Analysis of the thermoelastic response of a semi-space to a short laser pulse

Gary Renz
Analysis of the Thermoelastic Response of a Semi-Space to a Short Laser Pulse

by

Gary B. Renz

A thesis submitted in partial fulfillment of the requirements for the degree of

Master of Science

in

Mechanical Engineering

Approved by: PROFESSOR __________________________
Richard Hetnarski, Advisor

PROFESSOR __________________________
Hany Ghoneim

PROFESSOR __________________________
Josef Torok

PROFESSOR __________________________
Charles Haines, Department Head

Department of Mechanical Engineering
College of Engineering

ROCHESTER INSTITUTE OF TECHNOLOGY
Rochester, New York

May 13, 1997
Analysis of the Thermoelastic Response of a Semi-Space to a Short Laser Pulse

I, Gary B. Renz, hereby grant permission to the Wallace Memorial Library of the Rochester Institute of Technology to reproduce my thesis in whole or part. Any reproduction will not be for commercial use or profit.

________________________________________
Gary B. Renz

May 13, 1997
Acknowledgment

Through the course of the past three years, I have endeavored to do my very best in completing the required studies in the Master of Science Program in Mechanical Engineering at the Rochester Institute of Technology. It was a well thought out decision to pursue this degree, that would potentially have an impact on many people as I would attempt a cohesive balance between school, work, and home. I have hopefully succeeded in achieving that balance but know that it would not have been possible without the thoughtful caring and interest of many people.

The extensive list of individuals begins with my friends and colleagues at the Eastman Kodak Company, most notably my manager, Mr. W. Thomas Deever, for his endless support even in the most dismal of times, and from the corporation itself for the generous financial support through the course of the entire program.

For academic inspiration, I wish to thank all the R.I.T. professors who worked with me through specific course work in the program, but especially my advisor for this thesis, Dr. Richard B. Hetnarski, who stimulated my interest in thermoelasticity and whose sensitivity and patience through my learning process encouraged me to persevere.

My greatest inspiration and desire to succeed, comes from my family, including my mother and father and most importantly and influential, the love from my dear wife, Karen and son, Ryan who sacrificed many hours of family “quality time” to allow me to concentrate on my journey through this program. None of this would have been possible without their devoted love and support. It is to them that I dedicate this work.
Abstract

A study of the thermoelastic response of a semi-space medium to a short laser pulse generated heat is presented. This study uses the generalized thermoelasticity theory proposed by Green and Lindsay. This theory generalizes the classical theory of thermoelasticity by developing hyperbolic heat conduction equations and temperature-rate dependent constitutive equations. By so doing, an inherent paradox and physically unrealistic result in the classical theory of thermoelasticity that proposes infinite speed of thermal responses through a medium is corrected. Its validity is justified by the satisfaction of Fourier’s law by which classical thermoelasticity is based. The Green-Lindsay theory adopts two thermal relaxation times and a thermoelastic coupling constant as specific material parameters to account for finite speed thermoelastic waves.

The model presented in this study uses a semi-space medium that has imposed on its boundary a laser induced heat of the form of a product of an exponentially decreasing function of the semi-space depth and a skewed Gaussian temporal profile. A numerical analysis of the exact closed form solution is presented. This analysis reveals that for a fixed cross section of the semi-space depth, the stress-temperature response is represented by a pair of smooth transiental functions of time and display two distinct planar thermal wave fronts of finite speed.
## Table of Contents

**CHAPTER 1**  
**Introduction** .................................................................................. 1

**CHAPTER 2**  
**Thermoelasticity** ........................................................................... 4  
2.1  Introduction ..................................................................................... 4  
2.2  Classical Theory of Linear Thermoelasticity ......................... 6  
2.3  Generalized Thermoelasticity ..................................................... 11  
   2.3.1  Concept of Thermal Relaxation Time ......................... 14  
   2.3.2  Thermoelasticity with One Relaxation Time .............. 15  
   2.3.3  Thermoelasticity with two Relaxation Times ........... 16

**CHAPTER 3**  
**The General Solution** .................................................................. 19  
3.1  Introduction ................................................................................... 19  
3.2  Formulation of the Problem ...................................................... 20  
3.3  Green’s Function .......................................................................... 22

**CHAPTER 4**  
**Response to a Short Laser Pulse** .................................................. 25  
4.1  Introduction .................................................................................. 25  
4.2  Formulation of the Problem ...................................................... 26  
4.3  Development of the Closed Form Solution ......................... 27  
4.4  The Solution ................................................................................ 29  
   4.4.1  Stress-Temperature Formulations ............................. 30  
   4.4.2  Intermediate Function Formulations .................... 33  
4.5  Hypothesis .................................................................................. 35
<table>
<thead>
<tr>
<th>CHAPTER 5</th>
<th>Numerical Analysis</th>
<th>37</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1</td>
<td>Program Methodology</td>
<td>37</td>
</tr>
<tr>
<td>5.2</td>
<td>Parameter Values</td>
<td>39</td>
</tr>
<tr>
<td>5.3</td>
<td>Laser Profile</td>
<td>42</td>
</tr>
<tr>
<td>5.4</td>
<td>Evaluation of Primary Intermediate Functions</td>
<td>43</td>
</tr>
<tr>
<td>5.4.1</td>
<td>Intermediate Functions for ((\sigma_d, \theta_d))</td>
<td>44</td>
</tr>
<tr>
<td>5.4.2</td>
<td>Intermediate Functions for ((\sigma_w, \theta_w))</td>
<td>46</td>
</tr>
<tr>
<td>5.4.3</td>
<td>Intermediate Functions for ((\sigma_c, \theta_c))</td>
<td>50</td>
</tr>
<tr>
<td>5.5</td>
<td>Evaluation of Stress-Temperature Pairs</td>
<td>52</td>
</tr>
<tr>
<td>5.5.1</td>
<td>Diffusive Stress-Temperature Component</td>
<td>52</td>
</tr>
<tr>
<td>5.5.2</td>
<td>Wave Stress-Temperature Component</td>
<td>57</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CHAPTER 6</th>
<th>Discussion of Results</th>
<th>66</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.1</td>
<td>Introduction</td>
<td>66</td>
</tr>
<tr>
<td>6.2</td>
<td>The Total Stress Response</td>
<td>66</td>
</tr>
<tr>
<td>6.3</td>
<td>The Total Temperature Response</td>
<td>70</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CHAPTER 7</th>
<th>Conclusions</th>
<th>74</th>
</tr>
</thead>
<tbody>
<tr>
<td>References</td>
<td></td>
<td>77</td>
</tr>
<tr>
<td>Bibliography</td>
<td></td>
<td>78</td>
</tr>
<tr>
<td>Appendix A</td>
<td></td>
<td>79</td>
</tr>
<tr>
<td>PART A - Intermediate Function Evaluations</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Define parameters and initialize values</td>
<td>80</td>
<td></td>
</tr>
<tr>
<td>Create the necessary call files</td>
<td>81</td>
<td></td>
</tr>
<tr>
<td>Additional call files</td>
<td>83</td>
<td></td>
</tr>
<tr>
<td>Evaluate model parameters</td>
<td>85</td>
<td></td>
</tr>
</tbody>
</table>
Listing of parameter values ................................................................. 86
Evaluate intermediate functions $g_n^{(l)}$, $h_n^{(l)}$, $A_n^{(l)}$ ......................... 88
Evaluate functions $M_i(x,t)$, $N_i(x,t)$ .................................................. 95
Evaluate the laser profile ......................................................................... 99
Evaluate functions $\hat{R}_n^{(l)}(x_i,s;a), \hat{Q}_n^{(l)}(x_i,s;a), \hat{R}_n^{(l)}(x_i,s;a)$ .......... 103
Evaluate functions $P_i(x,t,a),Q_i(x,t,a),R_i(x,t,a)$ ...................................... 106
Evaluate functions $\hat{g}_n^{(l)}(t), \hat{Q}_n(x,t,a), \hat{R}_n(x,t,a)$ ........................... 115

PART B - Stress / Temperature Evaluations

Evaluate the diffusive stress responses $\sigma_d^{(l)}(x,t), \sigma_d(x,t)$ .................... 121
Evaluate the diffusive temperature responses $\theta_d^{(l)}(x,t), \theta_d(x,t)$ ............. 124
Evaluate the first wave stress response $\sigma_w^{(l)}(x,t), \sigma_w(x,t)$ ....................... 126
Evaluate the first wave temperature response $\theta_w^{(l)}(x,t), \theta_w(x,t)$ ............. 129
Evaluate boundary stress relationships $\sigma_0^{(l)}(0,t), \sigma_0^{(l)}(0,t)$ ................. 132
Evaluate boundary temperature relationships $\theta_0^{(l)}(0,t), \theta_0^{(l)}(0,t), \theta_0^{(l)}(0,t)$ .. 136
Evaluate total boundary stress / temperature relationships ............................. 141
Evaluate the “tilde” stress / temperature relationships $\tilde{\sigma}(0,t), \tilde{\theta}(0,t)$ ...... 144
Evaluate the second wave stress response $\sigma_e^{(l)}(x,t), \sigma_e(x,t)$ ................. 146
Evaluate the second wave temperature response $\theta_e^{(l)}(x,t), \theta_e(x,t)$ .......... 149
Evaluate the total stress response $\sigma(x,t)$ ............................................. 151
Evaluate the total temperature response $\theta(x,t)$ ..................................... 153
List of Tables and Figures

Tables:

5.2.1 Parameter Values ........................................................................................................... 41

Figures:

2.2.1 Parabolic Representation of Temperature Distribution .............................................. 10
2.3.1 Hyperbolic Representation of Temperature Distribution ......................................... 13
2.3.2 Theory of Thermoelasticity Flow Development ......................................................... 17
5.3.1 Laser Profile .................................................................................................................. 42
5.4.1 $R_i(0, t; a); i = 1$ ........................................................................................................ 45
5.4.2 $R_i(0, t; a); i = 2$ ........................................................................................................ 45
5.4.3 $M_i(x_i, t); i = 1$ .......................................................................................................... 47
5.4.4 $M_i(x_i, t); i = 2$ .......................................................................................................... 47
5.4.5 $P_i(x_i, t; a); i = 1$ ...................................................................................................... 48
5.4.6 $P_i(x_i, t; a); i = 2$ ...................................................................................................... 48
5.4.7 $Q_i(x_i, t; a); i = 1$ ...................................................................................................... 49
5.4.8 $Q_i(x_i, t; a); i = 2$ ...................................................................................................... 49
5.4.9 $N_i(x_i, t); i = 1$ .......................................................................................................... 51
5.4.10 \[ N_i(x_i,t_i);i = 2 \]

5.5.1 Diffusive Stress \( \sigma_d^{(i)}(x_i,t_i);i = 1 \)

5.5.2 Diffusive Stress \( \sigma_d^{(i)}(x_i,t_i);i = 2 \)

5.5.3 Total Diffusive Stress \( \sigma_d(x_i,t_i) \)

5.5.4 Diffusive Temperature \( \theta_d^{(i)}(x_i,t_i);i = 1 \)

5.5.5 Diffusive Temperature \( \theta_d^{(i)}(x_i,t_i);i = 2 \)

5.5.6 Total Diffusive Temperature \( \theta_d(x_i,t_i) \)

5.5.7 First Wave Stress \( \sigma_w^{(i)}(x_i,t_i);i = 1 \)

5.5.8 First Wave Stress \( \sigma_w^{(i)}(x_i,t_i);i = 2 \)

5.5.9 Total First Wave Component of Stress \( \sigma_w(x_i,t_i) \)

5.5.10 First Wave Temperature \( \theta_w^{(i)}(x_i,t_i);i = 1 \)

5.5.11 First Wave Temperature \( \theta_w^{(i)}(x_i,t_i);i = 2 \)

5.5.12 Total First Wave Component of Temperature \( \theta_w(x_i,t_i) \)

5.5.13 Second Wave Stress \( \theta_e^{(i)}(x_i,t_i);i = 1 \)

5.5.14 Second Wave Stress \( \theta_e^{(i)}(x_i,t_i);i = 2 \)

5.5.15 Total Second Wave Component of Stress \( \sigma_e(x_i,t_i) \)

5.5.16 Second Wave Temperature \( \theta_e^{(i)}(x_i,t_i);i = 1 \)

5.5.17 Second Wave Temperature \( \theta_e^{(i)}(x_i,t_i);i = 2 \)

5.5.18 Total Second Wave Component of Temperature \( \theta_e(x_i,t_i) \)

6.2.1 Total Stress Component \( (i = 1) \)

6.2.2 Total Stress Component \( (i = 2) \)
6.2.3 Total Stress Response $\sigma(x,t)$ ................................................................. 69
6.3.1 Total Temperature Component ($i = 1$) ......................................................... 72
6.3.2 Total Temperature Component ($i = 2$) ......................................................... 72
6.3.3 Total Temperature Response $\theta(x,t)$ ............................................................ 73
List of Symbols

\( A \) intermediate function
\( a \) inverse absorption coefficient for material
\( b \) laser shape parameter affecting pulse activation time
\( c \) specific heat
\( g \) intermediate function
\( H() \) Heaviside function
\( h \) intermediate function
\( h_i \) attenuation coefficient
\( K \) thermal conductivity
\( M \) power series of Neumann type
\( m \) laser shape parameter affecting positive skewness
\( N \) power series of Neumann type
\( n \) laser shape parameter affecting negative skewness
\( P \) intermediate function
\( \dot{P} \) intermediate function
\( Q \) intermediate function
\( \dot{Q} \) intermediate function
\( q \) heat flux
\( R \) intermediate function
\( \hat{R} \) intermediate function

\( r \) laser induced heat

\( S \) general stress response

\( S_c \) regular part of Green's function (stress)

\( S_N \) singular part of Green's function (stress)

\( T \) general temperature response

\( T_c \) regular part of Green's function (temperature)

\( T_N \) singular part of Green's function (temperature)

\( t \) time variable

\( t_f \) laser pulse activation time

\( t_r \) laser pulse rise time

\( t^d \) dimensionless thermal relaxation time

\( t_0 \) dimensionless thermal relaxation time

\( \tilde{t} \) dimensional thermal relaxation time

\( t_0^* \) reference thermal relaxation time

\( u_x \) displacement component

\( v_i \) wave velocity component

\( W \) internal heat source

\( x \) spatial variable

\( x_i \) \( i^{th} \) wave front location

\( Y \) laser profile

\( \alpha \) decomposition parameter

\( \alpha_i \) decomposition parameter

\( \hat{\alpha} \) non-dimensional parameter for a thermoelastic solid
\( \beta \)  
\( \hat{\beta} \)  
\( \Delta \)  
\( \varepsilon \)  
\( \varepsilon_{ij} \)  
\( \Gamma \)  
\( \hat{\gamma} \)  
\( \lambda \)  
\( \lambda_i \)  
\( \rho \)  
\( \Phi \)  
\( \sigma \)  
\( \sigma_{ij} \)  
\( \sigma_e \)  
\( \sigma_d \)  
\( \sigma_w \)  
\( \hat{\sigma}_w \)  
\( \sigma_0 \)  
\( \tilde{\sigma}_0 \)  
\( \Theta \)  
\( \Theta_0 \)  
\( \theta \)  
\( \theta_e \)  

decomposition parameter
non-dimensional parameter for a thermoelastic solid
decomposition parameter
thermoelastic coupling constant
strain tensor
central operator of the Green-Lindsay theory
non-dimensional parameter for a thermoelastic solid
decomposition parameter
convolution coefficient
mass density
thermoelastic potential
total stress response
stress tensor
second wave part of stress response
diffusive part of stress response
first wave part of stress response
wave part of stress response
boundary stress
contributing function for boundary stress
absolute temperature
initial uniform temperature
total temperature response
second wave part of temperature response
\( \theta_d \)    diffusive part of temperature response
\( \theta_w \)    first wave part of temperature response
\( \hat{\theta}_w \)  wave part of temperature response
\( \theta_0 \)  boundary temperature
\( \tilde{\theta}_0 \)  contributing function for boundary temperature
\( \tau \)    thermal relaxation time
\( \xi \)    laser profile validity constant
\( \zeta \)  \( i^{th} \) wave front; time offset
\( \nabla \)  gradient
\( \nabla^2 \)  second order operator \( (\nabla \cdot \nabla) \)
(\( \cdot \))  time derivative
CHAPTER 1

Introduction

Classical thermoelasticity has developed its theories from a useful coupling and generalization of the theories of rigid body heat conduction and the theory of elasticity. In this generalization, classical thermoelasticity theories have been found to be very effective in the analysis and solution to many practical problems where the effects of a thermomechanical loading on a body produces thermoelastic responses such as displacement, heat flux, stress, and temperature change. However, the classical theories have also been challenged as inexact, particularly in applications that involve sudden heat inputs, very short time spans, high heat fluxes, and low temperatures (~1K). This is because the classical formulations result in a parabolic heat conduction equation that predicts the effects of an external thermal loading to be felt instantaneously at any cross section far from the loading in a medium thus suggesting infinite speed of thermal response. To resolve this unrealistic result, many investigations (analytical and empirical) have sought to develop a generalization to the classical theory of thermoelasticity that would effectively predict finite wave speeds of a thermoelastic response.

This thesis is a study of generalized thermoelasticity with an objective of tracking a brief history of the development and usefulness of the generalized theory, providing an
introduction and foundation to the fundamental equations and constitutive parameters associated with the generalized theory, and finally providing a numerical analysis of an exact analytical, closed form solution for a specific model. It is the next step in studies published by Hetnarski and Ignaczak that have examined the generalized theories for a rigid thermoelastic body in a semi-space subjected to a planar heat source [7] and a short laser pulse [8] and previous thesis work by Gorman [3] who completed a one-dimensional numerical analysis for the thermoelastic body subjected to a planar heat source. This thesis concentrates on the numerical analysis for the case of a thermoelastic body in a semi-space subjected to a short laser pulse.

In beginning this study, Chapter 2 initiates a historical review of thermoelasticity concepts describing some of the significant work that has revolutionized this field of science since the late 1800’s. A developmental flow is established that highlights the foundations of the theory. An explanation of the unrealistic paradox of infinite thermal wave speed is introduced and the key equations that work to resolve that paradox, associated with the generalized theory, are introduced. Integral with this discussion, is an introduction of the concept of thermal relaxation times and the theories that effectively use them in the constitutive equations.

In Chapter 3, the general solution of thermoelasticity with two relaxation times is introduced and briefly describes the techniques and tools utilized to achieve the solution. These tools include the use of Green’s function and the decomposition theorem. It is in Chapter 3 that many of the key parameters are defined and formulated.

The closed form solution for the response to a short laser pulse in a semi-space is developed in Chapter 4. The laser profile is described and general problem constraints are stated. The chapter concludes with discrete formulations for the stress-temperature response and gives a hypothesis for the response through the range of values considered in this model.
Chapter 5 outlines the methodologies that were utilized in the construction of the numerical analysis. This part of the analysis details the key intermediate functions and their behaviors that are integral to the stress-temperature formulations based on a defined set of parameter values. Mathematica Version 3, by Wolfram Research is the programming application used for this analysis. The program, which includes plots of all the intermediate functions, is listed in its entirety in Appendix A. Chapter 5 concludes with graphical representations of the three components that make up the stress-temperature response pairs. It is in these representations that the evidence of the two wave fronts become apparent.

In Chapter 6, the total stress-temperature response is studied by combining the three components that were analyzed in Chapter 5. This chapter leads directly into Chapter 7 which provides a discussion of conclusions including a brief discussion of the applicability of this study to practical applications and possible directions for future work.

As previously stated, the Mathematica program, in its entirety, is listed in Appendix A. Since Mathematica is a high level interpreted language that uses program objects rather than program source code, and because it possesses an elegant output formatting structure, it was possible to incorporate the function plots for this thermoelastic model directly as part of the program script.
CHAPTER 2

Thermoelasticity

A Historical Review

2.1 - Introduction

Thermoelasticity is the study of an elastic body or material under the influence of non-uniform changes in its temperature field. It is a generalization of the theory of elasticity by accounting for thermal reactions as well as mechanical reactions. The theories of thermoelasticity (classical and modified) have been developed by a useful coupling of Fourier's law and the related Fourier's law of heat conduction with the standard formulations developed in the theory of elasticity. In the theory of thermoelasticity, the constitutive equations that describe the elastic behavior of a particular material include temperature dependencies and relationships derived from Fourier's laws that effectively relate the heat flux in a body with a local temperature gradient.

To further develop the understanding of thermoelasticity, it is instructive to examine the fundamental concepts that are developed independently by the theory of elasticity and the heat transfer theories associated with heat conduction.
As stated by Gould [4], "The theory of elasticity comprises a consistent set of equations which uniquely describe the state of stress, strain, and displacement at each point within an elastic deformable body." An elastic deformable body is one that may be described as a body that has imposed upon it a systematic "loading" that will "deform" its elements only to the extent that upon release of the "loading" the body will fully return to its original undeformed state. Interestingly, when a body undergoes such a loading that may either compress or dilate elements of the body, the elements undergoing the deformation will experience internal stresses that are directly related to the resulting strain but will also experience a temperature change that is related to the deformation. As such, an element will experience a temperature rise in the state of compression, and a temperature decrease in the state of dilation [4], albeit very small. These temperature changes are not treated in the theory of elasticity and are left for the formulations derived in the theory of thermoelasticity.

In the study of heat transfer, the main interest is in the rate of heat exchange that may take place in a given system. This is, of course, relevant any time a temperature gradient exists within a system; in particular a material body. In the study of heat transfer, a quantity referred to as the heat flux \( q \) is defined as the amount of heat transfer per unit area, per unit time and is readily determined from the law relating the heat flux to the temperature gradient. This is commonly referred to as Fourier's law named for the French mathematical physicist Joseph Fourier who used the theory in his analytic work in the theory of heat. The theory is further expanded into conductive heat transfer which defines the mode of heat transfer where energy exchange takes place from a region of high temperature to a region of low temperature by the kinetic motion or direct impact of molecules or electrons. In this mode, Fourier's law is expanded to account for relationships between a temperature gradient (heat gains by conduction) and a rate of change of temperature with respect to time. It is these heat transfer theories that are used as
a fundamental basis with the formulations from the theory of elasticity that establish the models used in classical linear thermoelasticity.

In this chapter, a historical review of the various theories of thermoelasticity will be reviewed. The review will begin with a study of the classical theory of linear thermoelasticity and will progress to a study of the generalized theory of thermoelasticity. The generalized theory seeks to resolve practical paradoxes inherent in the classical theory by incorporating into the theory the concept of second sound. The chapter will conclude with a review of previous work that has formulated both qualitative and numerical results.

2.2 - Classical Theory of Linear Thermoelasticity

The classical theory of linear thermoelasticity, like all other theories in the realm of continuum mechanics, is based on three basic laws; (1) the law of motion which includes kinematic and static equilibrium equations, (2) the conservation of mass, and (3) the conservation of energy, which in the theory of thermoelasticity is derived as the first law of thermodynamics. Overarching these three laws is the second law of thermodynamics which ensures the positive production of entropy.

It is important to note that the classical theory of linear thermoelasticity imposes additional constraints of validity on the medium under study in that only small deformations (relative to the dimensions of the medium), small temperature changes (relative to the absolute initial temperature of the medium), and gradual temperature changes are considered. Under these constraints it is reasonable to deduce that strain will depend linearly on stress and temperature and that Fourier’s equation of heat conduction will apply [11]. Additionally, since only gradual temperature changes are under consideration, it is reasonable to further deduce that inertial characteristics associated with the medium’s
deformation will be negligible thus reducing the equations of motion to the equations of equilibrium. This gradual motion under these temperature effects is termed quasi-static motion [11].

For a body conforming to linear elasticity, the theory of elasticity determines that a linear relationship must exist between stress and strain. In consideration of temperature effects of a body in an elastic state, it is convenient to define a thermoelastic potential (Φ) that may be introduced into the stress-strain relationship such that,

\[ \sigma_{ij} = \frac{\partial \Phi}{\partial \varepsilon_{ij}} \]  

(2.2.1)

where Φ depends on both strain components and, in contrast to isothermal elasticity, on temperature. If Φ is of quadratic form, then it is clear by observation that the generalized Hooke’s law will be valid, thus, the stress will be related linearly to the strain and will also be a function of temperature.

In determining solutions to problems of thermoelasticity, the first step is always to determine the temperature field that is ultimately generating the stress-strain response. To do this, the heat transfer theories of heat conduction, based on Fourier’s law for rigid bodies are utilized. Fourier’s law is stated as,

\[ q = -K \nabla \theta \]  

(2.2.2)

where,

\[ \theta = \Theta - \Theta_0 \]  

(2.2.3)

which relates the heat flux (q) to the temperature gradient (\( \nabla \theta \)). The additional terms are defined as, temperature difference (\( \theta \)), absolute temperature (\( \Theta \)), initial uniform
temperature \( (\Theta_o) \), and thermal conductivity \( (K) \). The minus sign indicates that heat will flow in the direction of decreasing temperature.

Combining equation (2.2.2) with the law of conservation of energy, thus balancing heat flow into a body with heat flow out of a body, the Fourier’s equation of heat conduction is derived. First, considering the case of uniform heat conduction with an internal heat source, the equation is stated as,

\[
\frac{\partial \theta}{\partial t} = A(\nabla^2 \theta + \frac{1}{K}W)
\]

(2.2.4)

where \( A \) is a constant for the material of the body defined as,

\[
A = \frac{K}{\rho c}
\]

(2.2.5)

given an internal heat source, \( (W) \), and material constants defined as mass density \( (\rho) \) and specific heat \( (c) \).

For the case where an internal heat source is absent, equation (2.2.4) reduces to,

\[
\frac{\partial \theta}{\partial t} = A\nabla^2 \theta.
\]

(2.2.6)

To establish a complete set of relationships for the theory of thermoelasticity, it is necessary to derive constitutive equations that are temperature dependent and include Fourier’s law. These constitutive equations are described as coupled equations due to their characteristics of fully describing the thermomechanical interaction of an external loading or heat source imposed on an elastic body. These coupled equations will include relationships between heat flux, displacement, stress, and temperature thus setting a firm foundation for the thermal and mechanical changes that may occur in an elastic body subjected to a thermal or mechanical loading [3]. In considering an ideal thermoelastic body, it is noted that the
mechanical deformations and thermal variants will vanish upon the removal of the external thermomechanical load.

The non-isothermal thermoelastic process is described by a set of field equations and inequalities defined as (taken from [9]):

a) Geometrical relations - relates the strain field to the displacement field.

b) The laws of balance of forces and moments.

c) The law of conservation of energy.

d) The constitutive relations - couples the temperature changes to the stress and strain tensors.

e) Fourier’s law.

The partial differential equation (2.2.6) represents the fundamental heat equation. This equation is characteristically a parabolic relationship. By virtue of this characteristic, the equation predicts that a temperature change \( \theta \) will be observed at any point distant from the application point in infinite time within a semi-space. In an example cited by Gorman [3], originally solved by Beck, et. al. (page 17 [1]), a semi-space \( (x > 0) \) is considered for \( (t \geq 0) \) with the following initial and boundary conditions.

Initial conditions:
\[
\theta(x,0) = 0 \quad \text{where; } \theta = \theta(x,t) \quad (2.2.7)
\]

Boundary conditions:
\[
\theta(0,t) = \Theta_0 \quad (2.2.8)
\]

In the solution of this example, Beck, et. al. uses a Green’s function. Green’s functions are useful as a basic solution to a specific differential equation with homogeneous boundary conditions. As an example, a Green’s function may be used to describe the temperature distribution response from an instantaneous thermal pulse. Green’s functions
have been used extensively in the solution of transient and steady-state linear heat conduction problems.

Through the use of a Green's function, the solution has the form of the complimentary error function (erfc), such that:

\[ \theta(x,t) = \theta_0 \text{erfc} \left( \frac{x}{2 \sqrt{At}} \right) \]  

(2.2.9)

The complimentary error function is defined as:

\[ \text{erfc}(x) = 1 - \text{erf}(x) \]  

(2.2.10)

where:

\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-z^2} dz \]  

(2.2.11)

and is graphically represented in Figure (2.2.1) across a range of \( x \) spatial values for a given fixed time \( t \). Figure (2.2.1) clearly shows that at any distance from the point of heat application, a temperature effect will be observed instantaneously as characterized by the error function's values asymptotically approaching the x-axis as \( x \) increases.

\[ \text{erfc}(x) \]

\[ \theta \]

\[ \theta_0 \]

x

**Figure 2.2.1 - Parabolic Representation of Temperature Distribution**
The solution to this problem, as shown above, presents a physical paradox that has been studied by many researchers over the past forty years. The physical paradox is understood by asking the question, “How can the effect of a thermal disturbance be observed instantaneously at distances infinitely far from the original thermal source given a continuous medium?” From an intuitive point of view, this effect is physically impossible and from a theoretical sense, inexact.

For most practical problems in heat conduction, the solutions produced by the classical formulations are likely to present a level of accuracy that will suffice the problem at hand. However, there does exist a range of problem types that would suffer by not using a modification to the classical solutions that minimize or eliminate the effects due to this paradox. The literature over the past thirty years has identified problem types that involve sudden heat inputs, very short time spans, high heat fluxes, and low temperatures (~1K) as those that will benefit most from a modified classical formulation. These may be closely linked to applications including laser penetration, laser welding, explosive bonding, and explosive melting [2].

2.3 - Generalized Thermoelasticity

To resolve the problem of infinite speed of thermal propagation presented by a parabolic solution in classical thermoelasticity, a hyperbolic solution is proposed in the form of a one-dimensional wave equation. This implies a formulation of the form:

\[
\frac{\partial^2 \theta}{\partial x^2} - \frac{\partial^2 \theta}{\partial t^2} = 0 \tag{2.3.1}
\]

A solution to this problem is also cited by Gorman [3]. Similar to the solution of the problem presented for the parabolic formulation, a case is considered on the semi-space
$x > 0$ and $t \geq 0$. For this problem, the following initial and boundary conditions are defined:

Initial conditions:

\[ \theta(x,0) = 0 \quad \text{where } \theta = \theta(x,t) \]  \hspace{1cm} (2.3.2)

Boundary conditions:

\[ \theta(0,t) = \theta_0[H(t)] \]  \hspace{1cm} (2.3.3)

where $H(t)$ is the Heaviside function defined by:

\[ H(t) = \begin{cases} 
0, & t < 0 \\
1, & t \geq 0 
\end{cases} \]  \hspace{1cm} (2.3.4)

A solution to this problem is:

\[ \theta(x,t) = \theta_0(t-x)[H(t-x)] \]  \hspace{1cm} (2.3.5)

such that:

\[ \theta = \begin{cases} 
0, & (t-x) < 0 \\
\theta_0(t-x), & (t-x) \geq 0 
\end{cases} \]  \hspace{1cm} (2.3.6)
The solution (2.3.5) to this hyperbolic formulation is graphically presented in Figure (2.3.1). It is clear from this representation that the solution presents a characteristic wave phenomenon rather than the diffusion phenomenon apparent in the parabolic solution. This is evident from the observation that for values of \( x > t_0 \), at any fixed time \( t_0 \), the temperature disturbance response vanishes. Thus, the disturbance travels through the medium with finite speed.

Since it appears that the hyperbolic wave equation will more adequately characterize a thermal disturbance response in a medium, much research has been dedicated to postulating reasonable and justifiable formulations that will properly represent the wave propagation phenomenon without violating Fourier's law. This research has lead to even higher levels of generalization of the thermoelasticity theories. The development of these are presented in the subsequent sections.
2.3.1 - Concept of Thermal Relaxation Time

Wave type heat flow was first postulated by Maxwell (1867) who suggested a modification to Fourier's law to accommodate the concept. Since Maxwell's first postulation, others have continued to speculate thermal wave propagation in mediums such as gases and liquids as well as solids. Primarily, research and experimentation has concentrated on mediums considered to be "good thermal conductors" most specifically at very low temperatures (i.e. 0 K - 3 K).

In the completion of research studies on super fluid helium, Landau (1941) described the wave type phenomenon as the propagation of phonon density disturbance that possessed a wave speed approximately 57% of the speed of "ordinary" sound at 0 K. Experimentally, this propagation was first detected in liquid helium at 1.4 K by Peshkov (1944). Later studies in the late 1960's and early 1970's observed the phenomenon in sapphire and other crystals. Due to the characteristic nature of the thermal wave propagation and its similarities to the wave characteristics of sound, the term second sound has been regularly used to describe the effect [2].

Cattaneo (1948) worked extensively to develop a theoretical basis to account for the existence of second sound. His proposal was to generalize Fourier's law; in effect formulating a modified Fourier's law. As discussed earlier, the modification was directed at creating a hyperbolic relationship in the heat conduction equation. This modification, commonly referred to as Cattaneo's equation adds a term that is the product of a non-negative constant (τ) and a flux-rate term \( \frac{\partial q}{\partial t} \). The non-negative constant is called the thermal relaxation time. Chandrasekharahaiah (1986) [2] described a physical interpretation for the thermal relaxation time as the time lag necessary for an element in a material to attain steady state heat conduction subsequent to the application of a sudden temperature gradient.
on the element. Tamma further characterized an understanding of the thermal relaxation
time by describing the transport of thermal energy as quantized electronic excitations (free
electrons) and vibrational energy (phonons) that experience collisions of a dissipative
nature causing a thermal resistance in a medium. The average communication time between
these collisions for the initiation of resistive flow is termed the thermal relaxation time [12].

2.3.2 - Thermoelasticity with One Relaxation Time

As first proposed by Cattaneo (1948) and later studied extensively by Lord and
Shulman (1967), a generalization of Fourier's law and the heat conduction equation to
include a single relaxation time was developed. In the case of an isotropic and
homogeneous material, the modified Fourier's law is given by:

\[ \tau \frac{\partial q}{\partial t} + q = -KV\theta \quad (2.3.7) \]

and the corresponding heat conduction equation:

\[ \frac{\partial \theta}{\partial t} + \tau \frac{\partial^2 \theta}{\partial t^2} = AV^2\theta \quad (2.3.8) \]

The equation (2.3.8) is hyperbolic and will therefore predict thermal responses that
propagates at a finite speed. It should be noted that as the thermal relaxation time (\( \tau \))
approaches zero, equation (2.3.7) resorts to equation (2.2.2) and equation (2.3.8) resorts
to equation (2.2.6) in the classical theory.

Throughout the development of the generalized thermoelasticity theory, the main
motivation has been to develop a hyperbolic heat conduction equation to allow prediction of
finite wave speeds in thermal response waves. This clearly addresses the obvious physical
interpretation but does not by itself allow acceptance based on a firm mathematical
foundation. Ignaczak and Bialy (1980) were successful in developing the desired solution
mathematically by proving a domain of influence theorem. The theorem asserts that for an unbounded body, a thermoelastic disturbance produced by a thermomechanical load on a bounded support cannot invade the entire body in finite time [10]. The theorem considers an initial-boundary value problem in terms of the temperature and an expression defined as a thermoelastic potential ($\Phi$), where (in one-dimensional space):

$$u_x = \frac{\partial \Phi}{\partial x}$$

(2.3.9)

An exploration of the domain of influence theorem and its proof is beyond the scope of this thesis. Suffice it to say, however, that by the proof of this theorem, the existence of the second sound phenomenon in thermoelastic disturbances has been established based on a firm mathematical foundation.

2.3.3 - Thermoelasticity with Two Relaxation Times

In the previous section, the generalized theory of elasticity with one relaxation time was developed on the basis that Fourier’s law and the resulting heat conduction equation assumed a modified “predetermined” form by the addition of a heat flux rate term. Müllcr (1971) considered an alternative to the predetermined assumptions in the heat conduction equation by formulating general constitutive equations for the entropy flux and entropy source and by using a generalized entropy inequality [2]. In this generalized theory, Müllcr developed a nonlinear theory of thermoelasticity that included temperature-rate among the constitutive variables thus also considering second sound effects. In 1972, Green and Lindsay [5] formulated a theory similar to Müllcr’s that was much less complex and more explicit. Green and Lindsay’s theory is based on an entropy production inequality proposed by Green and Laws (1972), includes the constitutive relations for entropy flux and entropy source, and also includes the temperature-rate as one of the constitutive
variables. Thus, the theory is described as Temperature-Rate Dependent Thermoelasticity. The characteristic of Green and Lindsay's theory that is of most interest is that it does not make any modification to the classical Fourier's Law [2].

As a summary of the discussions in this chapter, Figure (2.3.2) graphically shows the flow of the analytical work that has progressed, leading to the current generalized theories.

![Diagram showing the flow of Thermoelasticity theories](image)

**Figure 2.3.2** - Theory of Thermoelasticity Flow Development
In the next chapter, closed form solutions of generalized thermoelasticity will be discussed as studied by Hetnarski and Ignaczak (1993) [7]. These closed form solutions will then be applied to the specific study under consideration in this thesis; the response of a semi-space to a short laser pulse.
CHAPTER 3

The General Solution

Thermoelasticity with Two Relaxation Times

3.1 - Introduction

Hetnarski and Ignaczak, in [7], developed the closed form solutions to the initial-boundary value problem for the one-dimensional case of dynamic coupled thermoelasticity. In the article referenced, the following three requirements for the solutions are noted.

a) They are valid for a large range of independent variables and material variables.

b) They are useful in obtaining physically sound analytical solutions.

c) They may be relevant in experimentation on a thermoelastic body in which quasi-mechanical (first sound) and quasi-thermal (second sound) effects may be observed.

As discussed in the previous chapter, this study begins with the understanding that the generalized theory of thermoelasticity in the temperature-rate dependent case will make modifications by generalizing the constitutive relations for stress, temperature, and entropy by introducing two relaxation times \( (\tau^f, \tau^t) \) and a thermoelastic coupling constant \( (\varepsilon) \). The
relaxation times and thermoelastic coupling constant are dimensionless and are constrained to ranges defined by \((\ell > t_0 > 1)\) and \((\epsilon > 0)\).

To obtain the closed form solution, Hetnarski and Ignaczak presented a one-dimensional Green’s function corresponding to an instantaneous plane source of heat in an infinite body of the Green-Lindsay type, presented a similar analysis for a semi-space, and provided the details of completing the inverse Laplace transforms for both the infinite space and semi-space.

3.2 - Formulation of the Problem

The problem first to be considered is a semi-infinite solid (semi-space \(x \geq 0\)) of the Green-Lindsay type initially at rest, subject to a thermal disturbance in the form of an instantaneous heat source on the plane \(x = x_0 > 0\). The bounding plane of this semi-infinite solid is stress free and maintained at a zero temperature. A further assumption is that the thermoelastic wave response due to the heat source vanishes at infinity. The stress and temperature responses generated by the planar heat source are described in terms of a scalar potential \(\Phi = \Phi(x,t)\), such that (taken from [8]):

\[
S(x,t;x_0) = \frac{\partial^2 \Phi}{\partial t^2} \tag{3.2.1}
\]

\[
\left(1 + t^0 \frac{\partial}{\partial t}\right)T = \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)\Phi \tag{3.2.2}
\]

\[
\Gamma \Phi = -\left(1 + t^0 \frac{\partial}{\partial t}\right)r \tag{3.2.3}
\]
where \( (S) \) is defined as the stress field, \( (T) \) is defined as the temperature field, \( (r) \) is defined as the heat source, and \( (\Gamma) \) is the central operator of the one-dimensional, homogeneous, isotropic, temperature-rate dependent thermoelasticity defined as:

\[
\Gamma = \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right) \left( \frac{\partial^2}{\partial x^2} - t_0 \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} \right) - \varepsilon \left( \frac{\partial^3}{\partial x^2 \partial t} \right) \left( 1 + t^0 \frac{\partial}{\partial t} \right) \tag{3.2.4}
\]

where \((t^0, t_0)\) are the two thermal relaxation times and \((\varepsilon)\) is the thermoelastic coupling constant such that:

\[
t^0 \geq t_0 > 1, \quad \varepsilon > 0 \tag{3.2.5}
\]

Note that all quantities in the formulations are of dimensionless form.

In considering homogeneous initial and boundary conditions for the solution of this problem, the following is stated.

Initial conditions:

\[
\frac{\partial^k \Phi(x,0)}{\partial t^k} = 0 \quad k = 0, 1, 2, 3 \tag{3.2.6}
\]

\[
T(x,0) = \frac{\partial T(x,0)}{\partial t} = 0 \quad x > 0 \tag{3.2.7}
\]

Boundary conditions:

\[
\Phi(0,t) = T(0,t) = 0 \quad t \geq 0 \tag{3.2.8}
\]

As stated, the initial conditions verify that the semi-space is initially thermomechanically at rest and the boundary conditions verify that the plane \((x = 0)\) is stress free and at zero temperature for all \(t \ (t \geq 0)\).

The set of functions defined by equations [(3.2.1), (3.2.2), (3.2.3)] generated by the scalar potential \((\Phi)\) are one-dimensional Green’s function or temperature-rate dependent thermoelasticity for the semi-infinite solid [7].
3.3 - Green's Function

To complete this problem formulation, a closed form Green's function must be developed. This is a rigorous analysis that is well documented in [7]. In very basic terms, the closed form Green's function is generated by the field equations and homogeneous initial conditions that have been presented above. A Laplace transform technique is used to solve the initial-boundary value problem. An important mathematical tool that is used to complete the solution of the central equations is the use of a decomposition theorem. Fundamentally, decomposition will reduce the resulting partial differential equations from fourth order to second order thereby simplifying the solution process. The decomposition theorem states (from [9]):

\( \text{Let } \Phi = \Phi(x,t) \text{ be a solution of the equation } \)

\[ \Gamma \Phi = 0 \quad (x \geq 0) \quad (3.3.1) \]

\( \text{satisfying the homogeneous initial conditions } \)

\[ \frac{\partial^k \Phi}{\partial t^k}(x,0) = 0 \quad k = 0, 1, 2, 3 \quad (3.3.2) \]

\( \text{Then the function } (\Phi) \text{ admits the representation: } \)

\[ \Phi = \Phi_1 + \Phi_2 \quad x \geq 0 \quad (3.3.3) \]

\( \text{where } \)

\[ \Phi_1 \left[ f_1(v_1,h_1,\lambda_1,\alpha,\beta,J_1(\beta t),t) \right] = 0 \quad x \geq 0 \quad (3.3.4) \]

\[ \Phi_2 \left[ f_2(v_2,h_2,\lambda_2,\alpha,\beta,J_1(\beta t),t) \right] = 0 \quad x \geq 0 \quad (3.3.5) \]
where \( J_x = J_1(x) \) is the Bessel function of order 1 and of the first kind.

In the process of decomposition, the following parameters are defined.

\[
\alpha = \left[ (1 + \varepsilon)(t_0 + \varepsilon t^0) - (1 - \varepsilon) \right] \Delta^{-1} \tag{3.3.6}
\]

\[
\beta = 2\sqrt{\varepsilon \left[ 1 + (1 + \varepsilon)(t^0 - t_0) \right]} \Delta^{-1} \tag{3.3.7}
\]

\[
\Delta = \left( 1 - t_0 + \varepsilon t^0 \right)^2 + 4\varepsilon^2 t_0 t^0 \tag{3.3.8}
\]

and,

\[
\lambda_i = \lambda v_i^2 \quad i = 1, 2 \tag{3.3.9}
\]

\[
\lambda = \frac{1}{4} \beta^2 \Delta^{1/2} \tag{3.3.10}
\]

\[
h_i = \frac{1}{2} k_i v_i^2 \quad i = 1, 2 \tag{3.3.11}
\]

\[
v_i = \sqrt{2} \left( 1 + \varepsilon + \varepsilon t^0 \pm \Delta^{1/2} \right)^{-1/2} \quad i = 1, 2 \tag{3.3.12}
\]

\[
k_i = \frac{1}{2} \left( 1 + \varepsilon \mp \alpha \Delta^{1/2} \right) \quad i = 1, 2 \tag{3.3.13}
\]

where in equation (3.3.12) the plus, (minus) and in equation (3.3.13) the minus, (plus) is taken for \( i = 1, (i = 2) \).

The resulting Green’s functions that may be used in the solution of thermoelastic responses in a semi-infinite solid are:

**Stress:**

\[ S_{N}^{(0)}(x,t;x_0) \]

\[ S_{C}^{(0)}(x,t;x_0) \]

**Temperature:**

\[ T_{N}^{(0)}(x,t;x_0) \]

\[ T_{C}^{(0)}(x,t;x_0) \]
given by equations (58), (59), (95), and (96) in [7]. Here, \((S_N, T_N)\) is the singular part of the Green's function and \((S_c, T_c)\) is the regular part of the Green's function. As part of these formulations, additional parameters are defined as:

\[
\begin{align*}
\bar{t}_i &= t - x_i \\
x_i &= \frac{x}{v_i} \\
\omega_i &= h_i^2 + x_i \\
\alpha_i &= \alpha + h_i
\end{align*}
\]  

(3.3.14) \hspace{1cm} (3.3.15) \hspace{1cm} (3.3.16) \hspace{1cm} (3.3.17)

In the following chapter, the general formulations that have been developed here will be used to formulate a specific problem that derives an exact solution for the stress-temperature response of a semi-space subjected to a short laser pulse at its boundary.
CHAPTER 4

Response to a Short Laser Pulse
The Closed Form Solution for a Semi-Space

4.1 - Introduction

The problem to be considered is defined in [8] by understanding that when laser radiation is incident on an absorbing material, laser induced thermomechanical waves will occur through the material that may have a number of effects on the physical-mechanical properties of the material. These may include melting and vaporization of the absorbing material and may also include electron emission and thermal radiation from the laser heated material. As further defined in [8], the problem here is to analyze laser induced thermomechanical waves in an absorbing thermoelastic semi-space of the Green-Lindsay type such that the material itself remains in the elastic zone (no plastic deformation) and neither melting nor vaporization occurs.

In this chapter, a one-dimensional solution to the field equations for the temperature-rate dependent thermoelasticity for a semi-space subjected to a laser induced heat source will be reviewed [8]. The analysis will use the Green’s functions that were developed in the previous chapter for a semi-space subjected to a planar heat source. The
closed form solutions will be revealed and a hypothesis based on a qualitative analysis of the exact solution will be discussed. These discussions will lead into the discussions in Chapters 5 and 6 in review of a numerical analysis based on the exact solution derived here.

4.2 - Formulation of the Problem

In the consideration of this problem, an appropriate starting place is in understanding the characteristics of the laser profile. In this case, the laser induced heat has the form of a product of an exponentially decreasing function of the semi-space depth and a skewed Gaussian temporal profile such that:

$$r(x,t) = Y(t)\exp(-ax) \quad x \geq 0, t \geq 0$$

(4.2.1)

where $Y = Y(t)$ is the skewed Gaussian temporal laser profile such that:

$$Y(t) = Y_0 t^n \exp(-bt^m) \quad t \geq 0$$

(4.2.2)

and for all time derivatives of $Y(t)$,

$$Y^{(k)}(0) = 0 \quad k = 0, 1, 2, \ldots, N$$

(4.2.3)

where $(a^1)$ is an absorption coefficient $(a > 0)$, $Y_0$ is a material constant, and $b, m, n$ are positive constants that define the profile of the laser.

In reviewing the definition of the problem, it is important to ensure that an appropriate approach to the problem is engaged. In this case the temperature-rate dependent theory of thermoelasticity is proposed and is justified by [8]:

a) The light energy is considered to be instantaneously converted into heat into the absorbing body. This type of heat supply may be described by a temperature-rate analysis.
b) The instantaneous thermoelastic response of a body to a short laser pulse is sensitive to a temperature-rate due to the energy distribution of the laser pulse as described by the skewed Gaussian profile.

The analysis as discussed in [8] places some restrictions on the problem that should be noted. These are imposed on the problem to make it analytically tractable. These are:

a) The thermoelastic wave is one-dimensional. This may be stated if the dimensions of the laser beam are large compared to the depth of penetration of heat and if the laser beam power density is constant over the spot of incidence.

b) The semi-space is to be considered a homogeneous and isotropic thermoelastic material of the Green-Lindsay type. This implies that the thermoelastic properties of the material under consideration are independent of temperature and are the same in every direction of the body.

c) The propagating thermoelastic wave corresponds to homogeneous initial conditions and homogeneous stress-temperature boundary conditions. This condition is compliant with the initial conditions and boundary conditions defined in Chapter 3 [(3.2.6), (3.2.7), (3.2.8)].

4.3 - Development of the Closed Form Solution

Using the one-dimensional Green’s function for a semi-space as derived in [7] and reviewed in Chapter 3 as part of this thesis, the fundamental stress-temperature response equations to the heat supply defined by equation (4.2.1) are:
\[ S(x,t) = \int_0^t \mathcal{Y}(\tau) \left[ \int_0^\infty \exp(-ax_0) S^*(x,t-\tau;x_0) \, dx_0 \right] \, d\tau \quad (4.3.1) \]
\[ T(x,t) = \int_0^t \mathcal{Y}(\tau) \left[ \int_0^\infty \exp(-ax_0) T^*(x,t-\tau,x_0) \, dx_0 \right] \, d\tau \quad (4.3.2) \]

where \((S^*, T^*)\) represents the stress-temperature response (Green’s function) of a semi-space \((x \geq 0)\) to an instantaneous plane source of heat at the boundary \((x = x_0 = 0)\).

\((S^*, T^*)\) are given by the relationship:
\[ S^*(x,t;x_0) = S_N(x,t;x_0) + S_C(x,t;x_0) \quad (4.3.3) \]
\[ T^*(x,t;x_0) = T_N(x,t;x_0) + T_C(x,t;x_0) \quad (4.3.4) \]

As stated above, \((S_N, T_N)\) is the singular part of the Green’s function and \((S_C, T_C)\) is the regular part of the Green’s function. Further, from the decomposition theorem, the final Green’s function relationships are:
\[ S_N(x,t;x_0) = S_N^{(1)}(x,t;x_0) + S_N^{(2)}(x,t;x_0) = \sum_{i=1}^{2} S_N^{(i)} \quad (4.3.5) \]
\[ T_N(x,t;x_0) = T_N^{(1)}(x,t;x_0) + T_N^{(2)}(x,t;x_0) = \sum_{i=1}^{2} T_N^{(i)} \quad (4.3.6) \]

and
\[ S_C(x,t;x_0) = S_C^{(1)}(x,t;x_0) + S_C^{(2)}(x,t;x_0) = \sum_{i=1}^{2} S_C^{(i)} \quad (4.3.7) \]
\[ T_C(x,t;x_0) = T_C^{(1)}(x,t;x_0) + T_C^{(2)}(x,t;x_0) = \sum_{i=1}^{2} T_C^{(i)} \quad (4.3.8) \]

The full solutions to the Green’s function are documented in [8] equations (11) and (13). The resulting equations, however, are complex and not easily manipulated in this form. In [8], an alternate approach is taken whereby the Laplace transform of equations (4.3.1) and (4.3.2) is taken, the integration with respect to \(x_0\) is completed, and then the inverse
Laplace transform is taken to express the formulations back into the time domain. This procedure is well documented in [8].

In the final form of the solution, the following substitutions are made to determine the resulting stress-temperature relationships.

\[ \sigma(x,t) = \Delta^{1/2} S(x,t) \]  
\[ \sigma_0(x,t) = \Delta^{1/2} S_0(x,t) \ast Y(t) \]  
\[ \sigma_c(x,t) = \Delta^{1/2} S_c(x,t) \ast Y(t) \]

and

\[ \theta(x,t) = \Delta^{1/2} T(x,t) \]  
\[ \theta_0(x,t) = \Delta^{1/2} T_0(x,t) \ast Y(t) \]  
\[ \theta_c(x,t) = \Delta^{1/2} T_c(x,t) \ast Y(t) \]

where \( \ast \) represents the convolution operator.

4.4 - The Solution

From the development of the solution presented above, as documented in [8], the stress-temperature pair \((\sigma, \theta)\) depends on nine independent variables, such that,

\[ \sigma = \sigma(x, t; f', t_0, \epsilon; m, n, b; a) \]  
\[ \theta = \theta(x, t; f', t_0, \epsilon; m, n, b; a) \]
The independent variables are defined as:

\( x \)  spatial variable

\( t \)  time variable

\( t^0 \)  thermoelastic relaxation time

\( t_0 \)  thermoelastic relaxation time

\( \varepsilon \)  thermoelastic coupling constant

\( m \)  laser shape parameter affecting positive skewness [6]

\( n \)  laser shape parameter affecting negative skewness [6]

\( b \)  laser shape parameter affecting pulse activation time [6]

\( a \)  inverse absorption coefficient for the material

Further, the nine independent variables are constrained to the following inequalities.

\[
\begin{align*}
\alpha & \geq 0, \quad \beta \geq 0 \quad (4.4.3) \\
n^0 & \geq t_0 > 1, \quad \varepsilon > 0 \quad (4.4.4) \\
m > 0, \quad n > 0, \quad b > 0 \quad (4.4.5) \\
a > 0 \quad (4.4.6)
\end{align*}
\]

4.4.1 - Stress-Temperature Formulations

In the qualitative analysis completed in [8], the resulting stress-temperature formulations are first defined by a singular and regular part of a Green's function developed for this problem. These are \((\alpha_0, \beta_0)\) and \((\alpha, \beta)\) respectively. Alternatively, the resulting stress-temperature formulations are each defined by a diffusive part \((\alpha_d, \beta_d)\) and a wave part \((\hat{\alpha}_w, \hat{\beta}_w)\) where the wave part of the response is discretized to a first and second wave component \((\alpha_w, \beta_w)\) and \((\alpha_c, \beta_c)\) respectively. Therefore, the stress-temperature relationships are given by:
where:

\[ o-ad + av = d + aw \]

\[ o-ed + dw = dd + dw + ec \]

\[ 1 = 1 \]

The components for the stress-temperature relationships appearing in equations (4.4.9) and (4.4.10) are given by:

**Diffusive part:**

\[ \phi(x,t) = \exp\{-a(x-t)\} \int_0^\infty \left( \frac{t-Y}{2} \right) \left( 0, x; a \right) \, dx \]

\[ \phi(x,t) = +\exp(-a(x-t))^\frac{t-Y}{2} \left( 0, T; a \right) - a^2 \int_0^T t^2 \left( 0, T; a \right) \, dM \]

**First wave part component:**

\[ \phi(x,t) = +t^2 \left( \frac{\partial^2}{\partial x^2} \right) \int_0^\infty \exp(-a(x-t))^\frac{t-Y}{2} \left( 0, T; a \right) \, dx \]

\[ \phi(x,t) = -t \left( \frac{\partial}{\partial x} \right) \int_0^\infty \exp(-a(x-t))^\frac{t-Y}{2} \left( 0, T; a \right) \, dx \]

**Second wave part component:**

\[ \phi(x,t) = -t^2 \left( \frac{\partial^2}{\partial x^2} \right) \int_0^\infty \exp(-a(x-t))^\frac{t-Y}{2} \left( 0, T; a \right) \, dx \]

\[ \phi(x,t) = \int_0^\infty \exp(-a(x-t))^\frac{t-Y}{2} \left( 0, T; a \right) \, dx \]
\[
\theta^{(0)}_e(x, t) = -\frac{1}{2} H(\zeta)
\times \left\{ \exp(-h_i x_i) \theta_0(0, \zeta_i) + \int_x \left[ N_i(x, t) \theta_0(0, t - \tau) \pm M_i(x, \tau) \tilde{\theta}_0(0, t - \tau) \right] d\tau \right\} \tag{4.4.16}
\]

where:

\[
\tilde{\sigma}_0(0, t) = \hat{\alpha} \sigma_0(0, t) + \hat{\beta} \tilde{\sigma}_0(0, t) - \hat{\gamma} \left[ \hat{\theta}_0(0, t) + \tilde{\theta}_0(0, t) \right] \tag{4.4.17}
\]

\[
\tilde{\theta}_0(0, t) = \hat{\alpha} \theta_0(0, t) + \hat{\beta} \tilde{\theta}_0(0, t) + \epsilon \hat{\gamma} \sigma_0(0, t) \tag{4.4.18}
\]

where:

\[
\sigma_0(0, t) = \sum_{i=1}^{2} \sigma^{(0)}_0(0, t); \quad \dot{\sigma}_0(0, t) = \sum_{i=1}^{2} \dot{\sigma}^{(0)}_0(0, t) \tag{4.4.19}
\]

\[
\theta_0(0, t) = \sum_{i=1}^{2} \theta^{(0)}_0(0, t); \quad \dot{\theta}_0(0, t) = \sum_{i=1}^{2} \dot{\theta}^{(0)}_0(0, t); \quad \ddot{\theta}_0(0, t) = \sum_{i=1}^{2} \ddot{\theta}^{(0)}_0(0, t) \tag{4.4.20}
\]

It is to be noted that these stress-temperature relationships at the semi-space boundary are a simplified formulation that is only valid for the range \((n \geq 3)\). Although, per the inequality constraints listed by the inequalities (4.4.5), \((n)\) may accept values in the range \((n > 0),\) for \((0 < n < 3)\), additional time derivatives of intermediate functions not considered in this thesis must be formulated and utilized in solution of these boundary stress-temperature relationships (see [8]).

The additional terms required for the boundary stress-temperature formulations are presented below.

Boundary stress:

\[
\sigma^{(0)}_0(0, t) = \pm \frac{1}{2} \left[ \hat{\lambda}(t - \tau) + t^0 \hat{\lambda}(t - \tau) \right] Q_0(0, \tau; -a) d\tau \tag{4.4.21}
\]

\[
\dot{\sigma}^{(0)}_0(0, t) = \pm \frac{1}{2} \left[ \hat{\lambda}(t - \tau) + t^0 \hat{\lambda}(t - \tau) \right] Q_0(0, \tau; -a) d\tau \tag{4.4.22}
\]
Boundary temperature:

\[ \theta_0^{(0)}(0,t) = \frac{1}{2} \int_0^t \left[ \int_0^\tau \dot{Y}(u) \, du \right] \left[ M_i(0,\tau) + a P_i(0,\tau; -a) \right] + \dot{Y}(\tau) Q_i(0,\tau; -a) \, d\tau \]

(4.4.23)

\[ \dot{\theta}_0^{(0)}(0,t) = \frac{1}{2} \int_0^t \left\{ Y(t-\tau) \left[ M_i(0,\tau) + a P_i(0,\tau; -a) \right] + \dot{Y}(t-\tau) Q_i(0,\tau; -a) \right\} \, d\tau \]

(4.4.24)

\[ \ddot{\theta}_0^{(0)}(0,t) = \frac{1}{2} \int_0^t \left\{ \dot{Y}(t-\tau) \left[ M_i(0,\tau) + a P_i(0,\tau; -a) \right] + \ddot{Y}(t-\tau) Q_i(0,\tau; -a) \right\} \, d\tau \]

(4.4.25)

and

\[ \hat{\alpha} = (1 + \epsilon) \Delta^{-1/2} \]

(4.4.26)

\[ \hat{\beta} = (t_0 + \epsilon t_0 - 1) \Delta^{-1/2} \]

(4.4.27)

\[ \hat{\gamma} = 2 \Delta^{-1/2} \]

(4.4.28)

Also appearing in these formulations are the first, second, and third time derivatives of the laser profile \( Y(t) \), given by equation (4.2.2) and repeated here as equation (4.4.29). They are:

\[ Y(t) = Y_0 t^n \exp(-bt^n) \]

(4.4.29)

\[ \dot{Y}(t) = Y_0 t^{n-1}(n - m bt^n) \exp(-bt^n) \]

(4.4.30)

\[ \ddot{Y}(t) = Y_0 t^{n-2} \left[ n(n-1) - (2n + m - 1) m bt^n + b^2 m^2 t^{2n} \right] \exp(-bt^n) \]

(4.4.31)

\[ \dddot{Y}(t) = Y_0 t^{n-3} \left\{ n(n-1)(n-2) - mb \left[ (m-1)(m-2) + 3n(n+m-2) \right] t^n + 3m^2 b^2 (n+m-1) t^{2n} - b^3 m^3 t^{3n} \right\} \exp(-bt^n) \]

(4.4.32)
### 4.4.2 Intermediate Function Formulations

The functions formulated in this section are the intermediate functions that appear in the stress-temperature relations presented in the previous section which are derived as a consequence of consolidating terms during the integration process while the expressions are in the Laplace domain. These are:

\[ P_i(x_i,t;\alpha) = v_i \exp(h_i t) \sum_{n=0}^{\infty} \frac{(\mp \lambda_i)^n}{n!} \int_{x_i}^{t} g_n^{(i)}(t-s) \hat{P}_n^{(i)}(x_i,s;\alpha) ds \]  

(4.4.33)

\[ Q_i(x_i,t;\alpha) = v_i^2 \exp(h_i t) \sum_{n=0}^{\infty} \frac{(\mp \lambda_i)^n}{n!} \int_{x_i}^{t} g_n^{(i)}(t-s) \hat{Q}_n^{(i)}(x_i,s;\alpha) ds \]  

(4.4.34)

\[ R_i(x_i,t;\alpha) = v_i^2 \exp(h_i t) \sum_{n=0}^{\infty} \frac{(\mp \lambda_i)^n}{n!} \int_{x_i}^{t} g_n^{(i)}(t-s) \hat{R}_n^{(i)}(x_i,s;\alpha) ds \]  

(4.4.35)

where:

\[ \hat{P}_n^{(i)}(x_i,s;\alpha) = A_n^{(i)}(x_i,s) + a v_i \int_{x_i}^{s} \exp[a v_i(u-x_i)] A_n^{(i)}(u,s) du \]  

(4.4.36)

\[ \hat{Q}_n^{(i)}(x_i,s;\alpha) = \int_{x_i}^{s} \exp[a v_i(u-x_i)] A_n^{(i)}(u,s) du \]  

(4.4.37)

\[ \hat{R}_n^{(i)}(x_i,s;\alpha) = \int_{x_i}^{s} \cosh[a v_i(u-x_i)] A_n^{(i)}(u,s) du \]  

(4.4.38)

In addition, the following first derivatives with respect to time are necessary. They are:

\[ \dot{Q}_i(x_i,t;\alpha) = -h_i \dot{Q}_i(x_i,t;\alpha) + v_i^2 \exp(-h_i t) \]  

(4.4.39)

\[ \dot{R}_i(x_i,t;\alpha) = -h_i \dot{R}_i(x_i,t;\alpha) + v_i^2 \exp(-h_i t) \]  

(4.4.40)
The functions $M^x(t)$ and $N^j(t)$ are the power series of Neumann's type associated with the wave-like operator $L_t$ occurring in the decomposition theorem of the Green-Lindsay theory [7] (Theorem 5.2 in [9], p. 314). These functions are defined as:

$$M^j(t) = \exp(-\eta t)$$

$$N^j(t) = \exp(-\eta t) U(x, t) + \int_0^x \exp(-\eta s) U(x, s) ds$$

(4.4.41)

(4.4.42)

where:

$$A^j = \cos(\omega \sigma L)$$.  

$$n = -1, 0, 1, 2, 3, ...$$

(4.4.43)

In the above formulations, $J_n(t)$ is the Bessel function of the first kind, order $\alpha$, and $I_n(t)$ is the Modified Bessel function of the first kind, order $\alpha$.

4.5 - Hypothesis

In the qualitative analysis from [8], the expected shape of the resulting stress-temperature functions are reviewed. By inspecting the diffusive part of the response
\((\sigma_d, \theta_d)\), it is expected that since \(Y(t)\) and \(R(o, t; 0)\) and their first derivatives are smooth for \((x \geq 0), (t \geq 0)\), \((\sigma_d, \theta_d)\) will also be smooth through the same range of variables.

Likewise, because \(Y(t), M_i(x, r, t), P_i(x, r, t; a), \) and \(Q_i(x, r, t, a)\) are smooth over the region \((x \geq 0), (t \geq 0)\), \((\sigma_w, \theta_w)\) will be smooth in that region as well. For the final stress-temperature pair, \((\sigma_c, \theta_c)\), since \(\sigma_0(0, 0) = \dot{\sigma}_0(0, 0) = \theta_0(0, 0) = \dot{\theta}_0(0, 0) = 0\), and the functions \(N_i(x, r, t)\) and \(M_i(x, r, t)\) exhibit smooth behavior through the region of interest, this stress-temperature pair will also be smooth through the region \((x \geq 0), (t \geq 0)\).

Therefore, as concluded in [8], “for a fixed cross section of the semi-space \((x = \hat{x} > 0)\), the pair \([\sigma(\hat{x}, t), \theta(\hat{x}, t)]\) represents a pair of smooth functions for \((t \geq 0)\).” Further, it is expected that:

\[
\sigma(x, t) \to 0, \quad \theta(x, t) \to 0 \quad \text{as} \quad t \to +\infty \tag{4.5.1}
\]

The next two chapters, Chapter 5 and Chapter 6, will concentrate on the numerical analysis of the solutions presented in this chapter. Chapter 5 will discuss the program methodology that was used, will develop the algorithms and review the results for the intermediate functions, and will conclude with the development of the algorithms and review the results for the three components of the total stress-temperature response to the model. Chapter 6 will be devoted to a discussion of the results from the numerical analysis for the total stress-temperature response.
CHAPTER 5

Numerical Analysis

The Stress-Temperature Components

5.1 - Program Methodology

The program for this numerical analysis was written using Mathematica, Version 3, Student Version, by Wolfram Research. The program was executed on a Power Macintosh, 8500/150. Mathematica is considered a high level interpreted language as opposed to C or FORTRAN which are considered low level compiled languages. Mathematica is well suited for this application due to its fully integrated environment for technical computing using a symbolic computer language with a wide offering of core computational capabilities. Some of these capabilities include symbolic as well as numeric computation. Since the major goal in this program was to study the numerical solution, the advantages that Mathematica offered in the area of numerical computation were impressive. These include adaptive precision control to guarantee precision on results, high performance compilation of functional numerical operations, optimized algorithms for one and higher dimensional interpolation, and high dimensional numerical integration.
As the development of the program progressed, it became obvious that with the multiple levels of recursion necessary in the intermediate functions to satisfy the computations, particularly in computing the stress-temperature components, an approach to reduce recursion would be needed to optimize the computer run time. The adopted approach was to define interpolated functions for some of the more complex intermediate functions in the stress-temperature formulations. To do this, *Mathematica* uses a built in expression object, *InterpolatingFunction*, constructed by an *Interpolation* object. The interpolated functions will approximate a given function based on a set of discrete points that are calculated exactly from the function’s formulation. *Mathematica* will fit polynomial curves, of order three, between the discrete points calculated. In creating the interpolated function, *Mathematica* assumes the function to vary smoothly between the calculated points. For the program used in this thesis, those functions that exhibited inherent jumps or other discontinuities in their formulation were interpolated with a greater number of points. For smooth varying functions, points were selected in 1.0 increments of “time” while for functions with jumps, points were selected in 0.1 increments of “time.” Tests were conducted to compare the actual function values to the interpolated function values with errors well less than 1%.

The numerical integration that *Mathematica* performs is completed not by looking at the functional form of the integrand but instead by simply finding a sequence of numerical values of the integrand at particular points then tries to deduce from those points a good approximation to the integral. In order to obtain a definite result, then, *Mathematica* must make certain assumptions about the smoothness and other properties of the integrand. Care was taken to try and ensure that integrands varied smoothly over the integration ranges of interest to avoid precision errors in the results.
The remainder of this chapter will systematically follow the process that was used in the development of the Mathematica program for this analysis. The program itself is presented in Appendix A.

5.2 - Parameter Values

The numerical values assigned to the parameters used in this analysis are presented in this section. All parameters are used in a dimensionless sense although each may have a dimensional interpretation (i.e. time, distance, etc.). For example, the key material parameters ($f$, $t_0$, $\epsilon$) have been estimated under certain constraints. For the thermoelastic coupling constant, ($\epsilon$), values have been estimated for a classical thermoelastic solid (i.e. giving no consideration to the thermal relaxation time). These are (from [3] and [9])

\[
\epsilon = 3.56 \times 10^{-6} \quad \text{for aluminum}
\]

\[
\epsilon = 2.97 \times 10^{-4} \quad \text{for steel}
\]

\[
\epsilon = 1.68 \times 10^{-2} \quad \text{for copper}
\]

\[
\epsilon = 7.33 \times 10^{-2} \quad \text{for lead}
\]

The desire for this study is to set parameter values similar to those used in previous studies to adopt consistency for comparing results. Therefore, the values that will approximate an aluminum alloy for this model material will be used.

Similarly, thermal relaxation time values have been estimated, in true dimensional units, for rigid heat conductors for the theory proposed by Lord and Shulman which considers a single relaxation time. These are (from [3] and [9]):

\[
39
\]
\[ \tilde{t}_0 = 8.0 \times 10^{-12} \text{ (s)} \quad \text{for aluminum alloys at 25°C} \]
\[ \tilde{t}_0 = 1.6 \times 10^{-12} \text{ (s)} \quad \text{for carbon alloys at 25°C} \]
\[ \tilde{t}_0 = 1.5 \times 10^{-12} \text{ (s)} \quad \text{for uranium silicate at 25°C} \]
\[ \tilde{t}_0 = 2.0 \times 10^{9} \text{ (s)} \quad \text{for liquid helium at 0.25 K} \]

Thermal relaxation times have not been estimated for the Green-Lindsay type material with two relaxation times, therefore, the values presented above will be used for approximation. In order to determine the non-dimensional values that relate to the relaxation times given above, equations [(5.5.25) and (5.5.25)] from [9] are used. For aluminum alloy, the thermal relaxation time is computed as:

\[ t_0 = \frac{\tilde{t}_0}{\tilde{t}_0} = \frac{8.00 \times 10^{-12}}{2.15 \times 10^{-12}} = 3.72 \quad (5.2.1) \]

This analysis will consider the stress-temperature response at a given "depth" into the material model. The depth is indicated by the spatial variable (x). This problem model is being studied as one-dimensional.
The parameter values are shown in Table (5.2.1) and listed by their analytical names and program names.

**Table 5.2. 1 - Parameter Values**

<table>
<thead>
<tr>
<th>Analytical Name</th>
<th>Program Name</th>
<th>Parameter Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>x1</td>
<td>10.0</td>
</tr>
<tr>
<td>$f^0$</td>
<td>to</td>
<td>4.0</td>
</tr>
<tr>
<td>$t_0$</td>
<td>tso</td>
<td>4.0</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>eps</td>
<td>0.05</td>
</tr>
<tr>
<td>$Y_0$</td>
<td>y0</td>
<td>1.0</td>
</tr>
<tr>
<td>$m$</td>
<td>m1</td>
<td>2.0</td>
</tr>
<tr>
<td>$n$</td>
<td>n1</td>
<td>3.0</td>
</tr>
<tr>
<td>$b$</td>
<td>b1</td>
<td>1000.0</td>
</tr>
<tr>
<td>$a$</td>
<td>a1</td>
<td>0.1</td>
</tr>
</tbody>
</table>
5.3 - Laser Profile

The laser profile will be the first element of this analysis to be reviewed. The profile and its first, second, and third time derivatives were previously listed as equations (4.4.29), (4.4.30), (4.4.31), and (4.4.32). The laser profile is shown graphically in Figure (5.3.1) using the parameter values listed in Table (5.2.1).

\[ Y(t) \]

---

**Figure 5.3.1 - Laser Profile**

For these parameters, the pulse activation time is:

\[ t_f = 0.1 \]  \hspace{1cm} (5.3.1)

From [8], this laser profile is valid for the stress-temperature response formulations previously described if \( \xi \) is sufficiently small such that:

\[ \xi = t_f' \exp(-bt_f^a) \ll 1 \]  \hspace{1cm} (5.3.2)

For the parameter values to be used for this problem, \( \xi = 4.54 \times 10^{-8} \). Therefore, this laser profile can be deemed appropriate for this analysis.
The laser pulse rise time \((t_r)\) is calculated by setting the first derivative of the profile function to zero and solving for \((t)\). Therefore:

\[
t_r = \left( \frac{n}{mb} \right)^{1/m} \quad (5.3.3)
\]

and for this analysis:

\[
t_r = 0.0387 \quad (5.3.4)
\]

5.4 - Evaluation of Primary Intermediate Functions

This section will develop the evaluations for the primary intermediate functions as part of the direct formulations of the stress-temperature pairs \((\sigma_d, \theta_d)\), \((\sigma_w, \theta_w)\), and \((\sigma_c, \theta_c)\). The other intermediate functions that are integral to the formulations for these functions are presented as part of the Mathematica program in Appendix A.

The evaluations of the intermediate functions requires an analysis of the summation part of the functions in order to determine the finite value of \((n)\) that the summation will evaluate to. Theoretically, the summations are stated to progress from \((n = 0 \to \infty)\). For the program, a finite value of \((n)\) is determined for which summations beyond that value may be ignored. As part of a separate numerical analysis, it was found that the summation series’ converge very quickly and that, universally, values of \((n \geq 3)\) yielded accuracy to six significant digits. This accuracy is well within the accuracy magnitude necessary for this analysis. Therefore, all series summations are evaluated through the range \((n = 0 \to 3)\).
5.4.1 - Intermediate Functions for $(\sigma_d, \theta_d)$

In addition to the laser profile (presented above) and its first derivative, the intermediate function of interest for the stress-temperature pair, $(\sigma_d, \theta_d)$ is $R_i(0,t;a)$ and its first derivative with respect to time. The plots for $R_i(0,t;a)$ are presented in Figure (5.4.1) for $(i = 1)$ and Figure (5.4.2) for $(i = 2)$.

As was predicted in the hypothesis in Section 4.5, the functions $R_i(0,t;a)$ for $(i = 1)$ and $(i = 2)$ are smooth through the range of interest (as is the laser profile) and as such, it is expected that the stress-temperature pair $(\sigma_d, \theta_d)$ will also be smooth through this range of values. The first time derivatives of $R_i(0,t;a)$ and $Y(t)$ are shown in the Mathematica program in Appendix A.
Figure 5.4. 1 - $R_i(0, t; a); i = 1$

Figure 5.4. 2 - $R_i(0, t; a); i = 2$
5.4.2 - Intermediate Functions for \((\sigma_w, \theta_w)\)

For the first wave components of the stress-temperature pair, \((\sigma_w, \theta_w)\), the functions (in addition to the laser profile) of interest are \(M_i(x_i, t)\), \(P_i(x_i, t; a)\), and \(Q_i(x_i, t; a)\). The plots of these functions for the range of interest are plotted in Figures (5.4.3) and (5.4.4) for \(M_i(x_i, t)\), \((i = 1, 2)\), Figures (5.4.5) and (5.4.6) for \(P_i(x_i, t; a)\), \((i = 1, 2)\), and Figures (5.4.7) and (5.4.8) for \(Q_i(x_i, t; a)\), \((i = 1, 2)\). As noted in the earlier hypothesis, these three functions exhibit smooth characteristics through the range of interest and therefore it is expected that the stress-temperature pair will also exhibit smooth characteristics. The only exception to this is at the wave fronts defined by the imposition of the Heaviside function in the stress-temperature formulations. These wave fronts will be evident at \((t = x_i)\), where for the parameters defined for this study:

\[
x_1 = 20.643 \quad (5.4.1)
\]
\[
x_2 = 9.689 \quad (5.4.2)
\]

These wave fronts become evident in the plots for \(P_i(x_i, t; a)\) and \(Q_i(x_i, t; a)\). Note that in Figures (5.4.5) and (5.4.7) the plots are zero for all \(t\), \((0 \leq t \leq 20.643)\) and in Figures (5.4.6) and (5.4.8) the plots are zero for all \(t\), \((0 \leq t \leq 9.689)\). These wave fronts are established in these formulations from the inclusion of the function \(A^n(x, t)\) which imposes a constraint on two of its terms to prohibit imaginary terms. It is this constraint that forms the wave fronts in this function [see equation (4.4.43)].
Figure 5.4. 3 - $M_i(x_t, t); i = 1$

Figure 5.4. 4 - $M_i(x_t, t); i = 2$
Figure 5.4. 5 - $P_i(x, t; \alpha); i = 1$

Figure 5.4. 6 - $P_i(x, t; \alpha); i = 2$
Figure 5.4.7 - $Q_i(x_i, t; a); i = 1$

Figure 5.4.8 - $Q_i(x_i, t; a); i = 2$
5.4.3 - Intermediate Functions for \((\sigma_c, \theta_c)\)

For the stress-temperature pair, \((\sigma_c, \theta_c)\), the dominating functions determining their shape are \(N_i(x_i,t)\) and \(M_i(x_i,t)\) for \((i = 1, 2)\). The plots for \(N_i(x_i,t)\) through the range of interest are shown in Figures (5.4.9) and (5.4.10). The plots for \(M_i(x_i,t)\) were shown previously in Figures (5.4.3) and (5.4.4). As discussed in Section 5.4.2, the function \(N_i(x_i,t)\) shows the clear wave fronts due to the term \(A_n^{(i)}(x_i, t)\) in its formulation. Through the range, however, the plots of the intermediate functions are smooth and it is therefore expected that the resulting stress-temperature relationships, \((\sigma_c, \theta_c)\) will too be smooth.
Figure 5.4. 9 - $N_t(x_t, t); i = 1$

Figure 5.4. 10 - $N_t(x_t, t); i = 2$
5.5 - Evaluation of Stress-Temperature Pairs

In this section, the evaluations of the three stress-temperature pairs (diffusive component, first wave component, and second wave component) will be discussed. In Chapter 6, the wave components for the three stress-temperature components will be summed to reveal the final stress-temperature response, at a fixed depth in the semi-space, to a short laser pulse.

5.5.1 - Diffusive Stress-Temperature Component; $\sigma_d(x,t); \theta_d(x,t)$

The diffusive components of the stress-temperature response is a result of instantaneous conversion of the light energy from the laser into heat in the whole semi-space. If tested at a specific time, it is expected that the diffusive response would reveal an exponential decay along the semi-space depth that is similar to the dissipation of the laser induced heat [8]. Clearly, as shown in Figures (5.5.1), (5.5.2), and (5.5.3) for the stress, and Figures (5.5.4), (5.5.5), and(5.5.6) for the temperature, this diffusive response does not exhibit any wave characteristics and begins to appear immediately after the laser heat is induced on the semi-space surface.
Figure 5.5. 1 - Diffusive Stress $\sigma_d^0(x,t); i = 1$

Figure 5.5. 2 - Diffusive Stress $\sigma_d^0(x,t); i = 2$
Figure 5.5. 3 - Total Diffusive Stress $\sigma_d(x,t)$
\[ \theta_d^{(i)}(x, t) = \begin{cases} a(x, t) & i = 1 \\ b(x, t) & i = 2 \end{cases} \]

**Figure 5.5. 4** - Diffusive Temperature \( \theta_d^{(i)}(x, t); i = 1 \)

**Figure 5.5. 5** - Diffusive Temperature \( \theta_d^{(i)}(x, t); i = 2 \)
Figure 5.5. 6 - Total Diffusive Temperature $\theta_d(x,t)$
5.5.2 - Wave Stress-Temperature Components;

\[ \sigma_w(x,t); \theta_w(x,t) \text{ and } \sigma_e(x,t); \theta_e(x,t) \]

The wave components of the stress-temperature response represent a correction to the diffusive part, formulated such that the boundary conditions of the model are satisfied. The wave part of the response maintains the characterization of the behavior of wave-like equations with convolution associated with this model [8].

In Figures (5.5.7), (5.5.8), and (5.5.9) for the first wave stress component, Figures (5.5.13), (5.5.14), and (5.5.15) for the second wave stress component, Figures (5.5.10), (5.5.11), and (5.5.12) for the first wave temperature component, and Figures (5.5.16), (5.5.17), and (5.5.18) for the second wave temperature component, the wave fronts become more obvious. These are an effect of the Heaviside function that is imposed on both the stress and temperature formulations. As discussed previously, the wave fronts are distinguished at \( t_1 = x_1 = 20.643 \) and at \( t_2 = x_2 = 9.689 \).
Figure 5.5. 7 - First Wave Stress \( \sigma_{w}^{(0)}(x,t); i = 1 \)

Figure 5.5. 8 - First Wave Stress \( \sigma_{w}^{(0)}(x,t); i = 2 \)
Figure 5.5.9 - Total First Wave Component of Stress $\sigma_w(x,t)$
Figure 5.5. 10 - First Wave Temperature $\theta_w^0(x,t); i = 1$

Figure 5.5. 11 - First Wave Temperature $\theta_w^0(x,t); i = 2$
Figure 5.5. 12 - Total First Wave Component of Temperature $\theta_w(x,t)$
Figure 5.5. 13 - Second Wave Stress $\sigma_c^0(x,t); i = 1$

Figure 5.5. 14 - Second Wave Stress $\sigma_c^0(x,t); i = 2$
Figure 5.5. 15 - Total Second Wave Component of Stress $\sigma_c(x,t)$
Figure 5.5. 16 - Second Wave Temperature \( \theta_c^{(i)}(x,t); i = 1 \)

Figure 5.5. 17 - Second Wave Temperature \( \theta_c^{(i)}(x,t); i = 2 \)
Figure 5.5. 18 - Total Second Wave Component of Temperature $\theta_c(x,t)$
CHAPTER 6

Discussion of Results

The Total Stress-Temperature Response

6.1 - Introduction

As discussed in Chapter 4, the total stress-temperature thermoelastic response of a semi-space to a short laser pulse is defined by a diffusive part and a wave part and depends on nine variables; two independent variables, \((x,t)\), three dimensionless constitutive variables for the material, \((\dot{\theta}, t_0, \varepsilon)\), and four laser parameters, \((m, n, b, a)\). Further, it is expected that the total thermoelastic response at a fixed cross section of the semi-space depth is represented by a pair of smooth transietal functions of time [8].

6.2 - The Total Stress Response

The total stress response to the laser pulse in the fixed semi-space depth is determined by summing the diffusive part and the two wave component parts of the stress responses examined in Chapter 5. The total response components are graphically presented
in Figures (6.2.1) and (6.2.2) for \(i = 1\) and \(i = 2\) respectively and then in Figure (6.2.3) for the total response.

From Figure (6.2.1), \((i = 1)\), the plot shows from \(0 \leq t \leq 20.643\), the diffusive part of the response. A positive jump, due to the imposition of the Heaviside function in the formulation for \((\sigma_w)\) and \((\sigma_t)\) at \((t = 20.643)\) is evident representing the slower of the two thermal waves. As expected, the stress response also shows evidence of approaching zero as \((t \to \infty)\).

From Figure (6.2.2), \((i = 2)\), the plot shows from \(0 \leq t \leq 9.689\) the diffusive part of the response and as seen in the previous discussion with \((i = 1)\), a positive jump, also due to the imposition of the Heaviside function, this time occurring at \((t = 9.689)\) representing the faster of the two thermal waves. As \((t \to \infty)\), the plot for this component of the stress approaches zero as part of a smooth function.

In comparing the jump values at the two wave fronts for the stress response, the value of the faster wave \((i = 2)\) is approximately thirty times the value of the slower wave at \((i = 1)\). The total stress response for this model is graphically represented in Figure (6.2.3). This shows the summation of the total stress components for \((i = 1)\) and \((i = 2)\).
Figure 6.2. 1 - Total Stress Component \((i = 1)\)

Figure 6.2. 2 - Total Stress Component \((i = 2)\)
Figure 6.2. 3 - Total Stress Response $\sigma(x,t)$
6.3 - The Total Temperature Response

The total temperature response to the laser pulse in the fixed semi-space depth is determined by summing the diffusive part and the two wave component parts of the temperature responses examined in Chapter 5. The total response components are graphically presented in Figures (6.3.1) and (6.3.2) for \( i = 1 \) and \( i = 2 \) respectively and then in Figure (6.3.3) for the total response.

As was revealed and discussed in the previous section for the stress response, in Figure (6.3.1), \( (i = 1) \), the plot shows from \( 0 \leq t \leq 20.643 \), the diffusive part of the temperature response. In this case, the evaluation shows the diffusive response to be very minimal compared to temperature response for this component at \( t \geq 20.643 \). Like the equivalent stress response, this plot shows this temperature component to be a smooth transiental function with respect to time, and shows a jump at the wave front, \( t = 20.643 \) due to the imposition of the Heaviside function in the formulation of \( (\theta_\text{w}) \) and \( (\theta_\text{c}) \) thus representing the slower of the two thermal waves. However unlike the equivalent stress response, this temperature component does not approach zero as \( t \to \infty \). This result does not match what was expected nor does it appear logical. This, of course, raises concern that an anomaly exists either in the formulation of the temperature response equations, the program script for these formulations, or in the methodology used by the program language to evaluate the formulations in the way of imprecision or inaccuracy.

From Figure (6.3.2), \( (i = 2) \), the plot shows from \( 0 \leq t \leq 9.689 \) the diffusive part of the response and displays a positive jump at the wave front, \( t = 9.689 \) representing the faster of the two thermal waves. As was the case for the total temperature response component \( (i = 1) \), the plot of this component \( (i = 2) \) also appears to display an anomaly in that the response does not approach zero as \( t \to \infty \). The same possible sources that were
discussed in the previous case and precipitating this anomaly are valid for this case as well. Further study of this phenomenon is required in order to complete a full coverage of the model presented in this thesis.

The total thermoelastic temperature response shown as the summation of the two total temperature response components, \((i = 1)\) and \((i = 2)\) is graphically presented in Figure (6.3.3). As expected by the anomaly present, this total temperature response does not approach zero as \((t \to \infty)\).
Figure 6.3. 1 - Total Temperature Component ($i = 1$)

Figure 6.3. 2 - Total Temperature Component ($i = 2$)
Figure 6.3. 3 - Total Temperature Response $\theta(x,t)$
In this thesis, a study has been presented that reviews several aspects of the problem of examining the thermoelastic response of a semi-space to a short laser pulse. This study has included an overview of the historical fundamentals studied by many researches since the late 1800’s including the contemporary studies that have worked to generalize the classical theories to remove the physical paradox of infinite travel speed of thermal responses suggested by the classical theories. In this overall review, the flow of theories from the foundations of Fourier’s law and the theory of elasticity were described thus leading to the classical theory of thermoelasticity and finally to the generalized theories of thermoelasticity. Two generalizations to the classical theory were discussed including the Lord-Shulman theory starting with a modification to Fourier’s law and the introduction of a thermal relaxation time. The second generalization was that proposed by Green-Lindsay which made no modification to Fourier’s law, developed temperature-rate dependent constitutive equations, and introduced two thermal relaxation times and a thermoelastic coupling coefficient. Thermal relaxation time was defined as the time lag necessary for an element in a material to attain steady state heat conduction subsequent to
the application of a sudden temperature gradient on the element. Relaxation time has been
termed as a second sound in that it resembles the characteristics evident in first sound
(phonon) phenomenon. In so doing, for both theories, hyperbolic heat conduction
equations were instituted to replace the parabolic equations inherent in the classical theory
of thermoelasticity. This was needed in order to develop response formulations that did not
predict infinite speeds of thermal responses in a medium. Thus, response formulations
were established in a characteristic wave equation form.

From the concept of thermoelasticity with two relaxation times, the general solution
was presented and included, in fundamental terms, how the use of Green’s function leads
to the closed form solution. From this development, formulations for some of the key
parameters were revealed including damping, attenuation, and wave velocity.

The development of the closed form solution for the response to a short laser pulse
was discussed. The problem was formulated as an initial-boundary value problem where a
short laser pulse is introduced at a constrained stress free boundary of a semi-space \((x \geq 0)\)
described as an isotropic, homogeneous, thermoelastic body of the Green-Lindsay type.

A single formulation for the laser profile, representing a skewed Gaussian temporal
profile, was established. In addition, the specific constraints on the problem were listed
noting that the light energy from the laser is converted instantaneously to heat absorbed by
the body, that the thermoelastic response is sensitive to a temperature-rate due to the energy
distribution of the laser pulse, and that the thermoelastic wave is one-dimensional.

The solution to the problem revealed that the stress-temperature pair \((\sigma, \theta)\) depends
on two independent variables \((x, t)\), three constitutive material variables \((\ell, t_0, \varepsilon)\) and four
laser parameters \((m, n, b, a)\) and that the stress-temperature pair is a sum of three stress-
temperature components; a diffusive component and two wave components. It was
hypothesized that the stress-temperature pair will represent a pair of smooth transienal
functions with respect to time \((t \geq 0)\) at a fixed cross section of the semi-space. Further, it was hypothesized that the stress-temperature pair would approach zero as \((t \to \infty)\).

A numerical analysis was completed using the formulations established from the closed form solution to the problem. This analysis clearly showed the existence of two wave fronts in the stress-temperature response that traveled with (dimensionless) speeds \((v_1 = 0.484)\) and \((v_2 = 1.032)\) that arrive at the cross section of interest at \((t = x_1 = 20.643)\) and at \((t = x_2 = 9.689)\). The numerical analysis revealed two obvious positive jumps in the total stress response as hypothesized and degraded to zero as \((t \to \infty)\). The total temperature response also revealed two obvious jumps at the expected time values, however did not degrade to zero as \((t \to \infty)\). This is not a logical result which prompted an attempt to reveal the source of the anomaly. Clearly, the anomaly rests in the formulations themselves, the programming script, or in the methodologies used in the programming applications (e.g. numerical integration) thus inadvertently adding imprecision or inaccuracy into the result. Further study of this anomaly is necessary to ensure complete closure to this study and the numerical analysis.

As a final conclusion, it should be noted that the theories and formulations studied in the context of this thesis and the prior research work that this thesis is based on form an excellent foundation for practical applications. As an example, these applications may include extremely low temperature physics or even in manufacturing interests related to material processing. However, very few material processing applications utilize a single pulse laser model as discussed in this work. In most applications, such as laser machining, or laser welding for instance, a constant train of pulses are concentrated on a material surface thus creating localized heating and thermal responses that are much more severe and produce material phase change as well [6]. On going studies in these areas are currently in process to understand the thermoelastic response under these conditions.
References


Bibliography


Appendix A

*Mathematica* Program
Analysis of the Thermoelastic Response of a Semi-Space to a Short Laser Pulse

Mathematica Program for the Numerical Analysis

PART A - Intermediate Function Evaluations

Define parameters and initialize values:

- \( x_1 \) - space variable (x)
- \( t \) - time variable (t)
- \( t_0 \) - material parameter (t0)
- \( \varepsilon \) - material parameter (e)
- \( y_0 \) - material parameter (Yo)
- \( m_l \) - laser parameter (m)
- \( n_1 \) - laser parameter (n)
- \( b_1 \) - laser parameter (b)
- \( a_1 \) - laser parameter (a)

\[ W := \text{Off} \]

Printed by Mathematica for Students 80
In[2]:= to = 4.0;
tso = 4.0;
eps = .05;
y0 = 1.;
ml = 2.0;
nl = 3.0;
b1 = 1000.0;
al = .1;
x1 = 10.0;

Create the necessary call files

NOTE: To create the call files, this section must be manually run after the rest of the program is initialized.

- \( M_i(x_i,t) \); \( i=1,2 \) (\( mi[1,x[1]], mi[2,x[2]] \))

\[
\text{Do[}
\quad \text{If}[i == 1, \text{Interpolation[Table[\{t, mi[i, x[i], t]\}, \{t, 0, 100\}]]} >>
\text{"mi[1,x[1]]"}, \text{Interpolation[}
\quad \text{Table[\{t, mi[i, x[i], t]\}, \{t, 0, 100\}]]} >> \text{"mi[2,x[2]]"}],
\quad \{i, 1, 2\}]
\]

- \( M_i(0,t) \); \( i=1,2 \) (\( mi[1,0], mi[2,0] \))

\[
\text{Do[If}[i == 1, \text{Interpolation[Table[\{t, mi[i, 0, t]\}, \{t, r1, 100\}]]} >>
\text{"mi[1,0]"}, \text{Interpolation[}
\quad \text{Table[\{t, mi[i, 0, t]\}, \{t, r1, 100\}]]} >> \text{"mi[2,0]"}],
\quad \{i, 1, 2\}]
\]

- \( N_i(x_i,t) \); \( i=1,2 \) (\( ni[1,x[1]], ni[2,x[2]] \))

\[
\text{Do[If}[i == 1, 
\quad \text{Interpolation[Table[\{t, ni[i, x[i], t]\}, \{t, 0, 100, .1\}]]} >>
\text{"ni[1,x[1]]"}, \text{Interpolation[}
\quad \text{Table[\{t, ni[i, x[i], t]\}, \{t, 0, 100, .1\}]]} >> \text{"ni[2,x[2]]"}],
\quad \{i, 1, 2\}]
\]
\[ P_j(x_j, t, a_1); \ i=1,2 \ (p_i[1,x[1],a_1], \ p_i[2,x[2],a_1]) \]

\[
\text{Do}\left[\left\{\begin{array}{ll}
\text{If} & [i == 1, \\
\text{Interpolation} & [\text{Table}\left[[t, \ p_i[i, x[i], t, a_1]], \{t, 0, 100}\right]] \gg \\
& "p_i[1,x[1],a_1]", \\
\text{Interpolation} & [\text{Table}\left[[t, \ p_i[i, x[i], t, a_1]], \{t, 0, 100}\right]] \gg \\
& "p_i[2,x[2],a_1]", \\
i, \{1, 2\}\right]\right\}
\right]
\]

\[ P_j(0,t,a_1); \ i=1,2 \ (p_i[1,0,a_1], \ p_i[2,0,a_1]) \]

\[
\text{Do}\left[\left\{\begin{array}{ll}
\text{If} & [i == 1, \\
\text{Interpolation} & [\text{Table}\left[[t, \ p_i[i, 0, t, a_1]], \{t, 0, 100}\right]] \gg \\
& "p_i[1,0,a_1]", \text{Interpolation}[
\ \
\text{Table}\left[[t, \ p_i[i, 0, t, a_1]], \{t, 0, 100}\right]] \gg "p_i[2,0,a_1]"], \\
i, \{1, 2\}\right]\right\}
\right]
\]

\[ P_j(0,t,-a_1); \ i=1,2 \ (p_i[1,0,-a_1], \ p_i[2,0,-a_1]) \]

\[
\text{Do}\left[\left\{\begin{array}{ll}
\text{If} & [i == 1, \\
\text{Interpolation} & [\text{Table}\left[[t, \ p_i[i, 0, t, -a_1]], \{t, r1, 100}\right]] \gg \\
& "p_i[1,0,-a_1]", \text{Interpolation}[
\ \
\text{Table}\left[[t, \ p_i[i, 0, t, -a_1]], \{t, r1, 100}\right]] \gg "p_i[2,0,-a_1]"], \\
i, \{1, 2\}\right]\right\}
\right]
\]

\[ Q_j(x_j,t,a_1); \ i=1,2 \ (q_i[1,x[1],a_1], \ q_i[2,x[2],a_1]) \]

\[
\text{Do}\left[\left\{\begin{array}{ll}
\text{If} & [i == 1, \\
\text{Interpolation} & [\text{Table}\left[[t, \ q_i[i, x[i], t, a_1]], \{t, 0, 100}\right]] \gg \\
& "q_i[1,x[1],a_1]", \text{Interpolation}[
\ \
\text{Table}\left[[t, \ q_i[i, x[i], t, a_1]], \{t, 0, 100}\right]] \gg "q_i[2,x[2],a_1]"], \\
i, \{1, 2\}\right]\right\}
\right]
\]

\[ Q_j(0,t,-a_1); \ i=1,2 \ (q_i[1,0,-a_1], \ q_i[2,0,-a_1]) \]

\[
\text{Do}\left[\left\{\begin{array}{ll}
\text{If} & [i == 1, \\
\text{Interpolation} & [\text{Table}\left[[t, \ q_i[i, 0, t, -a_1]], \{t, r1, 100}\right]] \gg \\
& "q_i[1,0,-a_1]", \text{Interpolation}[
\ \
\text{Table}\left[[t, \ q_i[i, 0, t, -a_1]], \{t, r1, 100}\right]] \gg "q_i[2,0,-a_1]"], \\
i, \{1, 2\}\right]\right\}
\right]
\]
\[ R_i(0,t,a_1); \ i=1,2 \ (ri[1,0,a_1], \ ri[2,0,a_1]) \]

\[
\text{Do[If} [i == 1, \\
\quad \text{Interpolation[Table[\{t, ri[i, 0, t, a_1]\}, \{t, 0, 100\}]} \]
\]
\[
\quad > \text{"ri[1,0,a1]"}, \text{Interpolation[}
\quad \text{Table[\{t, ri[i, 0, t, a_1]\}, \{t, 0, 100\}]} \]
\]
\[
\quad > \text{"ri[2,0,a1]"}], \\
\quad \{i, 1, 2\}]
\]

\[ Q_i(x_i,t;a); \ i=1,2 \ (qid[1,x[1],a1], \ qid[2,x[2],a1]) \]

\[
\text{Do[If} [i == 1, \\
\quad \text{Interpolation[Table[\{t, qid[i, x[i], t, a_1]\}, \{t, 0, 100\}]} \]
\]
\[
\quad > \text{"qid[1,x[1],a1]"}, \text{Interpolation[}
\quad \text{Table[\{t, qid[i, x[i], t, a_1]\}, \{t, 0, 100\}]} \]
\]
\[
\quad > \text{"qid[2,x[2],a1]"}], \\
\quad \{i, 1, 2\}]
\]

\[ R_i(0,t;a); \ i=1,2 \ (rid[1,0,a1], \ rid[2,0,a1]) \]

\[
\text{Do[If} [i == 1, \\
\quad \text{Interpolation[Table[\{t, rid[i, 0, t, a_1]\}, \{t, 0, 100\}]} \]
\]
\[
\quad > \text{"rid[1,0,a1]"}, \text{Interpolation[}
\quad \text{Table[\{t, rid[i, 0, t, a_1]\}, \{t, 0, 100\}]} \]
\]
\[
\quad > \text{"rid[2,0,a1]"}], \\
\quad \{i, 1, 2\}]
\]

\[
\text{Interpolation[Table[\{t, rid[2, 0, t, a_1]\}, \{t, 0, 100\}]} \]
\[
\quad > \text{"rid[2,0,a1]"}
\]

**Additional call files**

**NOTE:** To create the call files, this section must be manually run after the rest of the program is initialized.

The following interpolating functions will be integrated with the dependent variable as \((t-tau)\) with an integration range of \(x[i] \to t\). The range over which the functions will be interpolated must first be established.
The lower limit will be \( \text{rl} \); the upper limit, \( \text{r2} \)

\[
\text{in}[82] := \text{If}[\text{N}[\text{x}[1]] > \text{N}[\text{N}[\text{x}[2]]], \text{rl} = \text{N}[\text{Round}[-\text{x}[1] - 1]],
\text{rl} = \text{N}[\text{Round}[-\text{x}[2] - 1]]];
\]

\[
\text{in}[83] := \text{If}[\text{N}[\text{x}[1]] > \text{N}[\text{x}[2]], \text{r2} = \text{N}[\text{Round}[100 - \text{x}[2] + 1]],
\text{r2} = \text{N}[\text{Round}[100 - \text{x}[1] + 1]]];
\]

\[
\text{in}[84] := \text{Print}[\"The lower limit is\", \text{rl}] \text{Print}[\"The upper limit is\", \text{r2}] \\
\{\text{The lower limit is, } -22.\} \\
\{\text{The upper limit is, } 91.\}
\]

\[\sigma_0(i)(0,t); i=1,2 \text{ (sigmoi[1], sigmoi[2])} \]

\[
\text{Do}[\text{If}[i == 1, 
\text{Interpolation}[\text{Table}[\{t, \text{sigmoi}[i, t]\}, \{t, \text{rl}, \text{r2}, 1\}]] >> 
\"\text{sigmoi}[1]\", \text{Interpolation}[ 
\text{Table}[\{t, \text{sigmoi}[i, t]\}, \{t, \text{rl}, \text{r2}, 1\}]] >> \"\text{sigmoi}[2]\", 
\{i, 1, 2\}]
\]

\[\sigma_0(i)(0,t); i=1,2 \text{ (sigmaid[1], sigmaid[2])} \]

\[
\text{Do}[\text{If}[i == 1, 
\text{Interpolation}[\text{Table}[\{t, \text{sigmaid}[i, t]\}, \{t, \text{rl}, \text{r2}, 1\}]] >> 
\"\text{sigmaid}[1]\", \text{Interpolation}[ 
\text{Table}[\{t, \text{sigmaid}[i, t]\}, \{t, \text{rl}, \text{r2}, 1\}]] >> \"\text{sigmaid}[2]\", 
\{i, 1, 2\}]
\]

\[\theta_0(i)(0,t); i=1,2 \text{ (thetaoi[1], thetaoi[2])} \]

\[
\text{Do}[\text{If}[i == 1, 
\text{Interpolation}[\text{Table}[\{t, \text{thetaoi}[i, t]\}, \{t, \text{rl}, \text{r2}, 1\}]] >> 
\"\text{thetaoi}[1]\", \text{Interpolation}[ 
\text{Table}[\{t, \text{thetaoi}[i, t]\}, \{t, \text{rl}, \text{r2}, 1\}]] >> \"\text{thetaoi}[2]\", 
\{i, 1, 2\}]
\]

\[\theta_0(i)(0,t); i=1,2 \text{ (thetaid[1], thetaid[2])} \]

\[
\text{Do}[\text{If}[i == 1, 
\text{Interpolation}[\text{Table}[\{t, \text{thetaid}[i, t]\}, \{t, \text{rl}, \text{r2}, 1\}]] >> 
\"\text{thetaid}[1]\", \text{Interpolation}[ 
\text{Table}[\{t, \text{thetaid}[i, t]\}, \{t, \text{rl}, \text{r2}, 1\}]] >> \"\text{thetaid}[2]\", 
\{i, 1, 2\}]
\]

Printed by Mathematica for Students
\[ \theta_0 (0,t); \text{i}=1,2 (\text{thetaidd}[1], \text{thetaidd}[2]) \]

\[
\text{Do}[\text{If}[\text{i} == 1, \\
\quad \text{Interpolation}[\text{Table}[\{\text{t}, \text{thetaidd}[[\text{i}, \text{t}]]\}, \{\text{t}, \text{r}_1, \text{r}_2, 1\}]] >>
\quad "\text{thetaidd}[1]"; \\
\quad \text{Interpolation}[\text{Table}[\{\text{t}, \text{thetaidd}[[\text{i}, \text{t}]]\}, \{\text{t}, \text{r}_1, \text{r}_2, 1\}]] >> "\text{thetaidd}[2]"], \\
\quad \{\text{i}, 1, 2\}]]
\]

Evaluate the model parameters

- Formulate \( \delta, \alpha, \beta \)

\[
\text{In}[6]:= \quad \delta = (1 - \text{tso} + \text{eps} \text{ to})^2 + 4 \text{eps} \text{ tso to}; \\
\quad \alpha = -\frac{(1 + \text{eps}) (\text{tso} + \text{eps} \text{ to}) - (1 - \text{eps})}{\delta}; \\
\quad \beta = \frac{2 \sqrt{\text{eps}} \sqrt{1 + (1 + \text{eps}) (\text{to} - \text{tso})}}{\delta};
\]

- Formulate \( k[i], v[i], h[i], x[i] \)

\[
\text{In}[9]:= \quad k[i_] := .5 \left(1 + \text{eps} + (-1)^i \alpha \sqrt{\delta} \right); \\
\quad v[i_] := \frac{\sqrt{2}}{\sqrt{1 + \text{tso} + \text{eps} \text{ to} - (-1)^i \sqrt{\delta}}}; \\
\quad h[i_] := .5 k[i] v[i]^2; \\
\quad x[i_] := \frac{x}{v[i]^2};
\]

- Formulate \( \alpha_1[i], \lambda_1[i], \omega_1[i], \zeta_1[i] \)

\[
\text{In}[13]:= \quad \alpha_1[i_] := \alpha + h[i]; \\
\quad \lambda_1[i_] := .25 \beta^2 \sqrt{\delta} v[i]^2; \\
\quad \omega_1[i_] := h[i]^2 + (-1)^i \lambda_1[i]; \\
\quad \zeta_1[i_, t_] := t - x[i];
\]
## Listing of parameter values

```mathematica
In[17]:= TableForm[
    {{{"to", to}, {"tso", tso}, {"eps", eps}, {"y0", y0}, {"m1", m1},
      {"n1", n1}, {"b1", b1}, {"a1", a1}, {"x1", x1}, {"delta", delta},
      {"alpha", alpha}, {"beta", beta}, {"k[1]", N[k[1]]},
      {"h[1]", N[h[1]]}, {"h[2]", N[h[2]]}, {"x[1]", N[x[1]]},
      {"x[2]", N[x[2]]}, {"alpha1[1]", N[alpha1[1]]},
      {"alpha1[2]", N[alpha1[2]]}, {"lamda1[1]", N[lamda1[1]]},
      {"lamda1[2]", N[lamda1[2]]}, {"omegal[1]", N[omegal[1]]},
      {"omegal[2]", N[omegal[2]]}, {"zetal[1,t]", N[zetal[1, t]]},
      {"zetal[2,t]", N[zetal[2, t]]}}, TableSpacing -> {1, 10},
    TableHeadings -> None, {StyleForm["PARAMETER",
      FontFamily -> "Helvetica", FontWeight -> "Bold"], StyleForm["VALUE",
      FontFamily -> "Helvetica", FontWeight -> "Bold"]})
```
<table>
<thead>
<tr>
<th>PARAMETER</th>
<th>VALUE</th>
</tr>
</thead>
<tbody>
<tr>
<td>to</td>
<td>4.</td>
</tr>
<tr>
<td>tso</td>
<td>4.</td>
</tr>
<tr>
<td>eps</td>
<td>0.05</td>
</tr>
<tr>
<td>y0</td>
<td>1.</td>
</tr>
<tr>
<td>m1</td>
<td>2.</td>
</tr>
<tr>
<td>n1</td>
<td>3.</td>
</tr>
<tr>
<td>b1</td>
<td>1000.</td>
</tr>
<tr>
<td>al</td>
<td>0.1</td>
</tr>
<tr>
<td>xl</td>
<td>10.</td>
</tr>
<tr>
<td>delta</td>
<td>11.04</td>
</tr>
<tr>
<td>alpha</td>
<td>-0.313406</td>
</tr>
<tr>
<td>beta</td>
<td>0.0405085</td>
</tr>
<tr>
<td>k[1]</td>
<td>1.04567</td>
</tr>
<tr>
<td>k[2]</td>
<td>0.00433119</td>
</tr>
<tr>
<td>v[1]</td>
<td>0.484426</td>
</tr>
<tr>
<td>v[2]</td>
<td>1.03215</td>
</tr>
<tr>
<td>h[1]</td>
<td>0.122693</td>
</tr>
<tr>
<td>h[2]</td>
<td>0.00230707</td>
</tr>
<tr>
<td>x[1]</td>
<td>20.643</td>
</tr>
<tr>
<td>x[2]</td>
<td>9.68853</td>
</tr>
<tr>
<td>alphal[1]</td>
<td>-0.190713</td>
</tr>
<tr>
<td>alphal[2]</td>
<td>-0.311099</td>
</tr>
<tr>
<td>lambdal[1]</td>
<td>0.000319869</td>
</tr>
<tr>
<td>lambdal[2]</td>
<td>0.00145212</td>
</tr>
<tr>
<td>omegal[1]</td>
<td>0.0147337</td>
</tr>
<tr>
<td>omegal[2]</td>
<td>0.00145744</td>
</tr>
<tr>
<td>zetal[1,t]</td>
<td>-20.643 + t</td>
</tr>
<tr>
<td>zetal[2,t]</td>
<td>-9.68853 + t</td>
</tr>
</tbody>
</table>
Evaluate intermediate functions $g_n^{(i)}, h_n^{(i)}, A_n^{(i)}$

Program variables will be defined as:

- $g_n^{(i)}(t) \Rightarrow g[n,i,t]$
- $h_n^{(i)}(t) \Rightarrow h1[n,i,t]$
- $A_n^{(i)}(u,s) \Rightarrow an[n,i,t]$

Formulate $g[n,i,t]$

\[ g[n, i, t] := \exp[\alpha_1] \left( \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{\alpha}{\beta}^{n-k} \text{BesselJ}[n+k, \beta] \right) \]

Plot the function $g_n^{(i)}(t); \ n=0; \ i=1$

\[ \text{Plot}[g[0, 1, t], \{t, 0, 50\}, \text{AxesLabel} \rightarrow \{"time", "g(t)"\}, \text{PlotLabel} \rightarrow \"Function g(t); n=0; i=1\", \text{PlotRange} \rightarrow \{0, 1\}] \]

- Graphics -
Plot the function $g_n^{(i)}(t)$; $n=0; i=2$

$$\text{Plot}[g[0, 2, t], \{t, 0, 50\}, \text{AxesLabel} \to \{"time", "g(t)"\},$$
$$\text{PlotLabel} \to "\text{Function } g(t); n=0; i=2\text{", PlotRange} \to \{0, 1\}]$$

- Graphics -

Plot the function $g_n^{(i)}(t)$; $n=1, 2, 3, 4; i=1$

$$\text{In[147]}:= \text{gplot1} = \text{Plot}\{[g[1, 1, t], g[2, 1, t], g[3, 1, t], g[4, 1, t]],$$
$$\{t, 0, 50\}, \text{PlotRange} \to \{-0.4, 0.2\}, \text{AxesLabel} \to \{"time", "g(t)"\},$$
$$\text{PlotLabel} \to "g(t); n=1, 2, 3, 4; i=1"\}$$

$$\text{In[148]}:= \text{Show}[\text{gplot1}, \text{Graphics}[\text{Text["n=1", \{6, -.35\}]], Graphics[}\text{Text["n=3", \{6, -.1\}]}, \text{Graphics}[\text{Text["n=4", \{6, .04\}]},$$
$$\text{Graphics}[\text{Text["n=2", \{6, .19\}]]\}$$

- Graphics -
Plot the function $g_n(t); n=1,2,3,4; i=2$

```
In[148]:= gplot2 = Plot[{g[1, 2, t], g[2, 2, t], g[3, 2, t], g[4, 2, t]},
{t, 0, 50}, PlotRange -> {.2, .1}, AxesLabel -> {"time", "g(t)"},
PlotLabel -> "g(t); n=1,2,3,4; i=2"]
Out[148]= -Graphics -
```

**Formulate $h_1[n,i,t]$**

```
In[19]:= h1[n_, i_, t_] := Exp[alpha[i] t]
    Sum[1/2^k Binomial[n, k] (-1)^k (alpha/beta)^n-k (n + k)
If[t == 0,
Which[n + k == 1, .5, n + k > 1, 0], BesselJ[n + k, beta t] / t];

Verify $h(t); n=1; i=1; t=0; \Rightarrow -3.868$

```

```
In[154]:= N[h1[1, 1, 0]]
Out[154]= -3.8684

Verify $h(t); n>1 \text{ (i.e. } n=2); t=0; \Rightarrow 0$

```

```
In[155]:= N[h1[2, 1, 0]]
Out[155]= 0
```
Plot the function $h_n(t); \, n=1,2,3,4; \, i=1$

```
In[163]:= hplot1 = Plot[{h[1, 1, t], h[2, 1, t], h[3, 1, t], h[4, 1, t]},
                   {t, 0, 50}, PlotRange -> All, AxesLabel -> {"time", "h(t)"},
                   PlotLabel -> "h(t); \, n=1,2,3,4; \, i=1"]
```

```
Out[163]= -Graphics-
```

Plot the function $h_n(t); \, n=1,2,3,4; \, i=2$

```
In[166]:= hplot2 = Plot[{h[1, 2, t], h[2, 2, t], h[3, 2, t], h[4, 2, t]},
                   {t, 0, 50}, PlotRange -> All, AxesLabel -> {"time", "h(t)"},
                   PlotLabel -> "h(t); \, n=1,2,3,4; \, i=2"]
```

```
Out[166]= -Graphics-
```
\textbf{Formulate an[n,i,u,s]}

\begin{verbatim}
In[20]:= an[n_, i_, u_, s_] := omegal[i]^n \left(\sqrt{\text{omegal}[i]} (s^2 - u^2)\right)^
\text{If}[\text{N}\left[u^2\right] > s^2, 0, 1] \text{BesselI}[n, \sqrt{\text{omegal}[i]} (s^2 - u^2)];
\end{verbatim}
Plot the function $A_n^1(u,s); n=1; i=1$

```mathematica
Plot[an[1, 1, x[1], t], {t, 0, 50}, PlotRange -> All,
AxesLabel -> {"time", "A"}, PlotLabel -> "A(u,s); n=1; i=1"]
```

- Graphics -

Plot the function $A_n^2(u,s); n=1; i=2$

```mathematica
Plot[an[1, 2, x[2], t], {t, 0, 50}, PlotRange -> All,
AxesLabel -> {"time", "A"}, PlotLabel -> "A(u,s); n=1; i=2"]
```

- Graphics -
Plot the function $A_n^{(1)}(u,s)$; $n=-1; i=1$

```
Plot[an[-1, 1, x[1], t], {t, 0, 50}, PlotRange -> All,
AxesLabel -> {"time", "A"}, PlotLabel -> "A(u,s); n=-1; i=1"]
```

```
Plot[an[-1, 2, x[2], t], {t, 0, 50}, PlotRange -> All,
AxesLabel -> {"time", "A"}, PlotLabel -> "A(u,s); n=-1; i=2"]
```
Evaluate functions $M_j(X_j,t)$, $N_j(X_j,t)$.

Program variables will be defined as:

- $M_j(X_j,t) \rightarrow m[i,x[i],t]$
- $N_i(x,t) \rightarrow n[i,x[i],t]$

Define the Heaviside Function $(hsf)$ for $\zeta_1[i]$

- $\zeta_1[i] := hsf[\zeta_\_] := \text{If}[\zeta > 0, 1, 0]$

Test Plot $\zeta[i,t]$ and $hsf[\zeta_1]$ for $i=1$

- $14174 := \text{Plot}[\zeta[i,t] \text{\_\_}, \{t, 0, 50\}, \text{PlotLabel \_\_} \rightarrow \text{AxesLabel \_\_} \rightarrow \{\text{time}, \zeta[i,t] \}_l$
\textbf{Formulate }\texttt{mi[i,xi,t]}\textbf{ }

\texttt{In[22]:= mill[i_, t_] := Exp[-h[i] t]}

\texttt{In[23]:= mil2[i_, t_] := g[0, i, zeta1[i, t]]}

\texttt{In[24]:= mil3[i_, xi_, t_] := xi \sum_{n=0}^{3} \frac{1}{n!} \left((-1)^n \text{lamda1[i]}\right)^n NIntegrate[g[n, i, t-s] an[n-1, i, xi, s], \{s, xi, t\}]}

\texttt{In[25]:= mi[i_, xi_, t_] := mill[i, t] (mil2[i, t] + mil3[i, xi, t])}
Plot the function $M_i(x_i, t); i=1$

```math
Plot[mi[1, x[1], t], {t, 0, 50}, PlotRange -> All,
AxesLabel -> {"time", "Mi"}, PlotLabel -> "Mi[xi,t]; i=1"]
```

- Graphics -

Plot the function $M_i(x_i, t); i=2$

```math
Plot[mi[2, x[2], t], {t, 0, 50}, PlotRange -> All,
AxesLabel -> {"time", "Mi"}, PlotLabel -> "Mi[xi,t]; i=2"]
```

- Graphics -

- **Formulate $n_i[i, x_i, t]$**

```math
n[i_, x[i_], t_] := x[i] Exp[-h[i] t]
```
in[27]:= nil2[i_, xi_, t_] := an[-1, i, xi, t]

in[28]:= nil3[i_, xi_, t_] := \(\sum_{n=1}^{3} \frac{1}{n!}((-1)^i lamda1[i])^n\)
        NIntegrate[h[n, i, t-s] an[n-1, i, xi, s],
                   \{s, xi, t\}]""

in[29]:= ni[i_, xi_, t_] := nil1[i, xi, t] (nil2[i, xi, t] + nil3[i, xi, t])

Plot the function \(N_i(x_i,t); i=1\)

Plot[ni[1, x[1], t], \{t, 0, 50\}, PlotRange -> All,
      AxesLabel -> \{"time", "Ni\}, PlotLabel -> "Ni[xi,t]; i=1"]
Plot the function $N_j(X_j,t)$.

$$\text{Evaluate}$$

The laser profile will be defined as:

$$Y(t) \rightarrow y[t]$$

$$Y(t) = y^\prime[t]$$

$$Y(t) = y^{\prime\prime}[t]$$

$$Y(t) = y^{\prime\prime\prime}[t] = y_0$$

$$a$$ is a material constant.

$Y=Y(t)$ represents a "skewed" Gaussian temporal profile.

Formulate $y[t], y^\prime[t], y^{\prime\prime}[t], y^{\prime\prime\prime}[t]$

$$y[t] := y_0 \frac{t}{n_l} \exp\left[-a b t^2\right]$$

$$y^\prime[t] := y_0 \frac{t}{n_l} \left(\frac{n_l}{m_l} - b f_1\right) \exp\left[-b t^2\right]$$

Printed by Mathematica for Students 99
\[ydd[t_] := y_0 t^{n_1-2} (n_1 (n_1 - 1) - (2 n_1 + m_1 - 1) (m_1 b_1 t^{m_1}) + b_1^2 m_1^2 t^{2m_1})\]
\[\text{Exp}[-b_1 t^{m_1}]\]

\[yddd[t_] := \text{If}[t == 0, 6, (y_0 t^{n_1-3}) (n_1 (n_1 - 1) (n_1 - 2) - (m_1 b_1) t^{m_1} ((m_1 - 1) (m_1 - 2) + 3 n_1 (n_1 + m_1 - 2)) + 3 m_1^2 b_1^2 t^{2m_1} (n_1 + m_1 - 1) - b_1^3 m_1^3 t^{3m_1})\]
\[\text{Exp}[-b_1 t^{m_1}]\]

- Plot the laser profile and its derivatives

Determine the valid laser pulse activation time to characterize the pulse as "short"

By definition: \(x_i \sim "\text{sufficiently small}"\) (e.g. \(< 0.001\) )

\[\text{xi}[t_] := t^{n_1} \text{Exp}[-b_1 t^{m_1}]\]

\[\text{Plot}[\text{xi}[t], \{t, 0, .1\}, \text{AxesLabel} \rightarrow \{"activation time", "xi"\}, \text{PlotLabel} \rightarrow "\text{Laser Pulse, } x_i(t)\", \text{PlotRange} \rightarrow \text{All}]\]

- Graphics -

The valid pulse activation time, therefore will be approximately \(0.1\)
Plot the laser profile; \( Y(t) \)

\[
\text{Plot}[y[t], \{t, 0, .1\}, \text{AxesLabel} \rightarrow \{\text{"time"}, \text{"Y(t)"}\}, \\
\text{PlotLabel} \rightarrow \text{"Laser Profile Y(t)"}]
\]

- Graphics -

Compute the Laser Pulse Rising Time (\( tr \))

\[
tr = \left( \frac{nl}{ml b1} \right)^{1/ml}
\]

0.0387298
Plot the laser profile first time derivative; \( \dot{Y}(t) \)

\[
\text{Plot}[yd[t], \{t, 0, .1\}, \text{AxesLabel} \rightarrow \{"time", "dY/dt"\}, \\
\text{PlotLabel} \rightarrow "\text{Laser Profile } dY/dt\", \text{PlotRange} \rightarrow \text{All}]
\]

\[\begin{array}{c}
\text{time} \\
0.02 \quad 0.04 \quad 0.06 \quad 0.08 \quad 0.1 \\
\end{array}\]

- Graphics -

Plot the laser profile second time derivative; \( Y''(t) \)

\[
\text{Plot}[ydd[t], \{t, 0, .1\}, \text{AxesLabel} \rightarrow \{"time", "d^2Y/dt^2\"\}, \\
\text{PlotLabel} \rightarrow "\text{Laser Profile } d^2Y/dt^2\"]
\]

\[\begin{array}{c}
\text{time} \\
0.02 \quad 0.04 \quad 0.06 \quad 0.08 \quad 0.1 \\
\end{array}\]

- Graphics -
Plot the laser profile third time derivative; \( Y(t) \)

\[
\text{Plot}\left[yddd[t], \{t, 0, .1\}, \text{AxesLabel} \to \{\text{"time"}, \text{"d}^3\text{Y/dt}^3\}\}, \text{PlotLabel} \to \text{"Laser Profile d}^3\text{Y/dt}^3\}\]

Evaluate functions \( \hat{P}_n^{(i)}(x_i, s; a_1) \), \( \hat{Q}_n^{(i)}(x_i, s; a_1) \), \( \hat{R}_n^{(i)}(x_i, s; a_1) \)

Program variable will be defined as:

\[
\begin{align*}
\hat{P}_n^{(i)}(x_i, s; a_1) & \Rightarrow \text{pcap}[n, i, x[i], s, a1] \\
\hat{Q}_n^{(i)}(x_i, s; a_1) & \Rightarrow \text{qcap}[n, i, x[i], s, a1] \\
\hat{R}_n^{(i)}(x_i, s; a_1) & \Rightarrow \text{rcap}[n, i, x[i], s, a1]
\end{align*}
\]

Formulate \( \text{pcap}[n, i, x[i], s, a1] \)

\[
\text{pcap}[n_, i_, x_, s_, a_] := \text{an}[n, i, x, s] + a[l v[i]] \text{NIntegrate}[\text{Exp}[a[l v[i]] (u-x[i])] \text{an}[n, i, u, s], \{u, x, s\}, \text{AccuracyGoal} \to 4, \text{PrecisionGoal} \to 4]
\]
Plot the function $\hat{P}_n^{(i)} (x_i, s; a_1); n=1, i=1$

```mathematica
Plot[pcap[1, 1, x[1], s, a1],
{s, 0, 50}, AxesLabel -> {"time", "Pcap(i,s,xi;al)"},
PlotLabel -> "Function Pcap(i,s,xi;al); n=1; i=1",
PlotPoints -> 10]
```

- Graphics -

Formulate qcap[n,i,x[i],s,a1]

```mathematica
qcap[n_, i_, xi_, s_, a1_] :=
NIntegrate[Exp[a1 v[i] (u - xi)] an[n, i, u, s], {u, xi, s},
AccuracyGoal -> 4, PrecisionGoal -> 4]
```
Plot the function \( \hat{Q}_n^{(0)}(x_i, s; a_1) \); \( n=1, i=1 \)

```mathematica
Plot[Qcap[1, 1, x[1], s, a1],
   {s, 0, 50}, AxesLabel -> {"time", "Qcap(i,s,xi;al)"},
   PlotLabel -> "Function Qcap(i,s,xi;al); n=1; i=1",
   PlotPoints -> 10]
```

- Graphics -

■ Formulate \( rcap[n,i,x[i],s,a1] \)

\[
\text{rcap}[n_, i_, xi_, s_, al_] := \\
\int_{xi}^{s} \cosh[al v[i] (u - xi)] an[n, i, u, s] \, du
\]
Plot the function $\hat{R}_n^{(i)}(x_i,s;a1); n=1, i=1$

```
Plot[rcap[1, 1, x[1], s, al],
    {s, 0, 50}, AxesLabel -> {"time", "Rcap"},
    PlotLabel -> "Rcap(i,s,xi;al); n=1; i=1", PlotRange -> All]
```

Evaluate functions $P_i(x_i,t;a1)$, $Q_i(x_i,t;a1)$, $R_i(x_i,t;a1)$

Program variables will be defined as:

$P_i(x_i,t;a1) \Rightarrow pi[i,x[i],t,a1]$

$Q_i(x_i,t;a1) \Rightarrow qi[i,x[i],t,a1]$

$R_i(x_i,t;a1) \Rightarrow ri[i,x[i],t,a1]$

---

Formulate $pi[i,x[i],t,a1]$

```
in[37]:= pi01[n_, i_, xi_, t_, al_] := pi01[n, i, xi, t, al] =
    NIntegrate[g[n, i, t - s] pcap[n, i, xi, s, al], {s, xi, t}]
```
\[ \pi[i, x_i, t, a1] := v[i] \exp[-h[i] t] \sum_{n=0}^{\infty} \frac{(-1)^n \lambda d a[i]}{n!} \pi_0[i, n, x_i, t, a1] \]

Plot the function \( P_i(x_i, t; a1); i=1 \)

Plot \[ \pi[1, x[1], t, a1], \{t, 0, 50\}, \text{PlotRange} \to \text{All}, \text{AxesLabel} \to \{\text{"time", "Pi"}, \text{PlotLabel} \to \text{"Pi}[x_i, t; a1]; i=1"] \]

Plot the function \( P_i(x_i, t; a1); i=2 \)

Plot \[ \pi[2, x[2], t, a1], \{t, 0, 50\}, \text{PlotRange} \to \text{All}, \text{AxesLabel} \to \{\text{"time", "Pi"}, \text{PlotLabel} \to \text{"Pi}[x_i, t; a1]; i=2"] \]
Plot the function \( P_i(x_i, t; a_1); i=1, x_i=0 \)

\[
\text{Plot}[p_i[1, 0, t, a_1], \\
\{t, 0, 50\}, \text{PlotRange} \to \text{All}, \text{AxesLabel} \to \{\text{"time"}, \text{"Pi"}\}, \\
\text{PlotLabel} \to \text{"Pi}[x_i, t; a_1]; i=1, x_i=0"]
\]

- Graphics -

Plot the function \( P_i(x_i, t; a_1); i=2, x_i=0 \)

\[
\text{Plot}[p_i[2, 0, t, a_1], \\
\{t, 0, 50\}, \text{PlotRange} \to \text{All}, \text{AxesLabel} \to \{\text{"time"}, \text{"Pi"}\}, \\
\text{PlotLabel} \to \text{"Pi}[x_i, t; a_1]; i=2, x_i=0"]
\]

- Graphics -
Plot the function $P_i(x_i, t; -a_1)$; $i=1$, $x_i=0$

Plot[$P_i[1, 0, t, -a_1],
\{t, 0, 50\}, PlotRange \rightarrow All, AxesLabel \rightarrow \{"time", "P_i"\},
PlotLabel \rightarrow "P_i[x_i, t; -a_1]; i=1, x_i=0"]

- Graphics -

Plot the function $P_i(x_i, t; -a_1)$; $i=2$, $x_i=0$

Plot[$P_i[2, 0, t, -a_1],
\{t, 0, 50\}, PlotRange \rightarrow All, AxesLabel \rightarrow \{"time", "P_i"\},
PlotLabel \rightarrow "P_i[x_i, t; -a_1]; i=2, x_i=0"]

- Graphics -
Formulate \( \text{qi}[i, x[i], t, a1] \)

\[
\text{qi01}[n_, i_, xi_, t_, al_] := \text{NIntegrate}[g[n, i, t - s] \text{qcap}[n, i, xi, s, al], \{s, xi, t\}]
\]

\[
\text{qi}[i_, xi_, t_, al_] := v[i]^2 \text{Exp}[-h[i] t] \sum_{n=0}^{\infty} \frac{((-1)^i \text{lamdal}[i])^n \text{qi01}[n, i, xi, t, al]}{n!}
\]

Plot the function \( Q_i(x_i, t; a1); i=1 \)

\[
\text{Plot}[\text{qi}[1, x[1], t, a1], \{t, 0, 50\}, \text{PlotRange} \to \text{All},
\quad \text{AxesLabel} \to \{"time", "Qi"}, \text{PlotLabel} \to \("Q_i[x_i,t;a1]; i=1"\}]
\]
Plot the function $Q_i(x_i,t; a_1); i=2$

Plot[$q_2(2, x[2], t, a_1), \{t, 0, 50\}, PlotRange \rightarrow All,$
AxesLabel -> {"time", "Qi"}, PlotLabel -> "Qi[xi,t;a1]; i=2"]

- Graphics -

Plot the function $Q_i(x_i,t; -a_1); i=1, x_i=0$

Plot[$q_1[1, 0, t, -a_1],$
{t, 0, 50}, PlotRange \rightarrow All, AxesLabel -> {"time", "Qi"},
PlotLabel -> "Qi[xi,t;-a1]; i=1, xi=0"]

- Graphics -
Plot the function $Q_i(x_i,t;-a_1)$; $i=2$, $x_i=0$

```math
Plot[q_i[2, 0, t, -a_1], 
   {t, 0, 50}, PlotRange -> All, AxesLabel -> {"time", "Q_i"}, 
   PlotLabel -> "Q_i[x_i,t;-a_1]; i=2, x_i=0"]
```

```
- Graphics -
```

- **Formulate r_i[i,x[i],t,a1]**

```
In[41]:= ri01[n_, i_, xi_, t_, a1_] := ri01[n, i, xi, t, a1] = 
   NIntegrate[g[n, i, t - s] rcap[n, i, xi, s, a1], {s, xi, t}, 
      AccuracyGoal -> 3, PrecisionGoal -> 3]

In[42]:= r_i[i_, xi_, t_, a1_] := (ν[i]^2) Exp[-h[i] t] Sum[
   (((((-1)^i) lamdal[i])^n)/Factorial[n]) ri01[n, i, xi, t, a1], 
   {n, 0, 3}]
```
Plot the function $R_i(x_i,t;a_1); \ i=1$

```mathematica
Plot[r1[1, x[1], t, a1], {t, 0, 50},
     PlotRange -> {0, 1.0}, AxesLabel -> {"time", "R_i[x_i,t; a_1]"},
     PlotLabel -> "Function $R_i(x_i,t;a_1); \ i=1$", PlotPoints -> 10]
```

- Graphics -

Plot the function $R_i(x_i,t;a_1); \ i=1, \ x_i=0$

```mathematica
Plot[r1[1, 0, t, a1], {t, 0, 50},
     PlotRange -> {0, 5}, AxesLabel -> {"time", "R_i"},
     PlotLabel -> "R_i[x_i,t;a_1]; \ i=1; \ x_i=0", PlotPoints -> 10]
```

- Graphics -
Plot the function $R_i(x_i, t; a1); i=2$

\[ Ri[x_i, t; a1] \]

Function $R_i[x_i, t; a1]; i=2$

- Graphics -

Plot the function $R_i(x_i, t; a1); i=2, x_i=0$

\[
\text{Plot}\left[\text{Evaluate}\left[R_i[2, 0, t, a1]\right],
\{t, 0, 50\}, \text{PlotRange}\to\text{All}, \text{AxesLabel}\to\{"time", "Ri"\},
\text{PlotLabel}\to"R_i[x_i, t; a1]; i=2; x_i=0", \text{PlotPoints}\to 10\]
\]

- Graphics -
Evaluate functions $g^{(i)}_n(t)$, $Q_i(x_i,t;a1)$, $R_i(x_i,t;a1)$

Program variables will be defined as:

\[
\begin{align*}
&g^{(i)}_n(t) \quad \Rightarrow \quad gd[n,i,t] \\
&Q_i(x_i,t;a1) \quad \Rightarrow \quad qid[i,x[i],t,a1] \\
&R_i(x_i,t;a1) \quad \Rightarrow \quad rid[i,x[i],t,a1]
\end{align*}
\]

**Formulate $gd[n,i,t]$**

```
In[43]:= gd[n_, i_, t_] := alpha[i] g[n, i, t] +
  .5 beta Exp[alpha[i] t] \[Sum]_{k=0}^{n} \frac{1}{2^k} \left( Binomial[n, k] (-1)^k \left( \frac{alpha}{beta} \right)^{n-k} \right.

(BesselJ[n+k-1, beta t] - BesselJ[n+k+1, beta t])
```

Printed by Mathematica for Students 115
Plot the function $g_n^{(i)} (t); n=1; i=1$

```
Plot[gd[1, 1, t], {t, 0, 50},
    PlotRange -> All, AxesLabel -> {"time", "dg(t)/dt"},
    PlotLabel -> "dg(t)/dt; n=1; i=1"]
```

- Graphics -

Plot the function $g_n^{(i)} (t); n=1; i=2$

```
Plot[gd[1, 2, t], {t, 0, 50},
    PlotRange -> All, AxesLabel -> {"time", "dg(t)/dt"},
    PlotLabel -> "dg(t)/dt; n=1; i=2"]
```

- Graphics -
Formulate qid[i, xi, t, a1]

\[ \text{in}[44]:= \text{qid01}[n_\_, i_\_, xi_\_, t_\_, a1\_] := \text{qid}[n, i, xi, t, a1] = \]
\[ \text{NIntegrate}[\text{gd}[n, i, t-s] \text{qcap}[n, i, xi, s, a1], \{s, xi, t}\] \n
\[ \text{in}[45]:= \text{qid}[i_, xi_, t_, a1\_] := \]
\[ -h[i] \text{qi}[i, xi, t, a1] + v[i]^3 \text{Exp}[-h[i] t] \left( \text{qcap}[0, i, xi, t, a1] + \right. \]
\[ \left. \sum_{n=0}^{3} \frac{((-1)^n \lambda \text{lamda}[i])^n \text{qid01}[n, i, xi, t, a1]}{n!} \right) \]

Plot the function \( Q_i(x_i; t; a1); i=1 \)

\[ \text{Plot}[\text{qid}[1, x[1], t, a1], \{t, 0, 50\}, \]
\[ \text{PlotRange} \to \text{All}, \text{AxesLabel} \to \{"time", "dQi/dt"}, \]
\[ \text{PlotLabel} \to \text{"dQi(xi,t;al)/dt; i=1"}] \]
Plot the function $Q_i(x_i,t;a_1); i=2$

```mathematica
Plot[qid[2, x[2], t, a1], {t, 0, 50},
PlotRange -> All, AxesLabel -> {"time", "dQi/dt"},
PlotLabel -> "dQi(xi,t;a1)/dt; i=2"]
```

- Graphics -

**Formulate rid[i,xi,t,a1]**

```
In[46]:= rid01[n_, i_, xi_, t_, a1_] :=
NIntegrate[gd[n, i, t - s] rcap[n, i, xi, s, a1], {s, xi, t}]

In[47]:= rid[i_, xi_, t_, a1_] :=
-h[i] ri[i, xi, t, a1] + v[i]^2 Exp[-h[i] t] (rcap[0, i, xi, t, a1] +
3 \sum_{n=0}^{\infty} \frac{(-1)^n \lambda[i]^n}{n!} rid01[n, i, xi, t, a1])
```
Plot the function $R_i(x_i,t; a_1); i=1$

```
Plot[rid[1, x[1], t, a1], {t, 0, 50},
     PlotRange -> Automatic, AxesLabel -> {"time", "dRi/dt"},
     PlotLabel -> "dRi(x_i,t;a1)/dt; i=1"]
```

- Graphics -

Plot the function $R_i(x_i,t; a_1); i=2$

```
Plot[rid[2, x[2], t, a1], {t, 0, 50},
     PlotRange -> Automatic, AxesLabel -> {"time", "dRi/dt"},
     PlotLabel -> "dRi(x_i,t;a1)/dt; i=2"]
```

- Graphics -
Plot the function $R_i(x_i,t;\alpha_1); \ i=1; \ x_i=0$

Plot[rid[1, 0, t, \alpha_1], {t, 0, 50},
     PlotRange \to \text{All}, AxesLabel \to \{"time", "dR_i/dt"},
     PlotLabel \to \"dR_i(x_i,t;\alpha_1)/dt; \ i=1; \ x_i=0\"
]

- Graphics -

Plot the function $R_i(x_i,t;\alpha_1); \ i=2; \ x_i=0$

Plot[rid[2, 0, t, \alpha_1], {t, 0, 50},
     PlotRange \to \text{Automatic}, AxesLabel \to \{"time", "dR_i/dt"},
     PlotLabel \to \"dR_i(x_i,t;\alpha_1)/dt; \ i=2; \ x_i=0\"
]
PART B - Stress / Temperature Evaluations

Evaluate the diffusive stress responses \( \sigma_d^{(i)}(x,t) \) and \( \sigma_d(x,t) \)

Program variables will be defined as:

\[
\begin{align*}
\sigma_d^{(i)}(x,t) & \quad \Rightarrow \text{sigmadi}[i,t] \\
\sigma_d(x,t) & \quad \Rightarrow \text{sigmad}[t]
\end{align*}
\]

Formulate \( \text{sigmadi}[i,t] \)

\[
\text{In}[48]:= \text{sigmadi} [i_, t_] := (-1)^{i+1} \text{Exp}[-\alpha_1 x] \int_0^t \gamma d[t - \tau] \text{If}[i == 1,
(<< "ri[1,0,\alpha]\[\tau]\"[\tau]) + to (<< "rid[1,0,\alpha]\[\tau]\"[\tau]),
(<< "ri[2,0,\alpha]\[\tau]\"[\tau]) + to (<< "rid[2,0,\alpha]\[\tau]\"[\tau])] \\
\text{dtau}
\]

Printed by Mathematica for Students
Plot the function $\sigma_d^{(0)}(x,t); \ i=1$

Plot[sigmadi[1, t], {t, 0, 50}, AxesLabel -> {"time", "sigmadi"},
PlotLabel -> "sigmadi(x,t); i=1", PlotRange -> All]

- Graphics -

Plot the function $\sigma_d^{(0)}(x,t); \ i=2$

Plot[sigmadi[2, t], {t, 0, 50},
PlotRange -> All, AxesLabel -> {"time", "sigmadi"},
PlotLabel -> "sigmadi(x,t); i=2"]

- Graphics -

Formulate $\text{sigmad}[t]$

\text{In}[48]:= \text{sigmad}[t_] := \text{sigmadi}[1, t] + \text{sigmadi}[2, t]
Plot the function $\sigma_d(x,t); i=1,2$

```mathematica
Plot[sigmad[t], {t, 0, 50},
PlotRange -> All, AxesLabel -> {"time", "sigmad"},
PlotLabel -> "sigmad(x,t); i=1,2"]
```

- Graphics -
Evaluate the diffusive temperature responses \( \theta_d^{(i)}(x,t) \) and \( \theta_d(x,t) \)

Program variables will be defined as:

\[
\begin{align*}
\theta_d^{(i)}(x,t) & \Rightarrow \text{thetadi}[i,t] \\
\theta_d(x,t) & \Rightarrow \text{thetad}[t]
\end{align*}
\]

Formulate \( \text{thetadi}[i,t] \)

\[
\text{In}[50]:= \ \text{thetadi}[i_, t_] := ((-1)^i) \ \text{Exp}[-a1 \ x1] \ \int_0^t y[t - \tau] \text{If}[i == 1, \\
\left( (\text{"rid}[1,0,a1]"[\tau]) - a1^2 \int_0^{\tau} (\text{"ri}[1,0,a1]"[u]) \, du \right), \\
\left( (\text{"rid}[2,0,a1]"[\tau]) - a1^2 \int_0^{\tau} (\text{"ri}[2,0,a1]"[u]) \, du \right) \, d\tau]
\]

Printed by Mathematica for Students
Plot the function $\theta_d^{(i)}(x,t); i=1$

```math
Plot[\theta_d[1, t], \{t, 0, 50\}, AxesLabel -\{"time", "\theta_d\"\},
PlotLabel -\"\theta_d(x,t); i=1\", PlotRange -\All\]
```

![Graph of $\theta_d^{(1)}(x,t)$]

- Graphics -

Plot the function $\theta_d^{(i)}(x,t); i=2$

```math
Plot[\theta_d[2, t], \{t, 0, 50\}, AxesLabel -\{"time", "\theta_d\"\},
PlotLabel -\"\theta_d(x,t); i=2\", PlotRange -\All\]
```

![Graph of $\theta_d^{(2)}(x,t)$]

- Graphics -

### Formulate $\theta_d[t]$

```math
In[51]:= \theta_d[t_] := \theta_d[1, t] + \theta_d[2, t]
```

Plot the function \( o \cdot w(i)(x,t) \) and \( o \cdot w(x,t) \):

- **Program variables will be defined as:**
  - \( o \cdot w(i)(x,t) \rightarrow \text{sigmawi}[i,t] \)
  - \( o \cdot w(x,t) \rightarrow \text{sigmaw}[t] \)

- **Formulate \( \text{sigmawi}[i,t] \):**
  
  ```mathematica
  \text{sigmawi}[i_,t_] := If[N[\text{hsf}[zeta[i,t]]] == 0, 0, (-1)^i \text{yd}[t-tau] If[i == 1, ((\text{qi[l,x[l]al]}[tau]) + Jx[i]to((\text{qid[l,x[l]al]}[tau])), ((\text{qi[2,x[2]al]}[tau]) + to((\text{qid[2,x[2]al]}[tau]))) dtau]
  ```

- **Printed by Mathematica for Students 126**
Create an interpolating function for $\sigma_w^{(i)}(x,t); i=1,2$
(sigmawi[1], sigmawi[2])

\begin{verbatim}
Do[
  If[i == 1, Interpolation[Table[{t, sigmawi[1, t]}, {t, 0, 50}]] >> "sigmawi[1]", Interpolation[
    Table[{t, sigmawi[2, t]}, {t, 0, 50}]] >> "sigmawi[2]", {i, 1, 2}]
\end{verbatim}

Plot the function $\sigma_w^{(1)}(x,t); i=1$

\begin{verbatim}
Plot[<< "sigmawi[1]"[t],
  {t, 0, 50}, AxesLabel -> {"time", "sigmawi"},
  PlotLabel -> "sigmawi(x,t); i=1", PlotRange -> All]
\end{verbatim}

- Graphics -
Plot the function $\sigma_w^{(i)}(x,t)$; $i=2$

```
Plot[<< "sigmawi[2]"[t],
  {t, 0, 50}, AxesLabel -> {"time", "sigmawi"},
  PlotLabel -> "sigmawi(x,t); i=2", PlotRange -> All]
```

![](image1.png)

- Graphics -

■ Formulate $\text{sigmaw}[t]$

```
In[53]:= sigmaw[t_] := << "sigmawi[1]"[t] + << "sigmawi[2]"[t]
```

Plot the function $\sigma_w(x,t)$; $i=1,2$

```
Plot[sigmaw[t], {t, 0, 50}, AxesLabel -> {"time", "sigmaw"},
  PlotLabel -> "sigmaw(x,t); i=1,2", PlotRange -> All]
```

![](image2.png)

- Graphics -
Evaluate the first wave temperature response component \( \theta_w^{(1)}(x,t) \) and \( \theta_w(x,t) \)

Program variables will be defined as:

\[
\begin{align*}
\theta_w^{(1)}(x,t) & \Rightarrow \text{thetawi}[i,t] \\
\theta_w(x,t) & \Rightarrow \text{thetaw}[t]
\end{align*}
\]

Formulate \( \text{thetawi}[i,t] \)

\[
\text{in}[54]:= \text{thetawi}[i_\_, t\_] := \text{If}[\text{N}[\text{hsf[zeta1}[i, t]]] == 0,
0, .5 (-1)^i \text{If}[i == 1, \int_{x[i]}^{t} \left( \int_{0}^{t-\tau} y[u] \, du \right)
\left( \text{<<"mi[1,x[1]]"}[\tau] + \text{al} \text{<<"pi[1,x[1],al]"}[\tau] \right)
\text{yd}[t-\tau] \text{<<"qi[1,x[1],al]"}[\tau]) \, d\tau,
\int_{x[i]}^{t} \left( \int_{0}^{t-\tau} y[u] \, du \right)
\left( \text{<<"mi[2,x[2]]"}[\tau] + \text{al} \text{<<"pi[2,x[2],al]"}[\tau] \right)
\text{yd}[t-\tau] \text{<<"qi[2,x[2],al]"}[\tau]) \, d\tau]
\]

Printed by Mathematica for Students
Plot the function $\theta_w^{(i)}(x,t); i=1$

\begin{verbatim}
in[100]:= Plot[theta[i,1,t], {t, 0, 50}, AxesLabel -> {"time", "theta[i,f]"}, PlotLabel -> "theta[i,f](x,t); i=1", PlotRange -> All]
\end{verbatim}

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{theta1.png}
\caption{Plot of $\theta_w^{(1)}(x,t)$ for $i=1$.}
\end{figure}

\textbf{Out[100]}= -Graphics -

Plot the function $\theta_w^{(i)}(x,t); i=2$

\begin{verbatim}
in[99]:= Plot[theta[i,2,t], {t, 0, 50}, AxesLabel -> {"time", "theta[i,f]"}, PlotLabel -> "theta[i,f](x,t); i=2", PlotRange -> All]
\end{verbatim}

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{theta2.png}
\caption{Plot of $\theta_w^{(2)}(x,t)$ for $i=2$.}
\end{figure}

\textbf{Out[99]}= -Graphics -

\textbf{Formulate} $\texttt{theta[t]}$

\begin{verbatim}
in[55]:= theta[t_] := theta[1, t] + theta[2, t]
\end{verbatim}
Plot the function $\theta_w(x,t); i=1,2$

\[
\text{Plot}[\theta_w[t], \{t, 0, 50\}, \text{AxesLabel} \to \{"time", "\theta_w"\}, \\
\text{PlotLabel} \to "\theta_w(x,t); i=1,2", \text{PlotRange} \to \text{All}]
\]

Evaluate parameters used in the temperature and stress wave part of the response; ( $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ )

- Formulate $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$

\[
\text{ln}[56]:= \text{alphacap} = \frac{1 + \text{eps}}{\sqrt{\text{delta}}}; \\
\text{betacap} = \frac{\text{tso} + \text{eps to } - 1}{\sqrt{\text{delta}}}; \\
\text{gammacap} = \frac{2}{\sqrt{\text{delta}}};
\]

131
Evaluate boundary stress relationships at $x = 0$;

$\sigma_0^{(i)}(0,t)$ and $\dot{\sigma}_0^{(i)}(0,t)$

Program variables will be defined as:

$\sigma_0^{(i)}(0,t) \Rightarrow \text{sigmaoi}[i,t]$  

$\dot{\sigma}_0^{(i)}(0,t) \Rightarrow \text{sigmaoid}[i,t]$

**Note:** Interpolating Functions for these functions are created in the "call file" section of this program.

**Formulate sigmaoi[i,t]**

```mathematica
In[59]:= sigmaoi[i_, t_] :=
(-1)^i .5 \int_0^t (yd[t - tau] + to ydd[t - tau]) (If[i == 1, 
  (<< "qi[1,0,-a1]"[tau]), (<< "qi[2,0,-a1]"[tau]])
  ) dtau
```

Printed by Mathematica for Students
Plot the function $\sigma_0^{(0)}(0,t); i=1$

```
Plot[<< "sigmaoi[1][t],
{t, -22, 41}, AxesLabel -> {"time", "sigmaoi"},
PlotLabel -> "sigmaoi(x,t); i=1", PlotRange -> All]
```

- Graphics -

Plot the function $\sigma_0^{(0)}(0,t); i=2$

```
Plot[<< "sigmaoi[2][t],
{t, -22, 41}, AxesLabel -> {"time", "sigmaoi"},
PlotLabel -> "sigmaoi(x,t); i=2", PlotRange -> All]
```

- Graphics -
Formulate $\sigma_{i,t}$

\[
\text{sigmoid}[i_, t_] := (-1)^{1+i} \cdot 0.5 \int_0^t (ydd[t - \tau] + t \cdot yddd[t - \tau]) \, d\tau
\]

If[$i == 1$, (\text{"qi}[1,0,-al]"[\tau]), (\text{"qi}[2,0,-al]"[\tau])]

Plot the function $\sigma_{0,1}(0,t)$; $i=1$

\[
\text{Plot}[\text{"sigmoid[1]"}[t],
\{t, -22, 41\}, \text{AxesLabel} \rightarrow \{"time", "sigmoid"\},
\text{PlotLabel} \rightarrow \text{sigmaoid}(x,t); i=1",
\text{PlotRange} \rightarrow \text{All}]
\]
Plot the function \( \sigma_0^0 (0,t); i=2 \)

```mathematica
Plot[
  \<< "sigmoid[2][t],
  
  \{t, -22, 41}, AxesLabel -> \{"time", \"sigmoid\"},
  
  PlotLabel -> \"sigmoid(x,t); i=2\", PlotRange -> All\]
```

- Graphics -
Evaluate boundary temperature relationships at \( x = 0 \);

Program variables will be defined as:

\[ \theta_{oi}[i,t] = \theta_{oi}[i,t] \]

\[ \theta_{oid}[i,t] = \theta_{oid}[i,t] \]

\[ \theta_{oidd}[i,t] = \theta_{oidd}[i,t] \]

Note: Interpolating functions for these functions are created in the "call file" section of this program.

Formulate

\[ \theta_{oi}[i,t](i) := \frac{1}{2}((-1)^i) \]

If \( i = l \),

\[ \frac{d\theta}{dt} = \frac{d\theta}{dt} \]

Printed by Mathematica for Students 136
Plot the function $\theta_0^{(i)}(0,t); i=1$

\[
\text{Plot}[[\theta_0^{(1)}[t],
\{t, -22, 41\}, \text{AxesLabel} \rightarrow \{"time", "\theta_0^{(1)}"\},
\text{PlotLabel} \rightarrow "\theta_0^{(1)}(x,t); i=1", \text{PlotRange} \rightarrow \text{All}]
\]

- Graphics -

Plot the function $\theta_0^{(i)}(0,t); i=2$

\[
\text{Plot}[[\theta_0^{(2)}[t],
\{t, -22, 41\}, \text{AxesLabel} \rightarrow \{"time", "\theta_0^{(2)}"\},
\text{PlotLabel} \rightarrow "\theta_0^{(2)}(x,t); i=2", \text{PlotRange} \rightarrow \text{All}]
\]

- Graphics -
Formulate \( \theta_{oid}[i, t] \)

\[
\text{thetaoid}[i_, t_] := \text{If}\[i == 1, \int_0^t (y[t - \tau] \nonumber \\
(\text{mi}[1, 0][\tau] + a1 (\text{pi}[1, 0, -a1][\tau]) + \nonumber \\
ydd[t - \tau] (\text{qi}[1, 0, -a1][\tau])) \nonumber \\
d\tau, \nonumber \\
\int_0^t (y[t - \tau] \nonumber \\
(\text{mi}[2, 0][\tau] + a1 (\text{pi}[2, 0, -a1][\tau]) + \nonumber \\
ydd[t - \tau] (\text{qi}[2, 0, -a1][\tau])) \nonumber \\
d\tau)\nonumber \\
]^{(i)} \nonumber \\
\text{Plot the function } \theta_0(0, t); i=1 \nonumber \\
\text{Plot}[[\text{thetaoid}[1][t], \nonumber \\
\{t, -22, 41\}, \text{AxesLabel} \rightarrow \{"time", "thetaoid"\}, \nonumber \\
\text{PlotLabel} \rightarrow \theta_{oid}(x, t); i=1", \text{PlotRange} \rightarrow \text{All}\nonumber \\
\text{Graphics} -
Plot the function $\theta_0 \ (0,t); \ i=2$

```
Plot[<< "thetaoid[2][t],
{t, -22, 41}, AxesLabel -> {"time", "thetaoid"},
PlotLabel -> "thetaoid\(x,t); \ i=2", PlotRange -> All]
```

Formulate thetaidd[i,t]

```
In[83]:= thetaidd[i_, t_] := (.5 (-1)^i) If[i == 1, \[Integral]_0^t (yd[t - tau]
( (<< "mi[1,0][tau]) + a1 (<< "pi[1,0,-a1][tau]) +
yddd[t - tau] (<< "qi[1,0,-a1][tau]) dtau,
\[Integral]_0^t (yd[t - tau]
( (<< "mi[2,0][tau]) + a1 (<< "pi[2,0,-a1][tau]) +
yddd[t - tau] (<< "qi[2,0,-a1][tau])
dtau]
```

```
Printed by Mathematica for Students
```

139
Plot the function $\theta_0 (0,t); i=1$

```
Plot[<< "thetaoidd[1]"[t],
   {t, -22, 41}, AxesLabel -> {"time", "thetaoidd"},
   PlotLabel -> "thetaoidd(x,t); i=1",
   PlotRange -> All]
```

```
Graphics
```

Plot the function $\theta_0 (0,t); i=2$

```
Plot[<< "thetaoidd[2]"[t],
   {t, -22, 41}, AxesLabel -> {"time", "thetaoidd"},
   PlotLabel -> "thetaoidd(x,t); i=2",
   PlotRange -> All]
```

```
Graphics
```

Printed by Mathematica for Students
Evaluate the total boundary stress/temperature relationships at $x=0; o-0(0,t), o-0(0,t), 0o(O,t), 0o(O,t), 0o(O,t)$.

Program variables will be defined as:

- $o-0(0,t) => \sigma_{o\ [t]}$
- $o-0(0,t) => \sigma_{o\ [d\ [t]}$
- $0o(O,t) => \theta_{o\ [t]}$
- $0o(O,t) => \theta_{o\ [d\ [t]}$
- $0o(O,t) => \theta_{o\ [d\ [d\ [t]}$

Formulate $\sigma_{o\ [t]}, \sigma_{o\ [d\ [t]}, \theta_{o\ [t]}, \theta_{o\ [d\ [t]}, \theta_{o\ [d\ [d\ [t]}$

1464: $\sigma_{o\ [t]} = \sigma_{o[i\ [1]}\ [t] + \sigma_{o[i\ [2]}\ [t]$
1465: $\sigma_{o\ [d\ [t]} = << \sigma_{o[i\ [1]}\ [t] + \sigma_{o[i\ [2]}\ [t]$
1466: $\theta_{o\ [t]} = \theta_{o[i\ [1]}\ [t] + \theta_{o[i\ [2]}\ [t]$
1467: $\theta_{o\ [d\ [t]} = \theta_{o[i\ [1]}\ [t] + \theta_{o[i\ [2]}\ [t]$
1468: $\theta_{o\ [d\ [d\ [t]} = \theta_{o[i\ [1]}\ [t] + \theta_{o[i\ [2]}\ [t]$

Printed by Mathematica for Students
Plot the function $\sigma_0(0,t); i=1,2$

```
Plot[\[Sigma]0[t], {t, -22, 41}, AxesLabel -> {"time", "\[Sigma]0"},
PlotLabel -> "\[Sigma]0(x,t); i=1,2", PlotRange -> All]
```

- Graphics -

Plot the function $\dot{\sigma}_0(0,t); i=1,2$

```
Plot[\[Sigma]0'[t], {t, -22, 41}, AxesLabel -> {"time", "\[Sigma]0'"},
PlotLabel -> "\[Sigma]0'(x,t); i=1,2",
PlotRange -> All]
```

- Graphics -
Plot the function $\theta_0(0,t); i=1,2$

$$\text{Plot[thetao}[t], \{t, -22, 41\}, \text{AxesLabel} \rightarrow \{"time", "thetao"}, \text{PlotLabel} \rightarrow \" \theta_0(x,t); i=1,2\", \text{PlotRange} \rightarrow \text{All}]$$

- Graphics -

Plot the function $\theta_0(0,t); i=1,2$

$$\text{Plot[thetaod}[t], \{t, -22, 41\}, \text{AxesLabel} \rightarrow \{"time", "thetaod"}, \text{PlotLabel} \rightarrow \" \theta_0(x,t); i=1,2\", \text{PlotRange} \rightarrow \text{All}]$$

- Graphics -

Printed by Mathematica for Students 143
Plot the function \( \theta_0(0,t); i=1,2 \)

\[
\text{Plot[thetaodd[t], \{t, -22, 41\}, AxesLabel \to \{"time", "thetaodd"\},}
\text{ PlotLabel \to \" thetaodd(x,t); i=1,2\",}
\text{ PlotRange \to \text{All}]}
\]

Evaluate the "tilda" stress and temperature relationships

Program variables will be defined as:

\[
\tilde{\sigma}_0(0,t) \Rightarrow \text{sigmaotilda[t]}
\]

\[
\tilde{\theta}_0(0,t) \Rightarrow \text{thetaotilda[t]}
\]

Formulate \( \text{sigmaotilda[t]} \)

\[
\text{ln[69]:=} \quad \text{sigmaotilda[t_] := alphacap (sigmao[t]) + betacap (sigmaod[t]) -}
\quad \text{gammacap ((thetaod[t]) + to (thetaodd[t])}}
\]
Plot the function $\tilde{\sigma}_0(0,t)$

\[
\text{Plot}[\sigma_{0\tilde{t}}[t], \{t, -22, 41\}, \text{AxesLabel} -> \{"time", "\sigma_{0\tilde{t}}\}, \text{PlotRange} -> \text{All}]
\]

- Graphics -

Formulate $\theta_{0\tilde{t}}[t]$

\[
\text{In[70]} := \theta_{0\tilde{t}}[t_] := \text{alphacap}(\theta_0[t]) + \text{betacap}(\theta_0d[t]) + \text{eps}(\text{gammacap})(\sigma_0[t])
\]

Plot the function $\tilde{\theta}_0(0,t)$

\[
\text{Plot}[\theta_{0\tilde{t}}[t], \{t, -22, 41\}, \text{AxesLabel} -> \{"time", "\theta_{0\tilde{t}}\}, \text{PlotRange} -> \text{All}]
\]

- Graphics -
Evaluate the second wave stress response component;
\( \sigma_c^{(i)}(x,t) \) and \( \sigma_c(x,t) \)

Program variables will be defined as:
\[ \sigma_c^{(i)}(x,t) \Rightarrow \text{sigmac}[i,t] \]
\[ \sigma_c(x,t) \Rightarrow \text{sigmac}[t] \]

**Formulate \( \text{sigmac}[i,t] \)**

\[
\text{sigmac}[i_, t_] := \text{If} [\text{N}[\text{hsf}[\text{zetal}[i, t]]] == 0, 
0, .5 \left( \text{Exp}\left[-h[i] x[i]\right] (\text{sigmao}[\text{zetal}[i, t]]) + 
\text{If}[i == 1, \int_{x[i]}^{t} ((<< "ni[1,x[1]]"[tau]) (\text{sigmao}[t - tau])) + 
(-1)^i ((<< "mi[1,x[1]]"[tau]) \text{sigmaotilda}[t - tau]) dt, 
\int_{x[i]}^{t} ((<< "ni[2,x[2]]"[tau]) (\text{sigmao}[t - tau])) + 
(-1)^i ((<< "mi[2,x[2]]"[tau]) \text{sigmaotilda}[t - tau]) 
\right)]
\]
Plot the function $\sigma_e^{(i)}(x,t)$; $i=1$

\begin{verbatim}
Plot[sigmac[1, t], {t, 0, 50}, AxesLabel -> {"time", "sigmac"},
PlotLabel -> "sigmac(x,t); i=1", PlotRange -> All]
\end{verbatim}

- Graphics -

Plot the function $\sigma_e^{(i)}(x,t)$; $i=2$

\begin{verbatim}
Plot[sigmac[2, t], {t, 0, 50}, AxesLabel -> {"time", "sigmac"},
PlotLabel -> "sigmac(x,t); i=2", PlotRange -> All]
\end{verbatim}

- Graphics -

Formulate \texttt{sigmac[t]}

\begin{verbatim}
in[72]:= sigmac[t_] := sigmac[1, t] + sigmac[2, t]
\end{verbatim}
Plot the function $\sigma_c(x,t); i=1,2$

```mathematica
In[81]:= Plot[sigmac[t], {t, 0, 50}, AxesLabel -> {"time", "sigmac"}, PlotLabel -> "sigmac(x,t); i=1,2", PlotRange -> All, PlotPoints -> 50]
```

```
Out[81]= Graphics
```
Evaluate the second wave temperature response component; $\theta_c^{(i)}(x,t)$ and $\theta_c(x,t)$

Program variables will be defined as:

\[
\begin{align*}
\theta_c^{(i)}(x,t) & \Rightarrow \text{thetaci}[i,t] \\
\theta_c(x,t) & \Rightarrow \text{thetac}[t]
\end{align*}
\]

- Formulate $\text{thetaci}[i,t]$

\[
\text{thetaci}[i_\_, t_] := \text{If}[\text{N[hsf[zeta1[i, t]]]} == 0, 0, \\
-.5 \left( \text{Exp}[-h[i] x[i]] (\text{thetao[zeta1[i, t]]}) + \text{If}[i == 1, \\
\int_{x[i]}^{t} (\text{<<ni1,x[1][tau]}} (\text{thetao[t-tau]} + (-1)^{i+1} \\
(\text{<<mi1,x[1][tau]}} \text{thetaotilda[t-tau]} \text{d}\tau, \\
\int_{x[i]}^{t} (\text{<<ni2,x[2][tau]}} (\text{thetao[t-tau]} + \\
(-1)^{i+1} (\text{<<mi2,x[2][tau]}} \text{thetaotilda[t-tau]} \text{d}\tau) \right)
\]
\]

Printed by Mathematica for Students 149
Plot the function $\theta_c^{(i)}(x,t); i=1$

\[
\text{Plot}\left[\text{thetaci}[1, t], \{t, 0, 50\}, \text{AxesLabel} \rightarrow \{\text{"time"}, \text{"thetaci"}\}, \right.
\text{PlotLabel} \rightarrow \text{"thetaci}(x,t); i=1\}, \text{PlotRange} \rightarrow \text{All}\]
\]

- Graphics -

Plot the function $\theta_c^{(i)}(x,t); i=2$

\[
\text{Plot}\left[\text{thetaci}[2, t], \{t, 0, 50\}, \text{AxesLabel} \rightarrow \{\text{"time"}, \text{"thetaci"}\}, \right.
\text{PlotLabel} \rightarrow \text{"thetaci}(x,t); i=2\}, \text{PlotRange} \rightarrow \text{All}\]
\]

- Graphics -

**Formulate thetac[t]**

\[
in[74]:= \text{thetac}[t_] := \text{thetaci}[1, t] + \text{thetaci}[2, t]
\]
Plot the function $\theta_c(x,t); i=1,2$

\begin{verbatim}
Plot[thetac[t], {t, 0, 50}, AxesLabel -> {"time", "thetac"},
PlotLabel -> "thetac(x,t); i=1,2", PlotRange -> All]
\end{verbatim}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{thetac.png}
\caption{Plot of $\theta_c(x,t); i=1,2$}
\end{figure}

Evaluate the total stress response; $\sigma(x,t)$

- Formulate $\sigma_1[t]; i=1$

\begin{verbatim}
\sigma1[t_] := (sigmadi[1, t]) + (sigmawi[1][t]) + (sigmaci[1, t])
\end{verbatim}
Plot the stress response; \( \sigma_1(x, t); i=1 \)

\[
\text{In}[96]:= \text{Plot}\left[\sigma_1[t], \{t, 0, 50\}, \text{AxesLabel} \rightarrow \{"time", "\sigma_1"\}, \right.
\]

\[
\text{PlotLabel} \rightarrow \text{"Stress Response; } \sigma_1(x, t); i=1\text{", PlotRange} \rightarrow \text{All, PlotPoints} \rightarrow 50\]

\[
\text{Out}[96]= \quad \text{- Graphics -}
\]

Formulate \( \sigma_2[t]; i=2 \)

\[
\text{In}[76]:= \sigma_2[t_\_] := (\text{sigmad}[2, t]) + (\text{"sigmawi}[2][t]) + (\text{sigmaci}[2, t])
\]

Plot the stress response; \( \sigma_2(x, t); i=2 \)

\[
\text{Plot}\left[\sigma_2[t], \{t, 0, 50\}, \text{AxesLabel} \rightarrow \{"time", "\sigma_2"\}, \right.
\]

\[
\text{PlotLabel} \rightarrow \text{"Stress Response; } \sigma_2(x, t); i=2\text{", PlotRange} \rightarrow \text{All}\]

\[
\text{Out}[76]= \quad \text{- Graphics -}
\]
Formulate $\sigma[t]$

\[ \sigma[t_] := (\sigma_1[t]) + (\sigma_2[t]) \]

Plot the total stress response; $\sigma(x,t)$

\[ \text{Plot}[\sigma[t], \{t, 0, 50\}, \text{AxesLabel} \rightarrow \{"time", "\sigma"\}, \text{PlotLabel} \rightarrow \text{"Total Stress Response; } \sigma(x,t)\", \text{PlotRange} \rightarrow \text{All}, \text{PlotPoints} \rightarrow 50] \]

Evaluate the total temperature response; $\theta(x,t)$

Formulate $\theta_1[t]; i=1$

\[ \theta_1[t_] := (\text{thetadi}_1[1, t]) + (\text{thetawi}_1[1, t]) + (\text{thetaci}_1[1, t]) \]
Plot the temperature response; \( \theta_1(x,t); i=1 \)

\[
\text{Plot}[\theta_1[t], \{t, 0, 50\}, \text{AxesLabel} \rightarrow \{"time", "\theta_1"\}, \\
\text{PlotLabel} \rightarrow "\text{Temp. Response; } \theta_1(x,t); i=1", \text{PlotRange} \rightarrow \text{All}]}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{plot1}
\caption{Temp. Response; \( \theta_1(x,t); i=1 \)}
\end{figure}

- Graphics -

- Formulate \( \theta_2[t]; i=2 \)

\[
\text{In[78]:= } \theta_2[t_] := (\text{thetadi}[2, t]) + (\text{thetawi}[2, t]) + (\text{thetaci}[2, t])
\]

Plot the temperature response; \( \theta_2(x,t); i=2 \)

\[
\text{Plot}[\theta_2[t], \{t, 0, 50\}, \text{AxesLabel} \rightarrow \{"time", "\theta_2"\}, \\
\text{PlotLabel} \rightarrow "\text{Temp. Response; } \theta_2(x,t); i=2", \text{PlotRange} \rightarrow \text{All}]}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{plot2}
\caption{Temp. Response; \( \theta_2(x,t); i=2 \)}
\end{figure}

- Graphics -

Printed by Mathematica for Students 154
Formulate $\theta[t]$

\[\theta[t_] := (\theta_1[t]) + (\theta_2[t])\]

Plot the total temperature response; $\theta(x,t)$

\[
\text{Plot}[	heta[t], \{t, 0, 50\}, \text{AxesLabel} \to \{"time", "\theta"\}, \\
\text{PlotLabel} \to \text{"Total Temp. Response; } \theta(x,t)\text{"}, \text{PlotRange} \to \text{All}]
\]