Numerical analysis of one-dimensional waves in generalized thermoelasticity

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Numerical Analysis of One-Dimensional Waves in Generalized Thermoelasticity

by

Mark J. Gorman

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mechanical Engineering

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1993
Numerical Analysis of One-Dimensional Waves in Generalized Thermoelasticity

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December 15, 1993
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Józef Ignaczak was on leave from the Institute of Fundamental Technological Research, Polish Academy of Sciences, Warsaw, Poland.
Abstract

Classical thermoelasticity theory is based on Fourier's Law of heat conduction, which, when combined with the other fundamental field equations, leads to coupled hyperbolic-parabolic governing equations. These equations imply that thermal effects are to be felt instantaneously, far away from the external thermomechanical load. Therefore, this theory admits infinite speeds of propagation of thermoelastic disturbances. This paradox becomes especially evident in problems involving very short time intervals, or high rates of heat flux.

Since infinite wave speeds are physically unrealistic in some situations, and since experiments have shown the existence of wave-type thermoelastic interactions, like in the observation of thermal pulses in dielectric crystals, "generalized" thermoelasticity theories have been developed. This thesis concentrates on one generalized thermoelasticity theory, proposed by Green and Lindsay, in which a generalized thermoelastic coupling constant, \( e \), and two relaxation times, \( t_0 \) and \( t^0 \), account for finite speed thermoelastic waves.

A numerical analysis of an exact analytical solution, involving an instantaneous plane source of heat in an infinite body, is performed. The analysis reveals two finite speed wave fronts for each of the four fields: displacement, stress, temperature, and heat flux. The results are complimentary to previous analyses, and improve upon them, because a large range of parameters is involved, and the exact solution to the problem has been used.
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List of Symbols

\( b \) - body force
\( c \) - specific heat capacity
\( C_\varepsilon \) - specific heat for zero deformation
\( C \) - elasticity tensor
\( E \) - strain tensor
\( f(\cdot) \) - internal heat source
\( H(\cdot) \) - Heaviside function
\( h_i \) - attenuation
\( k_i \) - damping
\( K \) - thermal conductivity
\( K \) - thermal conductivity tensor
\( M \) - stress-temperature tensor
\( \rho \) - mass density
\( q \) - heat flux
\( q \) - reduced heat flux
\( q \) - portion of heat flux associated with Dirac function
\( \mathbf{Q} \) - heat flux vector
\( r \) - heat supply
\( S \) - stress
\( S \) - reduced stress
\( S \) - portion of stress associated with Dirac function
\( \mathbf{S} \) - stress tensor
\( t \) - time
\( t_0, t^0 \) - nondimensional thermal relaxation times
\( t_0 \) - dimensional thermal relaxation time
\( t_0^* \) - time unit
\( u \) - displacement
\( \mathbf{u} \) - displacement vector
\( v \) - velocity of a thermoelastic disturbance
\( x, y, z \) - space coordinates
\( \Gamma \) - central operator of Green-Lindsay theory
\( \varepsilon \) - thermoelastic coupling coefficient
List of Symbols

continued

\( \varepsilon \) - minimum absolute value of \( g_n^{(i)}(t) \) or \( h_n^{(i)}(t) \), to be included in summation

\( \lambda_i \) - convolution coefficient

\( \Phi \) - thermoelastic potential

\( \tau \) - thermal relaxation time

\( \theta \) - temperature difference

\( \Theta \) - absolute temperature

\( \Theta_0 \) - reference temperature

\( \text{div} \) - divergence

\( \nabla \) - gradient

\( (\cdot) \) - time derivative
I. Introduction

Classical theories of rigid body heat conduction and of linear thermoelasticity have been challenged in numerous papers in the last forty years or so. The common parabolic equation for heat conduction can be viewed as only a very close approximation to the actual mechanical interactions taking place in a deformable solid under the influence of an external thermomechanical load. This stems from the fact that the parabolic equation allows for thermal effects to be felt instantaneously, far away from the external load. This paradox becomes especially evident in problems involving very short time intervals, or high rates of heat flux. The motivation to correct this physically unrealistic effect has prompted many investigations, and some experimentation. Solutions in generalized thermoelasticity have been developed using different techniques.

The objective of this thesis is to give a brief introduction to generalized thermoelasticity, and then provide a numerical analysis of an exact analytical solution. It examines the proposed thermoelastic reactions of a body to an external thermomechanical load, including the displacement, stress, temperature and heat flux fields. The graphical output also provides an easily understood physical interpretation of "second sound".

In 1977, Nayfeh [1] performed a numerical analysis of generalized thermoelasticity with one relaxation time (Lord-Shulman, or "L-S", theory.) This analysis showed a second sound response for nondimensional stress and temperature. In 1992, Tamma and Nambru [2] provided a finite element analysis for generalized thermoelasticity with two relaxation times (Green-Lindsay, or "G-L", theory.) They used an explicit computational architecture for both sudden and ramp-type temperature change models.

All of these investigations were performed for small times, or with some other severe restrictions. In this thesis, an analysis
of G-L theory is performed. However, exact analytical solutions, instead of approximate solutions, for the displacement, temperature, stress, and heat flux fields are used, to reveal the characteristics of second sound.

Section II of the thesis gives an introduction to classical linear heat conduction and thermoelasticity. The fundamental relations which govern thermoelasticity are introduced as a point of reference. The inherent contradiction of infinite wave speed is then discussed, and parabolic and hyperbolic types of equations are compared. Theories in generalized thermoelasticity, and the concept of a thermal relaxation time, is presented in Section III.

A particular approach, first developed by Green and Lindsay (G-L theory) [3], and extensively researched by Ignaczak in [4] and [5], Ignaczak and Mróówka-Matejewska in [6], and Hetnarski and Ignaczak in [7], is then concentrated on. In Section IV, the central operator for G-L theory is discussed in detail. The solutions for thermoelastic fields that are derived from this operator are then presented. The intermediate functions leading up to these fields are also listed, as a complete numerical analysis must include an examination of these functions.

A comprehensive numerical analysis of all of these functions follows in Section V. Evaluation of the functions at characteristic times, and their overall behavior, is performed. Each of the functions has been programmed in Fortran 77.

The results of this analysis are given in Section VI. Each thermoelastic field is found to travel in two separate waves, and each wave, and their sum, is presented graphically. Conclusions that can be drawn, and areas for future work, are given in Section VII.

Appendix A contains a discussion of the dimensionless units used in the analytical solution, and an approximation of the dimensional lengths and times that characterize a thermoelastic body's behavior in this type of problem. Graphs of the intermediate functions are available in Appendix B, and the Fortran source code is listed in Appendix C.
II. Classical Linear Thermoelasticity

A. Heat conduction in rigid bodies.

Classical theories of rigid body heat conduction and thermoelasticity are based on Fourier's Law. This law states that the heat flux vector $q$ is related to the temperature gradient by the equation

$$ q = -K \nabla \theta $$  \hspace{1cm} (1)

where $\theta = \theta(x, y, z, t)$ is the temperature and $K$ is the thermal conductivity.

This combines with the conservation of energy law to yield the classical equation of heat conduction,

$$ \frac{\partial \theta}{\partial t} = a^2 (\nabla^2 \theta + \frac{1}{K} f(x, y, z, t)) $$  \hspace{1cm} (2)

where $f$ is an internal heat source, and $a^2 = K/(\rho c)$ (thermal conductivity divided by mass density and specific heat capacity). In the absence of any internal heat generation, this reduces to the equation,

$$ \frac{\partial \theta}{\partial t} = a^2 \nabla^2 \theta . $$  \hspace{1cm} (3)
B. Thermoelasticity.

Expanding from rigid bodies to elastic bodies, we become concerned with mechanical fields, such as stress, strain, and displacement. Thermoelasticity describes the interaction of these mechanical fields with a temperature field [8]. The constitutive equations are temperature dependent, and also include Fourier's Law [8]. (The generalized theory discussed in later sections are both temperature and temperature-rate dependent.) Since Section III discusses a generalization of thermoelasticity relations, the equations used in conventional theory are briefly introduced here as a point of reference.

Imposing an external pressure or heat source on an elastic body produces a thermomechanical interaction. This interaction is described by coupled equations, which include relations among displacement, temperature, stress, heat flux, and other quantities. So, either a thermal load or an external force exerted on a body will result in both a mechanical and thermal change. Since the body is an ideal elastic solid, it will return to it's previous condition upon the removal of these external loads.

The thermoelastic process is based on the following fundamental relations (taken from [4]):

Geometrical relations:

\[ \mathbf{E} = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T) \]  (4)

Laws of balance of forces and moments:

\[ \text{div} \mathbf{S} + \mathbf{b} = \rho \ddot{\mathbf{u}}, \quad \mathbf{S} = \mathbf{S}^r \]  (5)
Law of conservation

of energy: \[ -\text{div} \mathbf{q} + r = C_e \dot{\theta} - \Theta_0 \mathbf{M} \cdot \mathbf{E} \] (6)

Constitutive

relation: \[ S = C[\mathbf{E}] + \mathbf{M} \theta, \quad \theta = \Theta - \Theta_0 \] (7)

Fourier's Law:

\[ q = -\kappa \nabla \theta. \] (8)

Here, \( \theta \) is the temperature difference from the constant reference temperature \( \Theta_0 \), and \( \kappa \) is the conductivity tensor. When considered together, these equations will characterize the nonisothermal deformation of an elastic body.

C. Problems associated with classical theory.

Inspection shows that Equation (3) is parabolic in nature. But parabolic formulations allow for any thermal disturbance to be felt instantaneously, at any distance from the disturbance. In effect, a thermal wave would travel at an infinite speed, and cause a change in temperature at an infinite distance.

1. To illustrate this, we draw an example from [9], page 17. Consider the half space \( x > 0 \), for temperature \( \Theta = \Theta(x,t) \), with time \( t \geq 0 \). The initial condition is \( \Theta(x,0) = 0 \). The boundary condition is \( \Theta(0,t) = \Theta_0 \). An instantaneous planar temperature is applied at the semi-space boundary. The temperature distribution for this problem will be governed by equation (3). The solution to this Initial Boundary Value Problem is generated by Beck, et. al. [9], with the use of Green's function, and has the form of the
complementary error function:

\[ \theta(x,t) = \theta_0 \text{erfc}\left(\frac{x}{2a\sqrt{t}}\right). \]  

(9)

Figure 1. Parabolic representation of temperature distribution at a fixed time.

For any finite \( t > 0 \), as \( x \) grows very large, \( \theta \) approaches zero, but still remains positive and nonzero. Thus, we are presented with a paradox; how can any physical effect travel with infinite speed in a continuous media? This solution gives a close approximation, but it must be inexact, considering the behavior as \( x \) grows large. This approximation generally suits most heat conduction situations, but in instances that deal with very short time spans, or high heat fluxes, the inexactness of this solution becomes an issue.
By contrast, consider an Initial Boundary Value Problem for a hyperbolic equation:

2. Find \( \theta \) such that

\[
\frac{\partial^2 \theta}{\partial x^2} - \frac{\partial^2 \theta}{\partial t^2} = 0
\]  

with \( x > 0, \ t \geq 0 \). The initial condition is \( \theta(x,0)=0 \), and the boundary condition is \( \theta(0,t)=\theta_0 \ H(t) \), where \( H \) is the Heaviside function and \( \theta_0 \) is the constant initial temperature. A solution to this equation is \( \theta=\theta_0(t-x) \ H(t-x) \). Then

\[
\theta = \begin{cases} 
\theta_0(t-x) , & (t-x) \geq 0 \\
0 , & (t-x) < 0 
\end{cases}
\]  

Figure 2. Hyperbolic representation of temperature distribution at a fixed time.
Clearly, the solution vanishes for every \( x \geq t \). For any finite \( t > 0 \), as \( x \) grows sufficiently large, the boundary disturbance \( \theta(0,t) \) does not arrive at \( x \). This means that \( \theta \) propagates with a finite velocity (\( v = 1 \)). This solution is shown in Figure 2.

This hyperbolic equation is the **one-dimensional wave equation**. It would be helpful to use an equation of this type, along with a parabolic equation, to model heat conduction and thermoelastic responses in instances where conventional theory provides physically unacceptable drawbacks. These cases would be, as mentioned earlier, when small intervals of time, high heat fluxes, or very low temperatures (close to absolute zero), are involved.

The solution presented in Section IV, and analyzed in the subsequent sections, uses a central operator that combines hyperbolic and parabolic operators. Also, the theory supporting the development of this operator does not violate Fourier's Law.
III. Generalized Thermoelasticity

A. Thermoelasticity with one relaxation time.

To describe thermoelasticity in a way that will include finite speeds, different formulations have been proposed. Postulation of wave-type heat flow in rigid bodies, in a hyperbolic form, had been examined as early as 1867 by Maxwell. A more recent theory, first developed by Cattaneo [10], starts with a modification (or "generalization") of Fourier's Law. The derivation involves introducing another term, so that

\[ \tau \frac{\partial q}{\partial t} + q = -k \nabla \theta \]  \hspace{1cm} (12)

where \( \tau \) is a non-negative constant called the thermal relaxation time [10]. This equation is commonly referred to as Cattaneo's equation. The corresponding heat transport equation is

\[ \frac{\partial \theta}{\partial t} + \tau \frac{\partial^2 \theta}{\partial t^2} = a^2 \nabla^2 \theta . \]  \hspace{1cm} (13)

This equation is hyperbolic in nature, and will predict a finite wave speed for heat propagation equal to

\[ v = a \tau^{-1/2} . \]  \hspace{1cm} (14)

Note that if \( \tau \) approaches zero, (13) approaches the original parabolic form, and the wave speed approaches infinity again.

Lord and Shulman developed this idea of including one thermal relaxation time in thermoelasticity in [11]. This is referred to
from now on in this paper as L-S theory.

Rubin has recently (1992) attempted to show that Cattaneo’s model can violate the second law of thermodynamics, by predicting that heat may flow from cold to hot regions during finite time periods [10]. A rigid heat conductor with suitably restricted parameters was used to prove the statement.

B. Thermoelasticity with two relaxation times.

In 1972, Green and Lindsay [3] proposed a system of equations of thermoelasticity with two relaxation times. This theory will now be referred to as G-L theory. Ignaczak has provided extensive research on G-L theory, and this thesis will focus on analyzing some of his results. The idea involves developing a central operator in G-L theory (which will include two thermal relaxation times), and using it to derive a closed form solution for the coupled behavior of such quantities as displacement, heat flux, temperature, and stress. The first two quantities were generated in his paper with Mrówka-Matejewska in 1990 [6], and the latter two in his paper with Hetnarski in 1993 [7].

An advantage of G-L theory over L-S theory is that in G-L theory, the classical Fourier Law of heat conduction is not violated. However, the free energy has been modified in comparison to the classical case, to include the temperature-rate as an independent variable. As a result, the classical Second Law of Thermodynamics (the dissipation inequality) has to be interpreted in a generalized sense in this theory.

C. Existence of relaxation times.

A thermal relaxation time, referred to by Rubin as the "nonnegative constant controlling the correction for hyperbolic heat conduction" [10], can be physically interpreted as the "time
lag needed to establish the steady state of heat conduction in an
element of volume when a temperature gradient is suddenly imposed
on that element" [12].

Generalized thermoelasticity is often referred to as
"thermoelasticity with second sound". The equations allow for two
distinct and separate wave fronts (thus, the "second sound") to
propagate through the medium. This is clearly shown in the graphs
in Section VI. The existence of one or more thermal relaxation
times, and second sound, has been associated with the fact that all
solid continua exhibit phonon-type excitations [2]. Phonons, or
quanta of vibrational energy, can be thought of as the transport
mechanism for thermal energy. These phonons will undergo
collisions of a dissipative nature, giving rise to a thermal
resistance [2]. The average communication time between these
collisions will be the thermal relaxation time. (A more thorough
investigation into the physics of the matter is beyond the scope of
this thesis.)

Experimentation with thermal pulses in solid sodium fluoride
[13] and bismuth [14], has shown the existence of second sound
phenomena at very low temperatures (below 20° Kelvin).
Chandrasekharaih provides some more details on experimentation in
[12]. Values of $\tau$ range from $10^{-10}$ seconds for gases to $10^{-14}$
seconds for metals. This implies that the additional terms
introduced by generalized thermoelasticity should not be neglected
when elapsed time is less than about $10^{-5}$ seconds, or when the heat
flux is greater than around $10^5$ W/cm$^2$ [12]. So, areas such as
nucleate boiling, crystalline temperature change in exothermic
catalytic reactions, laser penetration and welding, explosive
bonding and melting, and cold-temperature experimentation [12],
could all benefit from a thorough investigation of exact prediction
of heat conduction and the other thermoelastic quantities.
IV. Generalized (G-L) Solution

A. The Central Operator.

Consider an infinite body of the G-L type. The material is an isotropic, homogeneous solid, initially at rest and at uniform temperature (see [6]). An instantaneous heat source in the form of a plane is introduced, producing a propagating one-dimensional thermoelastic signal.

The central operator developed by Ignaczak is

$$\Gamma = \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right) \left( \frac{\partial^2}{\partial x^2} - t_0 \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} \right) - \varepsilon \left( 1 + t^0 \frac{\partial}{\partial t} \right) \frac{\partial^3}{\partial x^2 \partial t} .$$  (15)

Here, $t_0$ and $t^0$ are the two thermal relaxation times, and $\varepsilon$ is the thermoelastic coupling coefficient. The following examination of
Γ, applied to the scalar potential $\Phi = \Phi(x,t)$ described in [4], reveals the operator's characteristics:

1. If $\varepsilon = 0$, and

$$\Phi_1 = \left( \frac{\partial^2}{\partial x^2} - t_0 \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} \right) \Phi$$

then

$$\Gamma \Phi = \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right) \Phi_1 = 0 . \quad (16)$$

This equation describes one-dimensional isothermal waves propagating with speed $v = 1$.

2. If $\varepsilon = 0$ and $t_0 = 0$, and

$$\Phi_2 = \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right) \Phi$$

then

$$\Gamma \Phi = \left( \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t} \right) \Phi_2 = 0 . \quad (17)$$

This equation describes a one-dimensional diffusive heat conduction process.

3. If $\varepsilon = 0$, $t_0 > 0$, and
\[ \Phi_3 = \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial \tau^2} \right) \Phi \]  

(20)

then

\[ \Gamma \Phi = \left( \frac{\partial^2}{\partial x^2} - t_0 \frac{\partial^2}{\partial \tau^2} - \frac{\partial}{\partial \tau} \right) \Phi_3 = 0 . \]  

(21)

This equation describes a one-dimensional thermal wave propagating with velocity \( t_0^{(-1/2)} \), and with one thermal relaxation time introduced. This is L-S theory for rigid bodies.

4. If \( t_0 = t^0 = 0 \), but \( \varepsilon > 0 \), then we have

\[ \Gamma \Phi = \left[ \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial \tau^2} \right) \left( \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial \tau} \right) - \varepsilon \frac{\partial^3}{\partial x^2 \partial \tau} \right] \Phi = 0 . \]  

(22)

This is a hyperbolic-parabolic type of equation, and it describes one-dimensional thermoelastic disturbances. Note that solutions to (22) are not waves with clear fronts, as in case 1.

5. If \( t^0 = 0, \varepsilon > 0 \), and \( t_0 > 0 \), then

\[ \Gamma \Phi = \left[ \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial \tau^2} \right) \left( \frac{\partial^2}{\partial x^2} - t_0 \frac{\partial^2}{\partial \tau^2} - \frac{\partial}{\partial \tau} \right) - \varepsilon \frac{\partial^3}{\partial x^2 \partial \tau} \right] \Phi = 0 . \]  

(23)

This equation describes one-dimensional thermoelastic waves with two wave fronts (second sound), such that both temperature and mechanical deformation fronts propagate with finite speeds. This is L-S theory for a deformable solid.
6. For the final case, if \( t^0 \geq t_0 > 0 \), and \( \varepsilon > 0 \), the equation

\[
\Gamma \Phi = \left[ \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right) \left( \frac{\partial^2}{\partial x^2} - t_0 \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} \right) - \varepsilon \left( 1 + t^0 \frac{\partial}{\partial t} \right) \frac{\partial^3}{\partial x^2 \partial t} \right] \Phi = 0 \quad (24)
\]

describes one-dimensional thermoelastic waves with two wave fronts propagating at finite speeds. This is G-L theory for a deformable solid, the focus of this thesis. Note that the field equations and associated solutions discussed in this and following sections are written in a dimensionless form (see Appendix A).

B. Solutions for Thermoelastic Fields.

Using this central operator, a solution is developed for displacement and heat flux [6], and for stress and temperature [7], for \( i=1,2 \):

\[
u^{(i)}(x, t) = \pm \text{sgn} x \frac{1}{2\Delta^{1/2}} \left\{ t^0 M_i^i \rho_i, t \right\} + \int_{\rho_i}^t M_i^i \rho_i, \tau \right\} \delta(\xi_i) \quad (25)
\]

\[
\theta^{(i)}(x, t) = \frac{1}{4} \left\{ N_i \rho_i, t \right\} \pm \left[ \alpha M_i \rho_i, t \right\} + \beta M_i \rho_i, t \right\} \delta(\xi_i) \quad (26)
\]

\[
S^{(i)}(x, t) = \pm \frac{1}{2\Delta^{1/2}} \left\{ t^0 v_i \exp(-h_i \rho_i) \delta(\xi_i) \right\} + \left[ M_i \rho_i, t \right\} + t^0 M_i \rho_i, t \right\} \delta(\xi_i) \quad (27)
\]

\[
q^{(i)}(x, t) = \frac{1}{4} \text{sgn} x \left\{ (1 \pm \beta) \exp(-h_i \rho_i) \delta(\xi_i) \right\} \pm \left[ N_i \rho_i, t \right\} \delta(\xi_i) \right\} + \beta M_i \rho_i, t \right\} \delta(\xi_i) \right\} \quad (28)
\]
$M_i$, $N_i$, $M_i$, and $N_i$ are the power series of Neumann's type associated with the wave-like operator $L_i$ occurring in the decomposition theorem of the G-L theory, in [4], p. 314, Theorem 5.2:

\[
M_i(p_i, t) = \exp(-h_i t) \left[ g^{(i)}_0(\zeta_i) + \sum_{n=1}^{\infty} \frac{(-\lambda_i)^n}{n!} \right] \\
\cdot \int_{\rho_i}^t g^{(i)}_n(t-s) A^{(i)}_{n-1}(p_i, s) \: ds \\
\tag{29}
\]

\[
N_i(p_i, t) = \rho_i \exp(-h_i t) \left[ A^{(i)}_1(p_i, t) + \sum_{n=1}^{\infty} \frac{(-\lambda_i)^n}{n!} \right] \\
\cdot \int_{\rho_i}^t h^{(i)}_n(t-s) A^{(i)}_{n-1}(p_i, s) \: ds \\
\tag{30}
\]

\[
M_i(p_i, t) = \nu_i \exp(-h_i t) \sum_{n=0}^{\infty} \frac{(-\lambda_i)^n}{n!} \\
\cdot \int_{\rho_i}^t g^{(i)}_n(t-s) A^{(i)}_{n}(p_i, s) \: ds \\
\tag{31}
\]

\[
N_i(p_i, t) = \nu_i \exp(-h_i t) \left[ A^{(i)}_0(p_i, t) + \sum_{n=1}^{\infty} \frac{(-\lambda_i)^n}{n!} \right] \\
\cdot \int_{\rho_i}^t h^{(i)}_n(t-s) A^{(i)}_{n}(p_i, s) \: ds \\
\tag{32}
\]

$A^{(i)}_n$, $g^{(i)}_n$, and $h^{(i)}_n$ are defined by:

\[
A^{(i)}_n(u, s) = \omega_i^n \left[ \sqrt{\frac{\omega_i}{s^2-u^2}} \right]^n I_n \left( \sqrt{\frac{\omega_i}{s^2-u^2}} \right) \\
\tag{33}
\]

\[
(n = -1, 0, 1, 2, \ldots, \quad s \geq u)
\]
\[
g^{(i)}_n = \exp(\alpha_i t) \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{2^k} \left(\frac{\alpha}{\beta}\right)^{n-k} J_{n+k}(\beta t)
\]
\[
h^{(i)}_n = \exp(\alpha_i t) \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{2^k} \left(\frac{\alpha}{\beta}\right)^{n-k} (n+k) \frac{J_{n+k}(\beta t)}{t}.
\]

\(M_i, N_i, M_i, \text{and } N_i\) (and thus \(u^{(i)}, \theta^{(i)}, S^{(i)}, \text{and } q^{(i)}\)) have wave-like characteristics, such as

\[
\text{velocity } v_{1,2} = \sqrt{2} \left(1 + t_0 + \varepsilon t^0 \pm \Delta^{1/2}\right)^{-1/2}, \quad (v_2 > v_1 > 0)
\]

\[
damping \quad k_{1,2} = \frac{1}{2} (1 + \varepsilon \mp \alpha \Delta^{1/2})
\]

\[
\text{convolution coefficient } \lambda_i = \frac{1}{4} \beta^2 \Delta^{1/2} v_i^2
\]

\[
\text{and attenuation } h_i = \frac{1}{2} k_i v_i^2.
\]

Also, the other intermediate functions used are:

\[
\alpha = -[(1 + \varepsilon)(t_0 + \varepsilon t^0) - (1 - \varepsilon)] \Delta^{-1} \quad \alpha = (1 + \varepsilon) \Delta^{-1/2}
\]

\[
\beta = 2\sqrt{\varepsilon} \left[1 + (1 + \varepsilon)(t^0 - t_0)\right]^{1/2} \Delta^{-1} \quad \beta = (t_0 + \varepsilon t^0 - 1) \Delta^{-1/2}
\]

\[
\Delta = (1 - t_0 + \varepsilon t^0)^2 + 4 \varepsilon t_0 t^0 \quad \omega_i = h_i^2 \mp \lambda_i
\]
\[ \zeta_i = \tau - \rho_i \quad (46) \quad \rho_i = |x| v_i^{*1} \quad (47) \]

\[ \alpha_i = \alpha + h_i \quad (48). \]

As time is varied, the displacement, temperature, stress, and heat flux will each undergo a jump at a cross section of \( x \), as indicated by the Heaviside function. Additionally, the heat flux and stress suffer a discontinuity at the cross section, due to the Dirac function incorporated in them.
V. Numerical Analysis of Generalized Solution

All of the equations were programmed using Fortran 77 on the VAX system at Rochester Institute of Technology (RIT). Program listings are in Appendix C. Numerical output from the programs was plotted using DIS8, also on the VAX.

For all numerical calculations, values of $t_0 = t^0 = 4.0$, $\varepsilon = 0.05$ were used. Also, a fixed cross section of $x$ at $x = 10.0$ was observed, as time $t$ was varied. These values are all identical to the ones chosen in [6], so that graphical comparisons can easily be made. (See Appendix A for why these particular values are used.) They allow for jumps in the fields to occur at the dimensionless times 9.689 and 20.643, when observing the cross section of space at $x = 10.0$. A comparison of the numerical results presented here and those given in [6] is given in Section VI.

A. Evaluation of functions $g_n^{(1)}(t)$, $h_n^{(1)}(t)$, $A_n^{(1)}(u,s)$.

An examination of the functions $g_n^{(1)}(t)$, $h_n^{(1)}(t)$, and $A_n^{(1)}(u,s)$ is necessary for several reasons. Their behavior at zero and infinity must be determined. Also, their behavior as $n$ grows large must be examined, since eventually a truncation of their series' will be necessary in order to program them.

The relations given for $g_n^{(1)}$ and $h_n^{(1)}$ are summations of constants, times $J_n$, the Bessel function of the first kind and of order $n$. To program these Bessel functions in Fortran, use was made of the International Mathematical Statistics Library (IMSL), available on RIT's VAX system. It contains over 1650 user callable subroutines for Fortran programs. The IMSL subroutine "BSJNS" was used to calculate the Bessel functions of the first kind.

In the $h_n^{(1)}(t)$ function, the function $J_n(\beta t)/t$ for $t = 0,$
will be computed using the relations

\[
\lim_{t \to 0} \frac{J_1(t)}{t} = \frac{1}{2}, \quad \lim_{t \to 0} \frac{J_{n\ast}(t)}{t} = 0. \quad (49)
\]

For the \(A_n^{(1)}(u,s)\) function, the IMSL subroutine "BSINS" was used to program the modified Bessel function of the first kind, \(I_n\). This relation allows for \(n = -1, 0, 1, 2, 3\ldots\). Use was made of the fact that

\[
I_n(x) = I_n(x) \quad (50)
\]

in the program, so that only positive values for \(n\) were used as input for "BSINS".

1. Evaluation at \(t = 0\).
\(g_n^{(1)}(t=0)=0\) for \(n > 0\), and \(g_n^{(1)}(t=0)=1\) for \(n=0\):

\[
g_0^{(1)}(0) = \exp(0) J_0(0) = 1 \quad (51)
\]

\(h_n^{(1)}(t=0)=0\) for \(n > 1\), and \(h_n^{(1)}(t=0)= -3.868\) for \(n=1\):

\[
h_1^{(1)}(0) = \exp(0) \left[ \frac{1}{2} \frac{\alpha}{\beta} + 0 \right] = -3.686 \quad (52)
\]

\(A_n^{(1)}(0,0)=0\) for all \(n\).

2. Evaluation for \(0 < t < \infty\).
A criteria for a minimum was used to determine at what value
of \( n \) the summation in functions \( g_n^{(i)}(t) \) and \( h_n^{(i)}(t) \) can be ignored. Using

\[
| g_n^{(i)}(t) | < \varepsilon, \quad \text{and} \quad | h_n^{(i)}(t) | < \varepsilon, \tag{53}
\]

with \( \varepsilon = 0.05 \) at \( i = 1 \) and \( \varepsilon = 0.005 \) at \( i = 2 \) for both \( g_n^{(i)}(t) \) and \( h_n^{(i)}(t) \), orders of \( n > 6 \) can be ignored. So, the summations in these two functions will extend from \( k = 0 \) to \( k = n = 6 \).

Figures B-1 through B-6 in Appendix B show functions \( g_n^{(i)} \), \( h_n^{(i)} \), and \( A_n^{(i)} \), varied over time, at a fixed cross section \( x = 10.0 \).

3. Evaluation as \( t \to \infty \).

As \( t \) gets very large, \( g_n^{(i)} \) and \( h_n^{(i)} \) both become zero, while \( A_n^{(i)} \) gets very large. Inspection of equations (29) through (32) shows how \( A_n^{(i)} \) is multiplied by either \( g_n^{(i)} \) or \( h_n^{(i)} \), and thus extremely large values are not reached.

Functions \( g_n^{(i)} \) and \( h_n^{(i)} \) were checked using Mathematica, to ensure that the Fortran programs and the IMSL subroutines were functioning properly. The Mathematica results were identical to the Fortran output. The Mathematica program and graphs are shown in Appendix D.

B. Evaluation of functions \( M_i \), \( N_i \), \( M_i \), and \( N_i \).

The evaluation of \( M_i \), \( N_i \), \( M_i \), and \( N_i \) is necessary, because they will, along with their derivatives, dictate the behavior of the functions given for displacement, temperature, stress, and heat flux. As in the last section, an evaluation been made to choose the value of \( n \) at which an accurate series truncation can be made. Their behavior at the jumps and as time grows large is also checked.

The four functions have the form of a fairly complicated series of integrals. To compute the integrals numerically, the IMSL subroutine "QDAGS" was used. This uses adaptive quadrature to compute integrals, and allows for functions which may have an end point singularity. So, it is well suited for these functions.
1. Evaluation at the initial jump for $i = 1$ and $i = 2$.

$M_i$, $N_i$, and $N_i$ undergo discrete positive jumps at $t = t_1 = 20.643$, and at $t = t_2 = 9.689$. ($M_i$ has no initial jump.) These jumps are associated with $i = 1$ and $i = 2$, respectively. Since the wave front for $i = 2$ reaches the cross section of $x$ more quickly, it is the wave with the higher velocity. Using equation (36) to calculate velocity, $v_1 = 0.484$ and $v_2 = 1.032$.

2. Evaluation at $t \to \infty$.

All four functions describe smooth curves which asymptotically tend to zero as $t$ goes to infinity. Figures B-7 through B-14 show $M_i$, $N_i$, $M_i$, and $N_i$ varied over time, with their values at the jumps noted.

C. Evaluation of the displacement, temperature, stress, and heat flux fields.

1. Evaluation at the jump points for $i=1$ and $i=2$.

Ignaczak and Mrówka-Matejewska [6], and Hetnarski and Ignaczak [7], have evaluated the jumps of these four fields at $t = t_1$ and $t = t_2$ in a closed form. The values of the jumps predicted by Equations (56) and (66) in [6], and (136), (137), (153), and (154) in [7], match those retrieved from the analysis program. The finite jumps exhibited by displacement and temperature, and by the reduced stress and reduced heat flux, are shown on Figures 4 through 15.

The phrase "reduced" is used above because the total stress and heat flux fields include an infinite Dirac delta value at the discontinuities. Figures 7 - 9 and 13 - 15, naturally, show only the reduced fields, i.e. the fields (26) and (28), in which the Dirac delta terms are ignored.
2. Evaluation as $t \to \infty$.

Figures 4, 5, and 6 clearly show the displacement $u^{(i)}(t)$ converging to a finite value as $t$ goes to infinity. For large times, $u^{(1)}(t) \to -0.476$, and $u^{(2)}(t) \to +0.476$, and the total displacement $u(t) \to 0$.

Each of the two components ($i = 1, i = 2$) for the heat flux, temperature, and stress fields individually tend asymptotically to zero as $t$ grows large.

The time derivative of $M_i$ in the equation for heat flux was numerically computed using the IMSL subroutine "DERIV". The first and second time derivatives of $M_i$, in the equations for temperature and stress, are given in Appendix B of [7]. The computation of these derivatives was checked completely, by comparing them with the numerical solution using the "DERIV" subroutine.
VI. Results

A. Displacement/temperature pair.

1. The displacement, shown in Figures 4 - 6, suffers jumps at the two consecutive times \( t_1 \) and \( t_2 \) \((t_2 < t_1)\). The first jump is positive, and the second is negative. These two jumps correspond to each wave front passing through the cross section \( x = 10.0 \). The faster wave, \( u^{(2)}(t) \), which reaches the cross section at \( t_2 = 9.689 \), is 12 times greater in magnitude than the slower wave, \( u^{(1)}(t) \). \( u^{(1)}(t) \) reaches the cross section at \( t_1 = 20.643 \). The amplitude of each wave individually progresses to \( \pm 0.4747 \) as time goes to infinity. This is as predicted by Equation (68) of [6], which states

\[
\bar{u}_w = \frac{1}{2} \Delta^{-1/2} (\alpha^2 + \beta^2)^{-1/2}
\]  

(54)

The total displacement field \( u(t) = u^{(1)}(t) + u^{(2)}(t) \), therefore, reaches zero, as time goes to infinity. This total displacement is shown, varied over time, in Figure 6.

This graphical result differs somewhat from that given in [6]. Although the jump values at \( t_1 \) and \( t_2 \) are identical to those shown in [6], the displacement behavior of \( t > t_1 \) is different. Also, in [6], the value of \( \bar{u}_w \) was calculated to be \( \pm 0.2756 \), while here it is correctly calculated to be \( \bar{u}_w = \pm 0.4747 \). A comparison of the results with those shown in [6] is included in Figures 4-6.

2. The temperature field, \( \theta^{(1)}(t) \), is shown in Figures 7 - 9. It exhibits finite jumps at \( t_1 \) and \( t_2 \). The temperature jump at \( t = t_2 \) is one half that of the jump at \( t = t_1 \), and both jumps are positive in value. They match the values predicted by Equations
(137) and (154) in [7], as mentioned in Section V. The amplitude of each wave, and the sum of the two waves, approach zero as \( t \) goes to infinity.

The total temperature field is shown in Figure 9. This graph clearly shows the two distinct, positive jumps in temperature as the two thermoelastic waves pass through the cross section. (This field was not included in [6], so no comparisons are shown.)
Above graph is results from [6], at right is results from this thesis.

Figure 4.
Above graph is results from [6], at right is results from this thesis.

Figure 5.
Above graph is results from [6], at right is results from this thesis.

Figure 6.
Temperature component.

\( i = 1 \)

\[ \Theta(x, t) \]

(20.643, 0.0189)

\[ \Theta^{(1)} \rightarrow 0.0 \]

Figure 7.

29
Temperature component.

Figure 8.

(9.689, 9.31 \times 10^{-3})

\theta^{(2)} \rightarrow 0.0
Total temperature field,
B. Stress/heat flux pair.

1. The reduced stress field $S^{(i)}$ is defined in (115) of [7] as
   \[ \tilde{S}^{(i)}(t) = S^{(i)}(t) - S^{(i)}(t). \]
   It includes only the finite portion of the stress field defined in equation (27). So, it does not include the portion of the function associated with the Dirac function. $S^{(1)}(t)$ is shown in Figure 10, $S^{(2)}(t)$ in Figure 11, and their sum is shown in Figure 12. Similarly to the displacement field, the faster stress wave, $\tilde{S}^{(2)}(t)$, has a larger magnitude than the slower stress wave, $\tilde{S}^{(1)}(t)$. In fact, the jump of $\tilde{S}^{(2)}(t)$ at $t = t_2$ is 45 times greater than that of $\tilde{S}^{(1)}(t)$ at $t = t_1$, as shown in Figure 12. Each wave individually approaches zero as time goes to infinity. Also, the jump of $\tilde{S}^{(1)}(t)$ at $t_2 = 9.689$, is positive in value, and the jump at $t_1 = 20.643$, is negative. (The stress field was not included in [6], so no comparisons are shown.)

2. Like the reduced stress field, the reduced heat flux field, $\tilde{q}^{(i)}(t)$, does not include the portion of the function associated with the Dirac function. That is, $\tilde{q}^{(i)}(t) = q^{(i)}(t) - q^{(i)}(t)$. $\tilde{q}^{(1)}(t)$ is shown in Figure 13, $q^{(2)}(t)$ in Figure 14, and their sum is shown in Figure 15. $\tilde{q}^{(1)}(t)$ exhibits a greater amplitude at $t = t_1 = 20.643$. This amplitude is 1.5 times larger than that of the faster wave (which reaches the cross section at $t_2 = 9.689$).

This result appears to be more accurate than that shown in [6]. The results shown here match the jump values predicted by equations (56) and (60) in [6]. Figures 13-15 show the recalculated heat flux fields, compared to those given in [6].
Reduced stress component.

\[ S(x, t) \]

(20.643, \(-7.88 \times 10^{-4}\))

Figure 10.
33
Reduced stress component.

\( S(x, t') \)

Time

(9.689, 0.0356)
Reduced stress field.
i-1,2

Figure 12.
35
Above graph is results from [6], at right is results from this thesis.
Above graph is results from [6], at right is results from this thesis.

Figure 14.
Above graph is results from [6], at right is results from this thesis.
VII. Conclusion

An instantaneous plane heat source is introduced in an isotropic, homogeneous, infinite, thermoelastic body of G-L type, and the one-dimensional response of the body is examined. (Due to symmetry, only the response of the semi-space lying to the right of the plane source needs to be examined.) Using G-L theory, equations have been developed to characterize the thermoelastic response of the body. These coupled equations are generalized versions of classical linear thermoelastic theory. They include two relaxation times, and a thermoelastic coupling coefficient.

The response of the body, described by the displacement, temperature, stress, and heat flux fields, has the form of a sum of two plane waves propagating with finite speeds \( v_2 > v_1 > 0 \). In the numerical analysis, the velocity of the faster wave is \( v_2 = 1.032 \), and the slower wave travels at \( v_1 = 0.484 \). So, in this two-wave response, a "second sound" is revealed. The two waves travel from \( x = 0 \) towards \( x = +\infty \), with different velocities, magnitudes, and dampings.

When a fixed cross section of \( x \) is examined over time, each of the thermoelastic fields displays a jump. This jump corresponds to the arrival of each wave front at the cross section. The displacement field undergoes a positive, and then a negative jump. Each displacement component tends to a finite value as time goes to infinity, and the sum of the two components tends to zero. The temperature field undergoes two positive jumps at times \( t_2 \) and \( t_1 \), and each component tends to zero as time goes to infinity.

The stress and heat flux fields also exhibit jumps as each wave passes through the cross section. The reduced portions of these fields are graphically represented here; the total stress-heat flux response includes the Dirac functions indicated in equations (27) and (28).

The analysis of G-L theory presented here should be a helpful
foundation for further work in the area of temperature-rate dependent thermoelasticity. Research and experimentation in this field could prove to benefit many disciplines, such as laser engineering, low temperature physics, plasma coating processes, etc.
VIII. References


General References:


IX. Appendices
Appendix A

Dimensionless units used.
Thermoelastic coupling coefficient.

Values for the dimensionless thermoelastic coupling coefficient, \( \varepsilon \), have been estimated for a classical thermoelastic solid, i.e. when \( t_0 = t^0 = 0.0 \) [4]:

\[
\varepsilon = 3.56 \times 10^{-2} \text{ for aluminum},
\]

\[
\varepsilon = 2.97 \times 10^{-4} \text{ for steel},
\]

\[
\varepsilon = 1.68 \times 10^{-2} \text{ for copper},
\]

\[
\varepsilon = 7.33 \times 10^{-2} \text{ for lead}.
\]

Thermal relaxation times.

Values for thermal relaxation times, in dimensional units, have been estimated for rigid heat conductors in L-S theory (one thermal relaxation time) [4]:

\[
\tilde{t}_0 = 8.0 \times 10^{-12} \text{ for aluminum alloys at } 25^\circ \text{C},
\]

\[
\tilde{t}_0 = 1.6 \times 10^{-12} \text{ for carbon alloys at } 25^\circ \text{C},
\]

\[
\tilde{t}_0 = 1.5 \times 10^{-12} \text{ for uranium silicate at } 25^\circ \text{C},
\]

\[
\tilde{t}_0 = 2.0 \times 10^{-9} \text{ for liquid helium at } 0.25^\circ \text{K}.
\]

Since dimensional values for the two thermal relaxation times in G-L theory have not been estimated, the values defined above for L-S theory are used as an approximation. Then, the proper order of magnitude for the nondimensional parameters in this thesis' G-L theory problem can be used.

Using equations (5.5.25) and (5.5.26) in [4], the time unit is computed to be \( t_0' = 2.15 \times 10^{-12} \text{ s} \). The nondimensional thermal relaxation time for an aluminum alloy is then \( \tilde{t}_0 / t_0' = 3.72 \).

So, the nondimensional values used in the problem in this thesis are: \( \varepsilon = 0.05, t_0 = t^0 = 4.0 \).
Appendix B

Graphs of key equations.
Figure B-1.
This figure shows function $g_{n(i)}(t)$ for $n = 0$, and $i = 1, 2$. Note how $g_{0(1)}(t) = g_{0(2)}(t)$. $g_{0(i)}(\zeta)$ is responsible for the initial jump in the function $M_i$.

Figures B-2, B-3.
Function $g_{n(i)}(t)$ is shown here for $i = 1$ and $i = 2$. This function is a contributor to the $M_i$ and $M_i$ functions. Successive iterations of the summation in this function yield smaller and smaller contributions. For the final analysis, the series was truncated after $n = 6$.

Figures B-4, B-5.
Function $h_{n(i)}(t)$ is shown here for $i = 1$ and $i = 2$. We see that $h_{1(1)}(0) = h_{1(2)}(0) = -3.686$, as explained by equation (52). This function is a contributor to the jump in $N_i$ and $N_i$. As with $g_{n(i)}(t)$, the successive values of $n$ yield smaller and smaller contributions, so the series was truncated after $n = 6$.

Figure B-6.
This figure shows function $A_{n(i)}(u,s)$ for $i = 1$ and $i = 2$. This function grows very large as $t$ gets sufficiently large. But, it is multiplied by the very small values of $g_{n(i)}(t)$ and $h_{n(i)}(t)$. So, the product remains relatively small.

Figures B-7, B-8.
These figures show the function $M_i$ for $i = 1$ and $i = 2$. Note the sharp jumps at $t = t_1 = 20.643$ and $t = t_2 = 9.689$. This function is responsible for the finite jumps in the displacement field, and is partially responsible for the finite jumps in the heat flux field. The value of the jumps in $M_{i=1}$ and $M_{i=2}$ can be found using equation (55) of [6]. At $t_1$ and $t_2$, $M_i$ reduces to $M_i(t_i, t_i) = \exp(-ht_i)$. So, $M_1(t_1, t_1) = 0.079$ and $M_2(t_2, t_2) = 0.978$. $M_i$ approaches zero as $t$ approaches infinity.
Figures B-9, B-10.

Function \( N_i \) behaves similarly to \( M_i \). There is a finite positive jump for \( i = 1 \) and \( i = 2 \), at the times \( t_1 = 20.643 \) and \( t_2 = 9.689 \). \( N_2 \) has a greater damping than \( N_1 \) (like \( M_2 \) and \( M_1 \)), and this is reflected in the damping of the reduced heat flux waves. At the characteristic times, \( N_i \) can be reduced to: \( N_i(t_1, t_1) = 0.5 \omega_i t_i \exp(-h_i t_i) \). So, the value of the finite jumps of \( N_i \) are: \( N_1(t_1, t_1) = 0.012 \) and \( N_2(t_2, t_2) = 6.90 \times 10^{-3} \).

Figures B-11, B-12.

Function \( M_i \) is shown here for \( i = 1 \) and \( i = 2 \). It does not have any jumps. Its smooth curve, and its first and second derivatives, contribute to the stress and temperature functions. From equations (132) - (134) in [7], it follows that:

\[
\begin{align*}
M_i(t_1, t_1) &= 0, \\
M_i(t_1, t_1) &= v_i \exp(-h_i t_1), \\
M_i(t_1, t_1) &= v_i \exp(-h_i t_1) (\alpha + \omega_i t_1 - h_i).
\end{align*}
\]

So, the derivatives of \( M_i \) contribute to the finite jumps of stress and temperature at \( t_1 \) and \( t_2 \), while \( M_i \) itself does not. \( M_i \) and its derivatives tend to zero as \( t \) goes to infinity.

Figures B-13, B-14.

These figures show function \( N_i \) for \( i = 1 \) and \( i = 2 \). The finite jumps at \( t_1 = 20.643 \) and \( t_2 = 9.689 \) are partially responsible for the jumps in temperature at those times. Similarly to \( M_i(t_1, t_1) \), \( N_i(t_1, t_1) \) can be reduced to \( N_i(t_1, t_1) = v_i \exp(-h_i t_1) \). So, \( N_1 = 0.038 \) and \( N_2 = 1.009 \).
Function $g_i(t)$.

$n=0, i=1 \text{ or } 2$

Figure B-1.
Function $g(t)$.

$n=1, 2, 3, 4, \ldots, i-1$

Figure B-2.
Function $g_n^{(i)}(t)$.

$n=1,2,3,4, \ i=2$
Function $h_n(t)$.

$n=1, 2, 3, 4, \ i-1$

Figure B-4.
Function $h_n(t)$

$n=1,2,3,4, \ i=2$

Figure B-5.
Function $A^\omega_{n-1}(u,s)$.
$n-1, \, i=1,2$

Figure B-6.
Power series $N_i$: $10^{-2}$

$(9.689, 6.9 \times 10^{-3})$

Figure B.10.
Power series $M_i$.
Appendix C

Fortran programs.
FUNCTION ALPHAF

This function calculates Alpha.

REAL T0, T1, EPS
REAL DELTA, DELTAF
COMMON/BASICS/T0, T1, EPS, I

DELTA=DELTAF()
ALPHAF=-(1.0 + EPS) * (T0 + EPS * T1) - (1.0 - EPS))/DELTA

RETURN
END
FUNCTION ANF(RHOO, NN, T)

C

This function calculates A sub N for I = 1 or 2.
A can be used in the calculation of the M and N series.

INTEGER MAXSIZ
PARAMETER (MAXSIZ=20)

INTEGER I,NN
REAL T,RHOO,T0,T1,EPS
REAL ROOT,BESSI(MAXSIZ)
REAL OMEGAF
COMMON/BASICS/T0,T1,EPS,I

EXTERNAL BSINS

C

Ensure that (TIME**2) is greater than (RHOO**2),
or ROOT will be imaginary, which can’t be.
In that case, An = 0.
IF ((T**2) .LE. (RHOO**2)) THEN
   ANF = 0.0
   GO TO 900
ELSE
   ROOT= SQRT(OMEGAF() * (T**2-RHOO**2))
ENDIF

C

Note that I sub (-n) equals I sub (n), so calculate accordingly,
using the Absolute Value of (n).

CALL BSINS(ROOT,ABS(NN)+1,BESSI)

C

Calculate A sub N.
ANF=(OMEGAF()**(-NN))*(ROOT**NN)*BESSI(ABS(NN)+1)

900 RETURN
END
FUNCTION BETAF

This function calculates BETA.

REAL T0, T1, EPS
REAL DUMMY1, DELTA, DELTAF
COMMON/BASICS/T0, T1, EPS, I

DELTA = DELTAF(T0, T1, EPS)
DUMMY1 = 1.0 + (1.0 + EPS) * (T0 - T1)
BETAF = 2.0 * SQRT(EPS) * SQRT(DUMMY1) / DELTA

RETURN
END
FUNCTION BICOF(N,K)

PURPOSE: This function calculates the binomial coefficient of N and K. It does this by simple division; that is, no logarithms are used.

ARGUMENTS:
N: An integer, N => K => 0.
K: An integer, K => 0.

FACTLF: A function which calculates a factorial.

INTEGER N,K
REAL FACTLF

BICOF = FACTLF(N) / FACTLF(K) / FACTLF(N-K)

RETURN
END
FUNCTION DAMPF

This function calculates $K_{sub\ i}$. $i$ should equal either 1 or 2. $K$ is the damping coefficient.

INTEGER I
REAL T0, T1, EPS
REAL ALPHA, ALPHAF, DELTA, DELTAF
COMMON/BASICS/T0, T1, EPS, I

Calculate Alpha and Delta.
ALPHA = ALPHAF()
DELTA = DELTAF()

Now calculate $K_{sub\ i}$.
IF (I .NE. 2) THEN
    IF $i = 1$ (or 2), then compute $K_{sub\ 1}$ (or 2).
    DAMPF = $(1.0 + EPS - ALPHA * SQRT(DELTA)) / 2.0$
ELSE
    DAMPF = $(1.0 + EPS + ALPHA * SQRT(DELTA)) / 2.0$
ENDIF

RETURN
END
FUNCTION DELTAF

C This function calculates delta.

REAL T0,T1,EPS
REAL DUMMY
COMMON/BASICS/T0,T1,EPS,I

DELTAF= (1.0 - T0 + EPS * T1)**2 + 4.0 * EPS * T0 * T1

RETURN
END
REAL FUNCTION FACTLF(N)

C This function computes the value of N! (N factorial).

INTEGER N
INTEGER COUNT, DUMMYA(0:50)

C Set the first element of DUMMYA, which is zero factorial, to 1.
DATA DUMMYA(0)/1/

C Check to see if a negative value was entered.
IF (N .LT. 0) THEN
   PRINT *, ''
   PRINT *, 'ERROR: A NEGATIVE VALUE WAS INPUT IN THE FACTORIAL', *
     ' FUNCTION. NO FACTORIAL WAS COMPUTED.'
   PRINT *, ''
ELSE
   DO 100 COUNT=1,N,1
      DUMMYA(COUNT) = COUNT * DUMMYA(COUNT-1)
100 CONTINUE

C Compute N Factorial.
FACTLF=DUMMYA(N)

ENDIF

RETURN
END
FUNCTION GNF(T,NN)

C GNF calculates G sub N for I equals 1 or 2. G sub N can be used in calculating the series for M.

INTEGER MAXSIZ
PARAMETER (MAXSIZ=50)

INTEGER I,NN,K,L
REAL T0,T1,EPS,T
REAL ALPHAI,SUM,ALPHA,BETA,BETIME,BESS(MAXSIZ)
REAL ALPHAF,HF,BICOF,BETAF
COMMON/BASICS/T0,T1,EPS,I

EXTERNAL BSJNS

BETA=BETAF()
ALPHA=ALPHAF()
ALPHAI=ALPHA + HF()
BETIME=BETA * T

If N=0, there is no need to compute the Bessel Function, because J(order=0) for any argument equals 1.0.

IF (NN .EQ. 0) THEN
  SUM = 1.0
  GO TO 120
ENDIF

Call the Bessel Function subroutine, store results in array Bess.
CALL BSJNS(BETIME,2*NN+1,BESS)

C Perform summation.
SUM = 0.0
DO 100, K=0,NN,1
    SUM= SUM + BICOF(NN,K) * ((-1.)**K) / (2.**K) * (ALPHA / BETA)**(NN-K) * BESS(NN+K+1)
100 CONTINUE

C Calculate G sub N.
120 GNF= EXP(ALPHAI * T) * SUM

RETURN
END
FUNCTION HF

C This function calculates $H_{sub\ i}$. $i$ should equal either 1 or 2.

INTEGER I
REAL T0,T1,EPS
REAL DAMPF,VELOCF
COMMON/BASICS/T0,T1,EPS,I

C Calculate $H_{sub\ i}$.
HF = DAMPF() * (VELOCF()**2) / 2.0

RETURN
END
FUNCTION HAMDAF

C This function calculates Lamda (hat) sub i. i should equal 1 or 2.
C The expression HAMDA is used for Lamda to avoid confusion with FORTRAN
C assigning Lamda (L) as an integer.

INTEGER I
REAL T0,T1,EPS
REAL BETAF,DELTAF,VELOCF
COMMON/BASICS/T0,T1,EPS,I

C Now calculate Lamda.
HAMDAF=(BETAF()**2) * (VELOCF()**2) * SQRT(DELTAF())/4.0

RETURN
END
REAL FUNCTION HNF(M,TIME)

C HNF calculates H sub N for I equals 1 or 2. H sub N can be used in calculating the series for N.

INTEGER MAXSIZ
PARAMETER (MAXSIZ=20)

INTEGER I,M,K,L
REAL T0,T1,EPSTIME
REAL ALPHAI,SUM,ALPHA,BETA,BETIME,BESS(MAXSIZ)
REAL ALPHAF,HF,BICOFF,BETA
COMMON/BASICS/T0,T1,EPST,IAS
EXTERNAL BSJNS

BETA=BETAF()
ALPHA=ALPHAF()
ALPHAI=ALPHA + HF()
BETIME=BETAF() * TIME

C Call the Bessel Function subroutine, store results in array Bess.
CALL BSJNS(BETIME,2*M+1,BESS)

C Inside the following summation:
C Check to see if TIME equals zero. If so, a separate computation must be made to avoid division by zero.
C Note that Jn(0)/0 = 0.5 for n=1,
C and Jn(0)/0 = 0.0 for n > 1.

C Perform summation.
SUM = 0.0
DO 100, K=0,M,1
   IF (TIME.GT.0.0) THEN
      SUM=SUM+BICOFF(M,K)*((-1.0)**K)/(2.**K)*
      *(ALPHA/BETA)**(M-K)*(M+K)*BESS(M+K+1)/
      *TIME
   ELSE
      IF ((M+K).GT.1) THEN
         SUM=SUM+0.0
      ELSE
         SUM=SUM+BICOFF(M,K)*((1.0)**K)/(2.**K)*
         *(ALPHA/BETA)**(M-K)*(M+K)/2.
      ENDIF
   ENDIF
100 CONTINUE

C Calculate H sub N.
HNF= EXP(ALPHAI * TIME) * SUM

RETURN
END
REAL FUNCTION INT1(S)

The purpose of this function is to supply an expression for the integrand
used in computing the series for the variable N (script). It is declared
"external" in the calling program, and is called by the integration
subroutines.

INTEGER I,N
REAL X,TIME1,S,T0,T1,EPS
REAL GNF,RHO,ANF
COMMON/BASICS/T0,T1,EPS,I
COMMON/POINT/X,X0
COMMON/INTVAR/N,TIME4
COMMON/RHOVAR/RHO

INT1 = HNF(N,TIME4-S) * ANF(RHO,N,S)

RETURN
END
REAL FUNCTION INT2(S)

The purpose of this function is to supply an expression for the integrand used in computing the series for the variable M. It is declared "external" in the calling program, and is called by the integration subroutines.

INTEGER I,N
REAL X,TIME4,S,T0,T1,EPS
REAL GNF,RHO,ANF
COMMON/BASICS/T0,T1,EPS,I
COMMON/POINT/X,X0
COMMON/INTVAR/N,TIME4
COMMON/RHOVAR/RHO

INT2 = GNF(TIME4-S,N) * ANF(RHO,N-1,S)

RETURN
END
REAL FUNCTION INT3(S)

The purpose of this function is to supply an expression for the integrand used in computing the series for the variable N. It is declared "external" in the calling program, and is called by the integration subroutines.

INTEGER I,N
REAL X,TIME1,S,T0,T1,EPS
REAL HNF,RHO,RHOF,ANF
COMMON/BASICS/T0,T1,EPS,I
COMMON/POINT/X,X0
COMMON/INTVAR/N,TIME4
COMMON/RHOVAR/RHO

INT3 = HNF(N,TIME4-S) * ANF(RHO,N-1,S)

RETURN
END
REAL FUNCTION INT4(S)

The purpose of this function is to supply an expression for the integrand used in computing the series for the variable M (script). It is declared "external" in the calling program, and is called by the integration subroutines.

INTEGER I,N
REAL X,TIME1,S,T0,T1,EPS
REAL GNF,RHO,RHOHATF,ANF
COMMON/BASICS/T0,T1,EPS,I
COMMON/POINT/X,X0
COMMON/INTVAR/N,TIME4
COMMON/RHOVAR/RHO

INT4 = GNF(TIME4-S,N) * ANF(RHO,N,S)

RETURN
END
REAL FUNCTION INT5(S)

The purpose of this function is to supply an expression for the integrand used in computing the series for MDOT (script). It is declared "external" in the calling program, and is called by the integration subroutines.

INTEGER MAXSIZ
PARAMETER (MAXSIZ=50)

INTEGER I,N,K
REAL X,TIME4,S,T0,T1,EPS ,gnnf
REAL GDOT,RHO,ANF,GNF,BESS(MAXSIZ)
REAL TEMP,ALPHA,ALPHAF,BETA,BETAF,ALPHAI
REAL HF,BETIME,SUM,BICOF

COMMON/BASICS/T0,T1,EPS,I
COMMON/POINT/X,X0
COMMON/INTVAR/N,TIME4
COMMON/RHOVAR/RHO

EXTERNAL BSJNS

ALPHA=ALPHAF()
BETA=BETAF()
ALPHAI=ALPHA+HF()
BETIME=BETA*(TIME4-S)

CALL BSJNS(BETIME,2*N+2,BESS)

SUM=0.0
DO 100, K=0,N,1
   SUM=SUM + BICOF(N,K) * ((1.0)**K) / (2.**K)*
   * (ALPHA/BETA)**(N-K) * (BESS(N+K)-BESS(N+K+2))
100 CONTINUE

TEMP=EXP(ALPHAI*(TIME4-S))*SUM

GDOT=ALPHAI*GNF(TIME4-S,N)+.5*BETA*TEMP

INT5 = GDOT * ANF(RHO,N,S)

RETURN
END
REAL FUNCTION MF(TIME2)

C The purpose of this function is to compute M, the power series.
C M is a component of u (displacement) and q (heat flux).

INTEGER I,N
REAL TIME2,TIME3,T0,T1,eps,ZETAM,ZETAF,X,H,INT2
REAL SUM1,SIGNN,HAMDA,HAMDAF,FACTLF,HF
REAL RHO,AINTEGRAL,ERROR,GNF
COMMON/BASICS/T0,T1,eps,I
COMMON/POINT/X,X0
COMMON/INTVAR/N,TIME3
COMMON/RHOVAR/RHO

EXTERNAL INT2,QDAGS

TIME3 = TIME2

C Compute whether or not a plus or minus sign is needed
IF (I .EQ. 1) THEN
  SIGNN = -1.0
ELSE
  SIGNN = 1.0
ENDIF

H=HF()
ZETAM=ZETAF(TIME2)
HAMDA=HAMDAF()

IF (TIME2 .LT. RHO) THEN
  MF = 0.0
  GO TO 900
ENDIF

C Compute summation
SUM1 = 0.0
DO 200, N=0,6
  CALL QDAGS(INT2,RHO,TIME2,0.0,0.0001,AINTEGRAL,ERROR)
  SUM1 = SUM1 + (SIGNN*HAMDA)**N *
           FACTLF(N)*AINTEGRAL  
200 CONTINUE

C Compute M
N = 0
MF = EXP(-1.0*H*TIME2) * (GNF(ZETAM,N)
    + RHO * SUM1)

900 RETURN
END
REAL FUNCTION MDDF(TIME2)

C The purpose of this function is to compute the second time
derivative of (script) M, the power series. NEWMDD is a component
of s (stress).

INTEGER I,N
REAL TIME2,TIME3,T0,T1,EPS,X,H
REAL SUM1,SIGNN,HAMDA,HAMDAF,FACTLF,HF,VELOC
REAL RHO,AINTTEGRAL,ERROR,GNF,VELOCF
REAL ANF,ANFDOT,SUM2,SUM3,M,MDOT
REAL ALPHAF,ALPHAI

COMMON/BASICS/T0,T1,EPS,I
COMMON/POINT/X,X0
COMMON/INTVAR/N,TIME3
COMMON/RHOVAR/RHO

EXTERNAL INT6,ANFDOTF,QDAG,DERIV

TIME3 = TIME2

C Compute whether or not a plus or minus sign is needed
IF (I .EQ. 1) THEN
   SIGNN = -1.0
ELSE
   SIGNN = 1.0
ENDIF

H=HF()
ALPHAI = ALPHAF()+H
HAMDA=HAMDAF()
VELOC=VELOCF()

IF (TIME2 .LT. RHO) THEN
   MDDF = 0.0
   GO TO 900
ENDIF

C Compute summation
SUM1 = 0.0
SUM2 = 0.0
SUM3 = 0.0
DO 200, N=0,6
   CALL QDAG(INT6,RHO,TIME2,0.0,.0001,1,AINTTEGRAL,ERROR)
   SUM3 = SUM3 + (SIGNN*HAMDA)**N / 
           FACTLF(N)*AINTTEGRAL
200 CONTINUE

C Compute NEW M
M = NEWMF(TIME2)
C Compute NEW MDOT
   MDOT = MDTF(TIME2)

ANFDOT = DERIV(ANFDOTF, 1, TIME2, .2, .1)

C Compute MDD
MDDF = -1.0*(H**2)*M - 2*H*MDOT + VELOC*
* EXP(-1.0*H*TIME2) * ( ANFDOT + ALPHAI*
* ANF(RHO, 0, TIME2) + 0.5*ALPHAF() * SIGNN*
* HAMDA*ANF(RHO, 1, TIME2) + SUM3 )

900 RETURN
END
REAL FUNCTION MDOTF(TIME2)

The purpose of this function is to compute the time derivative of (script) \( M \), the power series. MDOT is a component of s (stress) and temp (temperature).

```fortran
INTEGER I,N
REAL TIME3,TIME2,T0,T1,EPS,ZETAM,ZETAF,X,H,INT4
REAL SUM1,SIGNN,HAMDA,HAMDAF,FACTLF,HF,VELOC
REAL RHO,AINTEGRAL,ERROR,VELOCF
REAL ANF,ANFDOT,INT5,SUM2,NEWM,HEAVI

COMMON/BASICS/T0,T1,EPS,I
COMMON/POINT/X,X0
COMMON/INTVAR/N,TIME3
COMMON/RHOVAR/RHO

EXTERNAL INT4,INT5,QDAGS

TIME3 = TIME2

C Compute whether or not a plus or minus sign is needed
IF (I .EQ. 1) THEN
   SIGNN = -1.0
ELSE
   SIGNN = 1.0
ENDIF

H=HF()
ZETAM=ZETAF(TIME2)
HAMDA=HAMDAF()
VELOC=VELOCF()

IF (TIME2 .LT. RHO) THEN
   MDOTF = 0.0
   GO TO 900
ENDIF

IF (TIME2 .EQ. RHO) THEN
   MDOTF = VELOC*EXP(-H*TIME2)
   GO TO 900
ENDIF

C Compute summation
SUM1 = 0.0
SUM2 = 0.0
DO 200, N=0,6
   CALL QDAGS(INT4,RHO,TIME2,0.0,.0001,AINTEGRAL,ERROR)
   SUM1 = SUM1 + (SIGNN*HAMDA)**N / 
   * FACTLF(N)*AINTEGRAL
   CALL QDAGS(INT5,RHO,TIME2,0.0,.0001,AINTEGRAL,ERROR)
   SUM2 = SUM2 + (SIGNN*HAMDA)**N3 / 
   * FACTLF(N)*AINTEGRAL
200 CONTINUE
```
C  Compute NEWM
NEWM = VELOC * EXP(-1.0*H*TIME2) * SUM1

C  Compute NEW MDOT
MDOTF = (-1.0 * H * NEWM) + VELOC *
       EXP(-1.0*H*TIME2) * ( ANF(RHO,0,TIME2)+SUM2 )

       MDOTF=MDOTF*HEAVI(ZETAM)

C
900 RETURN
END
REAL FUNCTION NF(TIME2)

C

The purpose of this function is to compute N, the power series. N is a component of \( q \) (heat flux).

INTEGER I,N
REAL TIME3,TIME2,T0,T1,EPS,X,H,INT3
REAL SUM1,SIGNN,HAMDA,HAMDAF,FACTLF,HF
REAL RHO,AINTEGRAL,ERROR,ANF
COMMON/BASICS/T0,T1,EPS,I
COMMON/POINT/X,X0
COMMON/INTVAR/N,TIME3
COMMON/RHOVAR/RHO

EXTERNAL INT3,QDAGS

TIME3 = TIME2

C

Compute whether or not a plus or minus sign is needed
IF (I .EQ. 1) THEN
   SIGNN = -1.0
ELSE
   SIGNN = 1.0
ENDIF

H=HF()
HAMDA=HAMDAF()

IF (TIME2 .LT. RHO) THEN
   NF = 0.0
   GO TO 900
ENDIF

C

Compute summation
SUM1 = 0.0
DO 200, N=1,6
   CALL QDAGS(INT3,RHO,TIME2,0.0,.0001,AINTEGRAL,ERROR)
   SUM1 = SUM1 + (SIGNN*HAMDA)**N /
         FACTLF(N)*AINTEGRAL
200 CONTINUE

C

Compute N
NF = RHO * EXP(-1.0*H*TIME2) *
*   (ANF(RHO,-1,TIME2) + SUM1)

900 RETURN
END
REAL FUNCTION NEWMF(TIME2)

The purpose of this function is to compute (script) M, the power series. NEWM is a component of u (displacement), q (heat flux), s (stress), and temp (temperature change).

INTEGER I,N
REAL LOCAL,TIME3,TIME2,T0,T1,eps,X,H,INT4,NEWM
REAL SUM1,SIGNN,HAMDA,HAMDAF,FACTLF,HF
REAL RHO,AINTTEGRAL,ERROR,GNF,VELOCF

COMMON/BASICS/T0,T1,eps,I
COMMON/POINT/X,X0
COMMON/INTVAR/N,TIME3
COMMON/RHOVAR/RHO

EXTERNAL INT4,QDAGS

TIME3=TIME2

C Compute whether or not a plus or minus sign is needed
IF (I .EQ. 1) THEN
  SIGNN = 1.0
ELSE
  SIGNN = 1.0
ENDIF

H=HF()
HAMDA=HAMDAF()

IF (TIME2 .LT. RHO ) THEN
  NEWMF = 0.0
  GO TO 900
ENDIF

C Compute summation
SUM1 = 0.0
DO 200, N=0,6
    CALL QDAGS(INT2,RHO,TIME2,0.0,.0001,AINTTEGRAL,ERROR)
    SUM1 = SUM1 + (SIGNN*HAMDA)**(N)/
           FACTLF(N)*AINTTEGRAL
200 CONTINUE

C Compute NEWM
NEWMF = VELOCF() * EXP(-1.0*H*TIME2) * SUM1

NEWMF = NEWMF*HEAVI(ZETAF(TIME2))

900 RETURN
END
REAL FUNCTION NEWNF(TIME2)

C The purpose of this function is to compute N (script), the power series. N (script) is a component of \( \Theta \) (TEMP).

INTEGER I,N
REAL TIME3,TIME2,T0,T1,EPS,X,H,INT1,NEWN
REAL SUM1,SIGNN,HAMDA,HAMDAF,FACTLF,HF
REAL RHO,AINTTEGRAL,ERROR,ANF,HEAVI

COMMON/BASICS/T0,T1,EPS,I
COMMON/POINT/X,X0
COMMON/INTVAR/N,TIME3
COMMON/RHOVAR/RHO

EXTERNAL INT1,QDAGS

C Compute whether or not a plus or minus sign is needed
IF (I .EQ. 1) THEN
  SIGNN = -1.0
ELSE
  SIGNN = 1.0
ENDIF

H=HF()
HAMDA=HAMDAF()
VELOC=VELOCF()

IF (TIME2 .LT. RHO) THEN
  NEWNF = 0.0
  GO TO 900
ENDIF

C Compute summation
SUM1 = 0.0
DO 200, N=1,6
  CALL QDAGS(INT1,RHO,TIME2,0.0,.0001,AINTTEGRAL,ERROR)
  SUM1 = SUM1 + (SIGNN*HAMDA)**N /
   FACTLF(N)*AINTTEGRAL
200 CONTINUE

C Compute N
NEWNF = VELOC * EXP(-1.0*H*TIME2) *
   (ANF(RHO,0,TIME2) + SUM1)

900 RETURN
END
FUNCTION OMEGAF

C This function calculates Omega sub i. i should equal either 1 or 2.

INTEGER I
REAL T0,T1,EPS
REAL HF,HAMDAF
COMMON/BASICS/T0,T1,EPS,I

C Now calculate Omega sub i.
IF (I .NE. 2) THEN
    OMEGAF= HF()**2 - HAMDAF()
ELSE
    OMEGAF= HF()**2 + HAMDAF()
ENDIF

RETURN
END
This function computes the heat flux, $q$, resulting from a thermoelastic wave front. It is a function of distance ($x$) and time ($time$). $i$ denotes either wave 1 or wave 2.

```fortran
FUNCTION Q(TIME1)

INTEGER I
REAL X,TIME,T0,T1,EPS
REAL BRACKET,MF,NF,SIGNN
REAL H,HF,ZETAQ,ZETAF,RHO,RHOF
REAL HEAVI,HEAVISIDE,HALPHAF,HBETAF,DERIVM
COMMON/BASICS/T0,T1,EPS,I
COMMON/POINT/X,X0
COMMON/RHOVAR/RHO

EXTERNAL MF,DERIV

C Compute H, Zeta, and Rho
H=HF()
ZETAQ=ZETAF(TIME1)
RHO=RHOF()
HEAVISIDE=HEAVI(ZETAQ)

C Decide whether or not a plus or minus sign is needed
IF (I .EQ. 1) THEN
    SIGNN = 1.0
ELSE
    SIGNN = 1.0
ENDIF

C Compute the derivative of M
DERIVM = DERIV(MF,1,TIME1,.2,.1)

C Compute the part in brackets
BRACKET = NF(TIME1) + SIGNN * HALPHAF() * MF(TIME1)
          + SIGNN * HBETAF() * DERIVM

C Compute the heat flux, q.
IF (HEAVISIDE .ne. 0.0) then
    Q = 0.25 * ( BRACKET * HEAVISIDE )
ELSE
    Q = 0.0
ENDIF

RETURN
END
```
REAL FUNCTION RHOF

C This function calculates Rho sub i.

INTEGER I
REAL X,T0,T1,EPS
REAL VELOCF
COMMON/BASICS/T0,T1,EPS,I
COMMON/POINT/X,X0

C Compute Rho sub i
RHOF = ABS(X) / VELOCF

RETURN
END
FUNCTION S(TIME1)

C This function computes the STRESS, S, resulting from a
C thermoelastic wave front. It is a function of distance (x)
C and time (time). i denotes either wave 1 or wave 2.

INTEGER I
REAL X,TIME1,T0,T1,EPS
REAL BRACKET,MF,NF,SIGNN,DELTA,DELTAF
REAL H,HF,ZETAS,ZETA,T,RHO,TEMP
REAL HEAVY,HEAVI,MDOT,MDOTF,MDD,VELOC

COMMON/BASICS/T0,T1,EPS,I
COMMON/POINT/X,X0
COMMON/RHOVAR/RHO

EXTERNAL DERIV,MDOT

C Compute H, Zeta, and Rho
RHO=RHOF()
H=HF()
ZETAS=ZETA(TIME1)
DELTA=DELTAF()
HEAVY=HEAVI(ZETAS)

C Decide whether or not a plus or minus sign is needed
IF (I .EQ. 1) THEN
  SIGNN = 1.0
ELSE
  SIGNN = -1.0
ENDIF

MDOT=MDOTF(TIME1)*HEAVY
MDD=DERIV(MDOTF,1,TIME1,.2,.1)*HEAVY

C Compute the part in brackets
BRACKET = MDOT + T1 * MDD

C Compute the stress, S
S = SIGNN/2./(DELTA**0.5) * BRACKET
S = S * HEAVY

RETURN
END
FUNCTION T(TIME1)

C This function computes the TEMP CHANGE, T, resulting from a
C thermoelastic wave front. It is a function of distance (x)
C and time (time). i denotes either wave 1 or wave 2.

INTEGER I
REAL X,TIME1,T0,T1,EPS
REAL BRACKET,MF,NF,SIGNN,HALPHA,HBETA
REAL H, HF, ZETAT, ZETAF, RHO, VELOCF
REAL HEAVI, MDOTF, NEWMF, NEWNF
COMMON/BASICS/T0, T1, EPS, I
COMMON/POINT/X, X0
COMMON/RHOVAR/RHO

C Compute Zeta
ZETAT=ZETAF(TIME1)

C Decide whether or not a plus or minus sign is needed
IF (I .EQ. 1) THEN
   SIGNN = 1.0
ELSE
   SIGNN = -1.0
ENDIF

IF (TIME1 .EQ. RHO) THEN
   T = .25*(1+SIGNN*HBETA())*VELOCF()*EXP(-HF()**TIME1)
ENDIF

C Compute the part in brackets
BRACKET = HALPHAF() * NEWMF(TIME1) + HBETA() * MDOTF(TIME1)

C Compute the temp change, T.
T = 0.25 * ( NEWNF(TIME1) + SIGNN*BRACKET ) * HEAVI(ZETAT)

900 RETURN
END
FUNCTION U(TIME1)

This function computes the displacement, u, resulting from a thermoelastic wave front. It is a function of distance (x) and time (time). i denotes either wave 1 or wave 2

INTEGER I
REAL X, TAU, TIME1, T0, T1, EPS
REAL BRACKET, MF, DELTAF, SIGNN
REAL AINTEGRAL, ZETAU, ZETAF, ERROR
REAL HEAVI, RHO
COMMON/BASICS/T0, T1, EPS, I
COMMON/POINT/X, X0
COMMON/RHOVAR/RHO

EXTERNAL MF, QDNG

C Compute Zeta and Rho
ZETAU = ZETAF(TIME1)
RHO = RHOF()

C Decide whether or not a plus or minus sign is needed
IF (I .EQ. 1) THEN
    SIGNN = -1.0
ELSE
    SIGNN = 1.0
ENDIF

C Compute the integral
IF (TIME1 .GT. RHO) THEN
    CALL QDNG (MF, RHO, TIME1, 0.0, 0.0001, AINTEGRAL, ERROR)
ELSE
    U = 0.0
    GO TO 900
ENDIF

C Compute the part in brackets
BRACKET = T1 * MF(TIME1) + AINTEGRAL

C Compute the displacement, u.
U = SIGNN*HEAVI(ZETAU)/2./((DELTAF())**(0.5))*BRACKET

900 RETURN
END
FUNCTION VELOCF

C This function calculates velocity sub i.
C i should equal either 1 or 2.

INTEGER I
REAL T0, T1, EPS
REAL DELTA, DELTAF
COMMON/BASICS/T0, T1, EPS, I

DELTA = DELTAF()

IF (I .NE. 2) THEN
    IF i = 1 (or 2) then calculate Veloc sub 1 (or sub 2).
    VELOCF = SQRT(2.0) / SQRT(1.0 + T0 + EPS * T1 + SQRT(DELTA))
ELSE
    VELOCF = SQRT(2.0) / SQRT(1.0 + T0 + EPS * T1 - SQRT(DELTA))
ENDIF

RETURN
END
REAL FUNCTION ZETAF(TIME5)

This function calculates Zeta.

INTEGER I,N
REAL X,T0,T1,EPS
REAL RHO,TIME5
COMMON/BASICS/T0,T1,EPS,I
COMMON/POINT/X,X0
COMMON/RHOVAR/RHO

Compute zeta
IF (TIME5 .GE. RHO) THEN
   ZETAF = TIME5 - RHO
ELSE
   ZETAF = 0.0
ENDIF

RETURN
END
Appendix D

Check of functions $g_n^{(1)}(t)$ and $h_n^{(1)}(t)$ using Mathematica program and graphs.
Mathematica (VAX/VMS) 1.2 (July 5 1990) [With pre-loaded data]
by S. Wolfram, D. Grayson, R. Ma\, H. Cejt\nS. Omohundro, D. Ballman and S. Ripper
with I. Rivin and D. Withoff
- VT240/340 graphics initialized

In[1]:=

Out[1]= \{\text{math.record, stdout}\}

In[2]:=(**)

In[3]:=(**)

In[4]:=(*)

In[5]:=(**)

In[6]:=(**)

In[7]:=(*)

This program will make plots of the functions g and h.
Alpha, alphai, and beta are precalculated constants. (*)

In[8]:=(**)

In[9]:=(**)

In[10]:=
alpha = -0.313406;

In[11]:=
alphai = -0.190713;

In[12]:=
beta = 0.0405085;

In[13]:=
a = alpha; ai = alphai; b = beta;

In[14]:=(**)

In[15]:=(**)

In[16]:=

rhs[t_, n_] := Binomial[n, k] ((1 - t)^(k - n))/((a/b)^(n - k) BesselJ[n + k, b t])

In[17]:=
\[ g_{t, n} := \text{Exp}[a_i t \sum \text{rhs}[t, n], \{k, 0, n\}] \]

\[ h_{t, n} := \text{Exp}[a_i t \sum \text{rhs}[t, n] (n+k)/t, \{k, 0, n\}] \]

\[ \text{In[18]} := \]
\[ \text{In[19]} := \]
\[ \text{In[20]} := \]
\[ \text{In[21]} := \]
\[ \text{In[22]} := \]
\[ \text{In[23]} := \]
\[ \text{g1picture} = \text{Plot}\{g[t,1], g[t,2], g[t,3], g[t,4]\}, \{t,0,40\}, \]
\[ \text{AxesLabel} \rightarrow \{"t","g(t)"\}, \text{PlotLabel} \rightarrow \text{"Function g, with i=1"} \]

\[ \text{Out[23]} = \text{-Graphics-} \]

\[ \text{Display["g1file.pic", \%]} \]

\[ \text{Out[24]} = \text{-Graphics-} \]

\[ \text{In[25]} := \]
\[ \text{In[26]} := \]
\[ \text{h1picture} = \text{Plot}\{h[t,1], h[t,2], h[t,3], h[t,4]\}, \{t,0.001,40\}, \]
\[ \text{AxesLabel} \rightarrow \{"t","h(t)"\}, \text{PlotLabel} \rightarrow \text{"Function h, with i=1"} \]

\[ \text{Out[26]} = \text{-Graphics-} \]

\[ \text{Display["h1file.pic", \%]} \]

\[ \text{Out[27]} = \text{-Graphics-} \]

\[ \text{In[28]} := \]
\[ \text{In[29]} := \]
\[ \text{In[30]} := \]
\[ \text{In[31]} := \]
\[ \text{alpha}_i = -0.311099; \]
\[ \text{In[32]} := \]
\[ \text{ai} = \text{alpha}_i \]

\[ \text{Out[32]} = -0.311099 \]
In[33]:= 
(**)

In[34]:= 
(**)

In[35]:= 

g2picture = Plot[ {g[t, 1], g[t, 2], g[t, 3], g[t, 4]}, {t, 0, 40}, AxesLabel -> {"t", "g(t)"}, PlotLabel -> "Function g, with i=2", PlotRange -> {-0.2, 0.1} ]

Out[35]= -Graphics-

In[36]:= 
Display["g2file.pic", %]

Out[36]= -Graphics-

In[37]:= 
(**)

In[38]:= 

h2picture = Plot[ {h[t, 1], h[t, 2], h[t, 3], h[t, 4]}, {t, 0.001, 40}, AxesLabel -> {"t", "h(t)"}, PlotLabel -> "Function h, with i=2" ]

Out[38]= -Graphics-

In[39]:= 
Display["h2file.pic", %]

Out[39]= -Graphics-

In[40]:= 
(**)

In[41]:= 
(**)

In[42]:= 
Exit
Function $g$, with $\lambda = 1$

- $n=1$
- $n=2$
- $n=3$
- $n=4$

Figure D-1.
Function \( g \), with \( \nu = 2 \)

- \( n = 2 \)
- \( n = 4 \)
- \( n = 3 \)
- \( n = 1 \)

Figure D-2.
Figure D-3.
Function \( h \), with \( n = 2 \)

Figure D-4.