a^4 D_{n1} + a^2 D_{n2}]}{2} i \sqrt{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} \frac{e^{-ibt} \delta(\omega+a) - e^{ibt} \delta(\omega-a)}{2(\omega^2 + \gamma^2)^2} d\omega \tag{J.14} \]

\[ X(x) = \frac{[a^4 D_{n1} - a^2 D_{n2}]}{4} i \left[ \frac{e^{-ax} e^{-ibt}}{(a^2 + \gamma^2)^2} - \frac{e^{ibt} e^{ax}}{(a^2 + \gamma^2)^2} \right] \tag{J.15} \]

\[ X(x) = \frac{[a^4 D_{n1} - a^2 D_{n2}]}{4} i \left[ \frac{\cos(ax + bt) - i \sin(ax + bt)}{(a^2 + \gamma^2)^2} - \frac{\cos(ax + bt) + i \sin(ax + bt)}{(a^2 + \gamma^2)^2} \right] \tag{J.16} \]

\[ X(x) = \frac{[a^4 D_{n1} - a^2 D_{n2}]}{2} \left[ \frac{\sin(ax + bt)}{(a^2 + \gamma^2)^2} \right] \tag{J.17} \]

Complete Solution

\[ \psi_0 = \psi_h + \phi \tag{J.18} \]

\[ \psi_h = X(x,t) \sum_{n=1}^{\infty} \sin(ny) \]

\[ X(x,t) = H \sin(ax + bt) \]

\[ H = \frac{[a^4 D_{n1} - a^2 D_{n2}]}{2(a^2 + \gamma^2)^2} \]

\[ \phi = -\frac{f}{6} y^3 + \frac{f}{2} y^2 + \left( \frac{1}{3} T - \frac{f}{3} \right) y \]
\[ f(x) = \sin(ax + bt) + 1 \]

Using the equations developed in Appendix Section A.4 for a small Deborah number perturbation solution and Mathematica to perform the calculus, an approximation to the UCM solution was found. In order to simplify calculations, after all derivatives with respect to time are taken, time is set to zero (we are not interested in plotting the solution as a function of time).

\[
\psi_1(x, y, t = 0) = \sum_{n=1}^{\infty} \frac{a^2 \cos(ax) \sin(y)}{48y^4(4a^4 + 5a^2\gamma^2 + \gamma^4)^2} \left[ \begin{array}{c} -36a^2b\gamma^2 - 72b\gamma^4 \ldots \\ + 9a^3(-53 + 4\gamma^2) \ldots \\ + a(-282\gamma^2 + 8\gamma^4) \\ + a\gamma^2\sin(ax) \\ + a^4(-438 + 64\gamma^2) \ldots \\ + a^2\gamma^2(-867 + 128\gamma^2) \end{array} \right] (J.19)
\]

\[
\psi = \psi_0 + \text{De} \psi_1 \\
(J.20)
\]

In order to quantify the effect of Deborah number, wavelength, and wavespeed on the solution, the phase shift away from the order one \( \psi_0 \) solution is calculated according to the equations below:

\[
\psi = \Lambda \sin(ax + \Theta) \\
= \Lambda [\sin(ax) \cos(\Theta) + \cos(ax) \sin(\Theta)] \\
= \Pi_1 \sin(ax) + \Pi_2 \cos(ax)
\]

\[
\Pi_1 = \Lambda \cos(\Theta) \\
\Pi_2 = \Lambda \sin(\Theta) \\
\Theta = \text{ArcTan} (\Pi_2 / \Pi_1)
\]

Terms which possess both a \( \cos(ax) \) and a \( \sin(ax) \) were including with \( \Pi_1 \). As a result, the phase shift \( \Theta \) is a function of \( x \). For purposes of plotting, a position of \( (0, 0.5) \) was chosen. The goal of plotting acquiring a phase shift is to understand the trend that the aforementioned parameters have on the flow; these trends can be established at any position within the flow.
Works Cited

19. Evans, C. M.; Koo, J. S., Airway mucus: The good, the bad, the sticky. Pharmacology & Therapeutics 2009, 121 (3), 332-348.
21. Erni, P.; Varagnat, M.; McKinley, G. H. In Little shop of horrors: Rheology of the mucilage of Drosera sp., a carnivorous plant, 15th International Congress on Rheology/80th Annual Meeting of the
Society-of-Rheology, Monterey, CA, Aug 03-08; Co, A.; Leal, L. G.; Colby, R. H.; Giacomin, A. J., Eds.
34. Taylor, Analysis of swimming microorganisms. 1951.


