Enhanced closed loop performance using non-dimensional analysis

Michael H. Wilson

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Enhanced Closed Loop Performance Using Non-Dimensional Analysis

By Michael H. Wilson

A Thesis Submitted in Partial Fulfillment of the Requirement for the

MASTER OF SCIENCE IN MECHANICAL ENGINEERING

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ROCHESTER INSTITUTE OF TECHNOLOGY

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Enhanced Closed-Loop Performance Using Non-Dimensional Analysis

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Date: 12/13/04 Signature of Author: Michael H. Wilson
ABSTRACT

This paper investigates the benefits of using non-dimension analysis to develop a control law for a flexible electro-mechanical system. The system that is analyzed consists of a DC motor connected to a load inertia through a set of gears. A state space system model is derived using LaGrange’s equation and then non-dimensionalized using a linear transformation. The resulting system model reveals the system character more clearly through the resulting dimensionless parameters. The parameters highlight the interaction between system properties and motor constants and demonstrate the benefits of a concurrent mechatronics design process. Open-loop behavior is analyzed and an optimal value for these parameters can be found by varying the gear ratio. Once the best possible gear ratio is determined, a PID control law is developed and the closed loop performance is analyzed. With the optimal gear ratio, the power required to control the system is minimized.

Also, dynamic inversion is applied to control the system. Dynamic inversion requires a square “B” matrix in the state space model. A new method to apply dynamic inversion to a system with a non-square “B” matrix is demonstrated. To make the matrix invertible, a linear transform is applied to the state space model. A Linear-Quadratic Regulator (LQR) design method is applied to find the transformation matrix values that will make the “B” matrix invertible. The power consumption of this control law is also minimized when the system contains the optimal gear ratio.
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# NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$B_L$</td>
<td>load friction coefficient</td>
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<tr>
<td>$B_M$</td>
<td>viscous-friction coefficient</td>
</tr>
<tr>
<td>$C_R$</td>
<td>damping of connectivity to rocket</td>
</tr>
<tr>
<td>$C_V$</td>
<td>damping of connectivity to vehicle</td>
</tr>
<tr>
<td>$i$</td>
<td>current</td>
</tr>
<tr>
<td>$J_L$</td>
<td>load inertia</td>
</tr>
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<td>$J_M$</td>
<td>motor shaft inertia</td>
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<td>rocket inertia</td>
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<td>$J_V$</td>
<td>launch vehicle inertia</td>
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<td>$K$</td>
<td>spring constant</td>
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<td>derivative gain</td>
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<td>back-emf constant</td>
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<td>torque constant</td>
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<td>spring constant of connectivity to rocket</td>
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<td>spring constant of connectivity to vehicle</td>
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<td>gear ratio</td>
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<tr>
<td>$Y$</td>
<td>output vector</td>
</tr>
<tr>
<td>$\Theta_D$</td>
<td>angular position variable for substitution</td>
</tr>
<tr>
<td>$\Theta_M$</td>
<td>angular position of motor</td>
</tr>
<tr>
<td>$\Theta_L$</td>
<td>angular position of load</td>
</tr>
</tbody>
</table>
\( \Theta_R \) angular position of rocket

\( \Theta_V \) angular position of vehicle

\( \omega_M \) motor angular velocity

\( \omega_n \) natural frequency

\( \omega_L \) load angular velocity

\( \omega_R \) rocket angular velocity

\( \omega_V \) vehicle angular velocity

**SUBSCRIPTS**

A motor armature

L load

M motor

R rocket

V vehicle

**SUPERSCRIPTS**

T transpose

. dot over symbol symbolizes first derivative

.. double dot over symbol denotes second derivative
I’d like to thank my advisor, Dr. Crassidis, for all his help and guidance, Dr. Kempski and Dr. Torok for their advice and criticism, and the entire Department of Mechanical Engineering Staff for all their support and hard work.
1.1 Introduction
The goal of the METEOR (Micro-Systems Engineering and Technology for the Exploration of the Outer Regions) program is to introduce a student led space program to the research community at RIT. One of the first objectives of the program is to deliver small payloads, or micro-satellites, to near earth orbit. The most important factor to consider in delivering a mass to orbit is weight. Weight and the cost of a mission are directly related. A movable launch platform was considered in order to increase the efficiency of the launch process. The ability to launch rockets from a movable platform in the mid to high ranges of the ionosphere presents a number of benefits that include: (1) less air resistance and drag, (2) the ability to change the latitude that the launch occurs and, (3) eliminating the need for ground launch equipment. Above an altitude of 80,000 ft the atmospheric density is less than 1% the density of the atmosphere at sea level. This results in negligible air resistance for any launches from a platform at that altitude and increases the efficiency of system due to the absence of parasitic drag on the rocket. Launching from altitude also eliminates the need for launch facilities on the ground. One of the challenges of this program is positioning the rocket before launch. In order to achieve the desired orbit, it is important to have the ability to have accurate angular orientation. Another consideration is power consumption. Power is supplied to the platform using onboard batteries. Limiting the required power of the system will reduce the amount of batteries needed thus lowering the weight of the system. The position of the rocket will be changed using a DC Motor. The nature of the connectivity of the system makes load positioning a challenge. Because the load, in this application a rocket, is connected to the motor using tethers there is very little damping of the system. By viewing this system as flexible, it places the problem in a well-studied category.

Flexible DC servomechanisms have been the subject of research for a number of years. Flexible motion of robotic manipulators was first considered by Book(1974). Dubowsky and Desforges(1979) studied adaptive control of robotic manipulators. Harokpos and Mayne(1985) investigated the interaction between an actuator and a compliant load using dimensionless parameters. Dimensionless parameters reduce the number of factors that need to be considered and make it easier to see the nature of the system. The analysis of the dynamic behavior was simplified by forming dimensionless
parameters that better described the character of the system. By performing nondimensional analysis on a servomechanism, the relationships between the system characteristics and motor parameters can be seen. One of these parameters contains a gear ratio, which when varied can affect the character of the system. By identifying an optimum value for this parameter, best open-loop performance can be achieved.

Panza and Mayne (1989) extended this approach to merge an actuator, a flexible load, and closed loop control in the analysis and design of compliant beam-like dynamic system. Sah (1990) considered how a set of lightweight gears connecting a DC motor to an aluminum beam affected a dynamical mechanical model. Sah showed that changing the gear ratio affects the level of interaction between the load and actuator. By varying the gear ratios and observing the behavior of the system, he was able to influence the performance of the system and achieve optimal results. Optimal open-loop performance can be achieved by selecting an appropriate gear ratio to regulate the damping of the system. Selecting a large gear ratio allows the beam dynamics to dominate the behavior of the whole system and is not desirable. Conversely, a small gear ratio there is very little beam loading on the motor. The gear ratio overly reduces the effect of the beam on the motor and the flexible beam begins to vibrate with any motion of the motor. Sah discovered that a desired level of interaction between the motor and the beam is achieved with a gear ratio of $r = .1$ which results in preferred open-loop response.

Panza and Mayne (1994) also used dimensionless parameters to improve closed loop performance of a hydraulically driven rotating flexible mechanism by optimizing certain parameters. They showed that a tuned system required less controller effort than a system that was not optimized. Ben-Tal (1996) demonstrated that simultaneous structural and control design iterations result in an optimal system performance. He developed dimensionless equations of motion for a flexible beam and studied how the system benefits from designing the structure and control system concurrently. Hermle and Eberhard (2000) presented a hierarchical control concept for flexible robot manipulators. The control parameters for this flexible system are found using parameter optimization.

Other topics that have been widely studied in control theory is Linear-quadratic regulator problems and dynamic inversion. In an attempt to optimally control a linear
system, a quadratic cost function is used to determine the values that populate a gain
matrix, $K$, which in turn defines the input to the system model. Rosen and Wang (1992)
studied the application of a discrete linear quadratic regulator problem and the associated
Riccati equation to a flexible beam. They also considered the effects of sampling time on
system stability. Dynamic inversion cancels out the dynamics of the system and forces
the response of the system to track a desired trajectory. Lin and Zhang (1993) applied a
model simplified using dimensionless parameters to a PUMA550 robot and demonstrated
the reliability of the simplification process. They used a nondimensionalization scheme
to simplify the dynamic formulation of the system and used the characteristic parameters
to optimize the system. They found that the simplified dynamic model greatly reduces
the burden of the inverse dynamics. Singh and Naidu (1995) used linear quadratic
regulator theory to stabilize a system that contained a flexible structure. They proposed a
method to place the eigenvalues of the system within a vertical strip that would make the
system response more desirable. Fer and Enns (1996) used dynamic inversion to stabilize
a triple inverted pendulum on a cart. They used a linear quadratic regulator controller to
obtain the gains of the desired dynamics for the dynamic inversion control law.
Combining dynamic inversion and a linear quadratic regulator, they were able to obtain a
desired response from an unstable system. Tadi (1997) applied the LQR control scheme
to a multicomponent distributed-parameter structure. He showed that convergent state
feedback control laws can be obtained for the flexible structure. Yuan (2000) proposed a
method to choose the suitable weighting matrices to force the system to have desired
closed loop poles. He also showed that the weighting matrices in a linear-quadratic
optimal control system are related to the transformation matrices. Maxwell and
Asokanthan (2003) presented a method for determining the optimal placement and
controller design for distributed actuators to reduce the vibration of flexible structures.
They included a linear-quadratic regulator controller as part of the optimization
procedure.

1.2 Objectives and Procedures

This work investigates the angular position control of the rocket using some of the
analytical tools described above. Because the METEOR program is in its infancy, the
launch vehicle that will ultimately deliver the rocket to launch altitude is in the preliminary design stages. This means that a number of the important values that make up the dynamic model are unknown. The goal of this research is to demonstrate a method to develop a viable control law for the launch rocket positioning system, derive a dynamic model of the system, and demonstrate the benefits of a concurrent design process using dimensionless analysis.

A flexible servomechanism consisting of a DC motor and an inertial load will be studied to demonstrate a method of controlling the system. Also, the benefits of introducing a set of lightweight gears into the system will be considered. The method in which these factors influence open-loop performance will be examined and a closed loop control law will be developed to improve system response and performance. A number of control methods will be applied to show that a system containing optimized dimensionless parameters and gear ratio yield the best closed loop results. Furthermore, a power consumption analysis will be performed to see how system design and parameter values affect the overall power required to control the load.

Once the control scheme is defined for a simplified model, and the benefits of dimensionless analysis are illustrated, a more accurate model of the METEOR positioning system will be developed and examined. A mathematical model of the system will be developed and some of the important parameter interactions will be uncovered using dimensionless analysis. By highlighting how the character of the system is affected by the properties of subsystems, a more efficient design process can be achieved. This model, and the demonstrated relationships of the subsystems, can be used by future METEOR teams once the launch vehicle is design becomes more finalized.
2.0 Analysis of simple System

There are a number of benefits that result from dimensionless parameters. As demonstrated by Harokopos and Mayne (1985), an understanding of how these parameters influence the character of a system can result in an optimal open-loop response and aid in motor selection. Non-dimensional manipulation of the state equations results in a deeper understanding of system character and can also demonstrate the relation of motor characteristics to the load. By applying this type of analysis to an electro-mechanical system, the benefits of an integrated mechatronics design process can be recognized more clearly. The goals of this application of non-dimensional analysis are to achieve better open-loop response, developing an appropriate control law to achieve the desired results and minimizing the amount of power required. This section will revisit the example used by Harokopos and Mayne and move forward to demonstrate the effects of dimensionless parameters on power consumption for a closed-loop system.

2.1 Modeling of simple System

The application of this process will focus on a simple DC servomechanism. A DC motor is used to position a rotational load where a voltage input results in a rotational position change of the load. By including a gear ratio in the system design, a designer can influence the system character and how the system responds to a pre-determined input.

2.1.1 Model of a DC Motor

A DC servomotor is a device used to drive rotational systems. A diagram of the circuit that powers the DC motor is shown in Figure 2.1 below
The motor output torque, $T$, is proportional to the motor current

$$T = K_M i \quad (2.1)$$

with $K_M$ representing the torque constant of the motor.

By applying Kirchoff’s voltage law to the system represented in Figure 2.1 we obtain

$$\frac{di}{dt} = \frac{1}{L_A} [V_A - R_A i - V_B] \quad (2.2)$$

The parameters $R_A, L_A$ and $i$ represent motor resistance, inductance, and current, respectively.

Voltage produced by the rotation of the current carrying armature in the magnetic field is called back electromotive force or, back-emf. This voltage opposes the current in the DC motor and is proportional to the angular velocity of the armature. The back-emf voltage can be described by

$$V_B = K_g \dot{\theta}_M \quad (2.3)$$

Where $K_g$ is the back-emf or generator constant.

Substituting equation 2.3 into 2.2 yields

$$\frac{di}{dt} = \frac{1}{L_A} [V_A - R_A i - K_g \dot{\theta}_M] \quad (2.4)$$

The equations of motion of the electro-mechanical system were derived using LaGrange’s energy method and are applied to the system shown in Figure 2.2.
The parameters $R_A, L_A, K_e, K_M, J_M, B_M$ and $B_L$ represent motor resistance, inductance, back-emf constant, torque constant, motor inertia, and viscous friction of the motor and load, respectively. $K$ and $J_L$ indicate the drive stiffness and load inertia and the gear ratio, $r$, is the relation of the radii of the gears. $V$, $\theta_M$, $\omega_M$, $\theta_L$, and $\omega_L$ symbolize the voltage input, motor position, motor speed, load position, and load speed. $\theta_D$ is a dummy variable used to define the gear ratio.

The gear train is assumed to be ideal. The gears are presumed to be rigid, mass less, frictionless, and with no energy loss.

### 2.1.2 Modeling of Servo-Mechanism

The mathematical dynamical model of the mechanical subsystem was derived using Lagrange's energy method.

$$
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{Q}_i} - \frac{\partial T}{\partial Q_i} + \frac{\partial U}{\partial Q_i} + \frac{\partial R}{\partial \dot{Q}_i} \right) = Q_{inc} \tag{2.5}
$$

These equations perform an energy balance of the system comparing its kinetic energy, potential energy, and energy dissipation. The kinetic energy of the system, $T$, is defined by

$$
T = \frac{1}{2} J_M \dot{\theta}_M^2 + \frac{1}{2} J_L \dot{\theta}_L^2 \tag{2.6}
$$

The potential energy of the system, based on the angular displacement of a rotational spring is represented by

Figure 2.2
Simple DC Servo-Mechanism
\[ U = \frac{1}{2} K(\Theta_D - \Theta_L)^2 \]  

Where \( \Theta_D = n*\Theta_M \)

\[ \therefore U = \frac{1}{2} Kr^2 \Theta_M^2 - Kr\Theta_M \Theta_L + \frac{1}{2} K\Theta_L^2 \]  

(2.8)

The Rayleigh energy dissipation term, a function of angular velocity, is described by

\[ R = \frac{1}{2} B_M \dot{\Theta}_M^2 + \frac{1}{2} B(\dot{\Theta}_D - \dot{\Theta}_L)^2 \]  

Where \( \Theta_D = n \Theta_M \)

\[ \therefore R = \frac{1}{2} B_M \dot{\Theta}_M^2 + \frac{1}{2} Br^2 \dot{\Theta}_M^2 - Br\dot{\Theta}_M \dot{\Theta}_L + \frac{1}{2} B\dot{\Theta}_L^2 \]  

(2.9)

A comparison of the kinetic and potential energies forms the Lagrangian equation.

\[ L = T - U \]

\[ \therefore L = \frac{1}{2} J_M \dot{\Theta}_M^2 + \frac{1}{2} J_L \dot{\Theta}_L^2 - \frac{1}{2} Kr^2 \dot{\Theta}_M^2 + Kr\Theta_M \Theta_L - \frac{1}{2} Kr^2 \Theta_L \]  

(2.11)

Partial derivatives of the Lagrangian energy balance results in:

\[ \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{\Theta}_M} \right) = J_M \ddot{\Theta}_M \]  

(2.12)

\[ \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{\Theta}_L} \right) = J_L \ddot{\Theta}_L \]  

(2.13)

\[ \frac{\partial L}{\partial \Theta_M} = -Kr^2 \Theta_M + Kn \Theta_L \]  

(2.14)

\[ \frac{\partial L}{\partial \Theta_L} = Kr\Theta_M - K\Theta_L \]  

(2.15)

\[ \frac{\partial R}{\partial \dot{\Theta}_M} = B_M \dot{\Theta}_M + Br^2 \dot{\Theta}_M - Br\dot{\Theta}_L \]  

(2.16)

\[ \frac{\partial R}{\partial \dot{\Theta}_L} = B\dot{\Theta}_L - Br\dot{\Theta}_M \]  

(2.17)

Collecting all the terms with corresponding coefficients yields the following dynamic model of the system.

\[ J_M \ddot{\Theta}_M + (B + Br^2) \dot{\Theta}_M + Kr^2 \Theta_M - Br\dot{\Theta}_L - Kr\Theta_L = U(t) \]  

(2.18)
\[ J_L \ddot{\Theta}_L + B \dot{\Theta}_L + K \Theta_L - Br \dot{\Theta}_M - Kr \Theta_M = 0 \] \hspace{1cm} (2.19)

Where \( U(t) = K_M i \), and equation 2.4 defines and \( i \) and the electrical subsystem behavior.

These equations can be placed in state space form by defining \( i(t), \dot{\Theta}_M(t), \omega_M(t), \dot{\Theta}_L(t), \) and \( \omega_L(t) \) as state variables. This form is better suited to the linear transformations required to find the dimensionless model of the system. After substitution, the system state equations can be written as:

\[ \dot{X} = AX + BU \] \hspace{1cm} (2.20)

Where

\[ A = \begin{bmatrix}
-\frac{R_A}{L_A} & 0 & -\frac{K_G}{L_A} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\frac{K_M}{J_M} & -\frac{Kr^2}{J_M} & \frac{(B_M + Br^2)}{J_M} & \frac{Kr}{J_M} & \frac{Br}{J_M} \\
0 & \frac{Kr}{J_M} & \frac{Br}{J_L} & -\frac{K}{J_L} & -\frac{B}{J_L}
\end{bmatrix} \] \hspace{1cm} (2.22)
\[
X = \begin{bmatrix}
i_{(t)} \\
\Theta_{M(t)} \\
\omega_{M(t)} \\
\Theta_{L(t)} \\
\omega_{L(t)}
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
1/L_A \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

and the input, \( U \), to the system is defined by

\[
U = V_A
\]  

\[\text{(2.23)}\]

\[\text{(2.24)}\]

\[\text{(2.25)}\]

### 2.1.3 Non-Dimensional Analysis

In order to better understand the character of the system, the relations of dimensions are removed from the system. Two transformation matrices are defined to nondimensionalize the state equations, one for the state vector and one for time. For the dimensionless analysis, it is assumed that the overall system damping comes from the motor actuator interaction, or the viscous friction coefficient of the motor, \( B_M \). The external damping term, \( B \), is set to zero for this analysis. Although the additional term could be used to better approximate the system, it would only cloud the effects of gear ratio selection.

Characteristic variables are first defined to develop the transformation matrices. Characteristic time is defined as the inverse of the load natural frequency, ie:
Characteristic current is defined as:

\[ i_c = \frac{K_G}{L_A} \]  

(2.27)

To nondimensionalize the dimensional state vector consider the following linear transformation

\[ \overline{X} = PX \]  

(2.28)

Where

\[
P = \begin{bmatrix}
\frac{1}{i_c} & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & t_c & 0 & 0 \\
0 & 0 & 0 & \frac{1}{r} & 0 \\
0 & 0 & 0 & 0 & \frac{t_c}{r}
\end{bmatrix}
\]  

(2.29)

Differentiating \( \overline{X} \) and solving for \( \dot{X} \) results in

\[ \dot{X} = P^{-1} \dot{\overline{X}} \]  

(2.30)

Solving equation 2.28 for \( X \) results in

\[ X = P^{-1} \overline{X} \]  

(2.31)

Substituting equation 2.30 & 2.31 into the original state space equation shown here

\[ \dot{X} = AX + BU \]  

(2.32)

and rearranging results in

\[ \dot{\overline{X}}_d = \frac{d\overline{X}}{dt} = PAP^{-1} \overline{X} + PBU \]  

(2.33)

To nondimensionalize time, the following linear transformation is now defined as

\[ \tau = \frac{t}{t_c} = \sqrt{\frac{K}{J_L}} \frac{t}{t_c} \]  

(2.34)

\[ d\tau = \sqrt{\frac{K}{J_L}} dt \]  

(2.35)
\[ dt = d\tau \sqrt{\frac{J_L}{K}} = d\tau P_1 \]

Where

\[
P_1 = \begin{bmatrix}
\sqrt{\frac{J_L}{K}} & 0 & 0 & 0 & 0 \\
0 & \sqrt{\frac{J_L}{K}} & 0 & 0 & 0 \\
0 & 0 & \sqrt{\frac{J_L}{K}} & 0 & 0 \\
0 & 0 & 0 & \sqrt{\frac{J_L}{K}} & 0 \\
0 & 0 & 0 & 0 & \sqrt{\frac{J_L}{K}} \\
\end{bmatrix}
\]

Substituting into equation 2.33 and re-arranging shows that

\[
\dot{X} = \frac{d\overline{X}}{d\tau} = P_1 P A P^{-1} \overline{X} + P_1 P B U
\]

Now the input to the system model, \( U \), must be nondimensionalized. The characteristic voltage is defined by

\[
V_C = \frac{L_A}{K_\gamma R_A}
\]

Using the following equation the input effort, \( V_A \) may be nondimensionalized

\[
U/V_C = \frac{V_A}{V_C} = \overline{U}
\]

Now, the state space model becomes

\[
\dot{X} = \frac{d\overline{X}}{d\tau} = P_1 P A P^{-1} \overline{X} + P_1 P B \frac{1}{V_C} U
\]

or

\[
\dot{X} = \overline{A} \overline{X} + \overline{B} \overline{U}
\]

The dimensionless state equations are represented by

\[
\dot{\overline{X}} = \frac{d\overline{X}}{d\tau} = \overline{A} \overline{X} + \overline{B} \overline{U}
\]
Where
\[
\bar{A} = P_1 P AP^{-1} \\
\bar{B} = P_1 PB \\
\bar{U} = \frac{U}{V_c} = \frac{V_A}{V_c}
\]

Matlab’s symbolic toolbox was used to carry out the matrix operations required to transform the state equations to a non-dimensional space. The “m-file” used to derive the dimensionless matrices can be found in Appendix A. This transformation results in

\[
\bar{A} = \begin{bmatrix}
\frac{R_A}{L_A} & 0 & -1 & 0 & 0 \\
-\frac{K}{\sqrt{J_L}} & 0 & 1 & 0 & 0 \\
J_L K_M K_G & -J_L r^2 & \frac{B_M}{J_M} & 0 & 0 \\
K J_M & J_M & \frac{K}{J_L} & J_M & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & -1 & 0
\end{bmatrix}
\]

or

\[
\bar{A} = \begin{bmatrix}
-C_e & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\frac{1}{C_D} & \frac{1}{C_D} & -J_r & -C_m & J_r \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & -1 & 0
\end{bmatrix}
\]

\[
\bar{B} = \begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix}
\]
$Ce$ represents the ratio of the electrical corner frequency of the motor and the natural frequency of the load.

$$Ce = \frac{\frac{R_{A}}{L_{A}}}{\sqrt[2]{\frac{K}{J_{L}}}}$$  \hspace{1cm} (2.47)

$Cm$ defines the mechanical corner frequency of the motor relative to the natural frequency of the load.

$$Cm = \frac{\frac{B_{M}}{J_{M}}}{\sqrt[2]{\frac{K}{J_{L}}}}$$  \hspace{1cm} (2.48)

$Jr$ is the ratio of the load inertia, transmitted through the reduction gears, to the motor inertia

$$Jr = \frac{J_{L}r^{2}}{J_{M}}$$  \hspace{1cm} (2.49)

The most important parameter to consider $C_D$, defines how the properties of the motor and the character of the system it is driving are related. Studying this term will show how concurrent mechatronic design can lead to optimum system performance.

$$C_D = \frac{\sqrt{\frac{K}{J_{L}}}J_{L}R_{A}r^{2}}{K_{M}K_{G}}$$  \hspace{1cm} (2.50)

$C_D$ can also be described by substituting the natural frequency of the load, $\omega_n = \sqrt{\frac{K}{J_{L}}}$ so that

$$C_D = \frac{J_{L}R_{A}r^{2}}{K_{M}K_{G}}\omega_n$$  \hspace{1cm} (2.51)
3.0 Simulation Set Up

The system was modeled using Matlab's Simulink toolbox. The "mfile" that sets the coefficient values and defines the state space matrices, as well as the open loop model, can be seen in Appendix B. An optimization routine was run to determine an optimum value for the gear ratio by varying the value of the gear ratio and comparing the open loop results to the desired position. A PID control law will also be developed and the closed loop performance of systems containing different gear ratios will be analyzed. The control parameters, $K_d$, $K_i$, and $K_p$, were also found using an optimization routine. The "m files" that found the optimum control parameters and system model can be found in Appendix C.

The values of the various coefficients that describe the character of the system can be found in Table 3.1.

<table>
<thead>
<tr>
<th>$K_m$</th>
<th>0.040625</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_g$</td>
<td>0.05538592</td>
</tr>
<tr>
<td>$R_A$</td>
<td>1.1 Ohms</td>
</tr>
<tr>
<td>$L_A$</td>
<td>0.0023 Henrys</td>
</tr>
<tr>
<td>$J_M$</td>
<td>2.864580E-05</td>
</tr>
<tr>
<td>$B_M$</td>
<td>4.973592E-06</td>
</tr>
<tr>
<td>$J_L$</td>
<td>0.1</td>
</tr>
<tr>
<td>$K$</td>
<td>0.6</td>
</tr>
</tbody>
</table>

Table 3.1
System Parameter Identification

3.1 Open-Loop Analysis

By varying the gear ratio of the system and tracking the open loop results, an optimal value for the gear ratio can be determined.

The "fminsearch" utility of Matlab was used to find the minimum value of a cost function. The cost function compared the open loop final value with the desired response and then squared the difference and can be seen below.
\[ J = \int_{t=0}^{t=\tau_f} (y - 1)^2 \, dt \] (3.1)

The gear ratio that resulted in the minimum value of the cost function, and therefore the optimal open loop response was .0864. The corresponding \( C_D \) value was 0.8941. Systems that had varying degrees of \( C_D \) were simulated to demonstrate that the value found using the optimization routine was truly optimal. The open loop responses of the three systems can be seen in Figure 3.1.

![Open loop comparison of different Cd values](image)

**Figure 3.1**  
Comparison of Varying Cd values

A system with a small gear ratio of \( r = .025 \) resulted in a \( C_D \) value of \( C_D = .075 \). This system was oscillatory and took a long time to reach a steady state value. Because the gear ratio was so small, there was very little mechanical feedback of the load inertia on the motor, resulting in the oscillatory character of the response.
When a larger gear ratio was implemented in the system, the oscillatory nature of the response was eliminated. A gear ratio of \( r = 0.3, C_D = 10.778 \), had no overshoot but was too slow of a response. These two cases show the relationship of \( C_D \) on the damping of the system.

When the system was constructed by selecting a gear ratio using an optimization routine, the response was remarkably better. A gear ratio of \( r = 0.0864, C_D = 0.894 \), settled in approximately 5 seconds with acceptable overshoot. This comparison of open loop response clearly shows how the system character can be effected, and improved by considering dimensionless parameters.

Harokopos and Mayne\(^1\) demonstrated that a \( C_D, R_D \) in their notation \( (C_D = 1/R_D) \), value of approximately 1 provided the optimal open loop response. The value of \( C_D = 0.894 \) \( (R_D = 1.1187) \) found using this cost function and optimization routine is very close to their findings.

### 3.2 Closed Loop Analysis

Again, Matlab’s Simulink toolbox was used to develop a closed loop model the system. By providing the system with feedback of the load position and motor speed, better steady state values and transient behavior can be achieved.

#### 3.2.1 Control Law Gains

A PI control system with a gain of \( K_D \) placed on the motor speed feedback was developed. A schematic of the control law can be seen in figure 3.2.

![PID Control Schematic](image-url)
Using the root locus design method, initial values for the control law coefficients were chosen to be $K_P = 1$, $K_i = 0$, and $K_D = .1$. Again, a cost function was optimized to determine the best values for $K_D$, $K_i$, and $K_P$. The control law gains were varied and the cost function that was minimized tracked the difference between the desired output and the steady state response of the system. Three different trials were run with different gear ratios. Gear ratios of $r = .3$, $r = .025$, and an optimum value of $r = .0846$ were analyzed.

### 3.2.2 Power Calculations

System power was calculated by multiplying the input voltage and the motor current

$$P = V_A \times i \quad (3.2)$$

Controller effort, $V_A$, is the input to the system model while current, $i$, is the first state. Average power and peak power values were both calculated for the simulation.

$$P_{\text{total}} = \int |V_A \times i| \, dt \quad (3.3)$$

$$P_{\text{peak}} = \text{Max}(|V_A \times i|) \quad (3.4)$$

This process was then repeated for different values of $C_D$ synchronous with $r = .3$, $r = .025$ and $r = .0846$. $C_D$ was varied by different magnitudes and the average and peak power values were recorded, as well as the resulting values for $K_D$, $K_i$, and $K_P$.

### 3.3 Close Loop Results

**Optimized Gear Ratio ($r= .0846$)**

With a gear ratio optimized to a value of .0864 the system achieved optimal results. The response had zero steady state error while the system reached a steady final value after approximately 12 seconds. The results of the control gain optimization and power analysis can be seen in Table 3.2 and the system response is shown in Figure 3.3.
Table 3.2
Optimized Gear Ratio

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>r=</td>
<td>0.0864</td>
</tr>
<tr>
<td>Cd=</td>
<td>0.894</td>
</tr>
<tr>
<td>Kp=</td>
<td>0.5845</td>
</tr>
<tr>
<td>Ki=</td>
<td>0.001</td>
</tr>
<tr>
<td>Kd=</td>
<td>-0.0098</td>
</tr>
<tr>
<td>Ptotal</td>
<td>0.1301</td>
</tr>
<tr>
<td>Ppeak</td>
<td>0.2558</td>
</tr>
</tbody>
</table>

Figure 3.3
Load Position for Optimized System
The motor current is shown in Figure 3.4

![Motor Current](image)

**Figure 3.4**
Motor Current for Optimized System

The input to the state space system model, or controller effort, can be seen in Figure 3.5

![Controller Effort](image)

**Figure 3.5**
Controller Effort for Optimized System
The power required to position the load is shown in Figure 3.6.

![Figure 3.6
Power Consumption for Optimized System](image)

**Larger gear ratio (r=.3)**

With a larger gear ratio, and therefore more mechanical feedback relating the load inertia and the motor, the system achieved similar response to the optimized gear ratio which can be seen in Figure 3.7. However, the power required to reach a stable steady state value was greater than in the optimized case, the results of which can be found in Table 3.3. With a larger gear ratio, it takes more a greater torque input to the system to achieve the desired results.

<table>
<thead>
<tr>
<th>r</th>
<th>0.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cd</td>
<td>10.7775</td>
</tr>
<tr>
<td>Kp</td>
<td>2.0264</td>
</tr>
<tr>
<td>Ki</td>
<td>-0.0391</td>
</tr>
<tr>
<td>Kd</td>
<td>0.2648</td>
</tr>
<tr>
<td>Ptotal</td>
<td>1.3044</td>
</tr>
<tr>
<td>Ppeak</td>
<td>1.1507</td>
</tr>
</tbody>
</table>
The motor current requisite to producing the required torque to position the load is shown in Figure 3.8.
The voltage input to the system model is shown in Figure 3.9.

![Figure 3.9](image)

**Figure 3.9**  
*Controller Effort with Large Gear Ratio*

The absolute value of the power required to position the load is shown in Figure 3.10. Notice that the magnitude of the power is significantly higher than in the optimized system.

![Figure 3.10](image)

**Figure 3.10**  
*Power Consumption with Large Gear Ratio*
Smaller gear ratio (r=.025)

When the gear ratio is made smaller, and the motor senses the load less, the results are consistent with the optimized result. The closed loop response for a small gear ratio can be seen in Figure 3.11. With a smaller gear ratio the peak power required to move the load is less than the optimum system. The power response can be seen in Figure 3.14. This is due to the lower amount of torque needed to move the system. The necessary torque is less because of the gear ratio. Because torque output is proportional to current the power consumed is lower. However, over the course of time, the system containing the optimum gear ratio consumes less energy. Because of the small gear ratio, the effects of the load are not as strong on the motor. This loss of mechanical feedback makes it harder to achieve a steady state and can result in a tendency to vibrate. This explains why the system would consume more power to achieve stable steady state. The power consumption is illustrated in Table 3.5

| Table 3.4 |
| Small Cd |
| r= | 0.025 |
| Cd= | 0.0748 |
| Kp= | 0.1908 |
| Ki= | 0 |
| Kd= | -0.2657 |
| Ptotal | 0.1494 |
| Ppeak | 0.1079 |
Closed Loop Response ($r = 0.025$)

![Graph of load position with small gear ratio.](image)

**Figure 3.11**
Load Position with Small Gear Ratio

The motor current that produced the necessary torque is shown in Figure 3.12.

![Graph of motor current with small gear ratio.](image)

**Figure 3.12**
Motor Current with Small Gear Ratio
The controller effort, or voltage input to the system model, is shown in figure 3.13.

![Figure 3.13 Controller Effort with Small Gear Ratio](image)

As stated earlier the peak power required to move the load is less than the optimized system for the case considering a small gear ratio. However, the overall power consumed
is less in the system containing the optimal gear ratio. Although the current required to produce the necessary torque is less with a small gear ratio, the system character is not as favorable due to the oscillatory nature of the response. The system requires more controller effort, or voltage, to achieve the desired results.

**Section Summary**

As shown in Table 3.5, the cumulative required power was at its lowest value when the values of $r$ and $C_D$ were optimal. An optimal value of $C_D$ means that the open loop response, and the inherent system character, is best. With optimal system character, it can be assumed that the system will be “easier to control”, therefore consuming less power.

<table>
<thead>
<tr>
<th>PID Control</th>
<th>Gear Ratio</th>
<th>$C_d$</th>
<th>Cumulative Power</th>
<th>Peak Power</th>
</tr>
</thead>
<tbody>
<tr>
<td>r=.025</td>
<td>0.0748</td>
<td>0.1494</td>
<td>0.1079</td>
<td></td>
</tr>
<tr>
<td>r=.0846</td>
<td>0.8941</td>
<td>0.1301</td>
<td>0.2558</td>
<td></td>
</tr>
<tr>
<td>r=.3</td>
<td>10.7775</td>
<td>1.3044</td>
<td>1.1507</td>
<td></td>
</tr>
</tbody>
</table>

**Table 3.5**

*Comparison of Power Consumption*
A comparison of the power required to position the load is shown in Figure 3.15

![Figure 3.15](image)

**Figure 3.15**

*Power Consumption Comparison*

The two smaller gear ratio systems performed significantly better than the system containing the large gear ratio. The system with the smallest gear ratio had the lowest peak power requirements but over time required more energy to achieve the desired steady state response.

By optimizing the control law gains an optimal steady state response can be achieved. Different gear ratios were tested and the results of the different cases were consistent. Also, by optimizing the character of the system by changing the gear ratio, power consumption over time can be minimized.
4.0 Dynamic Inversion

This section will explore a new control strategy in order to achieve similar closed loop response for the three gear ratio scenarios described in Chapter 3 as they track a single desired response. A new method to apply dynamic inversion to non-square “B” matrix systems will be theorized and demonstrated.

4.1 Derivation of Dynamic Inversion Control Law

Dynamic inversion will be applied to the system modeled in Chapter 2. The traditional form of a state space model is represented by

\[ \dot{X} = AX + BU \] (4.1)

Where A is a matrix defining system character and B defines a matrix which modifies the input to the system.

The output of the system can is defined by

\[ Y = CX \] (4.2)

Dynamic Inversion involves the definition of a control law such that the actual dynamics of the system are cancelled out so that the system tracks a desired trajectory. The first step is defining the inversion term, s.

\[ s = \bar{X} + \int \lambda X dt \] (4.3)

Where \( \bar{X} = X - X_D \)

\( X_D \) is the desired response of the system. The system response should closely resemble \( X_D \).

To ensure tracking of the desired response, “no movement” of \( s \) is allowed. In other words, the time rate of change of the inversion term should be zero or

\[ \dot{s} = 0. \]

\[ \dot{s} = \dot{X} - \dot{X}_D + \lambda \ddot{X} \] (4.4)

Substituting equation 4.1 into equation 4.4 yields

\[ \dot{s} = AX + BU - \dot{X}_D + \lambda \ddot{X} \] (4.5)

Solving for the controller effort, \( U \) that inverts the dynamic character and defines the controller effort as
\[
U = B^{-1}\left[\dot{X}_D - AX - \lambda \dot{X}\right]
\] (4.6)

However, for the case sown in Section 2.3.1 the B matrix, with dimensions \((5 \times 1)\) is not invertible because it is not a square matrix.

### 4.2 Making \(B\) invertible

To apply dynamic inversion to the system, a possible method to invert the B matrix must be found. If we return to the output equation, Equation 4.2, of the traditional state space model

\[Y = CX\]

And regard the C matrix as being a linear transform, the output equations takes on a new look

\[Y = TX\] (4.7)

Taking the time derivative shows us that

\[\dot{Y} = T\dot{X} = TAX + TBU\] (4.8)

The challenge now is to find a transformation matrix, \(T\), such that \(B\) is invertible. To be invertible, a matrix must be square and its determinate cannot be zero. Therefore, a matrix of dimension \((n \times 1)\) must be multiplied by a matrix of dimension \((1 \times n)\). In our application, \(B\) is a \((5 \times 1)\) matrix. To make \(B\) invertible, \(T\) must possess a dimension of \((1 \times 5)\). In order to find the values to populate the \(T\) transformation matrix a linear-quadratic regulator problem was considered. A quasi-feedback matrix is defined that would make the \(B\) matrix invertible, while maintaining system character. This will be discussed more thoroughly in Section 4.3

Once a transformation matrix, \(T\), is defined, the state space equation can be transformed as

\[\dot{X} = \hat{A}X + \hat{B}U\] (4.9)

Where

\[\hat{A} = TA\] (4.10)

And

\[\hat{B} = TB\] (4.11)
The B matrix has now been turned into a square matrix, which makes it invertible and able to have dynamic inversion applied to it. The A matrix has been modified as a result of the linear transformation. Substituting this new state space equation into the dynamic inversion surface equation 4.4 yields

$$\dot{s} = \hat{A}X + \hat{B}U - \hat{X}D + \hat{A}\hat{X}$$  \hspace{1cm} (4.12)

The controller effort input to the system model is now

$$U = \hat{B}^{-1}[\hat{X}D - \hat{A}X - \hat{A}\hat{X}]$$  \hspace{1cm} (4.13)

This input to the state space model cancels out the system dynamics and forces the response to track a desired trajectory.

### 4.3 Defining the Transformation Matrix

To make the B matrix invertible, a linear-quadratic regulator (LQR) control scheme was applied to define a suitable transformation matrix. Consider the traditional state space model, Eq 4.1.

$$\dot{X} = AX + BU$$

With a linear-quadratic regulator, the controller effort is defined as

$$U = -KX$$  \hspace{1cm} (4.14)

Where K is a gain matrix. The values that populate the K matrix are found by minimizing a cost function

$$J = \int_0^T X^T Q X + U^T R U dt$$  \hspace{1cm} (4.15)

Here Q and R are weighting matrices that are varied to minimize J. In a traditional linear quadratic regulator problem, the optimal value of K places the closed loop poles of the system in the left hand plane, making the system stable.

One way to apply a LQR to this problem is to replace K with the transformation matrix. Thus, by applying a LQR analysis a transformation matrix can be found that will allow the B matrix to be invertible. Instead of the gain matrix, K, the result of this analysis is the transformation matrix, T.
4.4 Simulation Set Up

Matlab was used to find the solution to the Linear-Quadratic regulator problem and find an appropriate transform to make the B matrix invertible. Once the state space model had been transformed, dynamic inversion could then be applied to the system. The control scheme was simulated using Simulink. The "m files" used to find the transformation matrix and system model can be found in Appendix D. A schematic of the dynamic inversion control scheme can be seen in Figure 4.1.

![Dynamic Inversion Control Schematic](image)

The system is driven to track a desired result and this is the input to the control law. Dynamic inversion attempts to cancel the dynamics of the system and forces the model to track the desired trajectory.

A smaller gear ratio needs less torque to position the load. In this case, we wanted the system to track the closed loop response of a system containing a gear ratio of r=0.025. Torque produced by a DC motor is described as

\[ T = K_M \times i \]

And power is defined as

\[ P = V \times i \]

If torque is minimized, the current needed to produce the torque is lower. This results in less power being required to generate the torque. Systems containing a gear ratios of r=0.3
and an optimal value of $r = 0.0846$ will be forced to track the response of the smaller gear ratio and the power consumption of all three systems will be considered.

The equations used to compare the power consumption of the system can be shown in Equation 4.16 and 4.17.

\[
P_{Total} = \int |V_A * i| \, dt \tag{4.16}
\]

\[
P_{Peak} = \text{Max}(|V_A * i|) \tag{4.17}
\]

Peak power represents the highest magnitude of energy needed to move the system while average power gives a good description of how much overall effort was required.

### 4.5 Results of Dynamic Inversion

**Optimized Gear Ratio (r=.0846)**

The transformation matrix for the system containing an optimized gear ratio is the solution to the LQR problem and is shown below

\[
T = \begin{bmatrix}
.1235 & 0 & 0 & 0 & 0 \\
0 & .2802 & 0 & 0 & 0 \\
0 & 0 & .0275 & 0 & 0 \\
0 & 0 & 0 & .7961 & 0 \\
0 & 0 & 0 & 0 & 1.621
\end{bmatrix}
\]

This matrix makes the B matrix invertible and allows dynamic inversion to be applied to the system. The closed loop response of the system can be seen in Figure 4.2.
Closed Loop Response

As Figure 4.2 illustrates, the system tracks the desired response and almost no deviation can be recognized.

The current generated by the armature to drive the motor is shown in Figure 4.3

The voltage input to the system model, or controller effort, is shown in Figure 4.4
Figure 4.4
Controller Effort with Optimized Gear Ratio
The energy required to position the motor is seen in Figure 4.5

Figure 4.5
Power Consumption with Optimized Gear Ratio
Larger gear ratio ($r=.3$)

The transformation matrix for the system containing the larger gear ratio is the solution to the LQR problem and is shown below

\[
T = \begin{bmatrix}
38.99 & 0 & 0 & 0 & 0 \\
0 & 64.51 & 0 & 0 & 0 \\
0 & 0 & 11.06 & 0 & 0 \\
0 & 0 & 0 & 243.9 & 0 \\
0 & 0 & 0 & 0 & 184.0 \\
\end{bmatrix}
\]

This matrix allows dynamic inversion to be applied to the system with the larger gear ratio. The closed loop response is shown below in Figure 4.6.

![Figure 4.6](Load Position with Large Gear Ratio)
The current required to produce the torque necessary to position the motor is shown in Figure 4.7

![Motor Current with Large Gear Ratio](image1)

**Figure 4.7**
Motor Current with Large Gear Ratio

The voltage input, or controller effort, to the state space dynamic model is shown in Figure 4.8

![Controller Effort with Large Gear Ratio](image2)

**Figure 4.8**
Controller Effort with Large Gear Ratio
The power required to position the load is shown below in Figure 4.9

![Power Consumption](image)

**Figure 4.9**

Power Consumption with Large Gear Ratio

**Smaller gear ratio (r=.025)**

The transformation matrix that enables a dynamic inversion control law to be applied to a system containing a small gear ratio is shown below

\[
T = \begin{bmatrix}
.1797 & 0 & 0 & 0 & 0 \\
0 & .6716 & 0 & 0 & 0 \\
0 & 0 & .0640 & 0 & 0 \\
0 & 0 & 0 & 5.009 & 0 \\
0 & 0 & 0 & 0 & 12.79
\end{bmatrix}
\]

The closed loop response of the system is shown below in Figure 4.10. This is the response that the other systems were driven to follow. This response was chosen because a system containing a smaller gear ratio is traditionally thought to consume less power because of the decreased torque requirements.
Figure 4.10
Load Position with Small Gear Ratio

The motor armature current is shown in Figure 4.11

Figure 4.11
Motor Current with Small Gear Ratio
The controller effort is the input to the dynamic system model and is shown in Figure 4.12

**Figure 4.12**  
Controller Effort with Small Gear Ratio

The overall power consumption of the system is shown in Figure 4.13

**Figure 4.13**  
Power Consumption with Small Gear Ratio
Summary of Results

As shown in Table 4.1, the system with the optimal gear ratio consumed the least power. Even though the system was forced to track the response of the smaller gear ratio, the optimal system’s closed loop performance was enhanced. Because the system character is optimal, it is therefore easier to control, requiring less controller effort. This reduced controller effort results in less power required to position the system. When used to analyze control systems, dimensionless analysis proves to be a useful tool. By eliminating the relationships of dimensions, the system character can be seen easier. By studying the interaction of the different subsystems, a better understanding of the nature of the system can be achieved.

<table>
<thead>
<tr>
<th>Gear Ratio</th>
<th>Cd</th>
<th>Cumulative Power</th>
<th>Peak Power</th>
</tr>
</thead>
<tbody>
<tr>
<td>r=.025</td>
<td>0.0748</td>
<td>3.6114</td>
<td>78.2215</td>
</tr>
<tr>
<td>r=.0846</td>
<td>0.8941</td>
<td>0.5564</td>
<td>6.6518</td>
</tr>
<tr>
<td>r=.3</td>
<td>10.7775</td>
<td>5.2144</td>
<td>13.4495</td>
</tr>
</tbody>
</table>

Table 4.1

Power Consumption of Dynamic Inversion
The power consumption differences can be easiest seen in Figure 4.14 where all three systems are considered and compared.

As you can see the system containing the optimum gear ratio \((r = .0864)\) performed better than the other two systems. It required less peak power and less overall power.
5.0 Formulations Applied to METEROR Concept Vehicle

This chapter presents the equations of motion for the conceptual METEOR launch vehicle and, through non-dimensional analysis, introduces a set of dimensionless parameters that show the dynamic relation of the subsystems. The system can be seen in Figure 5.1.

\[ \theta_v, \omega_v, \theta_m, \omega_m, \theta_R, \text{ and } \omega_R \text{ symbolize the angular position and velocity of the flight vehicle, motor shaft, and rocket respectively. } J_v \text{ represents the inertia of the flight vehicle, while } C_v \text{ and } K_v \text{ correspond to the damping and spring constants of the connection to the balloon. } J_m \text{ and } B_m \text{ indicate the inertia and viscous friction of the motor and the gear ratio, } r, \text{ is the relation of the radii of the gears. } J_R, K_R, \text{ and } C_R \text{ stand for the inertia of the rocket, spring constant of the connection to the motor, and the damping of the connection.} \]

5.1 System Modeling

The model of the DC motor developed in section 2.2.1 will also be used to influence the motion of the expanded model. The parameters \( R_A, L_A, K_G, K_M, J_M, \text{ and } B_M \) represent motor resistance, inductance, back-emf constant, torque constant, motor inertia, and viscous friction, respectively.

\[
\frac{dI}{dt} = \frac{1}{La}[-(Rai + Kg\dot{\theta}_M) + Va]
\] (5.1)
The mathematical dynamical model of the system was derived using LaGrange’s energy method.

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{Q}_i} \right) - \frac{\partial T}{\partial Q_i} + \frac{\partial U}{\partial Q_i} + \frac{\partial R}{\partial \dot{Q}_i} = Q_{ic} \tag{5.2}
\]

These equations perform an energy balance of the system comparing its kinetic energy, potential energy, and energy dissipation. The kinetic energy of the system, \( T \), is defined by

\[
T = \frac{1}{2} J_v \dot{\Theta}_v^2 + \frac{1}{2} J_M \dot{\Theta}_M^2 + \frac{1}{2} J_R \dot{\Theta}_R^2 \tag{5.3}
\]

The potential energy of the system, based on the angular displacement of a rotational spring is represented by

\[
U = \frac{1}{2} K_v \Theta_v^2 + \frac{1}{2} K_R (\Theta_R - \Theta_D)^2 \tag{5.4}
\]

Where \( \Theta_D = n \Theta_M \)

\[
\therefore U = \frac{1}{2} K_v \Theta_v^2 + \frac{1}{2} K_R \Theta_R^2 - K_n \Theta_M \Theta_L + \frac{1}{2} K_R \theta^2 \Theta_M^2 \tag{5.5}
\]

The energy dissipation terms, a function of angular velocity, are described by

\[
R = \frac{1}{2} B_M (\dot{\Theta}_M - \dot{\Theta}_v)^2 + \frac{1}{2} C_R (\dot{\Theta}_R - \dot{\Theta}_D)^2 + \frac{1}{2} C_v \dot{\Theta}_v^2 \tag{5.6}
\]

Where \( \Theta_D = n \Theta_M \)

\[
\therefore R = \frac{1}{2} B_M \dot{\Theta}_M^2 - \dot{\Theta}_M \dot{\Theta}_M B_M + \frac{1}{2} B_M \dot{\Theta}_v^2 + \frac{1}{2} C_R \dot{\Theta}_R^2 - C_R \dot{\Theta}_R \dot{\Theta}_M + \frac{1}{2} C_R r^2 \dot{\Theta}_M^2 + \frac{1}{2} C_v \dot{\Theta}_v^2 \tag{5.7}
\]

\[
L = T - U
\]

\[
\therefore L = \frac{1}{2} J_v \dot{\Theta}_v^2 + \frac{1}{2} J_M \dot{\Theta}_M^2 + \frac{1}{2} J_R \dot{\Theta}_R^2 - \frac{1}{2} K_v \Theta_v^2 - \frac{1}{2} K_R \Theta_R^2 + K_r \Theta_M \Theta_L - \frac{1}{2} K_R r^2 \Theta_M \tag{5.8}
\]

\[
\frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{\Theta}_v} \right) = J_v \ddot{\Theta}_v \tag{5.9}
\]

\[
\frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{\Theta}_M} \right) = J_M \ddot{\Theta}_M \tag{5.10}
\]
\[
\frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{\Theta}_R} \right) = J_R \ddot{\Theta}_R
\]  
(5.11)

\[
\frac{\partial L}{\partial \Theta_v} = -K_R^2 \Theta_M + K_R \Theta_L
\]  
(5.12)

\[
\frac{\partial L}{\partial \Theta_M} = -K_R^2 \Theta_M + K_R \Theta_R
\]  
(5.13)

\[
\frac{\partial L}{\partial \Theta_R} = K_R r \Theta_M - K_R \Theta_R
\]  
(5.14)

\[
\frac{\partial R}{\partial \Theta_v} = B_M \dot{\Theta}_v - B_M \dot{\Theta}_M
\]  
(5.15)

\[
\frac{\partial R}{\partial \Theta_M} = B_M \dot{\Theta}_M - B_M \dot{\Theta}_v + C_R r^2 \dot{\Theta}_M - C_R r \dot{\Theta}_R
\]  
(5.16)

\[
\frac{\partial R}{\partial \Theta_R} = C_R \dot{\Theta}_R - C_R r \dot{\Theta}_M
\]  
(5.17)

Collecting all the terms results in the following dynamical model

\[
J_v \ddot{\Theta}_v + (B_M + C_v) \dot{\Theta}_v + K_v \Theta_v - B_M \dot{\Theta}_M = 0
\]  
(5.18)

\[
J_M \ddot{\Theta}_M + (B_M + C_R r^2) \dot{\Theta}_M + K_R r \dot{\Theta}_M - B_M \dot{\Theta}_M - C_R r \dot{\Theta}_R - K_R \dot{\Theta}_R = U(t)
\]  
(5.19)

\[
J_R \ddot{\Theta}_R + C_R \dot{\Theta}_R + K_R \dot{\Theta}_R - C_R r \dot{\Theta}_M - K_R r \dot{\Theta}_M = 0
\]  
(5.20)

Where \( U(t) = K_M i \)

The dynamical model can be expressed in state space form by declaring \( i(t), \theta_v(t), \omega_v(t) \)
\( \theta_M(t), \omega_M(t), \theta_R(t), \) and \( \omega_R(t) \) as state variables. After substitution, the equations of
motion can be written in vector-matrix form as

\[
\dot{X} = AX + BU
\]  
(5.21)
Where

\[ \dot{X} = \begin{bmatrix}
\frac{di_{(t)}}{dt} \\
\frac{d\Theta_{M(t)}}{dt} \\
\frac{d\omega_{M(t)}}{dt} \\
\frac{d\Theta_{M(t)}}{dt} \\
\frac{d\omega_{M(t)}}{dt} \\
\frac{d\Theta_{M(t)}}{dt} \\
\frac{d\omega_{M(t)}}{dt}
\end{bmatrix} \]  

(5.22)

\[
A = \begin{bmatrix}
-R_A & 0 & \frac{K_G}{L_A} & 0 & -\frac{K_G}{L_A} & 0 & 0 \\
0 & L_A & 0 & 0 & 0 & 0 & 0 \\
-\frac{K_M}{K_V} & -\frac{K_V}{L_A} & \frac{(-B_V-C_{V_q})}{L_A} & 0 & \frac{B_M}{L_A} & 0 & 0 \\
J_V & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]  

(5.23)

\[
X = \begin{bmatrix}
i_{(t)} \\
\Theta_{V_{(t)}} \\
\omega_{V_{(t)}} \\
\Theta_{M_{(t)}} \\
\omega_{M_{(t)}} \\
\Theta_{R_{(t)}} \\
\omega_{R_{(t)}}
\end{bmatrix}
\]  

(5.24)
and the input to the system, $U$, is defined by

$$U = \begin{bmatrix} V_A \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$  \hspace{1cm} (5.26)$$

\section*{5.2 Non-Dimensional Analysis}

As shown in Section 2.2.3, the character of a system can be better understood by removing the relationships inherent in dimensions. By performing a linear transformation on the state space model, a non-dimensional dynamical model is produced. For the dimensionless analysis, it is assumed that the overall system damping comes from the motor actuator interaction, or the viscous friction coefficient of the motor, $B_M$. The external damping terms, $C_V$ and $C_R$, are set to zero for this analysis. Although the additional terms could be used to better approximate the system, it would only cloud the effects of gear ratio selection.

The first step in executing the linear transform on the system is defining characteristic variables to populate the transformation matrices.

Characteristic time is defined as the inverse of the load natural frequency, ie:

$$t_c = \frac{J_L}{K}$$  \hspace{1cm} (5.27)$$

Characteristic current is defined as:
\[
i_c = \frac{K_G}{L_A}
\]  

(5.28)

To nondimensionalize the dimensional state vector consider the following linear transformation

\[
\bar{X} = PX
\]  

(5.29)

Where

\[
P = \begin{bmatrix}
i_c & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \tau & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \tau & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{r} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{r} \tau
\end{bmatrix}
\]  

(5.30)

Differentiating \( \bar{X} \) and solving for \( \dot{X} \) results in

\[
\dot{X} = P^{-1} \dot{\bar{X}}
\]  

(5.31)

Solving equation 1 for \( X \) results in

\[
X = P^{-1} \bar{X}
\]  

(5.32)

Substituting equation 2 & 3 into the original state space equation shown here

\[
\dot{X} = AX + BU
\]  

(5.33)

and rearranging results in

\[
\dot{\bar{X}} = \frac{d\bar{X}}{dt} = PAP^{-1} \bar{X} + PBU
\]  

(5.34)

To nondimensionalize time, the following linear transformation is now defined as

\[
\tau = \frac{t}{t_r} = \sqrt{\frac{K}{J_L}} t
\]  

(5.35)

\[
d\tau = \sqrt{\frac{K}{J_L}} dt
\]  

(5.36)

so that

\[
dt = d\tau \sqrt{\frac{J_L}{K}} = d\tau P_1
\]  

(5.37)
Where

\[
P1 = \begin{bmatrix}
\sqrt{\frac{J_R}{K_R}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{\frac{J_R}{K_R}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{\frac{J_R}{K_R}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{\frac{J_R}{K_R}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{\frac{J_R}{K_R}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sqrt{\frac{J_R}{K_R}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{J_R}{K_R}}
\end{bmatrix}
\]

(5.38)

Substituting into equation 4 and re-arranging shows that

\[
\dot{\bar{X}} = \frac{d\bar{X}}{dt} = P_1PAP^{-1}\bar{X} + P_1PBU
\]

(5.39)

Now the input to the system model, U, must be nondimensionalized. The characteristic voltage is defined by

\[
V_C = \frac{L_A}{K_g R_A}
\]

(5.40)

Using the following equation the input effort, \(V_A\) may be nondimensionalized

\[
\frac{U}{V_C} = \frac{V_A}{V_C} = \bar{U}
\]

(5.41)

Now, the state space model becomes

\[
\dot{\bar{X}} = \frac{d\bar{X}}{dt} = P_1PAP^{-1}\bar{X} + P_1PB \frac{1}{V_C} U
\]

or

\[
\dot{\bar{X}} = AX + BU
\]

(5.43)
The dimensionless state equations are represented by

\[ \dot{\bar{X}} = \frac{\partial \bar{X}}{\partial \tau} = \bar{A}\bar{X} + \bar{B}\bar{U} \quad (5.44) \]

Where

\[ \bar{A} = P_A P A P^{-1} \]
\[ \bar{B} = P_A P B \]
\[ \bar{U} = \frac{U}{V_C} = \frac{V_A}{V_C} \]

Matlab's symbolic toolbox was used to carry out the matrix operations required to transform the state equations to a non-dimensional space. The "m-file" used to derive the dimensionless matrices can be found in Appendix E. This transformation results in

\[
\bar{A} = \begin{bmatrix}
\frac{R_A}{L_A} & 0 & 1 & 0 & -1 & 0 & 0 \\
\frac{K_R}{J_R} & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-\frac{J_R}{K_R} & \frac{K_M}{J_V} & \frac{K_G}{L_A} & -\frac{J_R}{K_R} & \frac{K_M}{J_V} & 0 & 0 \\
-\frac{J_R}{K_R} & \frac{K_M}{J_V} & \frac{K_G}{L_A} & -\frac{J_R}{K_R} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-\frac{J_R}{K_R} & \frac{K_M}{J_V} & \frac{K_G}{L_A} & -\frac{J_R}{K_R} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-\frac{J_R}{K_R} & \frac{K_M}{J_V} & \frac{K_G}{L_A} & -\frac{J_R}{K_R} & 0 & 0 & 0 \\
\end{bmatrix} \quad (5.45) 
\]
Or

\[
A = \begin{bmatrix}
-C_e & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-C_e J_r \frac{R_i}{C_D} & -R_0 & -C_{m_y} & 0 & C_{m_y} & 0 & 0 \\
C_e J_r \frac{1}{C_D} & 0 & C_{m_x} & -J_r & -C_{m_x} & J_r & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & -1 & 0
\end{bmatrix}
\]

(5.46)

\[
\bar{B} = \begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

(5.47)

5.3 Parameter Identification

\( C_e \) represents the ratio of the electrical corner frequency of the motor and the natural frequency of the load.

\[
C_e = \frac{R_A/L_A}{\sqrt{K_R/J_R}}
\]

(5.48)

\( C_{m_y} \) and \( C_{m_x} \) define the mechanical corner frequencies of the motor, respectively relative to the natural frequency of the load.

\[
C_{m_y} = \frac{B_M/J_v}{\sqrt{K_R/J_R}}
\]
\[ C_m = \frac{B_m}{J_M} \]  \hspace{1cm} (5.49)

\[ J_r = \frac{J_R r^2}{J_M} \]  \hspace{1cm} (5.50)

\[ R_i = \frac{J_M}{J_V} \]  \hspace{1cm} (5.51)

\[ R_{\omega} = \frac{J_R}{J_V} \frac{K_R}{K_V} \]  \hspace{1cm} (5.52)

The most important parameter to consider is \( C_D \), defines how the properties of the motor and the character of the system it is driving are related. Studying this term will show how concurrent mechatronic design can lead to optimum system performance.

\[ C_D = \frac{\sqrt{\frac{K_R}{J_R} J_R R_A r^2}}{K_M K_G} \]  \hspace{1cm} (5.53)

\( C_D \) can also be described by substituting the natural frequency of the load, \( \omega_r = \sqrt{\frac{K_R}{J_R}} \) so that

\[ C_D = \frac{J_R R_A r^2}{K_M K_G} \omega_r \]  \hspace{1cm} (5.54)

These dimensionless parameters, which describe the character of the system and highlight the relation of different subsystems, are a powerful design tool. As the METEOR program nears its goal and the requirements of the mission become clearer, these parameters can be used to improve the design process. By considering these factors
when selecting and designing subsystems the system can be optimized. This will increase the performance of the control system and minimize the power required to position the rocket. This is important because weight restrictions are a huge challenge in any space application. A more efficient system will require fewer batteries and conserve weight. This analysis will enable the METEOR program to design a more optimal system
6.0 Summary

This study presented the benefits of using dimensionless analysis on closed loop performance. Eliminating the inter-relations of dimensions enables a better understanding of the effects that motor selection has on the closed loop performance of a mechanical system. In chapter 2, a mathematical dynamical model of a flexible system was developed and then non-dimensionalized to gain a better view of system character.

Chapter 3 exhibited how optimal system character can be found by varying the gear ratio and the effects of different gear ratios on PID control performance. A gear ratio of \( r = .0846 \) was found to be optimal. This result is consistent with previous work that stated a gear ratio of approximately \( r = .1 \) achieved the best results. The power required to achieve steady state was evaluated for systems with gear ratios \( r = .3, r = .025 \), and an optimal value of \( r = .0846 \). Overall power consumption was lowest when the optimal gear ratio was used.

To further demonstrate the benefits of analyzing dimensionless parameters, a dynamic inversion control law was developed for the system in Chapter 4. In addition, a new way to apply dynamic inversion to systems with a non square B matrix was introduced. A linear quadratic regulator was used to determine a transformation matrix that made the B matrix invertible. Once dynamic inversion could be applied to the system, the power consumption of the control system was analyzed. All three gear ratio systems were forced to follow the trajectory of the smallest gear ratio and average and peak power were calculated for each case. Again, a system containing an optimized gear ratio had the lowest power consumption.

In chapter 5, a dynamical model was derived for the METEOR launch vehicle. This system model was also non-dimensionalized and the parameter interactions were highlighted. These relationships could prove to be extremely useful in future launch vehicle design.

As you can see, using dimensionless analysis as a tool to design a control law, total power consumption required can be minimized. An electro-mechanical system that has been optimized through gear ratio selection is demonstrated to have superior closed loop performance. The closed loop response of systems with different gear ratios has been
evaluated by applying a PID control law and through using dynamic inversion. The results of this study can are summarized in Table 6.1.

<table>
<thead>
<tr>
<th>Gear Ratio</th>
<th>Cd</th>
<th>Cumulative Power</th>
<th>Peak Power</th>
<th>Cumulative Power</th>
<th>Peak Power</th>
</tr>
</thead>
<tbody>
<tr>
<td>r = .025</td>
<td>0.0748</td>
<td>0.1494</td>
<td>0.1079</td>
<td>3.6114</td>
<td>78.2215</td>
</tr>
<tr>
<td>r = .0846</td>
<td>0.8941</td>
<td>0.1301</td>
<td>0.2558</td>
<td>0.5564</td>
<td>6.6518</td>
</tr>
<tr>
<td>r = .3</td>
<td>10.7775</td>
<td>1.3044</td>
<td>1.1507</td>
<td>5.2144</td>
<td>13.4495</td>
</tr>
</tbody>
</table>

Table 6.1 Summary of Results

6.1 Suggestions for future work

This work sets the stage for future development of the METEOR control system. A method for achieving accurate position control of the rocket load has been demonstrated. Also, a mathematical dynamical model of the system was developed that can be used for future control considerations.

More deliberation on the transformation matrices used to non-dimensionalize the system model could be useful. Special attention should be paid to the PI matrix used to apply dimensionless time to the system. The natural frequency of the flight vehicle might play an important role in determining the overall system character. These transforms would result in new dimensionless parameters. The effects of these relationships would need to be analyzed.

The most important step is to finalize the overall design of the system. Dimensionless analysis is extremely useful and if used in a concurrent design process could optimize the performance of the system. Once the system is defined, a control system can be developed. The final step in developing the control strategy is to test the performance in a laboratory setting. A rigorous testing procedure should be implemented to ensure accurate position control. Accurate positioning of the rocket is essential to mission success. The methods demonstrated here should prove useful in future years of the METEOR program.
REFERENCES


APPENDIX A
NON-DIMENSIONAL ANALYSIS
%m file to derive dimensions matrices

%declares the following variables as symbolic

syms Ra La Kg Km Jm K r Bm JL

%State Space Model
A=[-Ra/La 0 -Kg/La 0 0; 0 0 1 0 0; Km/Jm -(K*r^2)/Jm (-Bm)/Jm (K*r)/Jm 0; 0 0 0 0 1; 0 (K*r)/JL 0 -K/JL 0];

B=[1/La;0;0;0;0];

%-----Defines Transformation Matrix--Magnitude
P=diag([La/Kg 1 sqrt(JL/K) 1/r sqrt(JL/K)/r]);

%Defines Transformation Matrix--Dimensionless Time
P1=diag([JL/K)^(1/2) (JL/K)^((1/2) (JL/K)^((1/2) (JL/K)^(1/2) (JL/K)^(1/2)];

%-Characteristic Voltage
Uc=Kg*sqrt(K/JL);

%-----Defines inverse of P-----
Pinv=INV(P);

%---Performs Transformation Operations
Adim=P1*P*A*Pinv;
Bdim=P1*P*B*Uc;
APPENDIX B
OPEN-LOOP SIMULATION
global A B C D

\[ r = 0.1; \]
\[ Km = 0.040625; \]
\[ Kg = 0.05538592; \]
\[ Ra = 1.1; \]
\[ Jm = 2.864583333 \times 10^{-5}; \]
\[ La = 0.0023; \]
\[ Bm = 4.973591972 \times 10^{-6}; \]
\[ K = 0.6; \]
\[ JL = 0.1; \]
\[ Wa = \sqrt{K/JL}; \]
\[ Ce = (Ra/La)/Wa; \]
\[ Cm = (Bm/Jm)/Wa; \]
\[ Jr = (r*Jr*JL)/Jm; \]
\[ Cd = (r*r*Ra*JL*Wa)/(Km*Kg); \]

%%%---Defines State Space Matrices----
\[
A=[-Ra/La 0 -Kg/La 0 0; \\
   0 0 1 0 0; \\
   Km/Jm -(K*r^2)/Jm (-Bm)/Jm (K*r)/Jm 0; \\
   0 0 0 0 1; \\
   0 (K*r)/JL 0 -K/JL 0];
\]
\[
B=[1/La;0;0;0;0];
\]
\[
C=[0 0 0 0 1];
\]
\[
D=[0];
\]

%%%---Optimization of function---

%%%---Defines options of the optimization routine----
options=optimset;
optnew=optimset(options,'LevenbergMarquardt','on');

%%%---Optimizes x matrix values----
\[
x0=[r];
\]
\[
x=fminsearch('OpenloopDimensionlessforOPT_function_V2',x0,optnew)
\]

%sets gear ratio to result of optimization
\[
r=x
\]
function f=OpenloopDimensionlessforOPT_function(x)
global A B C D r

%---Declares Parameter Values---

r = x;
Km = 0.040625;
Kg = 0.05538592;
Ra = 1.1;
Jm = 2.864583333e-05;
La = 0.0023;
Bm = 4.973591972e-06;
K = 0.6;
JL = 0.1;

%---Define State Space variables to Run Simulation---

A=[-Ra/La 0 -Kg/La 0 0; 0 0 1 0 0; Km/Jm -(K*r^2)/Jm (-Bm)/Jm (K*r)/Jm 0; 0 0 0 0 1; 0 (K*r)/JL 0 -K/JL 0];
B=[1/La;0;0;0;0];
C=[0 0 0 0 1];
D=[0];

%---Run Simulation to get results---
[t,xsim,ysim]=sim('openloop_for_opt');

%---Evaluate Results using cost function---

h=ysim(length(t));
f=int(t,t.*(ysim-h).*(ysim-h))
cost=f
x
APPENDIX C
CLOSED-LOOP SIMULATION
%this m file calls the optimization routine and sets parameter values

global A B C D r Kp Ki Kd

%Declares Parameter Values
r = 0.3;
Km = 0.040625;
Kg = 0.05538592;
Ra = 1.1;
Jm = 2.864583333e-05;
La = 0.0023;
Bm = 4.973591972e-06;
K = 0.6;
JL = 0.1;
Wa = sqrt(K/JL);
Ce = (Ra/La)/Wa;
Cm = (Bm/Jm)/Wa;
Jr = (r*r*JL)/Jm;
Cd = (r*r*Ra*JL*Wa)/(Km*Kg);
Kp = .5;
Ki = 0;
Kd = .1;

%---Defines State Space Matrices---
A=[-Ra/La 0 -Kg/La 0 0;
   0 0 1 0 0;
   0 0 0 1 0;
   0 (K*m)/Jm 0 -K/JL 0];
B=[1/La;0;0;0;0];
C=diag([1 1 1 1]);
D=[0;0;0;0;0];

%---Optimization of function---
%------Defines options of the optimization routine------
options=optimset;
optnew=optimset(options,'LevenbergMarquardt','on');

%---Sets Initial Conditions---
x0=[Kp Ki Kd];

%---Calls Optimization routine---
x=fminsearch('ClosedLoopDimensionlessforOPT_function',x0,optnew)
%this m file defines and evaluates cost function value

function f=ClosedLoopDimensionlessforOPT_function(x)
global A B C D r Kp Ki Kd

%Declares Parameter Values
r = 0.3;
Km = 0.040625;
Kg = 0.05538592;
Ra = 1.1;
Jm = 2.864583333e-05;
La = 0.0023;
Bm = 4.973591972e-06;
K = 0.6;
JL = 0.1;
Kp=x(1);
Ki=x(2);
Kd=x(3);

%----Define State Space variables to Run Simulation----

A=[-Ra/La 0 -Kg/La 0 0;
   0 0 1 0 0;
   Km/Jm -(K*r^2)/Jm (-Bm)/Jm (K*r)/Jm 0;
   0 0 0 0 1;
   0 (K*r)/JL 0 -K/JL 0];

B=[1/La;0;0;0;0];

C=diag([1 1 1 1 1]);
D=[0;0;0;0;0];

%----Run Simulation to get results----

[t,xsim,ysim]=sim('closedloopPID_for_opt');

%----Evaluate Results using cost function----

f=int(t,t.*(ysim(:,4)-1).*(ysim(:,4)-1)+0*ysim(:,6))
cost=f
x
APPENDIX D
DYNAMIC INVERSION SIMULATION
%this m file defines open-loop dimensional state-space model
%and finds transformation matrix so that Dynamic Inversion can
%be applied to system model

global Ass Bss A B a1 a2 inv_B lamda timeopt theta_Lopt Q R

%---Gear Ratio Selection---
r  = 8.6406e-002;
r  = 0.3;
r  = 0.025;

%--Declares Parameter Values---
Km  = 0.040625;
Kg  = 0.05538592;
Ra  = 1.1;
La  = 0.0023;
Jm  = 2.864583333e-05;
Bm  = 4.973591972e-06;
K   = 0.6;
JL  = 0.1;
Wa  = sqrt(K/JL);
Ce  = (Ra/La)/Wa;
Cm  = (Bm/Jm)/Wa;
Jr  = (r*r*Jm)/Jm;
Cddim = (r*r*Ra*JL*Wa)/(Km*Kg);

%---Defines State Space Matrices---
Ass=[-Ra/La 0  0  0  0
     0  0  1  0  0
     0  0  0  1  0
     Km/Jm -(r*r*K)/Jm -Bm/Jm (r*K)/Jm 0
     0  0  0  0  1]
Bss=[1/La;0;0;0;0];
Css=[1 0 0 0 0
    0 1 0 0 0
    0 0 1 0 0
    0 0 0 1 0
    0 0 0 0 1];
Dss=[0;0;0;0;0];

%---Choose Initial Guesses Based on Gear Ratio Selection---
Q=diag([.01 .01 .01 50 .01]);R=0.1;  %r=8.6406e-002;
Q=diag([2.5 .1 100 1]);R=0.01;    %r=0.3;
Q=diag([.001 .001 .001 100 .001]);R=0.1;  %r=0.025;
T=lqr(Ass,Bss,Q,R);

A=T*Ass;
B=T*Bss;
inv_B=inv(B);

lamda= 25;

load theta_Lr025_lqry  \%r=0.025

x0=[diag(Q)' R];

options=optimset;
optnew=optimset(options,'LevenbergMarquardt','on');

x=fminsearch('motor_inertia_DI_lqr_opt_f_V2',x0,optnew)

sim('motor_inertia_DI_s')
figure(1),plot(time,theta_L,time,theta_Lopt,'r')
%this m file defines and evaluates the cost function

function f=motor_inertia_DI_lqr_opt_f_V2(x)
global Ass Bss A B a1 a2 inv_B lamda timeopt theta_Lopt Q R

x=abs(x);
Q = diag([x(1) x(2) x(3) x(4) x(5)]);
R = x(6);

%---Finds Transformation Matrix---
T=lqr(Ass,Bss,Q,R);

%---Defines Transformed System Model---
A=T*Ass;
B=T*Bss;
inv_B=inv(B);

%---Runs Simulation---
[t,xs,ys]=sim('motor_inertia_DI_s');

%---Evaluates System Using Cost Function---
theta_error=ys(:,1);
P=ys(:,2).*ys(:,3);
f=int(t,t.*(theta_error.*theta_error+0*P.*P));

x
cost=f
Dynamic Inversion Control Law
APPENDIX E
NON-DIMENSIONAL ANALYSIS OF BALLOON SYSTEM
syms X Y La Kg Jr Kr Tau Ra Km Jv Cv Jm Kr r Cr

\[
P = \begin{bmatrix}
La/Kg & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & (Jr/Kr)^{(1/2)} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & (Jr/Kr)^{(1/2)} & 0 \\
0 & 0 & 0 & 0 & 1/r & 0 \\
0 & 0 & 0 & 0 & 0 & (1/r)*(Jr/Kr)^{(1/2)}
\end{bmatrix};
\]

\[
P1= \begin{bmatrix}
(Jr/Kr)^{(1/2)} & 0 & 0 & 0 & 0 & 0 \\
0 & (Jr/Kr)^{(1/2)} & 0 & 0 & 0 & 0 \\
0 & 0 & (Jr/Kr)^{(1/2)} & 0 & 0 & 0 \\
0 & 0 & 0 & (Jr/Kr)^{(1/2)} & 0 & 0 \\
0 & 0 & 0 & 0 & (Jr/Kr)^{(1/2)} & 0 \\
0 & 0 & 0 & 0 & 0 & (Jr/Kr)^{(1/2)}
\end{bmatrix};
\]

\[
Am=\begin{bmatrix}
-Ra/La & 0 & Kg/La & 0 & -Kg/La & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
-Km/Jv & -Kv/Jv & (-Bm)/Jv & 0 & Bm/Jv & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
Km/Jm & 0 & Bm/Jm & (-Kr*r*r)/Jm & -(Bm)/Jm & (Kr*r)/Jm \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & (Kr*r)/Jr & 0 & -Kr/Jr
\end{bmatrix};
\]

\[
B=\begin{bmatrix}
1/La;0;0;0;0;0
\end{bmatrix};
\]

\[
Uc=Kg*sqrt(Kr/Jr);
\]

\[
Pinv=INV(P);
\]

\[
Adim=P1*P*Am*Pinv
\]

\[
Bdim=P1*P*B*Uc
\]