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Flanking numbers and its application to arankings of cyclic graphs

M. Daniel Short

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M. Daniel Short

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Approved by:

Dr. Darren Narayan, Principal Advisor

Prof. David Barth-Hart, Committee Member

Dr. Bonnie Jacob, Committee Member

Dr. Jobby Jacob, Committee Member

M. Daniel Short, Author

Department of Mathematics and Statistics
Rochester Institute of Technology
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1 Abstract

Given a graph G with a ranking function, $f : V(G) \to \{1, 2, \ldots, k\}$, the ranking is minimal if only if G does not contain a drop vertex. The arank number of a graph, $\psi_r(G)$, is the maximum k such that G has a minimal k-ranking. A new technique is established to better understand how to analyze arankings of various cyclic graphs, C_n. Then the technique, flanking number, is used to describe all arank properties of all cyclic graphs fully by proving the following proposition: $\psi_r(C_n) = \lfloor \log_2(n+1) \rfloor + \lfloor \log_2(\frac{n+2}{3}) \rfloor + 1$ for all $n > 6$.

2 Introduction

Definition A **labeling** $f\colon V(G) \to \{1, 2, \ldots, k\}$ is a k**-ranking** of a graph G if and only if $f(u) = f(v)$ implies that every $u - v$ path contains a vertex w such that $f(w) > f(u)$.

Let us first introduce some notation. Given a path, P_n, we use the notation $1-2-3-\ldots-k$ to represent the labels of each vertex in P_k. For a cyclic graph, C_n, we use the notation $1-2-3-\ldots-k-$ to represent the labels of each vertex in C_k. Note that in this case, there is a dash to the right side of k to illustrate that it cycles back to the first label in a cyclic graph. For example, the notation $1-2-1$ represents a path where the first and third vertex have a label of one, while the second vertex has a label of two.

Suppose we have a path with four vertices with the labels $1-2-1-3$. Then this path has a 3-ranking. However, if the path has the labels $2-1-2-3$, then the labeling is not a k-ranking, since the first vertex and the third vertex violate the definition of k-ranking. As a final example, a path with the labels $1-2-3-4$ is a 4-ranking of the path.

As an analogy, it might be helpful to think of all vertices on the path with a k-ranking as skyscrapers where no tower can see another tower of equal height. Shorter skyscrapers cannot see past the taller skyscrapers, which is why the labels $1-2-1-3$ on P_4 form a k-ranking, but the labels of $2-1-2-3$ do not.

Definition A ranking f is **minimal** if for all $x \in V(G)$ such that $f(x) > 1$, the function g defined on $V(G)$ by $g(z) = f(z)$ for z not equal to x, and $g(x) < f(x)$ is not a ranking.

This definition is from Ghoshal, Laskar, and Pillone [2]. Taking two valid labelings of P_4 from above, $1-2-1-3$ is minimal, but $1-2-3-4$ is not minimal. The latter is not minimal, since the labels of fourth vertex can be reduced such that we get the labels $1-2-3-1$.

To simplify the notation for the rest of the paper, we refer to k-ranking as "ranking", as the most of the time, k will only refer to the largest label of a given graph G.

Definition A ranking f has **drop vertex** x if the labeling defined by $g(v) = f(v)$ when $v \neq x$ and $g(x) < f(x)$ is still a ranking.

Taking the example of $1-2-3-4$, which is not a minimal ranking, we can say that the vertex with label of 3 is a drop vertex, since $1-2-1-4$ is another ranking. However, the ranking $1-2-1-3$, which is a minimal ranking, does not contain any vertex that is a drop vertex. It may not be surprising that there is a relationship between the minimal ranking and whether there is a drop vertex. Ghoshal, Laskar, and Pillone [2] proved that a ranking is minimal if and only if it contains no drop vertices. Further discussions about minimality of a graph can be found in an article from Issak, Jamison and Narayan [4]. For the purpose of this thesis, we assume that G has a minimal ranking unless noted otherwise.

There are two important varieties of minimal rankings – one with the smallest k, and one with the largest k. Recalling that k is the largest label in a ranking: the former is the **rank number of a graph**, $\chi_r(G)$, and the latter is the **arank number of a graph**, $\psi_r(G)$. As an example, let us take a cyclic graph, C_7. One possible ranking with the smallest k is $1-2-1-3-1-2-4-$, and the ranking with the largest k is $1-2-3-2-1-4-5-$. Note that the smallest k possible for a minimal ranking for C_7 is 4, and the largest k possible for a minimal ranking is 5.

Studies involving the rank number of a graph are motivated by its applications to designs of very large scale integration (VLSI) layouts [7], Cholesky factorizations and solution to Tower of Hanoi puzzle [6]. There is also a strong association between chromatic numbers and these rankings, with early results by Bodlaender et al. [1] showing that $\chi_r(P_n) = \lfloor log_2 n \rfloor + 1$. In addition to the rank number of a graph, the studies of arank number of a graph are motivated by the search for a certain bounding on possible chromatic numbers of a graph, especially since $\chi_r(G) \leq \psi_r(G)$. That is, if an arank of G is discovered, it might yield some new information about the bounds on the chromatic number of a graph.

3 Preliminaries

Definition For a graph G and a set $S \subseteq V(G)$, the **reduction** of G, denoted $G_S^\flat$, is a subgraph of G induced by $V - S$ with an extra edge uv in $E(G_S^\flat)$ if there exists a $u - v$ path in G with all internal vertices belonging to S.

Unless otherwise stated, the set S will consist of vertices labeled 1. To ease into notation for the rest of the paper, a reduction of G will always imply removal of all vertices with label 1 and $G_S^\flat$ will contain a ranking by which is produced by decrementing the remaining labels by 1. For example, given C_7, where the labels are $1-2-3-2-1-4-5-$, then the reduction, $(C_7)_S^\flat$, is $1-2-1-3-4-$. This can be taken further, using ideas from Laskar and Pillone [2], and defining the *reduction of the reduction* of G, using the notation $(G_S^\flat)_S^\flat$ [6]. The idea behind the redefinition is to show that we can take as many reductions as necessary to prove or establish a case.

Lemma 3.1. *Let G be a graph and let f be a minimal $\psi_r(G)$-ranking of G. Then a reduction of G yields a minimal $\psi_r(G_S^\flat)$-ranking of $G_S^\flat$.*

This lemma, from Ghoshal, Laskar, and Pillone's work [3], shows that if a ranking is minimal, then the reduction formed by removing all vertices with label 1 and decrementing all other labels will yield a graph with a minimal ranking. This result determines how one might be able to obtain a graph with minimal rankings from another graph with minimal rankings.

Definition Given a graph G, an **expansion** of G is a graph $G^\#$ such that $(G^\#)_S^\flat = G$.

Let us work with C_4 with the following labels, $1-2-1-3-$. An expansion of C_4 would yield $1-2-3-1-2-1-4-$, $1-2-1-3-1-$

$2-1-4-$, or $1-2-1-3-2-1-4-$. Note that in this case, expansion is limited to only raising all labels by one, then inserting some new vertices. However, it is possible to produce an expansion of an expansion of G. For aesthetic purposes, we will use *expand* to mean "produce an expansion of," and likewise for *reduce*, mutatis mutandis.

Lemma 3.2. *Let f be a minimal k-ranking of G with adjacent vertices u and v where $f(u) > 1, f(v) > 1$, and $f(u) \neq f(v)$. Let $G^{\#}$ be the graph created by subdividing (u, v) and inserting a vertex w between u and v. Then let the ranking $f^{\#}$ of $G^{\#}$ be defined so that $f^{\#}(w) = 1$ and $f^{\#}(x) = f(x)$ for all $x \neq w$. Then $f^{\#}$ is a minimal k-ranking of $G^{\#}$.*

Kostyuk and Narayan [6] presented this lemma as an approach in analysis of how one might be able to insert a vertex and still expect a minimal ranking of G. For example, suppose we have a C_5 with the following minimal ranking $1-2-1-3-4-$. Using the above lemma, we can insert a vertex with a label of 1 between the vertices with label 3 and 4 to get C_6 with a minimal ranking, $1-2-1-3-1-4-$. Kostyuk and Narayan [6] also presented the following:

Lemma 3.3. *Let f be a minimal k-ranking of G. A graph G' is created by subdividing edges of G and adding a set of vertices S that dominates G'. Then the labeling f' where $f'(x) = f(x) + 1$ for all $x \in V(G)$ and $f'(x) = 1$ for all $x \notin V(G)$ is a minimal (k+1)-ranking of G'.*

This lemma refers to how we might expand a graph G such that we still have a minimal $(k+1)$-ranking of G'. For example, let us use C_4 with minimal ranking $1-2-1-3-$. We expand this cyclic graph in accordance with above lemma first by raising the labels first to get $2-3-2-4-$, then inserting a set of vertices with label 1 such that it dominates each vertex to get $2-1-3-2-1-4-$. Note that it is possible to produce $1-2-3-1-2-4-$ as well.

However, this lemma does not tell us how we can dominate vertices in G', only that we must do so. In the next section, we will see a slight refinement of this lemma that makes it clear which vertices need to be dominated.

We take a look at three propositions that will establish many results in the upcoming sections:

Theorem 3.4. *Let $m \leq n$. Then $\psi_r(C_m) \leq \psi_r(C_n)$.*

This is the monotonicity property of arank number of cycles, proven by Kostyuk and Narayan [6], and it will serve as a major component for proving various properties about arank numbers of cyclic graphs.

Letting S_i denote the set of vertices labeled i in a ranking, Ghoshal, Laskar, and Pillone [2] presented the following lemma:

Lemma 3.5. *In any minimal k-ranking $|S_1| \geq |S_2| \geq \cdots \geq |S_k|$.*

That is, if there are more vertices with label 2 than vertices with label 1, then the rankings of graph G cannot be minimal. Simple, but quite powerful in our search for the arank numbers of various graphs. Kostyuk and Narayan [6] followed up with the following theorem:

Theorem 3.6. *For any minimal ranking of C_n, $|S_1 \cup S_2| \geq \frac{n}{2}$*

Later on in this paper, this theorem will have an impact on our choice of possible strategies for constructing graph with a minimal rankings.

Before moving on to the main part of the paper, we take a look at the main problem and the earlier results from many others' work. Given a cyclic graph, what is the arank number of cyclic graph of any length? This question is an open problem that has confounded many until now. Before going into details, let us introduce earlier works and their results. First, Kostyuk and Narayan [6] has produced results for certain families of cyclic graphs:

Lemma 3.7. *If $2^m - 2^{m-2} - 2 \leq n \leq 2^m - 2^{m-3} - 1$ then $\psi_r(C_n) = 2m - 2$.*

Lemma 3.8 (24)**.** *If $2^m - 1 \leq n \leq 2^m + 2^{m-2} - 1$ then $\psi_r(C_n) = 2m - 1$.*

As a follow up, Kostyuk and Narayan discovered a tight upper and lower bound of possible arank numbers of two families of cyclic graphs:

Lemma 3.9. *If* $2^m - 2^{m-3} \leq n \leq 2^m - 2$ *then* $\psi_r(C_n) = 2m-2$ *or* $2m-1$.

Lemma 3.10. *If* $2^m + 2^{m-2} \leq n \leq 2^m + 2^{m-1} - 3$ *then* $\psi_r(C_n) = 2m-1$ *or* $2m$.

However, this did not determine arank number for every cyclic graph, but it gave us new avenues to explore. The determination of arank number of two families of cyclic graphs is still an open problem. Prior to this paper, it is known whether all cyclic graphs with length between $2^m - 2^{m-3}$ and $2^m - 2$, inclusively, has an arank number of $2m-2$ or $2m-1$, but it is not known which graph has which arank number. Similarly for all cyclic graphs with length between $2^m + 2^{m-2}$ and $2^m + 2^{m-1} - 3$, inclusively, has arank number of $2m-1$ or $2m$.

Nathan Kaplan [5], a Ph.D student in mathematics at Harvard determined some of arank number of certain graphs in these families. He discovered the following fact: $\psi_r(C_{14}) = 6$, $\psi_r(C_{20}) = \psi_r(C_{21}) = 7$. As of February 2011, the smallest known case was $n = 28$. The paper will reveal how to solve for n = 28, 29, 30, and all other unknown cases. The technique introduced in the next section will enable us to solve this open problem of determining arank number of all cyclic graphs. Furthermore, this technique will uncover a very interesting property about the structure of arankings of cyclic graphs.

4 Introduction to Flanking Numbers

Definition Given a graph G with a minimal ranking, let G^+ be the graph where all labels of all vertices in G is raised by 1. We define the **flanking number** of a vertex to be 1 if its label cannot drop to label of 1 without violating the definition of a k-ranking in G^+, and 0 otherwise. More succinctly, let ζ be a function on a vertex v that defines its flanking number thus: $\zeta(v) \to \{0,1\}$. $\zeta(v) = 0$ if the label of v can drop to 1 in G^+, and 1 otherwise. This function can also act on a set of vertices, $\zeta(V(G)) \to \{0,1\}^n$ where n is the number of vertices in $V(G)$.

Given a cyclic graph C_7 with labels of $1-2-3-2-1-4-5-$, the function that defines flanking numbers, $\zeta(V(G))$ defines the flanking number of each vertex as follows: $(0-0-1-0-0-0-0-)$. Note the notation of using parentheses to help us note which is a ranking, and which is a set of flanking numbers.

Definition Given adjacent vertices w and u, with edge uw, we define **insertion** to be a process where by a vertex v is placed such that new edges vu and vw are formed, and edge uw is deleted.

In other words, we place a vertex on an edge between two vertices, splitting them. For the rest of the paper, we will consider all insertions to be insertions of vertices with label 1 into a raised-label graph, G^+, unless otherwise indicated. As an astute reader may have noticed, the insertion process is the exactly same as subdividing an edge into two edges and adding a new vertex. As a result, both processes can be used interchangably, but for the purposes of clarity, insertions will be used in the place of subdividing an edge. The reasons for this definition is made clearer by the next theorem.

Theorem 4.1. *Given a vertex v with $\zeta(v) = 0$ in a raised-label graph G^+, if v is not dominated by vertices with label 1 inserted into G^+, then the resulting $G^\#$ is not minimal.*

Proof. Since $\zeta(v) = 0$ in G^+, the vertex v is not flanked by anything. Let us move on to $G^\#$, and observe that if vertices with label 1 are not inserted next to v, the vertices adjacent to v will have labels greater than 1. Since $\zeta(v) = 0$, and no vertices adjacent to v have label 1, v can drop to a label of 1 without violating the definition of a ranking. Therefore, $G^\#$ is not minimal. $\square$

The significance of this theorem is that this provides us a method for understanding how arankings of various cyclic graph are obtained. Before we go into arankings, we need to understand how to construct rankings for cyclic graphs based on flanking numbers.

Proposition 4.2. *Let G be a graph with a minimal ranking and v be a vertex with $\zeta(v) = 0$. After an expansion of G, the flanking number of v in $G^{\#}$ is 0 if and only if v is not adjacent to at least two of the newly inserted vertices with label 1. Otherwise, $\zeta(v) = 1$.*

In other words, if vertex v has flanking number 0, it is not flanked by anything, and therefore, it can drop to label 1 in G^{+}. However, if we insert new vertices with labels of 1 in G^{+} to produce $G^{\#}$, and we happen to insert them such that at least two of those are adjacent to v, then v is "flanked" (hence the term "flanking number"), and v cannot drop to a label of 1 in any subsequent expansions.

Now we prove the proposition:

Proof. Proving in the forward direction, we shall prove by contrapositive: if $\zeta(v) = 0$ for some $v \in V(G)$, and during an expansion of G into $G^{\#}$, v is flanked by vertices with labels of 1 on either sides, then $\zeta(v) = 1$. This follows directly from the definition of a flanking number.

In the other direction, suppose that a vertex v has $\zeta(v) = 0$, which implies that v drops to label 1 in G^{+}. Now suppose that v does not gain vertices with label 1 on either side in the process of getting to $G^{\#}$. We apply Lemma 4.1 to show that $G^{\#}$ is not minimal. Now suppose only one vertex with label 1 is inserted on only one side of v. Since $\zeta(v) = 0$, one label of 1 is not sufficient to flank the vertex, as the v drops to label 1 in $G^{\#+}$. Therefore, v must have $\zeta(v) = 0$ in $G^{\#}$. □

Proposition 4.3. *In a given graph G, any vertex v such that $\zeta(v) = 1$ cannot become a vertex v such that $\zeta(v) = 0$ in $G^{\#}$ for any expansions of G.*

Proof. When a graph expands, all of labels of the vertices that the vertex is flanked by are raised by one, and all of the labels of the vertices that flanks the particular vertex will always remain above the labels of newly inserted vertices during the expansion of graphs. □

This proposition concerns the stability of flanking number. That is, once a vertex has a flanking number of 1, it will always be flanked by something in any future expansions, and thus will never become a vertex that can drop its label to 1 in subsequent expansions. The application of this idea becomes more evident later with flanking partition structure, but for now, we introduce some tools that let us readily identify flanking number for any given vertex on a graph G:

Proposition 4.4. *Given two adjacent vertices v and u on a path or a cyclic graph, if the label of v is smaller than the label of u, then $\zeta(v) = 0$*

Proof. Since the label of v is smaller than the label of u, u cannot flank v, as the label of u has no bearing on whether the label of v can drop. As a result, u cannot flank v, and thus $\zeta(v) = 0$. □

Corollary 4.5. *Adjacent vertices on a cyclic graph or a path cannot both have a non-zero flanking number.*

The above propositions motivate strategies for construction, especially Proposition 4.2. In fact, the idea of vertex domination of all vertices with flanking number 0 is the primary strategy for exploring different ways to construct arankings of cyclic graphs C_n from arankings of other cyclic graphs of lesser degree. Let us prove that C_7 has a unique labeling up to permutation of the top three labels through construction from scratch using flanking ideas. Before starting, let us define C_1 to be a single vertex with a label 1 where we begin construction of our graph, and let us define C_2 such that it is comprised of two vertices, u and v, and an edge, uv. For the rest, let us define C_n to be a normal cyclic graph.

The lone vertex of C_1 has flanking number of 0, and thus, only one possible way of inserting a vertex to produce C_2. With C_2, we have two distinct vertices, and $\zeta(C_2) = (0 - 0-)$. Since the flanking number of each vertex is zero, so the labels of both vertices can drop to label of 1 in C_2^+. The flanking numbers suggests that there are two possible insertion strategies - insert one vertex or two vertices with label one in G^+. In the former case, we have C_3 with labels $1 - 2 - 3-$ and $\zeta(C_3) = (0 - 0 - 0-)$. In the latter, we have C_4 with labels $1 - 2 - 1 - 3-$ and $\zeta(C_4) = (0 - 1 - 0 - 0-)$.

Working from C_3 to get to C_7, there are two possible strategies for domination of all zero flanking numbers, namely insertion of two or three vertices. These produce four possibilities:

$$1-2-1-3-4-$$
$$1-2-3-1-4-$$
$$2-1-3-1-4-$$
$$1-2-1-3-1-4-$$

The last case is different from all other cases, as it is C_6 where $\zeta(C_6) = (0-1-0-1-0-1-)$, and thus needs insertions of three vertices to dominate C_6 and is clearly not the route to reach C_7. Taking the three other cases, it is clear they differ only by the position of the top three labels, so we arbitrarily pick one, and demonstrate that we can get to C_7 by inserting only two vertices. Taking $1-2-1-3-4-$, we have $\zeta(C_5) = (0-1-0-0-0-)$. Given that C_5 has five vertices, we only can insert two more vertices into C_5^+ if there is any hope of reaching C_7. There is a unique way of dominating four vertices with flanking number of 0 with two insertions, and we get $1-2-3-2-1-4-5-$ for C_7.

Now we go back to the second possible expansion of C_2, i.e. C_4. We have the labels $1-2-1-3-$ and $\zeta(C_4) = (0-1-0-0-)$. Using Theorem 3.6, which states that at least half the vertices of C_7 must have label 1 or 2, it is clear that we cannot reach the arankings of C_7 from the aranking of C_4, but for the sake of understanding, we will check it anyway. It is clear that we need two insertions of vertices to dominate the three vertices with flanking number 0. However, it should be noted that the vertices produced by insertions will always have flanking number of 0 in the expanded graph. So if we try to flank a vertex v, two vertices with flanking numbers of 0 will take its place on either side of v. As a result, we cannot expect to insert one vertex in the expansion of C_6 to get minimal rankings of C_7 as it will violate Theorem 4.1. Since it is impossible to construct arankings of C_7 from arankings of C_4, the arankings of C_7 are unique up to the permutation of the top three labels.

A couple of observations above can be summarized in the following lemmas – one follows the nature of flanking numbers of newly inserted vertices, and the other concerns the monotonicity property of the number of vertices with flanking numbers of zero:

Lemma 4.6. *Given a graph G, if a vertex v with label 1 is inserted into the graph G^+ to produce an expansion of the graph, $G^\#$, then $\zeta(v) = 0$.*

Proof. Since vertex v has a label of 1, the lowest possible labeling in a ranking of any graph, vertex v cannot be flanked. $\square$

Lemma 4.7. *The number of vertices with flanking number of 0 in a cyclic graph with minimal ranking will always either remain equal or increase with subsequent new expansions of the graph.*

Proof. Applying Lemma 4.6, we know that all insertions will result in new vertices with flanking number 0. Note that it requires two insertions of vertices with flanking number 0 to force a vertex in G to have flanking number 1. Subsequent flankings will require one or two new vertices inserted, depending on which of the remaining vertices need to be flanked. As a result, it is impossible to have more flanked vertices than the number of vertices inserted into the graph, and thus, the number of vertices with flanking number of zero will always rise or remain equal – consider the example of C_2 to C_3, and C_2 to C_4 $\square$

Lemma 4.7 helps us understand how we construct minimal rankings of graphs from other minimal rankings graphs. When a vertex with label 1 is inserted, the number of vertices with flanking number of 0 either stays the same or increases. It increases only when the insertions occur next to a vertex that is not already dominated by another vertex. It stays the same only when the insertion of 1 is next to a vertex that is already dominated by another vertex.

Corollary 4.8. *The number of vertices insertions needed to dominate all vertices with flanking number of 0 on a cyclic graph will always remain equal or increase.*

Proposition 4.9. *Given a cyclic graph, vertices with flanking number 0 can be partitioned by vertices with non-zero flanking number. This in turn implies that we can focus on each partition independently of each other in expanded graphs.*

Proof. Given a cyclic graph G with two nonadjacent vertices, $u, v \in V(G)$, and $\zeta(u, v) = (1, 1)$, we observe that u and v will never contain flanking number 0 in any expansions of G. This is due to Proposition 4.3 about stability of flanking number. Now suppose we insert a vertex w into G^+, and we consider two possible paths from u to v – one that contains w and one that does not. Now observe that the label of w is smaller than the labels of both u and v, so it has no hope of flanking a vertex that is on the path uv without w. This implies that we can focus on each path segment between each vertex with flanking number 1 independently. □

The idea here is to make it simpler to approach the flanking number in approaches to discovering the aranking of larger cyclic graphs, such as C_{30}, or to understanding how to generate all possible arankings for C_{15}.

Corollary 4.10. *The number of vertices with flanking number 0 will always remain the same or increase in each partition.*

Proof. This corollary follows directly from Proposition 4.9 and Corollary 4.8, showing that inside each partition, the number of vertices that need to be dominated by insertions of vertices with label 1 will always increase or remain the same. □

Extending Theorem 4.1, and applying Lemma 4.9, we can use vertices with flanking number 1 to subdivide all vertices with flanking number 0 into sets of vertices that must be dominated by insertions of vertices into G^+. This allows us to look at each partition individually:

Lemma 4.11. *Given a cyclic graph G, if there exist m consecutive vertices with flanking number 0, then the expansion of the subgraph induced by m consecutive vertices must have at least $m/2$ insertions of vertices with label 1 in order to dominate all m vertices and achieve a minimal ranking for the expanded graph.*

Proof. If any vertex with flanking number 0 is not dominated by a vertex insertion during the expansion, then the graph $G^{\#}$ is not minimal due to Theorem 4.1. Also, since each vertex insertion is capable of dominating two vertices in the m consecutive vertices at the same time in a cyclic graph, then the number of insertions that will dominate all vertices is at least $\frac{m}{2}$. $\square$

This proposition gives us the tools that allow us to count roughly how many vertices we need to insert in order to dominate all vertices with flanking number 0 in order to preserve the minimality of a ranking.

Theorem 4.12. *All minimal rankings of cyclic graph C_n are constructed by expanding multiple times from C_1. Each expansion involves dominating all vertices with flanking number of 0 by inserting vertices with label 1.*

Proof. Suppose a minimal ranking of a cyclic graph is generated by dominating only some (not all) vertices with flanking number of 0 by inserting vertices with label 1. This immediately leads to a contradiction, as it was shown in Theorem 4.1 that in order for an expansion of a graph to have minimal ranking, all vertices with flanking number of 0 must be dominated. Now suppose a graph is not generated by expanding from C_1. If we reduce it to its lowest form, we will get either no label, which should not happen, and at least two vertices with the same highest labels, which violates the ranking rule. Therefore, all minimal cyclic graph C_n are constructed by expanding multiple times from C_1. $\square$

This implies that we can begin construction of all minimal cyclic graphs from a single vertex with label 1. This is the place from which we will begin to search for strategies that allows us to find arankings of cyclic graphs of various sizes.

5 Finding all arankings of C_{15}

A good start to understanding what kind of strategies we need to discover graphs is finding all possibilities for arankings of C_{15}. First, we know from previous works that it only can come from expanding C_7, since $\psi_r(C_7) = 5$

and $\psi_r(C_6) = 4$. In addition, we know from Theorem 3.6 that at least half of the vertices of C_{15} must have label 1 or 2. With these facts, we limit our search to C_7 as a starting point. We saw earlier that the labeling of C_7 is unique up to permutation of the top 3 labels. Now, for the first strategy, we try to insert three vertices with label 2 and five vertices with label 1 to obtain C_{15}, which means we want to insert three vertices with label 1 into C_7^+, then five vertices with label 1 into C_{10}^+. Let us look at one labeling of C_7 as follows:

$$3-2-1-4-5-1-2-$$

With $\zeta(C_7) = (1-0-0-0-0-0-0-)$, there is only one way of inserting three vertices with label 1 into C_7^+ such that all vertices with label 1 will dominate all vertices with flanking number zero. After insertion, we get the following labels and their respective flanking numbers:

$$\begin{gathered} 4-3-1-2-5-1-6-2-1-3- \\ (1-0-0-0-0-0-0-0-0-0-) \end{gathered}$$

Note that since all vertices inserted into C_7^+ did not dominate a single vertex twice, all flanking numbers except for the vertex with label 4 are 0. We now have 9 vertices with flanking number 0 in C_{10} that need to be dominated by the insertion of vertices with labels of 1 in C_{10}^+. There are 6 possible ways of dominating all 9 vertices with five 1's to get C_{15}, so we get the following cyclic graphs with minimal rankings:

$$\begin{gathered} 5-1-4-2-1-3-6-1-2-7-1-3-2-1-4- \\ 5-4-1-2-1-3-6-1-2-7-1-3-2-1-4- \\ 5-4-1-2-3-1-6-1-2-7-1-3-2-1-4- \\ 5-4-1-2-3-1-6-2-1-7-1-3-2-1-4- \\ 5-4-1-2-3-1-6-2-1-7-3-1-2-1-4- \\ 5-4-1-2-3-1-6-2-1-7-3-1-2-4-1- \end{gathered}$$

Now we try another strategy, where by we try to insert four vertices with label 1 and 4 vertices with label 2. This strategy implies that we first need to insert four vertices first to get C_{11}, then insert four more vertices in the second expansion to get C_{15}. There are ten ways of inserting four vertices with label 1 into C_7^+ such that they dominates all vertices with flanking number 0, and we shall investigate each:

a) 4-1-3-1-2-5-1-7-2-1-3- $\rightarrow$ (1-0-1-0-0-0-0-0-0-0-0-)
b) 4-1-3-2-1-5-1-7-2-1-3- $\rightarrow$ (1-0-0-0-0-1-0-0-0-0-0-)
c) 4-1-3-2-1-5-7-1-2-1-3- $\rightarrow$ (1-0-0-0-0-0-0-0-1-0-0-)
d) 4-1-3-2-1-5-7-1-2-3-1- $\rightarrow$ (1-0-0-0-0-0-0-0-0-0-0-)
e) 4-3-1-2-1-5-1-7-2-1-3- $\rightarrow$ (1-0-0-1-0-1-0-0-0-0-0-)
f) 4-3-1-2-1-5-7-1-2-1-3- $\rightarrow$ (1-0-0-1-0-0-0-0-1-0-0-)
g) 4-3-1-2-1-5-7-1-2-3-1- $\rightarrow$ (1-0-0-1-0-0-0-0-0-0-0-)
h) 4-3-1-2-5-1-7-1-2-1-3- $\rightarrow$ (1-0-0-0-0-0-1-0-1-0-0-)
i) 4-3-1-2-5-1-7-1-2-3-1- $\rightarrow$ (1-0-0-0-0-0-1-0-0-0-0-)
j) 4-3-1-2-5-1-7-2-1-3-1- $\rightarrow$ (1-0-0-0-0-0-0-0-0-1-0-)

Let us take each case and look at the number of labels with flanking number 0. Cases a,b,c,d,g,i, and j can be eliminated immediately, as there are nine or ten vertices with flanking numbers of zero, and we cannot dominate 9 vertices by inserting only 4 vertices on a cyclic graph. This leaves us with cases e,f, and h. We look at case e and h at the same time, and observe that the vertices with flanking number 0 are partitioned by vertices with flanking number 1 into three sets: 5 vertices, 1 vertex, and 2 vertices. Using lemma 4.11, we can look at each partition individually as a set of consecutive vertices with flanking number 0 separated by vertices with flanking number 1. For the partition with 5 vertices, we need 3 vertices to dominate all vertices in the partition. Also, one vertex insertion is needed for both of other partitions, bringing us to a total of five insertions of vertices with label 1 to dominate these partitions, thereby ensuring that the ranking would remain minimal. Therefore, these cases are impossible to use to get to C_{15}. Finally, we are left with only one case: f. This case has a set of vertices that are also partitioned by vertices with flanking number of 1, using lemma 4.9 in order to apply the ideas behind lemma 4.11 again. But this time, the vertices with flanking number of 0 are partitioned into 4 vertices, 2 vertices, and 2 vertices. For the respective sets, we only need 2 vertices with label 1 to dominate 4 vertices, and 1 vertex with label 1 to dominate each of the other 2 partitions, leaving us with case f as the only working case in this situation. After expanding, we get:

$$5-4-1-2-3-2-1-6-7-1-2-3-2-1-4-$$

With all the possibilities exhausted, we have shown that the below labelings of C_{15} are all the possible arankings of C_{15}, up to the permutation of the top three labels.

$$5-1-4-2-1-3-6-1-2-7-1-3-2-1-4-$$
$$5-4-1-2-1-3-6-1-2-7-1-3-2-1-4-$$
$$5-4-1-2-3-1-6-1-2-7-1-3-2-1-4-$$
$$5-4-1-2-3-1-6-2-1-7-1-3-2-1-4-$$
$$5-4-1-2-3-1-6-2-1-7-3-1-2-1-4-$$
$$5-4-1-2-3-1-6-2-1-7-3-1-2-4-1-$$
$$5-4-1-2-3-2-1-6-7-1-2-3-2-1-4-$$

The proof by constructing that there are only 7 possible labelings (up to permutation of the top 3 labels) presents an interesting potential for a strategy: the transformation of flanking number from zero to nonzero has a way of partitioning the graph into subsets by which we can independently focus on flanking number in each and come up with different strategies for determining arank of larger graphs.

Let us present a result from Kaplan's work [5]:

Theorem 5.1. $\psi_r(C_{14}) = 6$

It is proven by Kaplan that this is the case, but let us prove this again in the context of flanking numbers.

Proof. First, note that expanding the aranking of C_7 twice to get C_{14} is impossible, since the first expansion requires three vertex insertions to dominate C_7, producing C_{10}. Note that we cannot insert four vertices first, otherwise, the theorem 3.5 is violated since the second expansion will require three vertex insertions if there is any hope of reaching C_{14}. Looking at the cases of C_{10} from earlier, we have 9 consecutive vertices with flanking number 0. This implies that at least five vertex insertions are needed to dominate all vertices, annihilating any hope of producing $\psi_r(C_{14}) = 7$ from the arankings of C_7. Therefore, $\psi_r(C_{14}) = 6$. □

6 Arank of C_{30}

Proposition 6.1. $\psi_r(C_{30}) = 8$

Proof. Finally, let us look at C_{30}, as one of our main goals is to prove the arank number of this cyclic graph. In preparation for this proof, let us lay out the known facts about C_{30}. Using theorem 3.6, we know that at least half of the labels of C_{30} must be 1 or 2. As a result, we can focus only on C_{15} or smaller graphs. Since we know that the arank number of C_{14} is 6, this has the effect of limiting our strategies to only C_{15} in determining whether the arank number of C_{30} is 8 or 9. Since the arank number of C_{15} is 7, we want to use this to see if we can attain arank of 9 for C_{30}. First, we demonstrate that we can attain 8-ranking of C_{30} easily – simply insert 15 vertices into C_{15}.

We must consider all cases of C_{15} to make sure that we cover everything. To review, we have the following labels and their respective flanking numbers:

a) 5-1-4-2-1-3-6-1-2-7-1-3-2-1-4- $\rightarrow$ (1-0-0-0-0-0-0-0-0-0-0-0-0-0-0-)
b) 5-4-1-2-1-3-6-1-2-7-1-3-2-1-4- $\rightarrow$ (1-0-0-1-0-0-0-0-0-0-0-0-0-0-0-)
c) 5-4-1-2-3-1-6-1-2-7-1-3-2-1-4- $\rightarrow$ (1-0-0-0-0-0-1-0-0-0-0-0-0-0-0-)
d) 5-4-1-2-3-1-6-2-1-7-1-3-2-1-4- $\rightarrow$ (1-0-0-0-0-0-0-0-0-1-0-0-0-0-0-)
e) 5-4-1-2-3-1-6-2-1-7-3-1-2-1-4- $\rightarrow$ (1-0-0-0-0-0-0-0-0-0-0-1-0-0-)
f) 5-4-1-2-3-1-6-2-1-7-3-1-2-4-1- $\rightarrow$ (1-0-0-0-0-0-0-0-0-0-0-0-0-0-0-)
g) 5-4-1-2-3-2-1-6-7-1-2-3-2-1-4- $\rightarrow$ (1-0-0-0-1-0-0-0-0-0-0-1-0-0-0-)

First, observe that for all cases, we need to insert at least seven vertices to dominate all vertices with flanking number 0 in C_{15}, or we will not be able to get minimal rankings for C_{30}. Thus, our only possible strategy is to insert 7 vertices in the first expansion of C_{15}, and 8 vertices in the second expansion. Now, starting with case a and f: there are 14 vertices that need to be dominated. Since the only strategy is to insert 7 vertices in the first expansion, there is an unique domination of these fourteen vertices. This unique domination increases the number of vertex with flanking number 0 to 21, which implies that at least eleven additional vertices are needed to dominate the expansion of C_{15} in this case. Therefore, we cannot get a 9-ranking from these two cases.

The next cases, b and e, have two partitions of consecutive vertices with flanking number 0 which are separated by vertices with flanking number 1. In each case, the first partition has two vertices with flanking number 0, and the second partition has eleven vertices with flanking number 0. The first partition needs only one vertex to dominate it, and that implies that after the first expansion, we need two vertices to dominate this partition in order to get a cyclic graph with minimal ranking in the second expansion. Now we look at the second partition, which has 11 vertices that need to be dominated, and thus we need at least 6 vertex insertions. However, since 11 vertices need to be dominated, only one vertex can be dominated on the both sides, and thus the flanking number will be changed from 0 to 1 and we will have $11 + 6 - 1 = 15$ vertices that need to be dominated during the second expansion. This implies that we need 8 new vertices inserted to dominate that partition. Since we already need $8 + 2 = 10$ vertices, we can eliminate b and e in our quest for finding rankings of C_{30} where $k = 9$.

Looking at cases c and d, we have two partitions of consecutive vertices with flanking number 0. The partition with five consecutive vertices with flanking number 0 will require the insertion of three vertices to dominate all vertices. As for the partition with eight vertices, four vertices are required to dominate this particular partition. The insertion of four vertices into the partition of eight vertices is unique, which implies that there are twelve vertices with flanking number 0 in that particular partition in the expansion of C_{15}. In turn, this implies that six vertices are needed to dominate all vertices in this partition. However, when we look at the smaller partition, since it already needs 3 vertices to dominate for the next expansion, it is clear that the smaller partition needs at least 3 vertices to dominate completely. This leaves us needing at least 9 vertices for the second expansion, which makes it impossible to get to C_{30}.

Finally, we are left with the last case, g, which has three partitions with lengths 3, 6, and 3. It is clear that we need two vertex insertions to dominate the partitions with length 3, and three vertex insertions to dominate the partition of length 6. This gives us seven vertices for the first expansion. However, there is only one way to dominate the partition of length 6, and in the expansion of the C_{15}, we can see that the number of vertices with flanking number 0 in that particular partition increases to 9, which implies that it needs five vertices to dominate it completely. Since the two other partitions already need two vertices to dominate each, we are over 8 vertices needed to reach C_{30}. So this case fails as well.

There is no possible way to construct a minimal 9-ranking for C_{30}, and there exists a possible way to construct a minimal 8-ranking. Therefore, the arank of C_{30} is 8. $\square$

7 Strategies using Flanking Numbers

We note that above proof incorporated several strategies. Now the question is, how much more we can squeeze from the application of flanking numbers? First, observe that the amount of vertices with flanking number 0 will always increase, but there are cases where there are fewer flanking numbers of 0 in some expansions of a graph compared to other expansions of the same graph. Why does this happen? It is due to the vertex insertions that happen to fall on both sides of a vertex.

This observation motivates finding an intelligent approach to the problem. Now we establish a proposition:

Proposition 7.1. *Given $2m+b, b \in \{0,1\}$ consecutive vertices and $m+r+b$ (where $0 \leq r < m$) labels to insert in the graph during the expansion process, the maximum number of flankings that can occur is $2r+b$.*

Note that b reflects whether it is odd or even case. Now we move on to the proof:

Proof. Let us apply the pigeonhole principle for both the odd and the even case. First, observe that we are aiming to maximize the number of flankings possible, and that means we must insert two vertices on the inside of both the first vertex and the last vertex in the set of consecutive vertices with flanking number 0. Let us distribute $m + b - 2$ labels insertions such that every possible vertex with flanking number 0 is dominated. Note that in the $b = 1$ case, there will one flanking, as there will be one vertex that is dominated twice. Finally, we insert the remaining labels, and we get two flankings for each insertion. Therefore, the maximum number of flankings that can occur is $2r + b$. □

Proposition 7.2. *If the insertion of r vertices into a cyclic graph with m vertices with flanking number 0, creates s flankings, then the number of vertices that need to be dominated in the expansion of the expanded graph is $m + r - s$.*

Proof. Recall that flanking vertex v with flanking number 0 transforms v into a vertex with flanking number 1, removing it from the set of vertices with flanking number 0. So if we flank s vertices, then they cannot be included in the set of vertices with flanking number 0. All of the newly inserted vertices will have flanking number 0, so we add r to the number of vertices with flanking number 0, leaving us with $m + r - s$. □

The above two propositions give us a way of trying to minimize the number of vertices that we need to insert into the graph. After all, if we try to insert too many vertices, we will be prevented from accomplishing our quest for the aranking of a cyclic graph of a particular length. Also, note that if we do not insert enough vertices, we might actually be preventing ourselves from accomplishing arank status as well! It appears to be advantageous that we minimize how many insertions we make, maximize how many flankings we make, and minimize the number of vertices that need to be dominated in the next step.

As an example, let us look at a labeling of C_{15}:

a) 5-1-4-2-1-3-6-1-2-7-1-3-2-1-4 $\rightarrow$ 1-0-0-0-0-0-0-0-0-0-0-0-0-0-0

In this case, note that there is only one partition, with 14 vertices that need to be dominated. If we insert seven vertices to get to C_{22}, it is clear that the domination is unique, and also clear that there will be 21 vertices to dominate in the next expansion.

Now suppose we decide to insert between every single vertex except for those with non-zero flanking numbers. We will get thirteen vertex insertions, and 14 vertices to be dominated in the next expansion. In this case, since the domination is highly fragmented into many partitions, we will need 14 insertions total to dominate each partition in the next expansion, and we are already at C_{29}, with a 8-ranking, and it already does not appear to offer any strategy for finding arankings in any expansions.

To help us visualize how we can insert vertices and determine whether a vertex is flanked or not, we use the notation of a dot to represent a position in C_{15} where an insertion is desired:

$$(1-0-.-0-.-0-0-.-0-.-0-0-.-0-.-0-0-.-0-.-0-0-.-0-)$$

In this case, nine vertices are inserted into C_{15} to get C_{24}, and flanked five vertices in the process. This gives us $14+9-5=18$ vertices to dominate in the next round of expansion. This implies that we need at least nine vertex insertions in the next expansion. This seems much more efficient than both other methods, though it comes at small cost of increasing the length of cyclic graph.

In the context of arank, the above might not seem like much, but it compels a searching heuristic. First of all, the knowledge of a cyclic graph's arank number is useful, along with the Theorem 3.6.

As a brief interlude, since we are interested in partitions generated by vertices with flanking number 1, it becomes clear that we need a definition that simplifies our idea of how each partition functions:

Definition The **flanking partition structure** is a set of consecutive vertices with flanking number 0 separated by a vertex with flanking number 1. We establish that (m, r) notation on a cyclic graph implies that we have m consecutive vertices with flanking number 0, one vertex with flanking number 1, then r consecutive vertices with flanking number 0, then one vertex with flanking number 1.

For example, suppose we have the partition structure of one possible rankings of C_{15}, which is $(3, 6, 3)$. Then in this particular case, $\zeta(C_{15}) = (1-0-0-0-1-0-0-0-0-0-0-1-0-0-0-)$. Recall Proposition 4.9 which allows us to treat each partition independently; as in we can insert vertices in each partition and not worry about the other partitions until we need to include them in our analysis. We will use the above as a tool to help us determine the number of possible arankings of C_{31}.

First, we try to get to C_{31} through C_{22} from C_{15}, which means that we are inserting seven vertices and then inserting nine vertices. We work with the partition structure $(3, 6, 3)$, which means that we need to insert two vertices to dominate the first and the last partition, and insert three vertices to dominate the partition of length 6. That brings us to total of seven insertions in the first expansion. This presents us with the question of how to insert two vertices in the partition of length 3. Proposition 7.2 above suggests that we will have at least one flanking in both partitions if we are to maximize the number of flankings, thereby minimizing the number of vertex dominations in the next expansion. That suggests that the next partition structure after insertions is $(4, 9, 4)$. The partition of length 9 is predetermined, because the insertion of three vertices into the partition of length 6 is unique. Finally, the insertion of two vertices into each partition with length of three has only one possible way such that the insertion in the next expansion will require only two vertices per partition. This way involves one flanking, so after an insertion of two vertices, (3) will be transformed into $(2, 2)$. So the partition structure of the expanded graph is $(2, 2, 9, 2, 2)$, which implies we need to insert 1,1,5,1, and 1 vertices, respectively, bringing us to a total of 9 vertex insertions. After the insertions, we have six possible cases of C_{31}. How we arrive at six possible cases of C_{31} from this approach will be explained more clearly shortly.

Now we examine a completely different route, namely arriving at C_{31} via C_{23}. Let us look at the partition structure of $(3, 6, 3)$ again, and observe that if we try to put 3 labels in a 3 partition, we will not be able to achieve optimal strategy. The resulting partition structure would be $(1, 1, 1, 9, 2, 2)$, which implies $1 + 1 + 1 + 5 + 1 + 1 = 10$ insertions are needed to dominate all the vertices with flanking number 0 in the next expansion. This is not a way getting to C_{31}, as the number of required insertions exceeds the number of insertions determined for the second expansion. As a result, we must insert two vertices at both ends, and four vertices in the middle partition. As for the middle, Proposition 7.2 suggests that we should be able to get two flankings! Proceeding with the insertion (recall that we're doing the same thing with 3-partitions), we get $(3 + 2 - 1, 6 + 4 - 2, 3 + 2 - 1)$, or $([2, 2], [2, 4, 2], [2, 2])$ – partitions are grouped for clarity – which means we need two vertices for each partition on the ends, and four vertices for the middle partition, bringing the number of vertices needed to dominate this expansion to eight.

Notice that in the previous example, it is possible to simply use the language of flanking numbers to construct a graph, and also to help motivate the search for the arank number of a cyclic graph. It is a good exercise to use the ideas above, along with the seven cases delineated for C_{15} to demonstrate that there are only seven possible cases of C_{31} that can be reached from C_{15}. It should be noted that C_{16} is not a possible candidate to get to C_{31}, since C_{31} must be comprised by at least 16 labels of 1 or 2.

Our work thus far makes it easier for us to understand how to focus our efforts on each partition, as long as we focus on each partition independently. It is easier to focus on the number of vertices within a partition than focusing on each vertex. How we focus on each partition will become clearer with each proof of the Exhaustion Lemmas below. Before we move on to Exhaustion Lemmas, let us define the notation of "insertions" within a graph. Suppose one partition has three consecutive vertices with flanking number 0, then we write it out as:

$$0 - 0 - 0$$

Now suppose we wish to insert two vertices. Both insertions will occur between two 0's or at an end, and will be denoted with a dot. All possible insertions are as follows:

$$\begin{gathered} . - 0 - 0 - . - 0 \\ 0 - . - 0 - . - 0 \\ 0 - . - 0 - 0 - . \end{gathered}$$

This notation will help as we go through each Exhaustion Lemma. As to why they are called "Exhaustion Lemmas", notice that we exhausted all possible labelings for insertions of two vertices into the partition of size three. Also, notice that the first and last choice of insertion points have five vertices that need to be dominated in the subsequent expansion, but the second one has only four vertices that need to be dominated, yielding the partition structure of $(2, 2)$. This shows that different insertion strategies have a significant impact on future expansions. The Exhaustion Lemmas are designed to show precisely what the impact each partition in the partition structure has on the inserting vertices with label 1. Now we move on to the first one, which is already proven here.

Lemma 7.3. *Exhaustion Lemma* (3)*: Given a partition of length three, and two vertices to insert, then the optimal strategy is to produce the partition structure* $(2, 2)$*, as it will require only two vertices to dominate in the subsequent expansion. Otherwise, it will require three vertices to dominate all vertices in the subsequent expansion. Furthermore, if three vertices are inserted, then at least three vertices will be required to dominate all vertices in the subsequent expansion.*

Lemma 7.4. *Exhaustion Lemma* (6)*: Given a partition of length six, and three vertices to insert, then we will have nine vertices that need to be dominated by insertion of five vertices in the next expansion. However, if we have four vertices to insert into the partition structure, the optimal strategy is to insert such that we get the partition structure* $(2, 4, 2)$ *as it will require* *only four vertices to dominate in the subsequent expansion. Otherwise, it will require at least five vertices to dominate in the subsequent expansion.*

Proof. The case with three vertices inserted into the partition size of 6 is trivial, as each vertex insertion must dominate two vertices, and since we did not flank anything, we are left with partition structure of (9), which will require 5 vertices to dominate in the subsequent expansions. Now we turn to the insertion of four vertices. We exhaust over $(0-0-0-0-0-0)$, and give their respective partition structure based on the insertions:

$$(.-0-.-0-0-.-0-0-.-0) \to (1,8)$$
$$(.-0-0-.-0-.-0-0-.-0) \to (4,5)$$
$$(.-0-0-.-0-0-.-0-.-0) \to (7,2)$$
$$(.-0-0-.-0-0-.-0-0-.) \to (10)$$
$$(0-.-0-.-0-.-0-0-.-0) \to (2,1,5)$$
$$(0-.-0-.-0-0-.-0-.-0) \to (2,4,2)$$
$$(0-.-0-.-0-0-.-0-0-.) \to (2,7)$$
$$(0-.-0-0-.-0-.-0-.-0) \to (5,1,2)$$
$$(0-.-0-0-.-0-.-0-0-.) \to (5,4)$$
$$(0-.-0-0-.-0-0-.-0-.) \to (8,1)$$

Note that for all partition structures except for $(2,4,2)$, an insertion of five vertices are needed to dominate all vertices for the next expansion. With $(2,4,2)$, we only need four vertex insertions, and this insertion is unique. □

Lemma 7.5. *Exhaustion Lemma* (9)*: Given a partition of length nine, and five vertices to insert, there are six possible ways of inserting five vertices, giving us the following cases:* (14) *(twice),* $(2,11)$ *(permutable),* $(5,8)$ *(permutable).*

Proof. We start from the partition structure of (9), getting $(0-0-0-0-0-0-0-0-0)$. Now we attempt to insert five vertices with label 1, and give their respective partition structure.

$$(.-0-0-.-0-0-.-0-0-.-0-0-.-0) \to (14)$$
$$(0-.-0-.-0-0-.-0-0-.-0-0-.-0) \to (2,11)$$

$$(0 - . - 0 - 0 - . - 0 - . - 0 - 0 - . - 0 - 0 - . - 0) \to (5, 8)$$
$$(0 - . - 0 - 0 - . - 0 - 0 - . - 0 - . - 0 - 0 - . - 0) \to (8, 5)$$
$$(0 - . - 0 - 0 - . - 0 - 0 - . - 0 - 0 - . - 0 - . - 0) \to (11, 2)$$
$$(0 - . - 0 - 0 - . - 0 - 0 - . - 0 - 0 - . - 0 - 0 - .) \to (14)$$

As we can see, we have six partition structures, each of which we can handle independently for the next expansion. Intra-partition permutations brings the number of cases down to three.

□

Lemma 7.6. *Exhaustion Lemma for* (14),$(11, 2)$ *and* $(5, 8)$*: Given the partitions stated, inserting seven vertices with label 1 will require at least ten insertions of vertices in the next expansion. Also, inserting eight vertices will require at least nine vertices to insert in the next expansion.*

Proof. Apply Proposition 7.2 to determine the minimum number of vertices that can be inserted in the expansion, and we also can use Proposition 4.9 to do the same to each partition. We begin with (14) and seven vertices. Using the Proposition 7.2, we end up with (21), which implies that we need eleven vertex insertions to dominate the partition structure. Similarly, by inserting eight vertices, we can see that we get $14 + 8 - 2 = 20$ vertices that we need to dominate, resulting in a need to insert at least 10 vertices to dominate the expanded partition structure.

We move on to $(11, 2)$ and and insert seven vertices. First, observe that we need six vertices to insert into the partition of length 11 and one to dominate the partition of length two. This gives us $(11 + 6 - 1, 2 + 1)$, which means we need to dominate 16 vertices in one of the new partitions and three vertices in other. In total, we will need an insertion of at least eight vertices in one and two vertices in another. Therefore, we need at least ten vertices to dominate all in the next expansion. Looking at the insertion of eight vertices into $(11, 2)$, there are two possible ways: six vertices into (11) and two vertices into (2), or seven vertices into (11) and one vertex into (2). In the first case, we get $11 + 6 - 1$ vertices that need to be dominated along with two vertices, which implies that we need to insert at least ten vertices in the next expansion. In the other case, we get $11+7-3$ vertices that need to be dominated along with three vertices that also need to be dominated, which implies that we need $8+2$ vertex insertions to dominate all vertices.

Finally, we look at $(5,8)$ with seven vertices to insert. There is only one possible way of inserting seven vertices in this partition structure, which is three vertices into (5) and four vertices into (8). This results in $(5+3-1)$ and (12), which implies that we need $4+6$ vertices to dominate all vertices. Now we look at inserting eight vertices, and notice that there are two possible ways of inserting eight vertices into $(5,8)$. One way is to insert four vertices in (5), and four vertices in (8), and the other way is to insert three vertices into (5) and five vertices into (8). In the first case, we get $(5+4-3,12)$ which implies that we need at least nine vertices insertion in the next expansion. In the second case, we get $(5+3-1)$,$(8+5-2)$, which implies we need $4+6$ vertices to dominate all vertices in the next expansion. □

In order to fully prove the theorem, we will need to establish a special counting lemma, which will allow us to determine precisely the size of the cyclic graph that we are looking at by looking at the partition structure.

Lemma 7.7. *Given a set of partitions of vertices with flanking number 0 in a cyclic graph, the number of vertices with nonzero flanking number is equal to the number of partitions given. The number of vertices of a graph can be determined by adding the sizes of all partitions and the number of partitions in a graph.*

Proof. Each partition on a cyclic graph is separated by a vertex with non-zero flanking number. Thus, the number of vertices with non-zero flanking number in the flanking partition structure is determined by counting the number of partitions. Therefore, the number of vertices in a graph is determined by adding the lengths of all partitions and the number of partitions in a graph. □

8 Arank number of any size of Cyclic Graph

Now we are well on our way to proving arank number of any cyclic graphs. First, we prove the arank number of cyclic graphs that have size approximately equal to the average of two powers of 2.

Theorem 8.1. *Given $\psi_r(C_{2^m-2}) < \psi_r(C_{2^m-1})$ and the following partition structures of C_{2^m-1} for $m > 3$:*

(14, $(2^{m-2}-4)$ terms of 3's) (occurs twice)
(2,11, $(2^{m-2}-4)$ terms of 3's) (occurs twice)
(5,8, $(2^{m-2}-4)$ terms of 3's) (occurs twice)
(3,6,3, $(2^{m-2}-4)$ terms of 3's) (occurs once)

Then:

$$\psi_r(C_{2^m+2^{m-1}-3}) < \psi_r(C_{2^m+2^{m-1}-2})$$

Proof. Since $\psi_r(C_{2^m-2}) < \psi_r(C_{2^m-1})$, we will begin from $\psi_r(C_{2^m-1})$. We also take the partition structure as a given assumption.

Observe that we need to dominate all $2^{m-2}-4$ partitions of size 3, and we need two vertices insertions for each of the said partition, which implies that we need insertions of at least $2^{m-1}-8$ vertices. Finally, we look at each partition structures (14), $(2,11)$, $(5,8)$, and $(3,6,3)$ and we see that we need to insert at least seven vertices for each, giving us a total of at least $2^{m-1}-1$ vertex insertions into C_{2^m-1}, implying that we only can reach $C_{2^m+2^{m-1}-2}$. There is no way of reaching $C_{2^m+2^{m-1}-3}$ from C_{2^m-1}, since we need to dominate all vertices or the expansion does not have minimal ranking. Therefore, $\psi_r(C_{2^m+2^{m-1}-3}) < \psi_r(C_{2^m+2^{m-1}-2})$ for $m > 3$. □

This theorem is a small but important component of the next theorem, which has two parts – first, that a certain cyclic graph will always have distinct arankings, and distinct partition structures; second, that we can determine the structure of arank number in some cases of cyclic graphs.

Theorem 8.2. *a) C_{2^m-1} for $m > 3$, will always have seven possible instances with arank rankings (up to permutation of the top 3 labels), and the partition structures are as follows:*

(14, $(2^{m-2}-4)$ terms of 3's) (occurs twice)
(2,11, $(2^{m-2}-4)$ terms of 3's) (occurs twice)
(5,8, $(2^{m-2}-4)$ terms of 3's) (occurs twice)
(3,6,3, $(2^{m-2}-4)$ terms of 3's) (occurs once)

b) $\psi_r(C_{2^m-2}) < \psi_r(C_{2^m-1})$

Proof. We will prove by induction. We begin with base cases: we know from earlier in this paper that it is true for $\psi_r(C_{14}) < \psi_r(C_{15})$ and $\psi_r(C_{21}) < \psi_r(C_{22})$. Furthermore from earlier work, we know that C_{15} has only seven possible cases, and same goes for C_{31}. Also, we know that $\psi_r(C_{30}) < \psi_r(C_{31})$.

Now, due to the counting lemma, we can focus uniquely on the structure of the partitions of flanking numbers and determine the construction from there. From previous works, we know that for C_{15} and C_{31} there are only seven arankings for both up to permutation of the top three labels.

Now we move on to the induction part of the proof. First, we assume that all of the listed statements in the theorem is true for $m = i$, and we want to prove for $m = i + 1$. This means that $\psi_r(C_{2^m-2}) < \psi_r(C_{2^m-1})$ is true, and we only desire to work from C_{2^m-1} since it is our goal to maximize the arank number for all cyclic graphs.

Looking at partitions with (14), $(11, 2)$, or $(5, 8)$ component, $(2^{m-2} - 4)$ partitions of size 3 will require $(2^{m-1} - 8)$ vertices to dominate all such partitions. This gives us a partition structure of $(2, 2)$ for each (3) partition. This implies that we will need another $(2^{m-1} - 8)$ insertions of vertices, bringing us to a total of $(2^m - 16)$ vertices insertions just to dominate the particular set of partitions through both expansions. Since we are interested in determination of arank number of $C_{2^{m+1}-2}$, and $(2^m - 16)$ vertices are inserted into C_{2^m-1} to dominate all partitions of size 3, we are left with 15 more vertices needed to dominate the partition (14),$(11, 2)$ or $(5, 8)$ in order to reach $C_{2^{m+1}-2}$. Also observe that an equal number of vertices is used for both expansions, which has the effect of limiting us to an insertion of seven vertices into the partition (14), $(11, 2)$ or $(5, 8)$ for the first expansion, and an insertion of eight vertices for the second expansion. However, by the Exhaustion Lemma for all three partitions, we cannot insert eight vertices in the second expansion and expect to get a minimal cyclic graph. Note also that if we attempt to insert eight vertices, we still will not be able to reach an aranking of C_{2^m-1} with these partitions, thanks to the Exhaustion Lemma regarding (14), $(11, 2)$, and $(5, 8)$.

Now suppose we try to put one more vertex into one partition of size three, which will give us a partition structure of $(1, 1, 1)$. This implies that we will need three vertices to dominate the partition instead of two vertices, which has the eventual effect of making it clear that we cannot get to the arankings of $C_{2^{m+1}-2}$ from the arankings of C_{2^m-1} under assumptions that $\psi_r(C_{2^{m+1}-2}) = \psi_r(C_{2^m-2}) + 2$. Furthermore, this shows that we cannot reach $\psi_r(C_{2^{m+1}-2}$ with cases with partitions with (14), $(11, 2)$ or $(5, 8)$ component.

Let us focus on the last case, (3,6,3, $(2^{m-2} - 4)$ terms of 3's). This case is slightly different, but we begin in the same way. We begin by including the two partitions with size three from $(3, 6, 3)$ with all the other partitions with size of three, and dominate them all by inserting $(2^{m-1} - 4)$ vertices to produce $(2^{m-1} - 4)$ partitions of size two. These partitions will require another $(2^{m-1} - 4)$ insertions of vertices to dominate all partitions of size two. Now since we have inserted $(2^k - 8)$ vertices into C_{2^m-1}, and our goal is to prove for cases of $C_{2^{m+1}-1}$, we only can insert eight more vertices through two expansions. Now we focus on the last remaining partition, which is of size six. This presents us two possible strategies: insert three vertices in the first expansion, and then five vertices in the second expansion; or four vertices in the first expansion and four vertices in the second expansion.

We apply Exhaustion Lemma (6) to see that we can get the partition (9) with an insertion of three vertices, or partition $(2, 4, 2)$ with an insertion of four vertices. Also, by the Exhaustion Lemma (9), partition (9) requires five additional vertices to dominate all vertices with flanking number 0, and partition $(3, 6, 3)$ requires four additional vertices insertions to dominate all vertices with flanking number 0. Finally, by Exhaustion Lemma (9) with five vertices insertions, we get the partition structure (14), $(11, 2)$, $(8, 5)$, $(5, 8)$, $(2, 11)$, or (14), establishing six cases with arankings of $C_{2^m+1} - 1$. Finally, we insert four vertices to dominate all vertices with flanking number 0 into $(2, 4, 2)$, and we note that such domination is unique, and we get partition structure of $(3, 6, 3)$. Now we tack on other $(2^{m-1} - 4)$ partitions of size three to all seven partitions to get:

$$(14,\ (2^{m-1}-4) \text{ terms of 3's}) \text{ (occurs twice)}$$
$$(2{,}11,\ (2^{m-1}-4) \text{ terms of 3's}) \text{ (occurs twice)}$$
$$(5{,}8,\ (2^{m-1}-4) \text{ terms of 3's}) \text{ (occurs twice)}$$
$$(3{,}6{,}3,\ (2^{m-1}-4) \text{ terms of 3's}) \text{ (occurs once)}$$

This completes the proof of distinctiveness of structure in C_{2^m-1}. For the next component of the theorem, note that all of these are a result of smallest domination possible, and there is no way of constructing $C_{2^{m+1}-2}$, since the number of insertions which are forced by the need to dominate all vertices with the flanking number 0 are $(2^{m-1}-4)+(2^{m-1}-4)+3+5 = 2^m$ or $(2^{m-1} - 4) + (2^{m-1} - 4) + 4 + 4 = 2^m$, and thus never can be less than this value. It is impossible to reach $C_{2^{m+1}-2}$ from C_{2^m-1} with arank rankings. Since it is possible to reach $C_{2^{m+1}-1}$ from C_{2^m-1} with arank ranking, $\psi_r(C_{2^{m+1}-2}) < \psi_r(C_{2^{m+1}-1})$, thus completing the proof.

□

Corollary 8.3. $\psi_r(C_n) = \lfloor \log_2(n+1) \rfloor + \lfloor \log_2(\frac{n+2}{3}) \rfloor + 1$ *for all* $n > 6$

Proof. Since the ordering of arankings is included in the previous two theorems, this corollary is derived directly from that ordering and the fact that two expansions are needed between m and m+1 cases, according to the rules of the ordering of arankings. Furthermore, it should be noted that the theorem proving the order of arank numbers do not include early cases, but fortunately, those are well covered by the theory of flanking numbers. □

It should be emphasized that the above corollary closed the open problem of determining the arank number of all cyclic graph. Furthermore, as astute readers may have noticed, the fact that the collection of arankings of C_{2^m-1} has a very distinctive character – that there are only seven of them, up to permutation of top three labels! This result should come across as very surprising to many, especially when we go into large cyclic graphs, one would think that the number of possible arankings would increase quickly.

9 Future Directions

Now that all of the open cases of $\psi_r(C_n)$ have been resolved, we turn toward potential future fields of study. For instance, it might be interesting to reinterpret the results of $\psi_r(P_n)$ in terms of flanking numbers. Also, one might try to extend the application of the ideas behind flanking numbers to different families of graphs. Before doing so, observe that in non-cyclic graphs, it is possible to generate a graph where two adjacent vertices could both have flanking number 1, which will complicate the application of flanking numbers to the analysis. Also, it might be worth to attempt to characterize all families of graphs where the flanking numbers of two adjacent vertices are not 1.

One could attempt to count the number of all possible arankings for a given cyclic graph of length n. As we have seen, we have seven possible arankings, ignoring permutations of the top three labels for cases of C_{2^k-1} for $k > 3$. Now suppose we include permutations of the top three labels: how many distinct arankings do we have? How many arankings do we have for C_{2^k-2}? At first glance, it appears that this problem is trivial, but with due consideration of the number of ways to construct arankings of a cyclic graph with a given length, this problem can quickly turn nontrivial.

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