Nonlinear dynamics of a magnetoelastic system

Chetan O. Modi

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NONLINEAR DYNAMICS
OF A
MAGNETOElastic SYSTEM

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A Thesis Submitted in Partial Fulfillment
of the Requirements for the Degree of
Master of Science
in
Mechanical Engineering

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ACKNOWLEDGEMENT

First of all, special love and thanks to my wife Smita for being who she is and for every thing she has done for me. Thanks are also due to our parents for their support and to my brother-in-law Dilip for his help. Finally, I would like to express my appreciation to Dr. Joseph Torok for his valuable suggestions and help.
ABSTRACT

A dynamically buckled elastic beam is a physically realizable system exhibiting both periodic and chaotic behavior. The equations of motion are developed as a finite dimensional Galerkin approximation of an infinite degree of freedom system. Generalized eigenvalues or Lyapunov exponents are introduced as a quantitative characterization of chaos, i.e. unstable but bounded motion. A semi-discrete method for the estimation of the Lyapunov spectrum is used to investigate the influence of the forcing parameters on the system response. The equations of motion are then integrated numerically to correlate the steady state response with the value of the associated largest Lyapunov exponent.
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LIST OF SYMBOLS

$q_i$  generalized coordinates

( $i = 1, 2, \ldots, N$ )

$N$  number of modes

$q_{i}^{o}(t)$  partial derivative with respect to $t$

$w(x, t)$  beam deflection

$T$  total kinetic energy of the system

$V$  total potential energy of the system

$V_1$  elastic plus gravitational potential energy

$V_2$  magnetic potential energy

$D$  dissipation function

$m$  mass per unit length of the pipe-beam

$g$  gravitational constant

$v$  vertical longitudinal displacement of the beam

$E$  Young's modulus

$EI$  flexural rigidity or bending stiffness

$E'$  dynamic viscous modulus

$\phi_s(x)$  characteristic modal function of a uniform cantilevered beam

$\phi'_s(x)$  partial derivative with respect to $x$

$c$  viscous damping coefficient

$K_i$  magnetic stiffness coefficient
\( K_2 \) magnetic stiffness coefficient
\( L \) length of the beam
\( f \) forcing frequency
\( \omega_i \) natural frequencies corresponding to \( \phi_i \)
\( \zeta_i \) modal damping constants
\( \lambda_i \) constants appearing in the natural frequencies of a cantilevered beam
\( \alpha_i \) constants appearing in the mode shapes of a cantilevered beam
\( \psi_j \) nondimensional mode shapes for the cantilevered beam
\( \nu \) dimensionless Kelvin damping parameter
\( \theta_1 \) dimensionless parameter
\( \theta_2 \) dimensionless parameter
\( \tau \) dimensionless time parameter
\( \omega \) dimensionless frequency parameter
\( A_{is} \) dimensionless parameter
\( B_{is} \) dimensionless parameter
\( C_{is} \) dimensionless parameter
\( D_{is} \) dimensionless parameter
\( E_{is} \) dimensionless parameter
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CHAPTER 1

INTRODUCTION

In the recent past, a new field of engineering has been developed, which is concerned with the interaction between an elastic field and an electromagnetic one. This area of specialization is called magnetoelasticity. If a ferromagnetic material is subjected to a magnetic field, it will of course deform due to the magnetic forces. This is a result of an interaction between magnetic induction and induced magnetization. Magnetoelastic buckling, as one of those generic electro-magnetically forced problems, is a very fundamental and important subject. As Moon and Pao [1] have shown, magnetoelastic buckling of a cantilever beam inserted into a uniform magnetic field will occur when the acting magnetic field reaches a critical intensity.

New phenomena have also been investigated in all areas of nonlinear dynamics. Principal among these being chaotic vibrations. The terms random motion and chaotic motion can be distinguished as follows: Random motion is reserved for problems in which we truly do not know the input forces or we know only some statistical measure of the parameters. The term chaotic is reserved for those deterministic problems for which there are no random or unpredictable inputs or parameters, yet the response appears to be highly irregular.
A motivation for the recent development of the field was fostered by a growing interest in nonperiodic, steady-state solutions of nonlinear differential equations. These differential equations are derived from applications in atmospheric dynamics, electrical circuits, and structural analysis.

Chaotic oscillations of systems governed by Duffing's equation with a negative linear stiffness (as defined in Appendix-C) have been studied in the past. In theoretical and experimental studies by Moon [2], and Moon and Holmes [3], magnets were placed to the left and right of the free end of a cantilevered beam in order to buckle the beam. In their investigations, two stable equilibrium positions were found out of a total of three equilibria.

In a design setting, systems are usually modeled by linear differential equations, which can in principle be solved by analytical methods to give a closed-form solution expressible in terms of elementary functions (trigonometric, exponential, polynomial, etc.). These systems are deterministic, since their response is completely described by the governing equations and a set of initial conditions. Thus the physical solutions are predictable for very long times. It should be noted that such regular motion holds for only a relatively small number of idealized systems.

More realistically, physical systems are almost always described by nonlinear equations. These equations, under
reasonable conditions of regularity, or smoothness, are also deterministic. That is, a set of initial conditions determines a unique solution for all time. Even so, a deterministic system can still be unpredictable, since the response of a nonlinear system can be extremely sensitive to the choice of initial conditions.

In the past, the general understanding was that classical systems could be unpredictable only if they were influenced by some random internal or external forcing. Recent studies have shown that many nonlinear systems, that are in principle deterministic, can give rise to highly irregular and unpredictable behavior. The irregularity (meaning nonperiodicity) is attributable to the influence of the nonlinear terms in the differential equations. The unpredictability arises from the fact that these systems can display sensitive dependence to initial conditions. That is, a slight variation in the initial conditions can result in exponentially divergent trajectories. Quantification of this divergent behavior is the central theme of this investigation.

In linear systems theory the steady-state response, away from resonance, is proportional to the forcing term (with possibly a phase shift). Further, any error in the specification of initial conditions propagates at most linearly over the time history. These properties follow from the principle of superposition, which forms the basis of linear analysis. The steady-state response is directly correlated with the applied forcing. For stable systems, the long-term behavior is periodic or converges
asymptotically to an equilibrium state. These facts are well-known to scientists and engineers, but are all too often applied in the wrong situations.

Nonlinear systems, on the other hand, have no associated superposition principles. This is one reason why their analyses can be so formidable. Nonlinear terms in the governing equations can have quite unintuitive effects on the system inputs and hence on the long-term behavior. Consequently, even very small differences in initial conditions can be considerably magnified. This modulation is usually exponential in nature. Bounded systems, which are the ones observed physically, can possess an even more peculiar behavior. Due to exponential magnification of differences (stretching), the dynamics must be such that trajectories in the phase space are "folded" back onto themselves to preserve the bounded nature of the solutions. Thus two infinitesimally close initial conditions could give rise to totally uncorrelated trajectories. This stretching and folding of trajectories can be visualized as the mixing that occurs in a "taffy-pulling" machine. Such behavior is the principle reason for sensitivity to initial conditions.

The stretching (or magnification) along trajectories can be quantitatively analyzed by estimating the Lyapunov spectrum of the dynamical system. The Lyapunov characteristic exponents represent the average rate of deformation of n-dimensional hypercubes along a trajectory. The Lyapunov spectrum represents a direct generalization of the eigenvalues associated with linear
The purpose of this investigation is to develop an efficient method of Lyapunov exponent estimation and to illustrate its implementation in the analysis of a nonlinear magneto-elastic system.

The system under investigation is shown schematically in Fig. 1.1. A flexible pipe made of Tygon material is cantilevered and placed between two magnets. Tygon has higher material damping, larger mass per unit length and lower stiffness than typical metals. A steel (ferromagnetic) beam is placed inside this pipe. Thus a combined pipe-beam is constructed which has a very low lateral stiffness and high longitudinal stiffness. The pipe-beam is clamped at its upper end, and free at its lower end. Two permanent magnets are placed to the left and right of the free end of the pipe-beam, which is buckled by the magnetic forces. In Fig. 1.1, \( w_b \) represents the static displacement of the tip of the beam in a buckled state. This represents a stable equilibrium position. The state in which the beam is perfectly straight is clearly an unstable equilibrium configuration. The system is harmonically forced at a specified location.

The derivation of the governing partial differential equation and the associated modal approximation for the magnetically buckled beam system parallels the experimental work done by Tang and Dowell [4]. By choosing a set of convenient generalized coordinates, \( q_i(t) \), the equations of motion are obtained from variational principles by way of Lagrange's equations of motion. It is assumed that the beam's deflection, \( w(x,t) \), can be
considered as the sum of displacements in the $N$ lowest modes of a uniform cantilever beam. Given the mode shapes associated with the vibrations of a uniform cantilevered beam, $\varphi_i(x)$, the deflection is assumed to be of the form:

$$w(x,t) = \sum q_i(t) \varphi_i(x)$$  \hspace{1cm} \ldots (1.1)
Based on the above-mentioned expression for the beam deflection, the governing differential equations are derived by substituting equation (1.1) into Lagrange's equations of motion, as developed in Chapter 3. Furthermore, Lyapunov exponents are generated for the Rayleigh-Ritz modal approximation of the governing partial differential equation derived for the magnetically-buckled cantilevered beam. The Lyapunov exponents are generated using a semi-discrete formulation based on a forward-advance mapping. The Lyapunov exponents characterize the qualitative dynamics of the system. Since all Lyapunov exponents associated with this work are generated for the post-transient dynamics, a positive Lyapunov exponent associated with a steady-state attractor indicates chaotic trajectories.
[2.1] INTRODUCTION:

There is a definite need for nonlinear analysis in the modeling and design of modern structural systems. Today's technology requires the use of high-performance machines and structures. Modern systems necessitate an increased need for higher operating speeds. Simultaneously, materials are pushed into regimes near conditions that are deemed borderline as far as safe behavior is concerned. Traditional linear theory is inadequate for modeling many observed phenomena. It is thus necessary to perform nonlinear analysis, including higher order terms, of the equations that are utilized in the simulation of a physical system.

From an engineering point of view, an analyst must be aware of the physics governing the system, so that a reasonable model is deduced. Further, there must be some confidence in the inclusion or exclusion of various effects and factors in the development of the model. From a mathematical point of view, a thorough analysis must be performed. Since these systems are nonlinear, traditional methods are inadequate for the solution of the governing equations and the characterization of the associated dynamics.
Nonlinear dynamical systems theory provides an excellent framework in which to characterize and assess the complicated behavior exhibited by nonlinear systems. Another practical aspect of the modeling and simulation of a system is the physical interpretation of the results. Apparent irregular behavior need not be attributable to noise.

[2.2] ANALYSIS OF NONLINEAR SYSTEMS:

Until relatively recently, analysts were significantly restricted by necessity to limit analysis of structural and mechanical systems to linear techniques. The few available methods for nonlinear analysis, such as perturbation methods, equivalent linearization and the method of Krylov, Bogoliubov and Mitropolski (KBM), were impeded by computational tedium to systems admitting only several degrees of freedom (Minorsky [5]). Phase plane analysis was limited to two state variables [6-7]. Higher order systems were analyzed by iterative techniques. For the applications involved, the analyses were essentially satisfactory. Increased system performance and higher thresholds of material behavior render traditional linear analyses unacceptable. The advent of high-speed computing, coupled with the development of modern nonlinear dynamics theory, allowed scientists and engineers to approach nonlinear problems with a whole new point of view. New techniques were developed for the analysis and characterization of nonlinear physical, biochemical and ecological systems.
Most nonlinear systems fall outside of the domain of traditional closed-form analysis. The differential equations that govern the evolution of a dynamic system are typically integrated numerically to develop the time history. Difference equations governing discrete systems must be iterated to establish long-term behavior. The success of modern nonlinear dynamics theory in the analysis of system data parallels the development of new concepts and techniques for the characterization of the dynamics, both locally and globally, quantitatively and qualitatively. Nonlinear systems are not subject to the principle of superposition, and thus can lead to unintuitive and unexpected behavior. Phenomena not present in linear systems include subharmonic and superharmonic resonances, jumps, quasiperiodicity, intermittency and chaos.

The most interesting behavior exhibited by many nonlinear systems is that of chaotic behavior. Chaotic behavior essentially entails highly irregular motion, compounded by sensitivity to initial conditions. From a mathematical point of view, chaotic systems have been the subject of intense investigation for nearly two decades [8-15]. In particular physiological and ecological systems, chaotic behavior is necessary to maintain certain irregulatory functions in biological systems [15]. Periodic or regular behavior in some biological systems is highly undesirable. Chaotic behavior in physical systems is characterized by
an apparent erratic behavior and a loss of predictability due to sensitivity to initial conditions. From a mechanical/structural point of view, this is undesirable and the effects can be devastating.

Much of the development of modern nonlinear dynamics theory has been in the area of bifurcation theory and the enhancement of practical methods for the qualitative and quantitative analysis of data from experimental measurements and time evolution of the solutions to governing differential equations and mappings. Bifurcation theory focuses on the changes in equilibrium configurations of nonlinear systems as characteristic parameters are varied, and also on the stability of local structures such as equilibrium manifolds and attractors.

The development of global methods using new mathematical techniques has been the highlight of the modern approach to nonlinear dynamics. These include Poincare mapping (first introduced by Poincare in 1881), fractal geometry, symbolic dynamics, invariant measures, Lyapunov spectrum estimation, circle maps, cell-to-cell mapping and Melnikov theory. Most of these methods give qualitative information about dynamical systems. Analytical techniques such as the Melnikov method and the calculation of Lyapunov exponents have led to quantitative prediction of chaos criteria for certain nonlinear systems. Classical methods of analysis such as the measurement of Fourier spectra have also been enhanced to aid in the diagnosis of chaotic phenomena.

New concepts for the improved understanding of the diverse
behavior of nonlinear systems have been anchored on firm theoretical foundations. These concepts and principles include period-doubling, homoclinic orbits, horseshoe maps, strange attractors, Arnold tongues, fractal basin boundaries and center manifold theory [8-14]. These ideas are essential for a complete mathematical understanding of the rich behavior exhibited by nonlinear dynamical systems. Much theoretical work certainly remains to be done, but at the same time there is a vital need to apply these ideas and techniques to systems of practical physical significance.

[2.4] MECHANICAL AND STRUCTURAL SYSTEMS:

The area of applied mechanics abounds with nonlinear systems; in fact they are the rule rather than the exception. The following are examples of observed physical phenomena that can only be modeled by nonlinear governing equations. The characteristics of motion are extraordinary in the sense that they often contradict expectations based on linear theory.

Tung and Shaw [16] have analyzed the dynamic performance of impact print hammers used extensively by the computer industry. The speeds of printers are often limited by measured chaotic vibrations at high speeds. They develop criteria to demonstrate the limitations on printer speed and propose a control method that is able to increase that speed. The criteria are quite general and may be used for more complicated models and controllers.
Vibrating systems with impact constraints possess extremely complicated dynamical behavior. A simple but interesting problem of a ball bouncing on a vibrating table was investigated by Holmes [17] and Torok and Wissinger [18]. Similar systems consist of mechanical linkages with free play. Mechanisms such as these were analyzed by Shaw [19] and Thompson and Ghaffari [20]. Since mechanical/structural systems with rigid and elastic amplitude constraints play a significant role in the generation of industrial noise and vibration, it is of interest to study their dynamics.

An autonomous structural system exhibiting chaotic behavior is the flutter resulting from fluid flow over a buckled elastic plate. Panel flutter occurred in the outer skin of Saturn rockets in the early 1970s. Dowell [21] and his coworkers have performed extensive numerical simulations and observed stable motions for certain sets of conditions and chaotic vibrations for other sets of loading conditions. The characterization of the response of such systems is essential in the design of safe and reliable aeroelastic structures.

Meijaard and De Pater [22] investigated the motion of a railway wheelset moving with constant forward velocity. The lateral motion is limited by flanges and the rails have sinusoidal lateral deviations. They found that at elevated speeds, chaotic vibrations are possible when large rail deviations are present. Analysis of similar transportation-related mechanical/structural systems is
necessary for the prevention of costly accidents.

Perhaps the most well-known examples of chaotic behavior in structural systems, for which chaos has been experimentally observed, are associated with the vibration of buckled beams and columns. Chaotic solutions for a periodically-forced buckled beam were first studied by Holmes [23]. Later, experiments were performed by Moon and Holmes [24], which supported the mathematical analysis predicting the existence of an associated strange attractor. Simiu [25] and his coworkers at NIST investigated the mechanics of a modified Stoker column. They analyzed the governing differential equation for periodic as well as chaotic motion. A mechanical device was also built which experimentally verified the predicted behavior, qualitatively duplicating the time histories and broadband spectrum which is characteristic of chaotic motion.

The present investigation is also concerned with vibrations of a magnetically buckled beam. The system will be approximated by a finite number of degrees of freedom. The resulting finite dimensional dynamic system will be analyzed quantitatively from the point of view of calculating its Lyapunov exponents. It will be shown that a numerically efficient procedure can be developed and that there is a correlation between the sign of the largest Lyapunov exponent and the steady-state behavior of the system.
CHAPTER 3

THE GOVERNING DYNAMICAL SYSTEM

[3.1] DERIVATION OF GOVERNING EQUATIONS:

The equations of motion governing the magneto-elastic system described in Chapter One are obtained by the implementation of Lagrange's equations in the form:

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial q_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial q_i} = Q_i
\]

( \( i = 1, 2, 3, \ldots , N \) ) ....(3.1)

where,

- \( T \) = kinetic energy of the system
- \( q_i(t) \) = generalized coordinates
- \( V \) = total potential energy of the system
- \( D \) = dissipation function
- \( Q_i \) = generalized force not derivable from \( V \) or \( D \)

We assume that the beam deflection, \( w(x,t) \), can be expressed as the sum of displacements in the \( N \) lowest modes of a uniform cantilevered elastic beam. That is

\[
w(x,t) = \Sigma_s [q_s(t) \varphi_s(x)]
\]

( \( s = 1, 2, 3, \ldots , N \) ) ....(3.2)
This procedure is known as Galerkin's method, which is used extensively to approximate infinite degree of freedom systems by an equivalent finite dimensional system (Reddy[26]).

The boundary conditions for the shape functions are:

(i) \( \phi_i(0) = 0 \)
(ii) \( \phi_i'(0) = 0 \)
(iii) \( \phi_i''(L) = 0 \) \((i = 1,2,3)\)
(iv) \( \phi_i'''(L) = 0 \) \((i = 1,2,3)\)

From equation (3.2) it follows that:

\[
\frac{\partial w}{\partial t} = \sum_s \left[ q_s(t) \phi_s(x) \right]
\]

\[
\frac{\partial w}{\partial x} = \sum_s \left[ q_s(t) \phi_s'(x) \right]
\]

\[
\frac{\partial^2 w}{\partial x^2} = \sum_s \left[ q_s(t) \phi_s''(x) \right]
\]

Now the kinetic energy of the system is given as:

\[
T = 0.5m \int_0^L (\partial w/\partial t)^2 \, dx
\]

Hence,

\[
T = 0.5m \int_0^L \left[ \sum q^0_s(t) \phi_s(x) \right]^2 \, dx \quad \ldots (3.3)
\]
\[
\begin{align*}
\frac{\partial T}{\partial q^i} &= m \sum_s \int_0^L \left[ q^s(t) \sigma_s(x) \sigma_i(x) \right] dx \\
\frac{\partial T}{\partial q_i} &= 0 \quad \ldots (3.4)
\end{align*}
\]

The vertical longitudinal displacement of the beam is:

\[
v(x) = 0.5 \int_0^x (\partial w/\partial x)^2 dx \quad (0 \leq x \leq L) \quad \ldots (3.5)
\]

The combined elastic plus potential energy is then:

\[
V_1 = 0.5EI \int_0^L (\partial^2 w/\partial x^2)^2 dx + 0.5mg \int_0^L \left[ \int_0^x (\partial w/\partial x)^2 dx \right] dx
\]

The magnetic forces and couples due to the magnetic field acting on the system are assumed to be concentrated at the free end of the beam. The magnetic energy potential can be found (see Moon [2]) in terms of the displacement at the free end and it is given by:

\[
V_2 = 0.5 K_1 w^2(L, t) + 0.25 K_2 w^4(L, t)
\]

The total potential energy is:

\[
V = V_1 + V_2 \quad \ldots (3.6)
\]

Hence,
\[ V = 0.5EI \int_0^L \left[ \sum q_s(t) \varphi_{ss}(x) \right]^2 \, dx \]
\[ + 0.5mg \left( \int_0^L \left[ \int_0^L \left( \sum q_s(t) \varphi_s'(x) \right)^2 \, dx \right] \, dx \right) + 
\[ 0.5K_1 \left[ \sum q_s(t) \varphi_s(L) \right]^2 + 0.25K_2 \left[ \sum q_s(t) \varphi_s(L) \right]^4 \]

It follows that:

\[ \frac{\partial V}{\partial q_i} = EI \int_0^L \left\{ \sum q_s(t) \varphi''_s(x) \varphi''_i(x) \right\} \, dx + 
\[ mg \left( \int_0^L \left[ \int_0^L \left( \sum q_s(t) \varphi'_s(x) \varphi'_i(x) \right) \, dx \right] \, dx \right) + 
\[ K_1 \left[ \sum q_s(t) \varphi_s(L) \varphi_i(L) \right] + K_2 \left[ \sum q_s(t) \varphi_s(L) \right]^3 \varphi_i(L) \] (3.7)

The corresponding dissipation function in Lagrange's equations is expressed as:

\[ D = 0.5c_0 \int_0^L (\partial w/\partial t)^2 \, dx = 0.5c_0 \int_0^L \left[ \sum q^o_s(t) \varphi_s(x) \right]^2 \, dx \]

Hence,

\[ \frac{\partial D}{\partial q^o_i} = c_0 \int_0^L \sum q^o_s(t) \varphi_s(x) \varphi_i(x) \, dx \] ... (3.8)

Again from equation (3.4), we get

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial q^o_i} \right) = \sum \int_0^L m q^o_s(t) \varphi_s(x) \varphi_i(x) \, dx \] ... (3.9)

On substituting equations (3.4), (3.7), (3.8) and (3.9) into equation (3.1), the equations of motion are deduced
as follows:

\[
\sum s_0 \int_L \left[ m q_s(t) \varphi_s(t) \varphi_1(x) + EI q_s(t) \varphi^"_s(x) \varphi^"_1(x) + c q_s(t) \varphi_s(x) \varphi_1(x) + mg q_s(t) \int x \varphi'_{s}(x) \varphi'_{1}(x) \, dx \right] + \\
k_1 \left[ \sum q_s(t) \varphi_s(t) \right] \varphi_1(L) + k_2 \left[ \sum q_s(t) \varphi_s(L) \right]^3 \varphi_1(L) = Q_i(t) \\
( i = 1,2,3,\ldots,N ) \ldots(3.10)
\]

The dissipation of energy in the system induced by damping is very complex. In the present system, there are two possible types of damping to be considered, namely viscous damping and Kelvin damping. Viscous damping is often used in models of vibrating systems and Kelvin damping is widely accepted as representing internal damping in viscoelastic materials. Its properties are expressed by a dynamic viscous modulus coefficient \( E^* \). In Lagrange's equations, the Young's modulus \( E \) will be simply replaced by the expression:

\[
E + E^* \partial/\partial t \ldots(3.11)
\]

The exciting force is assumed to be a concentrated force applied at point \( x = x_F \). Moreover, the virtual work done by exciting force is:

\[
\delta W = F(t) \delta w(x_F,t)
\]
\[ \delta W = F(t) \sum_s \left[ \delta q_s(t) \, \varphi_s(x_F) \right] \quad \ldots (3.12) \]

Thus, the generalized force, \( Q_i \), in equation (3.10) can be given as:

\[ Q_i = F \, \varphi_i(x_F) \quad (i = 1, 2, \ldots, N) \quad \ldots (3.13) \]

Applying the boundary conditions, after integration by parts, results in

\[ \int_0^L \varphi''_s(x) \, \varphi''_i(x) \, dx = \int_0^L \varphi'^i_s(x) \, \varphi_i(x) \, dx \quad \ldots (3.14) \]

Similarly, the term in equation (3.10) with double integrals can be replaced by the following

\[ \int_0^L \int_0^x \left[ \varphi'_s(x) \, \varphi'_i(x) \right] \, dx = \int_0^L \left[ \varphi'_s(x) \, \varphi_i(x) - (L-x) \varphi''_s(x) \, \varphi_i(x) \right] \, dx \quad \ldots (3.15) \]

Thus, on substituting equations (3.11), (3.13), (3.14), and (3.15) into equation (3.10), we obtain:
\[
\sum_{s} \int_{0}^{L} \left[ m q_s(t) \phi_s(x) + EI q_s(t) \phi_{s}''(x) + E*I q_s(t) \phi_{s}'(x) \right] dx + c q_s(t) \phi_s(x) + m g q_s(t) \phi_s'(x) - m g (L-x) q_s(t) \phi''_s(x) \right] \phi_s(x) dx \\
+ K_1 \left[ \sum q_s(t) \phi_s(L) \right] \phi_1(L) + K_2 \left[ \sum q_s(t) \phi_s(L) \right] ^3 \phi_1(L) \\
= F(t) \phi_1(x_F) \\
( i = 1, 2, 3, \ldots , N ) \quad \ldots (3.16)
\]

[3.2] NONDIMENSIONAL PARAMETERS:

The following dimensionless parameters and coordinates are introduced in order to put equation (3.16) into a dimensionless form.

\( x = \frac{x}{L} \)
\( \psi_s = \frac{\phi_s(Lx)}{L} \)
\( a_s = \frac{q_s}{L} \)
\( f = \frac{F_o L^2}{EI} \)
\( w = \frac{w}{L} \)
\( \sigma = \frac{mgL^3}{EI} \)
\( \zeta = \frac{cL^2}{(EI m)^{0.5}} \)
\( \nu = (E^*/L^2E) \{ EI/m \}^{0.5} \)
\( \theta_1 = \frac{K_1 L^3}{EI} \)
\( \theta_2 = \frac{K_2 L^5}{EI} \)
\( \tau = \left[ \frac{EI}{mL^4} \right]^{0.5} t \)

Consequently, equation (3.16) can be rewritten in a more
compact dimensionless form as follows:

\[
\sum \left[ A_{is}a^{\infty} + B_{is}a^s + C_{is} \right] + \theta_2 \left[ \sum a_s \psi_s(1) \right]^3 \psi_i(1)
\]

\[= f \psi_i(x_F) \quad (i = 1, 2, 3, \ldots, N) \quad \ldots(3.17)\]

where:

\[A_{is} = 0, \quad (i \neq s)\]
\[A_{ii} = 1\]
\[B_{is} = 0, \quad (i \neq s)\]
\[B_{ii} = 2\xi_i \lambda_i^2\]
\[C_{ii} = \lambda_i^4 + \sigma D_{i1} + \theta_1 E_{ii}\]
\[C_{is} = \sigma D_{is} + \theta_1 E_{is} \quad (i \neq s)\]
\[D_{is} = G_{is} \left[ t_i^2(2 + G_{is}) - t_i^2(\sigma_i \lambda_i - \sigma_s \lambda_s) - (-1)^{i+s}(4 + G_{is}) \right]\]
\[D_{ii} = 0.5 \alpha_i \lambda_i^2 - \alpha_i \lambda_i + 2\]
\[E_{is} = \psi_i(1) \psi_s(1)\]
\[t_{is} = \lambda_i / \lambda_s \quad (i \neq s)\]
\[G_{is} = 4 / (t_i^4 - 1) \quad (i \neq s)\]

[3.3] MODAL ANALYSIS OF A CANTILEVERED BEAM:

In the above equations, \(\lambda_i\) are the constants appearing in the natural frequencies of a uniform cantilevered beam in free vibration. The \(\alpha_i\) are the constants identified with the characteristic mode shapes of cantilevered beam vibration (See Appendix - A for a derivation and
More specifically,
\[ \omega_N = \lambda^2_N (EI/ml^4)^{0.5} \]  \hspace{1cm} (3.18)

and
\[ \phi_i(x) = \cosh \lambda_i x - \cos \lambda_i x - \alpha_i (\sinh \lambda_i x - \sin \lambda_i x) \]  \hspace{1cm} (3.19)

in which
\[ \alpha_i = \frac{\cos \lambda_i L + \cosh \lambda_i L}{\sin \lambda_i L + \sinh \lambda_i L} \]  \hspace{1cm} (3.20)

The modal functions are normalized so that:
\[ \int_0^L [\phi_i(x)]^2 \, dx = L \quad (i = 1, 2, 3, \ldots, N) \]  \hspace{1cm} (3.21)

The values of these constants are computed numerically as
\[ \alpha_1 = 0.7340955 \quad \lambda_1 = 1.8751 \]
\[ \alpha_2 = 1.0184673 \quad \lambda_2 = 4.6941 \]
\[ \alpha_3 = 0.9992244 \quad \lambda_3 = 7.8548 \]

Now, if the system is subjected to a concentrated harmonic force \( F_0 \sin \omega t \), the dimensionless forcing frequency is:
\[ \Omega = \omega [ml^4/EI]^{0.5} \]  \hspace{1cm} (3.22)
Therefore the forcing function can be written as:

\[ F = F_0 \sin(\omega \tau) \] \hspace{1cm} \ldots (3.23)

Thus equation (3.17) can be rewritten as:

\[ \sum_s \left[ A_{is} a^{0s} + B_{is} a^{os} + C_{is} \right] + \theta_2 \left[ \sum a_s \psi_s(1) \right]^3 \psi_i(1) \]

\[ = F \psi_i(x_F) \]

( \( i = 1, 2, 3, \ldots \), N ) \hspace{1cm} \ldots (3.24)

[3.4] NUMERICAL VALUES OF DIMENSIONLESS PARAMETERS AND CONSTANTS:

Using the above mentioned equations and definitions, numerical values of dimensionless parameters and constants are calculated. These numerical values are listed as shown in Appendix-B. The physical constants used in numerical computations and numerical simulations follow directly from an experiment performed by Tang and Dowell [4]. These physical constants are also listed in Table-1 of Appendix-B.

[3.5] GOVERNING EQUATIONS FOR \( N = 1, 2, \) AND 3:

Further, equation (3.17) can be rewritten for selected values of \( N \) and \( i \) as follows:
[A]  \( N = 1: \text{Single Mode Approximation} \)

\[
(A_{11}a_0^0 + B_{11}a_0 + C_{11}a_1) + \theta_2 \left[ a_1\psi_1(l) \right]^3 \psi_1(l) = F\sin(\omega_1 t) \psi_1(x_F)
\]

[B]  \( N = 2: \text{Two Mode Approximation} \)

\[
\left[ (A_{11}a_0^0 + B_{11}a_0 + C_{11}a_1) + (A_{12}a_0^2 + B_{12}a_0 + C_{12}a_2) \right] +
\theta_2 \left[ a_1\psi_1(l) + a_2\psi_2(l) \right]^3 \psi_1(l) = F\sin(\omega_2 t) \psi_1(x_F)
\]

and

\[
\left[ (A_{21}a_0^0 + B_{21}a_0 + C_{21}a_1) + (A_{22}a_0^2 + B_{22}a_0 + C_{22}a_2) \right] +
\theta_2 \left[ a_1\psi_1(l) + a_2\psi_2(l) \right]^3 \psi_2(l) = F\sin(\omega_2 t) \psi_2(x_F)
\]

[C]  \( N = 3: \text{Three Mode Approximation} \)

\[
\left[ (A_{11}a_0^0 + B_{11}a_0 + C_{11}a_1) + (A_{12}a_0^2 + B_{12}a_0 + C_{12}a_2) +
(A_{13}a_0^3 + B_{13}a_0 + C_{13}a_3) \right] + \theta_2 \left[ a_1\psi_1(l) + a_2\psi_2(l) + a_3\psi_3(l) \right]^3 \psi_1(l) = F\sin(\omega_3 t) \psi_1(x_F)
\]

and

\[
\left[ (A_{21}a_0^0 + B_{21}a_0 + C_{21}a_1) + (A_{22}a_0^2 + B_{22}a_0 + C_{22}a_2) +
(A_{23}a_0^3 + B_{23}a_0 + C_{23}a_3) \right] + \theta_2 \left[ a_1\psi_1(l) + a_2\psi_2(l) + a_3\psi_3(l) \right]^3 \psi_2(l) = F\sin(\omega_3 t) \psi_2(x_F)
\]
and

\[
\left[ (A_3a_0^1 + B_3a_0^1 + C_3a_1) + (A_3a_0^2 + B_3a_0^2 + C_3a_2) + (A_3a_0^3 + B_3a_0^3 + C_3a_3) \right] + \vartheta_2 \left[ a_1\psi_1(1) + a_2\psi_2(1) + a_3\psi_3(1) \right]^3 \psi_3(1) = F\sin(\omega_3 t) \psi_3(x_F) \quad \ldots (3.25)
\]

These governing equations of motion will be integrated numerically for various values of the forcing parameters.
CHAPTER 4
CALCULATION OF LYAPUNOV EXPONENTS

[4.1] DEFINITION OF LYAPUNOV EXPONENTS:

Lyapunov exponents are a generalization of the eigenvalues of a dynamical system at an equilibrium point. They are used to determine the stability of any type of steady-state behavior, including chaotic solutions. More specifically, Lyapunov exponents measure the exponential rates of divergence or convergence associated with nearby trajectories.

Figure 4.1
For periodic motion, the spectrum of exponents contains only zero or negative values, indicating convergence to a highly predictable motion. At the other extreme, a chaotic system will exhibit at least one positive exponent. A positive exponent is significant, because it gives an indication of the rate at which one loses the ability to predict the system response. This is closely related to the property of sensitive dependence on initial conditions, which is characteristic of chaotic systems. Therefore, one way to determine if a system is behaving in a chaotic manner is to calculate the Lyapunov exponents. A further motivation for calculating these exponents is that a knowledge of the entire spectrum of Lyapunov exponents can be used to calculate an approximate value of the fractal dimensions of the attractor [12]. Unfortunately, there is no general analytical way to determine the Lyapunov exponents for a general system of equations. The Lyapunov spectrum must be numerically approximated.

[4.2] LINEAR SYSTEMS:

It is quite simple to calculate the Lyapunov exponents of a linear system. Linear differential equations can in principle be solved exactly, and thus the exponents can be determined by an inspection of the solution. For example, the unforced problem

\[ q^{\circ \circ} + 5q^{\circ} + 6q = 0 \]

has the solution
\[ q = A e^{\lambda_1 t} + B e^{\lambda_2 t} \]

in which the Lyapunov exponents, \( \lambda_1 \) and \( \lambda_2 \), are equal to \(-2\) and \(-3\), respectively.

Even for the case of a forced system such as

\[ q^{\circ} + 5q^{\circ} + 6q = F \cos(\omega t) \]

the same exponents are obtained, but with an additional exponent that has a zero real part, i.e. purely imaginary. This fact is obvious from the general solution of the problem, which is easily computed as

\[ q = A e^{\lambda_1 t} + B e^{\lambda_2 t} + C(e^{i\omega t} + e^{-i\omega t}) \]

It is interesting to note that the exponents in this example are constant, in that they do not depend on initial conditions. Furthermore, since the forcing function does not alter the complementary solution, \( \lambda_1 \) and \( \lambda_2 \) are independent of \( F \) and \( \omega \). This makes the computation of Lyapunov exponents for linear systems very easy.

[4.3] NONLINEAR SYSTEMS:

Nonlinearities can cause repeated stretching and folding of even a small region of initial conditions as it evolves in its state space [14]. This causes the locally
Fig 4.2

Fig 4.3
determined Lyapunov exponents to vary considerably over a trajectory. Thus, it is necessary to examine the long-time average of the exponents. Furthermore, unlike the case for linear systems, the Lyapunov exponents change with forcing amplitude and frequency. This dramatic effect is the motivation of this investigation.

Even for a relatively simple nonlinear differential equation, such as the Duffing equation (see Appendix-C), periodic as well as chaotic response can be obtained for a given set of system parameters merely by changing the forcing amplitude and frequency. For example, the forced Duffing equation

\[ q^{\circ} + 0.1q^{\circ} - q + q^3 = 3.2\cos(\omega t) \]

yields a periodic response for \( \omega = 0.4817 \), but exhibits chaotic behavior for \( \omega = 0.475 \), as shown in Figures 4.2 and 4.3. This indicates that one of the exponents has shifted from negative to positive over a very small change in the forcing frequency \( \omega \).

[4.4] **ESTIMATION OF LYAPUNOV EXPONENTS:**

In higher-order systems, Lyapunov exponents are determined by examining the long-term evolution of an infinitesimal n-sphere of initial conditions, with 'n' being the order of the dynamical system. As the system evolves, the initial n-sphere will expand or contract along its n principal axes, resulting in a deformed n-ellipsoid.
The general method of numerically calculating the exponents of a dynamical system proceeds in the following manner. First, a point that lies near the steady-state attractor of the system of interest is selected. This means that the system must first be integrated well into its post-transient state. Initially, a vector of magnitude $\varepsilon$ and arbitrary direction is chosen and placed with its base at the point on the trajectory. It is important that $\varepsilon$ be small because as pointed out above, a repeated stretching and folding can take place in the state space.
Only the stretching of the space is of interest here. A small test vector is better able to avoid any effects of folding. A second vector, perpendicular to the first, but equal in magnitude, is also constructed at the test point (see figure 4.5).

Additional vectors are added in a similar fashion until the vector set forms an orthogonal basis \( \{b_i\} \) for the state space in the region of the test point. The test point and nearby initial conditions determined by the vector set are then integrated a short time into the future.
After integration, the largest vector is used to calculate the largest local exponent from the equation

$$\lambda_1 = \left( \frac{1}{\Delta t} \right) \ln \left( \frac{l_f}{\varepsilon} \right)$$

where $\Delta t$ is the time interval over which the system is integrated and $l_f$ is the length of the largest vector after integration. This vector will automatically tend toward the direction of maximum divergence (or convergence). The second vector is not free to tend toward the direction associated with the second largest exponent because of the stretching effect of the largest exponent upon its evolution. Instead, the second exponent is determined from the calculation of the sum of the first two exponents, which measures the rate of contraction of an area in state space. This is obtained from the equation

$$\lambda_1 + \lambda_2 = \left( \frac{1}{\Delta t} \right) \ln \left( \frac{A_f}{\varepsilon^2} \right)$$

in which $A_f$ is the final area of the space determined by the first two vectors. Subsequent exponent sums are computed in a similar fashion for higher dimensional systems [14].

The largest vector is then normalized to a magnitude of $\varepsilon$, while its direction is preserved, enabling this vector to continue converging to the direction associated with the largest exponent.
Additional vectors are again constructed perpendicular to the first. This is repeated over a long time in the simulation and the exponents are then calculated as a long-time average over the steady-state motion. This long-time average is extremely important, as even very close trajectories leading to periodic motion can temporarily diverge from each other over short time intervals. This phenomenon is what characterizes transient chaos.

This procedure is based directly on the definition of the Lyapunov spectrum. The one drawback is that the collection of initial conditions must be repeatedly integrated over short time intervals. The evolution of an n-sphere must be carefully tracked to determine the divergence (convergence) rates along the attractor. In the following section, a more convenient method is proposed based on the concept of a forward-advance mapping.

[4.5] COMPUTATION OF LYAPUNOV EXONENTS:

The determination of the Lyapunov exponents for the differential equations (3.25) will be accomplished using a semi-discrete formulation. The dynamics of the system will be discretized by the introduction of a time-advance mapping. Consequently, Lyapunov exponents can be calculated by using techniques that work well for discrete dynamical systems, as addressed below.

Consider a nonlinear point mapping defined on \( R^N \) by the equation
$$x_{n+1} = F(x_n), \ n > 1 \quad \ldots (4.1)$$

Then to first order,

$$\delta x_{n+1} = F(x_n + \delta x_n) - F(x_n) \quad \ldots (4.2)$$

$$= [DF]_n \delta x_n \quad \ldots (4.3)$$

in which $[DF]_n$ is the Jacobian of $F(x)$ evaluated at $x = x_n$. From the usual definition

$$DF_{ij} = \frac{\partial F_i}{\partial x_j} \quad \ldots (4.4)$$

Thus equation (4.3) gives an estimate of the variation at the $(n+1)^{st}$ step of the iteration in terms of the variation at the $n^{th}$ step. Repeatedly applying equation (4.3) results in

$$\delta x_{n+1} = DF_n*DF_{n-1}*DF_n-2* \ldots \ldots *DF_1(\delta x_1) \quad \ldots (4.5)$$

Equation (4.5) represents the variation at the $(n+1)^{st}$ step of in terms of the initial variation $\delta x_1$. In equation (4.5), the product of the Jacobians acts on the initial variation vector, $\delta x_1$. To avoid divergence of the vector norms, due to repeated application of the Jacobian matrices, it is more convenient to express equation (4.5) as
\[ \delta x_{n+1} = D\!F_n \cdot [D\!F_{n-1} \cdot \cdots \cdot D\!F_1 (\delta x_1)] \] .. (4.6)

In equation (4.6), the latest Jacobian is applied to the current set of vectors. To avoid divergence of the variations, due to stretching of orbits, a Gram-Schmidt orthonormalization procedure is applied at each step to accomplish two things:

(i) estimate the local growth rate of the vectors

(ii) replace the vectors with a renormalized set as described in the previous section.

The vector with a largest growth rate is always renormalized and used as the first replacement vector. It should be noted that the growth rate of the local basis vectors is governed by the absolute values of the eigenvalues at each iteration. Denote the eigenvalues of \([D\!F]_n\) at the \(n^{th}\) iteration by

\[ \Lambda_1(n), \Lambda_2(n), \ldots, \Lambda_n(n). \] .. (4.7)

For non-degenerate cases, the magnitude of each eigenvalue can be expressed as

\[ \Lambda_i(n) = \exp(\lambda_{loc,i}(n)) \] .. (4.8)

in which
\[ \lambda_{loc,i}(n) = \log \Lambda_i(n) \] \hspace{1cm} \ldots (4.9)

represents the local convergence (or divergence) rate.

The Lyapunov exponents are computed as a long-time average, consequently the global behavior of the mapping is determined by the eigenvalues of the product Jacobian

\[ [JP] = [DF_N][DF_{N-1}] \ldots [DF_1] \] \hspace{1cm} \ldots (4.10)

as \( n \to \infty \). Thus the Lyapunov exponents for mappings such as (4.1) are defined as

\[ \lambda_i = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \log \Lambda_i(k) \] \hspace{1cm} \ldots (4.11)

In order to apply the above to continuous dynamical systems, the flow must be discretized. That is, the governing differential equations must be integrated forward, for arbitrary initial conditions, to a specified time in order to construct a time-advance mapping. The procedure is outlined as follows.

An initial value problem defined by a system of differential equations

\[ \dot{x} = X(x,t), \quad x(0) = x_0 \] \hspace{1cm} \ldots (4.12)

is integrated from the initial condition \( x_0 \). Denote this solution as \( x(t;x_0) \). Since the initial condition, \( x_0 \), is arbitrary, a sequence of points is defined inductively as
\[
x_{n+1} = F(x_n) \quad \cdots (4.13)
\]

in which

\[
x_{n+1} = x(h; x_n) \quad \cdots (4.14)
\]

That is, \( x_{n+1} \) is obtained by computing the time-advance of the solution starting at \( x_n \). At each step, the initial point \( x_0 \) in (4.12) is replaced by the current state vector \( x_n \). That is, \( x_{n+1} \) is obtained by advancing the solution of (4.12) over a time \( \Delta t = h \), from an initial point \( x_n \).

Since most differential equations cannot be integrated analytically, the time-advance mapping must be constructed with the aid of numerical integration schemes. The drawback of using a numerical routine such as Runge-Kutta, is that an explicit form of the time-advance mapping (4.13) is not obtained.

The standard method of estimating local divergence (or convergence) rates entails the comparison of the time-evolution of two trajectories at neighboring points [11]. The neighboring trajectories are tracked by solving the associated variational equation

\[
d/dt (\delta x) = [J] \delta x
\]

\[
\delta x(0) = \delta x_0 \quad \cdots (4.15)
\]
simultaneously with the original differential equation (4.12). The variational equations (4.15) are obtained by applying the variational operator to equation (4.12).

Although equation (4.15) is linear, the coupled system is now twice the order of the original system (4.12). This makes the numerical integration more tedious. Essentially, the differential equations along with the coupled set of local variational equations must be numerically integrated over small displacements in the phase space. The local divergence (or convergence) rates are determined by analyzing the evolution of the variations $\delta x$.

A more attractive approach is to construct an approximate, yet explicit, form of the time-advance map

$$x_{n+1} = F(x_n) \quad \ldots (4.16)$$

The objective is to develop an explicit mapping with no worse of an error than a corresponding numerical integration scheme. Since errors are typically based on Taylor series expansions, a truncated Taylor expansion of the solution will not introduce any more errors than numerical integration of the differential equation.

Series solutions to an initial value problem can be conveniently and efficiently developed from the theory of continuous transformation groups [25]. Briefly, the coefficients of the series expansion can be computed from the associated infinitesimal generator of the system of
differential equations. Specifically, given an initial value problem

\[
\begin{align*}
\dot{x}_1 &= X_1(x_1, x_2, \ldots, x_N), \quad x_1(0) = x_{01} \\
\dot{x}_2 &= X_2(x_1, x_2, \ldots, x_N), \quad x_2(0) = x_{02} \\
&\vdots \\
\dot{x}_N &= X_N(x_1, x_2, \ldots, x_N), \quad x_N(0) = x_{0n} \quad \ldots (4.17)
\end{align*}
\]

the infinitesimal generator is defined as the operator

\[
U = X_1 \frac{\partial}{\partial x_1} + X_2 \frac{\partial}{\partial x_2} + \ldots + X_N \frac{\partial}{\partial x_N} \quad \ldots (4.18)
\]

in which the coefficients in (4.18) are given as the right hand sides of equation (4.17). It can be shown that the series representation of the solution \( x_i^t = x_i(t) \) is given by

\[
x_i^t = x_i + (Ux_i) t + (U^2x_i) \frac{t^2}{2} + \ldots + (U^k x_i) \frac{t^k}{k} + \ldots
\]

in which the variable, \( x_i \), represents an arbitrary initial value.

Thus, the explicit time-advance mapping is developed as

\[
(x_i)_{n+1} = F((x_i)_n) = x_i^h \quad \ldots (4.19)
\]

That is,
\[(x_i)_{n+1} = x_i^h\]
\[= x_i + (Ux_i)h + (U^2x_i)h^2/2 + \ldots \quad \ldots (4.20)\]

in which \((x_i)_n\) is substituted for the initial condition \(x_i\).

As an example, consider the problem

\[x^o = x^2, \quad x(0) = x_o\]
\[y^o = xy, \quad y(0) = y_o \quad \ldots (4.21)\]

The infinitesimal generator associated with this system of differential equations is given as the operator

\[U = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y} \quad \ldots (4.22)\]

The powers of the operator \(U\) are computed successively as

\[Ux = x^2 \quad \quad Uy = xy\]
\[U^2x = 2x^3 \quad \quad U^2y = 2x^2y\]
\[U^3x = 6x^4 \quad \quad U^3y = 6x^3y\]
\[U^4x = 24x^5 \quad \quad U^4y = 24x^4y\]
\[\ldots (4.23)\]
and the series representations of the solutions are

\[ x^t = x + x^2t + 2x^3t^2/2 + 6x^4t^3/3 + 24x^5t^4/4 + \ldots \] \hspace{1cm} (4.24)  

\[ y^t = y + xyt + 2x^2yt^2/2 + 6x^3yt^3/3 + 24x^4yt^4/4 \] \hspace{1cm} (4.25)  

in which \( x \) and \( y \) are arbitrary initial conditions.

By inspection, the solutions (4.24) and (4.25) are found to converge to

\[ x^t = x/(1-xt) \] \hspace{1cm} (4.26)  

\[ y^t = y/(1-xt) \] \hspace{1cm} (4.27)

As mentioned above, the variables \( x \) and \( y \) in equations (4.26) and (4.27) represent arbitrary initial conditions. Hence substituting the initial conditions from equations (4.21), results in the solutions

\[ x(t) = x_0/(1-x_0t) \] \hspace{1cm} (4.28)  

\[ y(t) = y_0/(1-x_0t) \] \hspace{1cm} (4.29)  

Furthermore, the time-advance solution for arbitrary \( h \) is given by

\[ x^h = x_0/(1-x_0h) \] \hspace{1cm} (4.30)
\[ y^h = y_0 / (1 - x_0 h) \] \hfill (4.31)

Alternatively, since \( x_0 \) and \( y_0 \) are arbitrary, the time-advance mapping is explicitly deduced as

\[ x_{n+1} = x_n / (1 - hx_n) \] \hfill (4.32)

\[ y_{n+1} = y_n / (1 - hx_n) \] \hfill (4.33)

Hence the continuous dynamics of the system (4.12) are equivalently, yet precisely, represented by the discrete mapping (4.32) and (4.33). Thus point mapping techniques are applicable, eliminating the cumbersome numerical tracking of trajectories.

Most nonlinear systems, however do not admit solutions which are expressible in terms of elementary functions as in (4.28) and (4.29). In such cases, a truncated series representation is used to construct the time-advance mapping. Using this discretized version of the dynamics, equations (4.1) through (4.11) are used to estimate the Lyapunov spectrum.
CHAPTER 5
RESULTS AND CONCLUSIONS

One and two-mode approximations for a magnetically buckled elastic beam were derived using a Galerkin approximation. This allows the analysis of an infinite degree of freedom system by replacing it with projected lower order systems admitting only finite degrees of freedom. Such a reduction also forms the basis of the finite element method. The principles of dynamical systems theory can thus be applied to the deduced finite dimensional approximations of the model.

Lyapunov exponents were estimated using the proposed semi-discrete formulation described in the previous chapter. Forward advance mappings were constructed from the equations of motion. By expanding the series solutions out to the fourth-order term, fifth order accuracy was obtained as guaranteed by Taylor's theorem. The continuous systems, defined by initial value problems, were replaced by discrete systems in the form of forward-advance mappings. Lie series were used to construct the forward-advance mappings. These discrete approximations map the state space into itself and are defined for arbitrary initial conditions. Lyapunov exponents were estimated from the resulting forward-advance mappings.

The largest Lyapunov exponents were estimated for various values of the forcing parameters. The dimensionless
forcing amplitude was varied from 0.1 to 1.00. The dimensionless forcing frequency was varied from 1.0 to 12.0. The resulting estimates of the two largest Lyapunov exponents are displayed in Table 1 through Table 14, for both the one and two-mode approximations. The largest Lyapunov exponents are also displayed graphically in Figures 5.1 through 5.14, for selected values of the forcing parameters.
### TABLE - 1 \( \omega = 1.0 \)

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### TABLE - 12  \( \omega = 10.0 \)

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**TABLE - 14** \( \omega = 12.0 \)

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LYAPUNOV EXPONENT
1-MODE & 2-MODE CASES, FREQ. = 2.00

FIGURE 5.2
LYAPUNOV EXPONENT
1-MODE & 2-MODE CASES, FREQ. - 3.00

MAGNITUDE

FORCE AMPLITUDE

FIGURE 5.3
LYAPUNOV EXPONENT
1-MODE & 2-MODE CASES, FREQ. - 4.00

FIGURE 5.4
FIGURE 5.5

LYAPUNOV EXPONENT

1-MODE & 2-MODE CASES, FREQ. = 4.5

MAGNITUDE

FORCE AMPLITUDE

0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

0.5 1.0
LYAPUNOV EXPONENT
1-MODE & 2-MODE CASES, FREQ. - 5.00

FIGURE 5.6
LYAPUNOV EXPONENT
1-MODE & 2-MODE CASES, FREQ. = 6.00

FIGURE 5.7
LYAPUNOV EXPONENT
1-MODE & 2-MODE CASES, FREQ. = 7.00

FIGURE 5.8
LYAPUNOV EXPONENT
1-MODE & 2-MODE CASES, FREQ. = 7.5

FIGURE 5.9
LYAPUNOV EXPONENT
1-MODE & 2-MODE CASES, FREQ. = 9.00

FIGURE 5.11
LYAPUNOV EXPONENT
1-MODE & 2-MODE CASES, FREQ. - 10.00

FIGURE 5.12
LYAPUNOV EXPONENT
1-MODE & 2-MODE CASES, FREQ. - 12.00

FORCE AMPLITUDE

MAGNITUDE

FIGURE 5.14
It is quite apparent that the estimates of the largest Lyapunov exponents differ considerably for the one and two-mode approximations. For the specific levels of the forcing parameters, both positive and negative values of the largest Lyapunov exponents are possible, depending on the particular order of the Galerkin approximation. This suggests that an additional consideration should be taken when replacing a dynamical system by a lower-dimensional approximation. For linear systems, particularly systems with dissipation, finite dimensional approximations are adequate for the analysis and prediction of the dynamic behavior. In fact, techniques such as equivalent linearization and harmonic balance can be successfully applied to replace nonlinear systems (at specified amplitudes) with an equivalent linear model possessing the same dynamical characteristics [31].

It should be noted that in a chaotic regime, a nonlinear system cannot be replaced by a linear one. As previously pointed out, the Lyapunov exponents of a linear system are constant. For linear systems, the Lyapunov spectrum essentially consists of the eigenvalues of the system, which in principle can be explicitly computed. For bounded behavior, which is what is observed physically, the eigenvalues must have negative real parts. Thus chaotic behavior is impossible for linear systems.

The results of this investigation indicate that when chaotic behavior is likely, the number of degrees of freedom used to approximate the physics has a definite bearing on the parameter values that can likely induce a
chaotic response. Hence care should be taken in the choice of truncated approximations used to reduce the number of degrees of freedom of a model.

The equations of motion were integrated numerically to generate phase plots. In all cases, the dynamic variable was taken as the deflection of the tip of the free end of the beam. For the one-mode approximation, the tip displacement is represented as

\[ w = \phi_1(L)q_1(t) \]

For the two-mode approximation, the displacement of the tip is given by

\[ w = \phi_1(L)q_1(t) + \phi_2(L)q_2(t) \]

The generated phase plane plots graphically display the tip velocity versus tip displacement. In each case, the equations were first integrated over twenty-five forcing periods to eliminate the transient response. These phase plane plots graphically illustrate the correlation between the estimated Lyapunov exponents and the qualitative behavior of the systems. It should be noted that when the largest Lyapunov exponent is positive, the dynamics of the beam is no longer periodic. In such cases, the convoluted phase diagrams are indicative of sensitive dependence on initial conditions.
One Rod, \( \omega = 10 \), \( \Gamma = .9 \)
Two Modes, \( \alpha = 10.0, F = 0 \)
REFERENCES


APPENDIX - A

ADDITIONAL LIST OF SYMBOLS:

\( y(x,t) \) \text{ transverse displacement of a typical segment of the beam, located at the distance } x

\( DX \) \text{ element length}

\( V \) \text{ shearing force}

\( M \) \text{ bending moment}

\( X \) \text{ deflection}

\( X' \) \text{ slope}

\( X'' \) \text{ bending moment}

\( X''' \) \text{ shear force}

Figure A-1
As shown in Figure A-1, consider a prismatic beam in the xy plane, which is assumed to be a plane of symmetry for any cross section. Figure A-2 shows a FBD of an element of length $DX$ with internal and inertial action upon it.
When beam is vibrating transversely, the dynamic equilibrium condition for forces in y direction is:

\[ V - V - (\partial V/\partial x)dx - \rho A \ dx \ (\partial^2 y/\partial t^2) = 0 \quad \ldots [a] \]

and the moment equilibrium condition gives

\[-V \ dx + (\partial M/\partial x)dx = 0 \quad \ldots [b] \]

Substitution of \( V \) from equation [b] into equation [a] produces:

\[(\partial^2 M/\partial x^2)dx = -\rho A \ dx \ (\partial^2 y/\partial t^2) \quad \ldots [c] \]

From elementary flexural theory we have the relationship:

\[ M = EI \ (\partial^2 y/\partial x^2) \quad \ldots [d] \]

Using this expression in equation [c], we obtain

\[ \partial^2/M/\partial x^2 \ [EI \ (\partial^2 y/\partial x^2)] \ dx = -\rho A \ dx \ (\partial^2 y/\partial t^2) \quad \ldots [e] \]

which is the general equation for transverse free vibration of a beam. In the particular case of a prismatic beam, the flexural rigidity \( EI \) does not vary with \( x \), and we have:

\[ EI \ (\partial^4 y/\partial x^4) \ dx = -\rho A \ dx \ (\partial^2 y/\partial t^2) \quad \ldots [f] \]
This equation may also be written as:

\[ \frac{\partial^4 y}{\partial x^4} = - \left( \frac{1}{a^2} \right) \left( \frac{\partial^2 y}{\partial t^2} \right) \]  

...[g]

In this application the symbol a has the definition:

\[ a^2 = \frac{E I}{\rho A} \]  

...[h]

When a beam vibrates transversely in one of its natural modes, the deflection at any location varies harmonically with time, as follows:

\[ y = X(A \cos pt + B \sin pt) \]  

...[i]

Substitution of equation [i] into equation [g] results in:

\[ \frac{d^4X}{dx^4} - \left( \frac{p^2}{a^2} \right) X = 0 \]  

...[j]

In order to solve this fourth-order differential equation, let us introduce the notation

\[ \frac{p^2}{a^2} = k^4 \]  

...[k]

and rewrite equation [j] as:

\[ \frac{d^4X}{dx^4} - k^4 X = 0 \]  

...[l]

To satisfy equation [l], we let \( X = e^{kx} \) and obtain:
Thus, the values of \( n \) are found to be \( n_1 = k, \) \( n_2 = -k, \)
\( n_3 = jk, \) and \( n_4 = -jk, \) where \( j = (-1.0)^{0.5}. \) The general
form of the solution for equation [1] becomes:

\[
X = C e^{kx} + d e^{-kx} + E e^{jkx} + F e^{-jkx}
\]

which may also be written in the equivalent form

\[
X = C_1 \sin kx + C_2 \cos kx + C_3 \sinh kx + C_4 \cosh kx
\]

This expression represents a typical normal function for
transverse vibrations of a prismatic beam.

The constants \( C_1, C_2, C_3, \) and \( C_4 \) are determined from
boundary conditions at the ends of the beam.

At fixed end the deflection and slope are equal to zero.

i.e.

\[
X = 0 \text{ and } X' = 0
\]

At the free end of the beam, the bending moment and the
shear force both vanish.

i.e.

\[
X'' = 0 \text{ and } X''' = 0
\]

Furthermore, it is useful to write the general expression
for a normal function \{ equation [o] \} in the following equivalent form:

\[
X = C_1 (\cos kx + \cosh kx) + C_2 (\cos kx - \cosh kx) \\
+ C_3 (\sin kx + \sinh kx) + C_4 (\sin kx - \sinh kx) \quad \text{[p]}
\]

Now assuming that the left end \((x = 0)\) is built in, we have the boundary conditions:

\[
(X)_{x=0} = 0, \quad (dX/dx)_{x=0} = 0, \quad (d^2X/dx^2)_{x=L} = 0, \quad (d^3X/dx^3)_{x=L} = 0
\]

From the first two condition we conclude that \(C_1 = C_3 = 0\) in the general solution \{ equation [o] \}, so that the general form of the mode shapes is given by equation:

\[
X = C_2 (\cos kx - \cosh kx) + C_4 (\sin kx - \sinh kx) \quad \text{[q]}
\]

The remaining two conditions result in the following frequency equation:

\[
\cos kL \cosh kL = -1
\]

The consecutive roots of this equation are as follows:

\[
\begin{align*}
  k_1L & \quad k_2L & \quad k_3L & \quad k_4L \\
  1.8751 & \quad 4.6941 & \quad 7.854755 & \quad 10.99554
\end{align*}
\]

The normal modes or shape functions for a cantilevered beam with end \(x = 0\) of the beam is fixed and the end \(x = L\) is free are as follows:
\[ X_i = ( \cosh k_i x - \cos k_i x ) - \alpha_i ( \sinh k_i x - \sin k_i x ) \]

where

\[ \alpha_i = \frac{\cosh k_i L + \cos k_i L}{\sinh k_i L + \sin k_i L} \]

Using the roots of the frequency equation for this beam, we obtain:

\[
\begin{align*}
  k_1 L &= 1.8751 \quad \alpha_1 = 0.7340955 \\
  k_2 L &= 4.6941 \quad \alpha_2 = 1.0184673 \\
  k_3 L &= 7.854755 \quad \alpha_3 = 0.9992244 
\end{align*}
\]
APPENDIX - B

TABLE 1

Physical Constants used in the analysis:

\[ L = 54.5 \text{ cm} \]
\[ m = 1.254 \times 10^{-6} \text{ kg s}^2 \text{ in kg force/cm} \]
\[ EI = 49.338 \text{ kg cm}^2 \]
\[ K_1 = -11.508 \times 10^{-3} \text{ kg/cm} \]
\[ K_2 = 2793 \times 10^{-6} \text{ kg/cm}^3 \]
\[ \xi_1 = 0.0191 \]
\[ \xi_2 = 0.0339 \]
\[ \xi_3 = 0.437 \]

[1]

\[ \begin{align*}
A_{11} &= A_{22} = A_{33} = 1 \\
A_{12} &= A_{21} = A_{13} = A_{31} = A_{23} = A_{32} = 0
\end{align*} \]

i.e.

\[ A_{13} = I = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 
\end{bmatrix} \]

[2]

\[ B_{11} = 0.1343112 \]
\[
B_{22} = 1.4939442 \\
B_{33} = 5.3923332 \\
B_{12} = B_{21} = B_{13} = B_{31} = B_{23} = B_{32} = 0
\]

i.e.
\[
\begin{bmatrix}
B_{11} & B_{12} & B_{13} \\
B_{21} & B_{22} & B_{23} \\
B_{31} & B_{32} & B_{33}
\end{bmatrix} =
\begin{bmatrix}
0.1343112 & 0.0 & 0.0 \\
0.0 & 1.4939442 & 0.0 \\
0.0 & 0.0 & 5.3923332
\end{bmatrix}
\]

\[
[C] = 132.32803 \\
C_{22} = 369.39028 \\
C_{33} = 3756.2017 \\
C_{12} = C_{21} = 149.32778 \\
C_{23} = C_{32} = 158.67190 \\
C_{13} = C_{31} = -154.94624
\]

i.e.
\[
\begin{bmatrix}
C_{11} & C_{12} & C_{13}
\end{bmatrix} =
\begin{bmatrix}
-132.32803 & 149.32778 & -154.94624 \\
149.32778 & 369.39028 & 158.67190 \\
-154.94624 & 158.67190 & 3756.2017
\end{bmatrix}
\]

\[
[D] = 1.5708771 \\
D_{22} = 8.6471765 \\
D_{33} = 24.952091
\]
\begin{align*}
D_{12} &= D_{21} = -0.4223294 \\
D_{23} &= D_{32} = 1.8901106 \\
D_{13} &= D_{31} = -0.9674723 \\
\text{i.e.} & \\
D_{14} &= \begin{bmatrix}
1.5708771 & -0.4223294 & -0.9674723 \\
-0.4223294 & 8.6471765 & 1.8901106 \\
-0.9674723 & 1.8901106 & 24.952091
\end{bmatrix} \\
\end{align*}

[3]

\begin{align*}
E_{11} &= 3.999976 \\
E_{22} &= 4.000066 \\
E_{33} &= 4.0005384 \\
E_{12} &= E_{21} = -4.000021 \\
E_{23} &= E_{32} = -4.0003022 \\
E_{13} &= E_{31} = 4.0002572 \\
\text{i.e.} & \\
E_{14} &= \begin{bmatrix}
3.999976 & -4.000021 & 4.0002572 \\
-4.000021 & 4.000066 & -4.0003022 \\
4.0002572 & -4.0003022 & 4.0005384
\end{bmatrix} \\
\end{align*}

[6]

\begin{align*}
G_{12} &= -4.104508 \\
G_{21} &= 0.104508 \\
G_{23} &= -4.5847874
\end{align*}
$G_{32} = 0.5847874$
$G_{13} = -4.0130329$
$G_{31} = 0.0130329$

\[7\]

$t_{12} = 0.3994589$
$t_{21} = 2.5033865$
$t_{23} = 0.5976125$
$t_{32} = 1.673325$
$t_{13} = 0.2387216$
$t_{31} = 4.1889793$

\[8\]

$\tau = (2.1117816)t$

\[9\]

$\omega_1 = 7.4250241 \quad \text{For } N = 1$
$\omega_2 = 46.53221 \quad \text{For } N = 2$
$\omega_3 = 130.29096 \quad \text{For } N = 3$

\[10\]

$\omega_1 = 3.516 \quad \text{For } N = 1$
\( \Omega_2 = 22.034575 \) \; \text{For} \; N = 2

\( \Omega_3 = 61.697176 \) \; \text{For} \; N = 3

[11]

\( \psi_1(1) = 1.999994 \) \; \text{For} \; N = 1

\( \psi_2(1) = -2.0000165 \) \; \text{For} \; N = 2

\( \psi_3(1) = 2.0001346 \) \; \text{For} \; N = 3

[12]

\( \psi_1(x_F) = 0.0404675 \) \; \text{For} \; N = 1

\( \psi_2(x_F) = 0.2202579 \) \; \text{For} \; N = 2

\( \psi_3(x_F) = 0.5334204 \) \; \text{For} \; N = 3

[13]

\( \sigma = 4.036217 \)

[14]

\( \theta_1 = -37.757899 \)

\( \theta_2 = 27218.984 \)
\[ f = (60.202075)F_0 \times \sin(7.4250241 \times t) \quad \text{For } N = 1 \]
\[ f = (60.202075)F_0 \times \sin(46.532210 \times t) \quad \text{For } N = 1 \]
\[ f = (60.202075)F_0 \times \sin(130.29096 \times t) \quad \text{For } N = 3 \]

From experiment, modal damping constants are:

\[ \zeta_1 = 0.0191 \]
\[ \zeta_2 = 0.0339 \]
\[ \zeta_3 = 0.0437 \]
APPENDIX C

GLOSSARY OF TERMS IN NONLINEAR DYNAMICS OF MAGNETOElastic BEAM

[1] NONLINEAR:

A property of an input-output system or mathematical operation for which the output is not linearly proportional to the input.

[2] DUFFING'S EQUATION:

A second-order differential equation with a cubic nonlinearity and harmonic forcing

\[ x^{\ddot{}} + ax^{\dot{}} + bx + cx^3 = f_0 \cos \omega t \]

named after G. Duffing.

[3] DETERMINISTIC:

Deterministic refers to a dynamic system whose equations of motion, parameters, and initial conditions are known and are not stochastic or random. However, deterministic systems may have motions that appear random.
[4] STOCHASTIC PROCESS:

Stochastic process often refers to a type of irregular motion found in conservative or nondissipative dynamical systems subjected to random loading.

[5] ATTRACTOR:

A set of points or a subspace in phase space toward which a time history approaches after transients die out. For example, equilibrium point or fixed points in maps, limit cycles, or a toroidal surface for a quasiperiodic motions, are all classical dynamical attractors.

[6] QUASIPERIODIC:

A vibration motion consisting of two or more incommensurate frequencies.

[7] LYAPUNOV EXPONENTS:

Numbers that measure the exponential attraction or separation in time of two adjacent trajectories in phase space with different initial conditions. In other words Lyapunov exponents are defined as the rates of divergence or convergence on tangent planes relative to a given position of a nonlinear vector field. A positive Lyapunov exponent indicates a chaotic motion in a dynamical system with bounded trajectories.
[8] CHAOTIC:

Chaotic denotes a deterministic process that is sensitive to changes in initial conditions. A motion for which trajectories starting from slightly different initial conditions diverge exponentially. A motion with a (locally) positive Lyapunov exponent.