The design and analysis of a rapid transit speed regulator

Frank Svet
THE DESIGN AND ANALYSIS OF A RAPID TRANSIT SPEED REGULATOR

by

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A rapid transit speed regulator is examined utilizing classical control theory and digital computer simulation. The transient response to velocity step input commands are examined under constraints placed on percent overshoot, steady state error, acceleration and jerk limiting. The effect of each constraint is examined for various weight trains. A means of loop compensation that will meet the desired design objectives concerning the transient response is then analyzed and simulated. A detailed discussion of both vehicle and motor dynamics is also presented.
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INTRODUCTION

This paper was motivated by the need to establish an orderly and efficient approach to the design of rapid transit speed regulating systems.

A rapid transit speed regulating control system generally consists of a controller and a propulsion system as illustrated in Figure 1. The controller outputs a command signal containing some form of distance, velocity and acceleration information to the transit vehicle propulsion system. The vehicle propulsion system interprets this information and produces an appropriate response, causing the vehicle to "move" in some prescribed manner. Some form of a feedback interface, transmits the new distance, velocity and acceleration information back to the controller, which then modifies the next control command in light of the new information. If the system is a calibrated or open loop system, there is no feedback interface, as illustrated in Figure 2. The controller in this case would follow some preprogrammed set of instructions.

At the present time there are many methods used for controlling the velocity, acceleration and rate of change of acceleration of rapid transit vehicles. The approaches range from the simple calibrated or open loop system mentioned above to very complex central computing systems employing many decisions trees and complex control
algorithms. In addition, there are also numerous types of transit car propulsion systems running the gamut from large D.C. motors to linear induction machines. Thus, it is not difficult to see that a great many controller-propulsion control systems are, at least theoretically, possible.

This paper focuses on one of these many combinations. The form of the controller, though not the controller gain constants, is fixed and is illustrated in Figure 3. This controller form was chosen because of its inherent simplicity of design and its ease for hardware implementation in either a digital or analog scheme. The propulsion system is a synchronous motor. The synchronous motor was chosen over induction and D.C. motors because:

1. Speed control of a synchronous machine is easier than that of an induction machine. By varying the frequency of the applied motor voltage, the speed of the motor is effectively controlled in a linear fashion.
2. Power rectification problems and D.C. heating losses are minimized.

Motor operation was not specified by a transfer function or differential equations. Rather, the motor performance characteristics were given in the graphical form illustrated in Figures 4 and 5. The motor performance characteristics were manipulated into a form compatible with
data describing the transit vehicle's static and dynamic properties. The overall system block diagram, discussed later, was then created and used in the analysis and simulation of the control system. The system block diagram is shown in Figure 6. As will become apparent later, feedback was necessary in order to satisfy the design objectives discussed next.
DESIGN OBJECTIVES

Rapid transit systems must function in a safe and efficient manner. Safety is the prime concern since ultimately human lives are at stake. For this reason all public rapid transit systems in operation have many redundancies and checks built into them. The speed regulating system discussed in this paper operates in conjunction with "supervisory" and safety checking systems. These safety oriented systems create a much higher degree of safe system operation than would exist without them, but do impose slight reductions in system speed and efficiency. Efficiency is also of concern in the design of rapid transit systems. Most public rapid transit systems in operation today were built because they offer a degree of efficiency greater than that of private automobiles and buses. Obviously the term efficiency can be applied to a number of items common to rapid transit vehicles and other forms of transportation, i.e. fuel consumption, degree of pollution, etc. The term efficiency in this paper shall apply only to system capacity, that is, the number of passengers transported from point A to point B per unit of time. The greater the system efficiency, the more passengers transported per unit of time.

The design objectives generally specified by contractors evolve around system safety and efficiency. Two design objectives generally specified are the vehicle's transient
response to a velocity step input command and the steady state error in response to the velocity step command. It is usually specified that the transient response be critically damped and that the steady state error in response to the velocity step command be zero.

A critically damped transient response to a velocity step input is necessary because in the overall rapid transit scheme there exists the separate and distinct safety speed governing mechanisms (the supervisory system mentioned earlier), that overrides the controller when the actual vehicle velocity exceeds the governing level. This governing level is independent of the vehicle and is imposed as a speed limit on parts of the roadway or guideway on which the transit vehicle travels. This is very much like the speed zone areas one is familiar with on highways.

Suppose, for example, that we have a 40 mile per hour step input command to the controller in a 40 mile per hour speed limit zone. If the vehicle's velocity exceeds this limit for some reason in its transient or steady state response to the 40 mile per hour step input command, the overspeed governing mechanism immediately removes propulsion from the vehicle and imposes a penalty brake application. The penalty braking immediately slows the vehicle below the governing speed limit thereby lengthening the amount of time required to cover the distance from point A to point B. Obviously, any increase in transit time over the
prescribed minimum dictated by the speed limit results in a decrease in system capacity (the number of people transported per unit of time), which is undesirable. Therefore, the transient velocity response of the vehicle should not exceed the governing limit. In fact, maximum allowable system capacity will result if the transit vehicle runs at exactly the governing velocity at all times.

It should now be more apparent why the steady state error to a velocity step command input should be zero. The velocity step command input cannot exceed the upper governing level if there is to be no penalty braking due to the transient response. The velocity step command can, however, equal the governing level. In order to minimize transit time and increase system capacity without invoking a penalty brake application the vehicle must theoretically run at the governing speed level in the steady state.

Running the transit vehicle exactly at the governing speed level would require an extremely fast loop response if a feedback control approach is used for speed regulation. The transit vehicle regulator would be required to immediately regulate the level of propulsion in order to adjust for external disturbances on the vehicle such as tailwinds and downgrades. Disturbances such as these would cause the vehicle to speed up before the propulsion power could be reduced to a new quiescent point sufficient
to maintain zero acceleration and the desired velocity. Instantaneous loop response is, of course, impossible. An alternate means of preventing a penalty brake action is necessary. One alternative often employed trades off system capacity in return for an unnecessary penalty braking (due to external disturbances). Stated another way this simply means that the vehicle is run at a velocity slightly under the governing limit. How much is, of course, a function of how much the external disturbances increase the vehicle velocity and also how fast the loop responds. Typically, a rapid transit system will have five or six fixed governing velocity limits that can be imposed as necessary on the transit vehicle's guideway or right of way. Six typical velocity limits might be: 0 mph, 15 mph, 25 mph, 35 mph, 50 mph, 70 mph. The vehicle is usually run at no more than 95% the governing velocity limit but, as mentioned above, this is a function of the rapid transit guideway, environment and control loop.

The third and fourth design objective often specified in the design of a rapid transit speed regulating system addresses vehicle ride quality. The maximum vehicle acceleration and the rate of change of acceleration, known as jerk, are two commonly specified ride quality parameters. They, in effect, dictate how smooth the transient response of the vehicle will be. The penalty paid for this smoothness in terms of system performance is time. Stated another way the cost is a decrease in capacity or system efficiency.
The maximum acceleration to be permitted in this paper is 3.5 mph/sec while the maximum jerk allowed is 2.5 mph/sec$^2$.

Limiting the maximum vehicle acceleration effectively establishes the maximum rise time or response time of the transient velocity response. This introduces a pure time delay into the loop response as will be shown later. Restricting the maximum vehicle jerk effectively limits the bandwidth of the transient acceleration response. The effect of this constraint on system performance will also be examined later.

In summation, the Rapid Transit Speed Regulating System must operate to fulfill four design objectives. The transient response will be initiated by a velocity step input that is approximately 95% of the governing speed limit. The response to this input must be critically damped, exhibiting no overshoot. The slope of the transient velocity response must be less than some prescribed limit, the maximum vehicle acceleration. The transient velocity response must also exhibit no discontinuities or have a second derivative whose magnitude is greater than some prescribed limit, the maximum vehicle jerk. Finally, the steady state error to a velocity step input must be zero. These design objectives must be met with the specified controller form illustrated in Figure 3 and the synchronous motor specified for the propulsion system.
DISCUSSION OF VEHICLE AND MOTOR PARAMETERS

The synchronous motor performance characteristics illustrated in Figures 4 and 5 define the operation of the motor to be used to propel the vehicle. These characteristics are essential in order to design the controller to fulfill the desired design objectives. The converse, however, is not necessarily true. That is to say, one need not know the form or type of controller in order to specify the vehicle motor. What is necessary for specification of the vehicle motor is information about the vehicle and its environment. This information is also necessary for analysis of the total speed regulating control system. Therefore, a discussion of vehicle characteristics, undertaken next, is necessary to substantiate whether the motor choice was proper for meeting the desired design objectives.

For motor performance calculations the assigned vehicle weights are:

- $AW_1 =$ Empty Car -------72,000 pounds
- $AW_2 =$ Car plus 100 Passengers @ 150 lbs. each...87,000 pounds (normal max. load all seated)
- $AW_3 =$ Car plus 240 Passengers 108,000 pounds (overload with standees)

Other basic requirements are:

a. Two motors per car for propulsion
b. Maximum desired motor speed -------6,000 RPM
c. Diameter of a vehicle wheel (new condition)-------28 inches
d. Maximum sustained speed on level, tangent (no banking) track into a 15 mph headwind and 26 inch diameter (worn wheels)------80 mph

e. Rotating inertia of motor and drive train is assumed to equal 5% of the empty car mass or \( \frac{0.05 \text{ (AW1)}}{32.2 \text{ ft/sec/sec}} \).

GEAR RATIOS

To achieve a speed of 80 mph with 26 inch wheels and a motor rotating at 6000 rpm may require a gear reduction. This gear ratio, if necessary, can be found as follows, where \( V \) is translational velocity, \( \omega \) is rotational velocity and \( r \) is the wheel radius.

a. \( V = \omega r \) \hspace{1cm} \text{EQ. 1}

b. \( \omega = \frac{V}{r} = \frac{(80 \text{ mph})(5280 \text{ ft/mile})(12 \text{ in./ft})}{(3600 \text{ sec/hr})(13 \text{ inches})} \text{ rad/sec} \) \hspace{1cm} \text{EQ. 2}

c. \( \text{RPM} = \frac{\text{(rad/sec)}(60 \text{ sec/min})}{2\pi \text{ rad/deg}} \) \hspace{1cm} \text{EQ. 3}

d. Therefore, \( \omega = \frac{(80)(5280)(12)(60)}{(3600)(13)(2\pi)} = 1034 \text{ revs/min} \) \hspace{1cm} \text{EQ. 4}

e. Since we must reduce 6000 rpm to 1034 rpm we need a rear reduction of 6000/1034 = 5.8/1.

The gear ratio will be accomplished in two gear passes. If each pass is approximately 97% efficient then the overall gear ratio efficiency will be the product of the individual efficiencies or 94%. The effect of gear ratio efficiency shown in later calculations is to reduce the available torque of the vehicle wheels.
FORCES ACTING ON THE VEHICLE

The net tractive effort or force required to accelerate the vehicle at the rate of 3 miles/hour/sec can be determined from a simple summation of forces on the vehicle. In addition to overcoming vehicle inertia, both translational and rotational, the motor must overcome the vehicle's rolling resistance and aerodynamic drag. Additional forces acting on the vehicle that may enter into the calculation depending on the guideway layout are those due to grade and curves. A study of train resistance was published in 1926 by W. J. Davis, Jr. Although these equations were based on train shapes and speeds prevalent nearly fifty years ago, subsequent studies conducted on conventional rapid transit vehicles are in excellent agreement with the Davis equations. Detailed equations and their discussion appear in the American Railway Engineering Association (AAR) Handbook. Only a brief discussion of the equation will be presented here.

The Davis train resistance, (T.R.), equation, \(T.R. = Fa + Fm\), is a summation of two components, mechanical drag, \(Fm\), and aerodynamic drag, \(Fa\), acting as resisting forces on the vehicle. The term tractive effort refers to the summation of these and other forces acting on the vehicle such as the effects of guideway curves, grades and the desired acceleration. The units of tractive effort are pounds-force. Thus a net positive tractive effort
will accelerate the vehicle.

The mechanical drag, \( F_m \), in the Davis equation is expressed as:

\[
F_m = (29)(AX) + (1.3 + 0.45(V/10)W_t)
\]  
EQ. 5

where \( AX \) = the number of axles per vehicle (in our case, 4)
\( W_t \) = the vehicle weight in tons.
\( V \) = vehicle velocity in miles per hour.

The aerodynamic drag, \( F_a \), in the Davis equation is expressed as:

\[
F_a = C(A/100)(V/10)^2
\]  
EQ. 6

where \( C \) = air resistance coefficient.
\( V \) = vehicle velocity in mph.
\( A \) = cross sectional area of the vehicle front in \( ft^2 \).

The air resistance coefficient is a function of the vehicle shape. Wind tunnel tests of vehicles similar to the proposed vehicle indicate that \( C \) is:

\[
C = 3 + 20/(N) \frac{1 \text{ lbs.}}{\text{(ft}^2\text{)} \text{(mph}^2\text{)}}
\]  
EQ. 7

where \( N \) = number of vehicles per train. \((N \geq 1)\)

The second term in the above equation accounts for all trailing vehicles in a train of vehicles.

The resultant Davis equation is arrived at by adding \( F_m \) to \( F_a \).

\[
T.R. = \text{Train Resistance (lb/car)} = F_m + F_a
\]
EQ. 8

\[
T.R. = (29)(AX) + (1.3 + .45(V/10)W_t) + \frac{(3 + 20/N)AV^2}{10,000}
\]  
EQ. 9
But:

\[ AX = 4 \text{ axles per train} \]
\[ W_t = \frac{W}{2000} \quad \text{where } W = \text{Total Train weight in lbs.} \]
\[ A = 100 \text{ ft}^2 \quad (\text{the cross sectional area of the front of the vehicle}) \]

Thus:

\[ T.R.(lb/car) = 116 + \frac{1.3W}{2000} + \frac{.045WV}{2000} + \frac{(3 + 20/N) V^2}{100} \quad \text{EQ. 10} \]

**Multiplying** through by \( N \), the number of cars/train, one obtains

\[ T.R.(lbs) = \frac{116N + 1.3W}{2000} + \frac{.045WV}{2000} + (.2 + .03N) V^2 \quad \text{EQ. 11} \]

Equation 11 is the train resistance in pounds-force for a train moving in a zero wind environment. If the atmospheric winds are greater than zero then \( V^2 \) becomes \( kV^2 \) where \( k \) is defined as follows:

<table>
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<tr>
<th>Description of wind</th>
<th>( \phi ), angle between vectors of train speed, ( V ), &amp; absolute wind speed, ( V_S )</th>
<th>( k )</th>
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<tr>
<td>Head Wind</td>
<td>( \phi = 180^\circ )</td>
<td>( k = (1 + VS/V)^2 )</td>
</tr>
<tr>
<td>Tail Wind</td>
<td>( \phi = 0^\circ )</td>
<td>( k = (1 - VS/V)^2 )</td>
</tr>
<tr>
<td>Most Adverse Wind</td>
<td>( 110^\circ &lt; \phi &lt; 125^\circ )</td>
<td>( k \approx (1 + VS/V)^3 )</td>
</tr>
<tr>
<td>Averaged for Head &amp; Tail Wind</td>
<td>( \phi = 180^\circ ) and ( \phi = 0^\circ )</td>
<td>( k = 1 + \left(\frac{VS}{V}\right)^2 )</td>
</tr>
<tr>
<td>Averaged for all Directions</td>
<td>( 0^\circ &lt; \phi &lt; 360^\circ )</td>
<td>( k \approx 0.6 + \frac{3VS}{V} ) or ( \text{if } \frac{VS}{V} \leq 0.134, \ k=1 )</td>
</tr>
</tbody>
</table>

The equations in the above table are suited for use in
specific calculations. Simplifications can be made to facilitate routine calculations. The k values can be calculated utilizing a constant VS (wind speed) value. The following wind speed distribution table gives some insight for choosing a satisfactory VW constant:

<table>
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<th>Percent of total time</th>
<th>25 50 75 90 95 99 ≤ 100</th>
</tr>
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<tr>
<td>VS (mph)</td>
<td>7 10 14 19 22 28 85</td>
</tr>
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</table>

The value selected for VS in this problem is 15 mph. The selection of 15 mph as opposed to say 28 mph is reasoned because of windbreaking wayside structures along the vehicle guideway that decrease the wind velocity effect on the vehicle. In addition, a headwind was to be expected a majority of the time. Thus, the \( V^2 \) appearing in Equation #11 becomes \( (k \cdot V)^2 \) where \( k \) is defined as:

\[
k = \left(1 + \frac{VS}{V}\right)^2
\]

and VS is selected as a constant 15 mph.

Substituting \( V^2 = kV^2 \) into Equation #11 one obtains:

\[
T.R.(lbs) = 116N + 1.3W + \frac{(0.045V)W}{2000} + \frac{0.03N(V+VS)^2}{2000}
\]

EQ. 12

Since VS is to be a headwind, in the nomenclature \( VS = V_H \) is used. Thus train resistance becomes:

\[
T.R.(lbs) = 116N + 1.3W + \frac{(0.045V)W}{2000} + \frac{0.03N(V+V_H)^2}{2000}
\]

EQ. 13

The inclusion of train resistance in the closed loop is
illustrated in Figure 6 which will be discussed later.

The effects of grade, \( F_g \), curve, \( F_c \), and acceleration must be considered in addition to train resistance in calculating the net tractive effort.

Grade resistance can be calculated by referring to the diagram in Figure 7. Grade is normally expressed in percent. This percent is really the tangent of the angle, \( \alpha \), between the base and the hypotenuse of the illustrated grade triangle. A 7\% grade corresponds to 4° of "tilt" while a 100\% grade corresponds to 45° of "tilt". In the force vector diagram illustrated in Figure 7, side "b" is the vector force due to gravity less the force component directed down the incline. Side "a" is the gravity component and side "c" is the incline component. The loss or gain of tractive effort due to the incline is component "c". Component "c" can be determined by geometry as follows:

\[
F_g = C = (a) \sin \alpha \text{ (in pounds-force where } a \text{ is the vehicle weight in pounds and } \alpha \text{ is the grade angle)} \quad \text{EQ. 14}
\]

For an 87,000 lb. vehicle and a +7\% grade, \( C \) becomes (87,000)

\[
(\sin 4^\circ) = +6090 \text{ lbs.}
\]

Curve resistance is usually assumed to be:

\[
F_c = 0.2 \times \text{(degree curvature)} \times \text{vehicle weight. EQ. 15}
\]

If curvature is specified in terms of the curve radius in feet then \( F_c \) becomes:

\[
F_c = (4600w)/\text{radius of curve in feet. EQ. 16}
\]

Since no curves or grades were specified in this problem
both $F_c$ and $F_g$ are zero. The information was merely presented for completeness of this analysis.

The force required to overcome vehicular inertia to achieve an acceleration rate of 3 mph/sec is the final force to be analyzed. From classical physics one has the force-mass-acceleration relationship $F = ma$. Manipulating this relationship into the proper units is as follows:

$$F_1 \text{ (lbs)} = \frac{W \text{ (lb)}}{g \text{ (ft/sec}^2)} \times a \text{ (ft/sec)} \cdot \text{ EQ. 17}$$

$$F_1 \text{ (lbs)} = \frac{W \text{ (lbs)} \times a \text{ (mph/sec)}}{g \text{ (ft/sec}^2) \times 3600/1 \text{ (sec/hr)} \times 1/5280 \text{ (miles/ft)}} \cdot \text{ EQ. 18}$$

$$F_1 \text{ (lbs)} = \frac{W \text{ (lbs)} \times a \text{ (mph/sec)}}{g \text{ (ft/sec}^2) \times 0.6818 \text{(mile-sec/ft-hr)}} \cdot \text{ EQ. 19}$$

An additional force is necessary to overcome rotational inertia due to rotating mechanical parts such as drive shafts, gear trains or flywheels. This rotational inertia is assumed to be equal to 5% of the empty car mass or $(0.05AW_1)/32.2$ as mentioned earlier. To account for this additional inertia the weight, $W$, in equation 19 is modified to $W'$ where $W'$ is defined as:

$$W' = W + 0.05AW_1 \cdot \text{ EQ. 20}$$

Thus the force to overcome all vehicular inertia, $F_A$, is:

$$F_A = \frac{W'a}{g \text{ (0.6818)}} \text{ (lbs)} \cdot \text{ EQ. 21}$$

where: $W'$ is vehicle weight $+ (0.05)X$ (empty vehicle weight)

$a$ is the vehicle acceleration (3 mph/sec)
and $g$ is the acceleration of gravity 32.2 ft/sec$^2$.

The equation for determining net tractive effort is therefore:

$$F_t = T.R. + F_A (\text{lbs}) \quad \text{EQ 22}$$

$$F_t = 116N + \frac{1.3W}{2000} + \frac{0.045VW}{2000} + (0.2 + 0.03N)(V+V_H)^2$$

$$+ \frac{W'a}{g(0.6818)} (\text{lbs}) \quad \text{EQ 23}$$

where:

- $N =$ number of cars per train
- $W =$ total weight of train in pounds
- $V =$ train velocity
- $V_H =$ 15 mph headwind constant
- $a =$ 3 mph/sec train acceleration
- $g =$ acceleration of gravity 32.2 ft/sec$^2$
- $W' \approx (1.05) \times W$ in lbs.

The tractive effort required to develop an initial acceleration of 3 mph/sec with an 87,000 lb. vehicle may be found from Equation $\#23$ where actual velocity ($V = 0$) and $V_H = 15$ mph:

$$F_t = (116)(1) + \frac{(1.3)(87,000)}{2000} + 0 + (0.23)(15)^2$$

$$+ \frac{(1.05)(87,000)(3)}{(32.2)(0.6818)} \quad \text{EQ. 24}$$

$$= 116 + 56.66 + 0 + 51.75 + 12,182.94 \quad \text{EQ. 25}$$

$$= 12,707.24 \text{ lbs-force} \quad \text{EQ. 26}$$

Since there are to be two tractive motors per vehicle the net tractive effort each would have to provide is
6353.62 lbs-force. The motor torque required if the vehicle wheels are 28 inches in diameter may be found from the equation:

\[
\text{Torque (ft-lbs)} = \frac{\text{(Ttractive effort)(Wheel Radius)}}{\text{(Gear Ratio)(Gear Efficiency)}}
\]

EQ 27

Notice that the 94\% gear efficiency causes the required torque to be greater than if the efficiency were 100\%. Therefore: Torque = 1357.66 ft-lbs per motor

EQ 28

**DISCUSSION OF THE MOTOR**

The propulsion engineer uses the motor torque as a starting point in his motor selection. A motor is chosen that will provide at least the calculated starting torque. Calculations identical to those performed above are made to determine the torque at other vehicle speeds, both to accelerate the vehicle at some prescribed rate and also to maintain the vehicle velocity at some fixed level. These torque calculations are then compared with the actual motor torque-speed curve of a particular motor to determine if the motor selection is adequate over the entire speed range desired.

Examining the motor performance curve, Figure 4, for the acceleration mode, illustrates that during the initial acceleration period from 0 to 1667 rpm, the motor torque is constant at 1420 ft-lbs. A motor speed of 1667 rpm corresponds to a vehicle speed of 24 mph as follows:
\[ V = \omega r \text{ in mph} \quad \text{EQ 29} \]
\[ = 2\pi f \quad \text{EQ 30} \]
\[ f = 1667 \text{ rev/min} \quad \text{EQ 31} \]
\[ V = \frac{(2)(1667 \text{ rev/min})(60 \text{ min/hr})(1.165 \text{ ft})(1/5280 \text{ mile/ft})}{5.8 \text{ gear ratio}} \quad \text{EQ 32} \]
\[ V(\text{mph}) = (\text{rpm})(0.0143) \text{ for 28" wheels and 5.8:1 gear ratio.} \quad \text{EQ 33} \]
\[ V = 23.92 \text{ mph} \quad \text{EQ 34} \]

The applied motor voltage is proportional to speed from 0 to 1667 rpm. At 24 mph/1667 rpm the voltage is 447 volts rms (line to line). The constant current level of 500 amperes indicates a constant motor flux. The torque and the current are therefore, proportional. Output power can be determined from the product of torque and motor speed in radians per second. Notice, that since the torque is constant the power is proportional to motor speed.

From 1667 rpm/24mph to 6000 rpm/85.8 mph the motor power controllers program the applied motor voltage to be proportional to the square root of the motor speed. The motor will operate with constant power but with torque inversely proportional to motor speed. Since power is constant and the applied voltage increases in proportion to the square root of speed, the motor current must decrease in proportion to the inverse of the square root of speed.

The motor selected is also suitable for dynamic braking of the vehicle. This technique is used in cases where a
motor must be decelerated at a faster rate than the load friction can provide. In addition friction brake wear is reduced. The dynamic braking feature found on this and many rapid transit propulsion systems consists of a load resistor which is automatically switched across the motor armature anytime braking is called for. The dynamic brake depends on the armature back emf to provide the needed brake current. The braking force is greater at high speeds and less effective at lower speeds.

In decelerating from 6000 rpm/85.8 mph to 1667 rpm/24 mph the motor must develop a braking torque established by vehicle mass, inertias and frictional losses. Frictional braking of the wheels is to be avoided if unnecessary. A motor torque of -1200 ft-lbs. is sufficient to produce a deceleration rate of 3 mph/sec. Since braking power = torque x speed, braking power decreases with speed if the torque remains constant. By controlling the field excitation in the propulsion controller, the motor output voltage is programmed to drop in proportion to the square root of the speed. Since the value of the dynamic resistor is constant, the motor current also decreases in proportion to the square root of speed. The net result is a linearly decreasing braking power as the speed drops.

From 1667 rpm/24 mph to 0 mph, the motor flux is constant since the field excitation has saturated. With a constant flux, the motor voltage and current decrease linearly
with speed. Braking torque now decreases linearly, since it has become proportional to motor current in the constant flux mode. Braking power which is the product of torque x speed decreases in a square law fashion since both torque and speed are decreasing. The friction brakes on the vehicle now enter into the picture and provide the additional negative torque to produce a constant net -1200 ft-lbs, the value needed to decelerate at 3 mph/sec. Thus the torque remains constant over the entire decelerating speed range.

When braking ceases at 0 rpm/0 mph, very abruptly, a sizable jerk is experienced. There are in existence controller schemes (not analyzed in this paper) that can reduce the large jerk at the end by releasing the friction brakes seconds before the vehicle velocity reaches 0 mph. This causes the rate of deceleration to be reduced somewhat which causes the velocity of the vehicle to "flare" in toward 0 mph. The net result is a smooth almost jerkless stop.

It should be noted that dynamic braking (linear braking power) can be extended to speeds below 1667 rpm/24 mph by switching in a lower value of dynamic resistor. This would prevent saturation of the motor flux field at 1667 rpm extending it to some lower speed. The procedure of switching to a lower dynamic resistor at flux saturation can be repeated many times within the limits of the motor and the result will be linear braking power. The inclusion
of the motor dynamics in the closed loop is illustrated in Figure 6, which will be discussed later.
THE SPEED REGULATOR

The form of the controller specified is shown in Figure 3. Assume that the vehicle is at rest and that a velocity step input \( \text{Vref}/S \) is introduced at the \( \text{Vref} \) input of this controller. The actual velocity, \( \text{Va} \), is subtracted from the reference velocity, \( \text{Vref} \).

The velocity error \((\text{Vref}-\text{Va})\) is integrated at a rate determined by integrator gain constant \( K_2 \). The integrated error signal is then summed with a portion of the actual velocity signal \((K_3\text{Va})\) in an anticipatory measure and the result outputted to the propulsion motor controller. The command signal \( Y \), expressed in terms of Figure 3 is:

\[
Y(S) = K_2 \frac{Vref - \text{Va}}{S} - K_3 \text{Va} \quad \text{EQ 35}
\]

\[
= K_2 \frac{Vref}{S} - \left(1 + \frac{K_3 S}{K_2}\right) \text{Va} \quad \text{EQ 36}
\]

Let \( \tau = \frac{K_3}{K_2} \)

\[
Y(S) = K_2 \frac{Vref - K_2 (1 + \tau S) \text{Va}}{S} \quad \text{EQ 37}
\]

\[
Y(S) = K_2 \frac{(Vref - \text{Va} - (\tau S) \text{Va})}{S} \quad \text{EQ 38}
\]

Figure 8 is a block diagram of Equation 38. Figure 8 and Equation 38 are easily recognized as a basic feedback scheme of velocity plus acceleration. The acceleration is, of course, the derivative of the actual velocity. Velocity feedback is necessary because we desire a velocity control system. Acceleration feedback is necessary for "anticipation" in the velocity loop. The subtraction of the rate of change
of velocity (acceleration) from the algebraic summation of the actual and desired velocity will generate a negative offset that, as will be shown later, reduces overshoot and ringing in the transient response to step input velocity commands. In effect, acceleration feedback provides additional damping in the control loop.

Equations #35 and #38 result in an identical command, \( Y \), to the vehicle motor. However, generating a derivative in hardware and software as shown in Figure 8 is no easy feat since noise at the input of the differentiator complicates matters. Figure 3 has, however, avoided the necessity of a derivative. The term \( (K_3)V_a \) has the effect of an acceleration feedback signal. The derivative of the actual velocity signal and the velocity error integration have "nullified" each other resulting in \( K_3V_a \). When the velocity error is zero, \( Y \) remains constant at some value. This value is that needed to establish a level of motor power in order to maintain the desired velocity in the face of: vehicle inertia, rolling resistance, wind and motor inefficiency.
THE SYSTEM BLOCK DIAGRAM

Combining the information known about the vehicle and motor together with information about the speed regulator allows one to generate a basic closed loop model or block diagram of the system. This block diagram is shown in Figure 6.

The basic block diagram is composed of three portions: speed regulator, motor dynamics, and vehicle dynamics. In the speed regulator portion, the following sequence of events occurs. The reference velocity signal (Vref) is summed with the actual velocity signal (Va) producing a velocity error (V). V is then integrated at a rate determined by integrator gain K2. The result is called the integrated error (Pn). Pn is summed with a portion of the actual velocity signal (K3Va) as an "anticipating" measure. This sum is called the Demand Tractive Effort (TE1).

TE1 is the input to the motor-controller, as illustrated in the "motor dynamics" portion of Figure 6. Notice that the "OR" gate at the output of the motor-controller portion has three inputs. The mode "switch" allows us to be either in the acceleration or deceleration mode. The motor torque-speed characteristics are such that a constant maximum torque of 1420 ft-lbs is produced by each motor during accelerations at 3 mph/sec to 24 mph and a torque inversely proportional to speed above 24 mph. The decision diamond labeled "Va>Vb" represents a switch that is a function of
velocity. Below 24 mph, the constant NC1 is switched in during the acceleration mode. Above 24 mph, the inverse torque-speed relationship is switched in and the constant relationship is switched out. Thus, the torque-speed motor characteristic, (discussed below), is exactly modeled in 2 portions.

The constant C1 represents the maximum tractive effort available over the speed range 0 to 24 mph. Remember that maximum tractive effort is simply:

\[ \text{TE max.} = \frac{(\text{Torque/\text{max.}})(\text{Gear Ratio})(\text{Gear Efficiency})}{\text{Wheel Radius}} \]  

EQ 39

A torque of 1420 ft/lbs. is a tractive effort of 6,650 lbs for a 28" diameter wheel, 5.8:1 gear ratio and 94% efficiency. Since there are two motors per vehicle C1 becomes simply 2 X 6,650 lbs, or 13,300 lbs-force. The term N represents the number of vehicles per train.

The torque or tractive effort versus speed characteristic above 24 mph is represented by the multiplier and the inverter block. The multiplier block shown is utilized in a feedback path around the inverting amplifier. The "output" to "input" relationship of the multiplier and inverting amplifier is as follows:

\[ \text{Input} = \text{Output} \times \frac{\text{Va}}{(C_1)(N)(Vb)} \]  

\[ \frac{\text{Output}}{\text{Input}} = \frac{(C_1)(N)(Vb)}{\text{Va}} \text{ (lbs.-force)} \]  

EQ 40

EQ 41

where C1 = maximum tractive effort per vehicle (13,300 lbs)

N = number of vehicles per train
Vb = 24 mph

Va = actual vehicle velocity in mph (≥ 24 mph)

As Equation #41 points out, torque or tractive effort is proportional to the inverse of speed in the acceleration mode for vehicle velocities greater than or equal to 24 mph.

The second mode of motor operation is the deceleration mode. In this mode the motor exhibits a constant negative torque by means of dynamic braking down to the velocity determined by the value of the dynamic load resistor. Below this vehicle velocity the dynamic motor torque falls linearly with speed. However, friction braking of the vehicle wheels maintains the torque constant. A blending circuit on the vehicle increases frictional braking in the same proportion as the decrease of dynamic braking. The net braking torque thus remains constant from 86 mph to 0 mph. The value of this constant tractive effort, C2, is -11,231.5 lbs per vehicle. Once again, N represents the number of vehicles per train.

TE2, the motor-brake tractive effort, is the output of the "motor/brakes dynamics" portion of the block diagram. TE2 is the input to the vehicle dynamics portion of the block diagram. TE2 is initially summed with two terms, K1W and K8N. These terms are two of the rolling resistance terms of the Davis Train Resistance Equation (Equation #13). The constant K1 is equal to 1.3/2000, K2 is equal to 116,
W is the train weight in lbs, and N represents the number of cars per train.

The terms of $TE_2$, $K_1W$ and $K_8N$ are algebraically summed with terms that are functions of the actual vehicle velocity. The first term, $K_7WVa$ is the rolling resistance (third) term of the Davis Train Resistance Equation. The constant $K_7$ is equal to 0.045/2000, while $W$ is the train weight in lbs. Note that $Va$ is an input variable representing actual vehicle velocity. The second term $(Vw)^2(K_4+K_5N)$ is the aerodynamic drag term of the Davis Train Resistance equation, (EQ 13). The constant $K_4$ is equal to 0.2, $K_5 = 0.03$ and $N$ is the number of vehicles per train. The variable $Vw^2$ is the square of the sum of actual vehicle velocity, $Va$, and $V_H = 15$ mph, the headwind.

The total summation of tractive effort is called $TE_{net}$. This net tractive effort force acts to overcome the vehicle inertia represented by the term $(g)(K_6)/W'$, where $g =$ acceleration of gravity (32.2 ft/sec$^2$), $K_6$ is a conversion constant 0.6818, sec/hr ft/mile and $W'$ is the train weight + 0.05 AWLN. The result is the acceleration of the vehicle in mph/sec. The acceleration is then integrated once to yield actual velocity which when fed back to the various points indicated earlier closes the block diagram loop.
The system transfer function

The transfer function of a linear system is defined as the ratio of the LaPlace Transform of the output variable to the LaPlace Transform of the input variable, with all initial conditions assumed to be zero. Referring to Figure 7, it is clearly evident that in spite of the approximations used to arrive at the configuration indicated, it is nonlinear. Some additional simplification (linearization) is necessary in order to study the system via LaPlace Transforms.

Figure 7 may be linearized without losing too much identity with its present configuration by analyzing the dynamics below 24 mph and ignoring losses due to vehicle drag such as rolling resistance and aerodynamics. By constraining the analysis to speeds below 24 mph, the maximum tractive effort available for acceleration is the constant, \( C_1 \). Eliminating the vehicle drag terms rids Figure 7 of the other nonlinearity present in the loop. The linearized closed loop system is shown in Figure 9. It should be noted that the above simplifications are required only for the paper and pencil analysis. The computer simulation is not restricted to velocities below 24 mph and includes the vehicle drag losses.

Utilizing Mason's Loop Rule for signal flow graph reduction (indicated below), the transfer function of Figure 9 can be rewritten. Mason's Loop Rule states:
\[ T_{ij} = \sum_{k} P_{ijk} \frac{\Delta_{ijk}}{\Delta} \quad \text{EQ. 42} \]

where \( T_{ij} \) is the ratio of the LaPlace Transform of the output variable \( j \), divided by the LaPlace Transform of the input variable, \( i \).

\( P_{ijk} \) is the transmittance or gain of the path between the output variable \( j \) and the input variable, \( i \).

\[ \Delta \text{ is equal to: } 1 - (L_1 - L_2 \cdots L_n) + (L_1 L_2 \cdots L_n) \]

where \( L_1, L_2 \cdots L_n \) are loop gains of the individual loops.

\[ \Delta_{ijk} \text{ is equal to } \Delta \text{ with all the loops touching path } P_{ijk} \text{ set equal to zero.} \]

Apply Mason's Loop Rule to Figure 9 one obtains:

\[ T_{ij} = \frac{V_{\text{ACTUAL}}}{V_{\text{REF}}} \quad \text{EQ. 43} \]

\[ P_{12} = \frac{N K_2 C_1 g K_6}{W S^2} \quad \text{EQ. 44} \]

\[ L_1 = -N C_1 g K_6 K_3 \quad \text{EQ. 45} \]

\[ L_2 = -N K_2 C_1 g K_6 \quad \text{EQ. 46} \]

\[ T_{12} = \frac{P_{12}(1)}{1-(L_1 + L_2)} = \frac{N K_2 C_1 g K_6}{W S^2} \frac{1+N C_1 g K_6 K_3}{W S} + \frac{N K_2 C_1 g K_6}{W S^2} \]

Simplifying:

Let \( K_{10} \equiv \frac{N C_1 g K_6}{W} \quad \text{EQ. 48} \)
Then $\frac{V_{\text{ACTUAL}}}{V_{\text{REF}}}$ becomes
\[
\frac{V_{\text{ACTUAL}}}{V_{\text{REF}}} = \frac{K_2 K_{10}}{S^2 + (K_{10} K_3) S + K_2 K_{10}} \quad \text{EQ. 49}
\]

Equation 49 is the linearized loop transfer function.

This transfer function is of the classic second order form
\[
\frac{\text{OUTPUT}}{\text{INPUT}} = \frac{\omega n^2}{S^2 + (2 \omega n) S + \omega n^2} \quad \text{EQ. 50}
\]

This form will allow one to perform a simple straightforward loop analysis.
LOOP ANALYSIS - SYSTEM STABILITY

The transfer function obtained in Equation #49 gives the interrelationship between the speed regulator motor and vehicle. Since only the form of the speed regulating controller has been specified, and not the gain constants, it is desirable to examine the range over which the constants can vary without causing an unstable loop. Knowing this information will then enable an easier selection of values for these constants in order to meet the desired design objectives.

Clearly, in order to obtain a bounded (stable) response, the poles of the closed loop system must be in the left hand portion of the "S" plane. Thus, a necessary and sufficient condition for stability of a control system is that all the poles of the system transfer function have negative real parts. The Routh-Hurwitz Stability Criterion provides an answer to the question of stability by considering certain aspects of the characteristic equation of the system. It is the characteristic equation, whose roots are the poles of the closed loop system, that determines system stability for a range of loop gain. Requisites of the Routh-Hurwitz Stability Criterion are:

1. All coefficients of the characteristic equation polynomial in "S" must have the same sign. Also it is necessary that all the coefficients be nonzero.
2. The number of roots of the characteristic equation with positive real parts is equal to the number of changes in sign of the first column of the array. It is a requirement for system stability that there be no changes in sign in the first column of the array.

For the characteristic equation: \( S^2 + K_3 K_10 S + K_10 K_2 \), the Routh-Hurwitz array becomes:

\[
\begin{array}{cccc}
S^2 & 1 & K_10 K_2 & 0 \\
S^1 & K_3 K_10 & 0 & 0 \\
S^0 & A & & \\
\end{array}
\]

\( A \) is equal to \( (1)(0) - (K_3 K_10)(K_10 K_2) = +K_10 K_2 \). \( \text{Eq. 51} \)

Applying the requisites of the Routh-Hurwitz Criterion one finds that for a stable system \( K_3 K_10 \) must be greater than zero and "A", \( (K_10 K_2) \), must be greater than zero.

We recall from Equation \#48 that \( K_{10} \) is equal to:

\[ K_{10} = \frac{NC_1 g K_6}{W'} \] \( \text{Eq. 52} \)

We find that \( K_{10} \) is a positive number since \( N, C_1, g, K_6, \) and \( W' \) are all positive numbers. If \( K_{10} \) is positive then both \( K_2 \) and \( K_3 \) are required to be positive in order that \( K_{10} K_2 \) and \( K_{10} K_3 \) be positive. Thus, a stable system results as long as \( K_2 \) and \( K_3 \) are positive if \( K_{10} \) is positive or if \( K_{10} \) is negative, \( K_2 \) and \( K_3 \) are negative.
LOOP ANALYSIS - STEADY STATE ERROR

One of the desired design objectives outlined earlier was that the vehicle exhibit a zero steady state error. That is, it is important to know whether the vehicle will actually be at the desired velocity after a long period of time (typically 10 loop time constants or more) or whether it will be at some other speed. The steady state error to a step reference velocity input may be found as shown below:

\[
\text{Ess}(t) = \lim_{t \to \infty} (E(t)) \quad \text{EQ. 53}
\]

where \( E(t) \) is the transient response of the control loop to a velocity step input.

Since we already have the transfer function of the system expressed in "S" notation, it is more convenient to investigate the steady state error in the "S" plane.

Thus, utilizing the Final Value Theorem, the steady state error to a velocity step input, \( A/S \), may be found from:

\[
\text{Ess}(t) = \lim_{S \to 0} \frac{S(A)}{1 + G(S)} \quad \text{EQ. 54}
\]

where \( V_{\text{ref}}(S) = \frac{A}{S} \) is the applied step input.

\( G(S) \) equals the open loop transfer function obtained by rearranging the system transfer function to include unity feedback as shown in Figure 10.
Thus:

\[
\text{Ess}(t) = \lim_{S \to 0} \left( \frac{S(A)}{S(1 + \frac{K_1 S + K_2}{S(S + K_3 S + K_4 S)})} \right) = \lim_{S \to 0} \frac{SA}{S + \frac{K_1 S + K_2}{S + K_3 S + K_4}} = 0
\]

The steady state step input error is found to be zero. This means that the vehicle after a period of time will travel at a steady 40 mph for a velocity command step input of 40 mph.

If, on the other hand, a velocity command ramp, \(A/S^2\), is the reference input, the steady state error becomes \(K_3 A/K_2\) as shown below:

\[
\text{Ess}(t) = \lim_{S \to 0} \frac{S(A)}{1 + G(S)} = \lim_{S \to 0} \frac{S(A)}{S(1 + \frac{K_1 S + K_2}{S(S + K_3 S + K_4 S)})} = \frac{A}{0 + K_2} = \frac{K_3 A}{K_2}
\]

This means that for a ramp input of velocity such as could occur with a reference velocity ramp generator, the vehicle velocity will lag the desired velocity by \(K_3 A/K_2\), where \(A\) is the slope of the ramp.
LOOP ANALYSIS - THE EFFECT OF DISTURBANCES

It is important to know how external disturbances will affect the loop's control ability. For example, it is important to know how much a sudden gust of headwind or tailwind will slow down or speed up the vehicle, and how long it takes the loop to recover to its quiescent state after the disturbance passes. A second form of disturbance may be due to upgrades or downgrades in the vehicle's guideways. They cause the vehicle to speed up or slow down depending on the type of grade.

Figure 11 illustrates the introduction of a step disturbance of magnitude $K_d$ into the loop. At the point of its introduction the units are those of tractive effort, lbs-force. If the disturbance was a sudden 7% upgrade, the magnitude of the disturbance for an 87,000 lb. vehicle would be: (See Equation #14)

$$K_d \approx (87,000) \sin 4^\circ \approx (87,000)(0.07) \approx 6090 \text{ lbs.}$$

If $T_d$ is a step disturbance, $K_d$, its effect on the actual vehicle velocity can be found by utilizing "Mason's Loop Rule" again where:

$$P_1 = -\frac{gK_6}{W'S} \quad \text{EQ. 58}$$

$$L_1 = -NC_1gK_6K_3 \quad \text{EQ. 59}$$

$$L_2 = -NK_2C_1gK_6 \quad \text{EQ. 60}$$
then \[ \frac{V_A}{T_d} = -\frac{gK_6}{W} \]

\[ \text{EQ. 61} \]

\[ \frac{W_\text{S}}{1+NC_1gK_6K_3} \]

\[ + \frac{NK_2C_1gK_6}{WS^2} \]

which reduces to:

\[ \frac{V_A}{T_d} = -\frac{gK_6}{W} \frac{S}{S^2 + \left(\frac{NC_1gK_6K_3}{W}\right)S + \frac{NK_2C_1gK_6}{W}} \]

\[ \text{EQ. 62} \]

but \[ K_{10} = \frac{NC_1gK_6}{W} \]

\[ \text{EQ. 63} \]

thus:

\[ \frac{V_A}{T_d} = -\left[ \frac{K_{10}}{NC_1} \right] \frac{S}{S^2 + \left(\frac{K_{10}K_3}{S+K_2K_{10}}\right)S + K_2K_{10}} \]

\[ \text{EQ. 64} \]

Thus for a step disturbance, \[ K_d \], the effect on the actual velocity, \[ V_a \], becomes:

\[ V_a = -\frac{K_{10}K_d}{NC_1} \frac{S}{S^2 + \left(\frac{K_{10}K_3}{S+K_2K_{10}}\right)S + K_2K_{10}} \]

\[ \text{EQ. 65} \]

If the loop had been critically damped in order to meet one of the design objectives then \[ \omega_n \] is equal to 1.0. If \[ \omega_n = 1.0 \], then \[ K_{10} \] becomes \[ \frac{K_3K_{10}}{2} \] in Equation #65. Since the denominator of Equation #65 is of the second order form, \[ S^2 + 2\omega_nS + \omega_n^2 \] one can rewrite the expression for \[ V_a \] substituting for \[ \omega_n \] the term \( (\frac{K_3K_{10}}{2})^2 \). Thus, \[ V_a \] becomes:

\[ V_a = -\frac{K_{10}K_d}{NC_1} \frac{S}{S^2 + \left(\frac{K_{10}K_3}{S+K_2K_{10}}\right)S + \left(\frac{K_3K_{10}}{2}\right)^2} \]

\[ \text{EQ. 66} \]
Factoring the denominator one arrives at:

\[ V_a = \frac{-K_10 K_d / N C_1}{(S + \frac{K_{10} K_3}{2})} \text{ mph (with } \mathcal{L} = 1.0) \]  

EQ. 67

The inverse Laplace Transform of Equation 67 becomes:

\[ V_a(t) = (-K_10 K_d) t \exp(-K_{10} K_3 t / (2NC_1)) \text{ mph (with } \mathcal{L} = 1.0) \]  

EQ. 68

A plot of Equation 68, \( V_a \) versus time, is shown in Curve #20, to be discussed later in this paper.
SELECTION OF THE CONTROLLER GAIN CONSTANT $K_2$

Looking at Figure 9, one can see that integrator gain $K_2$ determines the integration rate of the velocity error. The value of $K_2$ can be defined knowing the maximum vehicle velocity and the acceleration desired. If the maximum velocity is 86.2 mph as determined by a 6000 rpm motor speed, 28" diameter wheel and 5.8:1 gear ratio and the desired acceleration is 3 mph/sec, $K_2$ can be found as follows:

$$P_n = \left( \frac{V}{\sqrt{K_2}} \right) \quad \text{EQ. 69}$$

If $V$ is a constant in Equation 69 then $P_n$ in the time domain becomes $P_n = VK_2t$ where $K_2$ has the dimensions of $\text{sec}^{-1}$. The integrator "time constant" is then $1/K_2$ seconds. Thus for a desired maximum acceleration of 3 mph/sec to a terminal velocity of 86.2 mph from 0 mph we require $86.2 \text{ mph}/3 \text{ mph/sec} = 28.7 \text{ seconds}$. This means that $K_2$ is $1/28.7$ seconds or 0.0349 seconds$^{-1}$.

SELECTION OF THE CONTROLLER GAIN CONSTANT $K_3$

A design objective was that the system transient response exhibit no overshoot. This condition is referred to as a critically damped system. In a critically damped second order system the damping ratio, $\zeta$, is unity. Comparing the classic second order system transfer function:

$$\omega_n^2 \frac{S^2}{S^2 + 2\zeta \omega_n S + \omega_n^2}$$
with our system transfer function

\[ \frac{K_{10}K_2}{S^2 + (K_{3}K_{10})S + K_{10}K_2} \]

one can solve for \( \omega_n \) and \( \zeta \). \( \omega_n \), the system's natural frequency, becomes \( \sqrt{K_{10}K_2} \). \( \zeta \), the system's damping ratio, becomes \( K_3K_{10}/2\omega_n \). The constant term \( K_{10} = \frac{NC_1gK_6}{W'} \)

becomes equal to 3.36 mile/hr-sec. for \( N=1 \), \( C_1=13,300 \) lbs. \( g = 32.2 \) ft/sec\(^2 \), \( K_6 = 0.6818 \) sec/hr/ft/mile and \( W' = 87,000 \) lbs. Substituting \( K_{10} = 3.36 \) and \( K_2 = 0.0349 \) sec\(^{-1} \) into \( \omega_n = \sqrt{K_{10}K_2} \), one obtains \( \omega_n = 0.342 \) rad/second.

If \( \zeta \) is unity then \( K_3 \) becomes equal to \( 2\omega_n/K_{10} \). For \( \omega_n=0.342 \) rad/sec, and \( K_{10}=3.36 \), \( K_3 \) becomes equal to 0.204 (dimensionless). Substituting the values obtained for \( K_{10}, K_2 \) and \( K_3 \) into the system transfer function one obtains:

\[ \frac{V_a}{V_{ref}} = \frac{0.1172}{S^2 + 0.685S + 0.1172} \]  \hspace{1cm} \text{EQ. 70}

Factoring the denominator of this transfer function one obtains:

\[ \frac{V_a}{V_{ref}} \approx \frac{0.1172}{(S+0.344)(S+0.344)} \]  \hspace{1cm} \text{EQ. 71}

Thus the real roots of the critically damped system are \( S_1 \approx -0.344 \) and \( S_2 \approx -0.344 \). The transfer function of Equation 71 should provide the control specified by the design objectives. It should be again noted that the transfer function represents the linearized block diagram and not the block diagram of Figure 6. However, the gain constants of the controller found using the transfer
function should prove to be good estimates for the more exact (better approximation of the real system) computer simulation. The transfer function analysis was presented so that a design "feel" could be obtained before any simulation is begun. A discussion of the system simulation is next in order.
THE CLOSED LOOP SPEED REGULATOR SIMULATION

Up until this point any information that is known about the loop characteristics was found using the linearized block diagram of Figure 9. Remember that this block diagram was obtained by disregarding the nonlinearities associated with the vehicle dynamics and constraining our analysis to velocities below 24 mph so as to utilize the constant torque portion of the tractive effort curves.

The simulation performed was digital in nature. The reason for this choice stems from the two multipliers that are present in the block diagram of Figure 6. It is easier to obtain a digital product rather than an analog one. Digital simulations, however, have their shortcomings owing to quantization error. In addition, according to C. E. Shannon the continuous signal, \( f(t) \), can be theoretically reproduced from the sampled signal \( f^*(t) \) by linear filtering if and only if the sample period is half the period of the highest frequency contained in the continuous system. A sampled signal will be very representative of the continuous signal, if the sample interval is at least 1/10th the system's dominant time constant. The dominant time constant of this problem becomes evident if one examines the unity feedback block diagram of Figure 10. If the \( K_{10^2} \) block is

\[
\frac{K_{10^2}}{\frac{S(S+K_{10^2})}{3}}
\]
rearranged in "time constant" form, \((\tau S+1)\), one obtains:

\[
\frac{K_2/K_3}{S \left( \frac{1}{K_3 K_10} S+1 \right)}
\]

\text{EQ. 72}

Equation 72 is the rearranged open loop second order transfer function \(Wn/2\zeta\) where the time constant \(\tau\) is equal to \(\frac{1}{2\zeta Wn}\). Earlier, \((\lambda)_{n}\) for an 87,000 lb. vehicle was found to be .344, while \(\zeta\) is equal to 1. The dominant loop time constant, \(\tau\), is therefore \(1/(2)(.344) = 1.45\) seconds. The sample interval used in the simulation is 0.02 seconds (for convenience). This interval is \(1/70\)th that of the dominant loop time constant so there will be no large quantization error.

The flow chart, Figure 12, can best be followed by frequent referrals to the block diagram of Figure 6. The flow chart begins by initializing \(Vref, \ V_n, \ Vh, \ P_n, \ N, \ W, \ K_3, \ K_2\) and \((\text{time})_{n}\). \(Vref\) is initialized to the desired step velocity input, i.e. 40 mph. \(\ V_n\) is initialized to 0 mph, in the majority of cases, because we are starting from rest. \(\ Vh\) is initialized to 15 mph for the reasons discussed earlier. \(\ P_n,\) the integrator output, is initialized to 0 if \(\ V_n\) is initialized to 0. If \(\ V_n\) is not initialized to 0, then the value of \(\ P_n\) required to maintain the vehicle at some specified \(\ V_n\) in a steady state condition must be determined. One method of
determining Pn is to start the loop from a Va_n of 0 and then by inputting a Vref step input equal to the new desired Va find the Pn value that will just maintain the desired Va_n. This value can then be used in all future initializations of Pn when starting the Loop at that particular Va_n. N is the number of cars per train. W is the total train weight. K_3 is the gain of the "acceleration feedback" block of the regulator. K_2 is the gain of the regulator integrator. Time_n refers to "real time", as opposed to "machine operate time", and is in units of seconds. The n subscript designates the n
th sample interval.

The execution of operations in the simulation as shown in the flow chart follows their occurrence in the loop from left to right. The summation of the reference velocity Vref and the negative actual velocity, Va_n, produces the error, V. The computer simulation next integrates the velocity error with an integrator gain, K_2. Digitally, the integration is accomplished by incrementing the "old integrator output", Pn-1, by 0.02 times K_2V, where 0.02 is the length of the sampling period. The result of this "update" is the new integrator output, Pn. The new integrator output, Pn, is then summed with -K_3Va_n which represents the "anticipation" or acceleration feedback. The result of this summation is
$TE_1$, the tractive effort demand. The demand is then outputted to the traction motors controller.

Simulations of the motor dynamics for the acceleration mode involves a velocity dependent decision block. (Note that the deceleration mode does not involve a velocity dependent decision block.) If the vehicle velocity, $V_a$, is below 24 mph the "no" path is followed. Here, the motor tractive effort, $TE_2$, is only a function of $TE_1$, $N$, and $C_1$, where $C_1$ is the maximum per car accelerating tractive effort. However, if the vehicle velocity is greater than 24 mph, the "yes" path is taken. Here, $TE_2$ is inversely related to velocity, $V_a$, and directly related to $TE_1$, $N$, $C_1$, and the normalizing velocity, 24 mph. Notice that at exactly 24 mph, both paths produce the same $TE_2$, so that no discontinuities exist in the determination of $TE_2$.

$TE_{net}$ is the summation of 5 terms. The first term, $TE_2$, is the tractive effort demand force. From $TE_2$ is subtracted the drag and aerodynamic losses described in the discussion of the Davis Train Resistance Equation. The second, third and fourth terms are the drag terms. The fifth term is the aerodynamic loss term. $TE_{net}$ is the net force that overcomes the vehicle inertia causing acceleration.

In Figure 12, Page B, vehicle acceleration, $A$, results from the product of $TE_{net}$ and the term $gK_6/W'$. 
The term $gK_g/W'$ represents the vehicle inertia (both translational and rotational). Actual velocity, $V_{a_n}$, is obtained by integrating the acceleration, $A$. This is again accomplished "digitally" using the relationship:

$$V_{a_n} = V_{a_{n-1}} + 0.02A \quad \text{EQ. 73}$$

Equation 73 illustrates how the new actual velocity, $V_{a_n}$, is obtained by adding 0.02A to the old actual velocity, $V_{a_{n-1}}$. $A$ is the acceleration just determined, (the product of $T_{E_{net}}$ and $gK_g/W'$).

For each iteration of the flow chart main loop, where an iteration is defined as the executable statements occurring between two successive "print" operations, "time" is incremented the 0.02 seconds. In Figure 12, Page B, this is illustrated as $T_{ime_{n}} = T_{ime_{n-1}} + 0.02$. The "new" loop time $T_{ime_{n}}$ is equal to the old loop time, $T_{ime_{n-1}}$ plus 0.02 seconds. This time base is used as reference for all dynamic responses occurring in the loop simulation. Since there is little need for knowing the response values as often as 50 times in a second, the next decision block is a scheme designed to allow print out of only integer time values. The scheme in actual machine programming depends on the computer language. In Fortran IV, for example, this is accomplished as follows:

$$J = T_{ime_{n}}$$

$$Z = J$$
If (Z.NE. Time_n) Go to 3
Write (6,100) Time_n, A, Va_n
3 CONTINUE

In the above programming example, equating J and Time_n causes Time_n to become an integer (J is a fixed point variable). Equating Z to J transforms the fixed point integer to a floating point integer so that the following "IF" statement does not contain a mixed mode (i.e. 2 becomes 2.0). The "IF" statement causes the execution of the "Write" statement if and only if Time_n is an integer. Thus, we have a print out only every integer second. This was found to be more than sufficient. It should be noted that there is no "Stop" or "END" statement in the Flow chart, Figure 12, Page B. This is intentional since in almost all of the simulation runs there is no advanced way of knowing how long to run the simulation. In the computer used and in most all computer facilities a simple "call Exit" or "Stop" statement typed on the teletype console will cause the computer to stop looping.

The computer used to simulate the control loop was the Hewlett Packard 9100B programmable calculator. The machine has capability for 392 program operations if there is no stored data. Each piece of stored data decreases the number of program operations by 14. Yet with even these limitations the machine proved more than adequate for solving a fairly complex simulation problem.
A useful feature was the ability to singly step through the simulation program as an aid in debugging the program.
DISCUSSION OF SIMULATION RESULTS

The response curves accompanying this report were all generated by simulating the Basic Closed Loop Regulator of Figure 6 on the digital computer according to the flow chart of Figure 12 or variations of it which will be discussed as needed. The response curves are organized systematically to show the effect of particular constraints imposed on the Basic Closed Loop System. Note that the constraints appear on each response curve. Where possible comparison is made between the mathematical analysis of the linearized model and the simulated model of the speed regulator.

Response Curves #1-#5 illustrate the transient response of various weight vehicles and trains to a 40 mph velocity step input. No acceleration or jerk limits were imposed on the control loop at this time. The gain constants used in the simulation are: $K_3 = 0.204$ (dimensionless), $K_2 = 0.0349 \text{ secs}^{-1}$ and $V_h = 15 \text{ mph}$. Notice that these are the gain constants derived earlier in this paper for the linearized model. A critically damped loop response can be observed in Curve #3. Near critical damping is obtained in response Curve #4. The reason for the slight overshoot in Curve #4 is that $K_2$ and $K_3$ were selected for a vehicle weight of 87,000 lbs. Examining Curve #4 first note that it is for a 108,000 lb. single vehicle. Consider the system transfer function:
\[
\frac{V_{\text{act}}}{V_{\text{ref}}} = \frac{K_{10}K_2}{S^2 + K_3K_{10}S + K_{10}K_2} = \frac{\omega n^2}{S^2 + 2\omega nS + \omega n^2}
\]

EQ. 74

where \( \zeta \), the damping ratio equals \( \frac{K_3K_{10}}{2\omega n} \)

The overshoot, I believe, is caused by the decrease in \( K_{10} \) which equals \( \frac{(NC_1g)(K_6)}{W'} \). If the vehicle weight, \( W' \), increases, \( K_{10} \) decreases, which in turn, causes a decrease in \( \zeta \). The decrease in \( \zeta \) causes the overshoot since \( \zeta \) will now be less than 1.0. The loop response of Curve #3 represents a critically damped system response. Curve #1 is actually overdamped since \( K_{10} \) causes a \( \zeta \) greater than 1.0; Curve #2 is also overdamped. The 144,000 lb. total train weight is actually 72,000 lb. per car. However, in a 2 car train there are also 2 more traction motors to generate the additional tractive effort required by the added mass. Thus, for all purposes, Curve #2 is identical to Curve #1 because the weight and power additions are in proportion. The differences shown result from the Davis Train Resistance Formula where train resistance increases for a multiple car train in a nonlinear fashion.

Curve #5 is similar to Curve #4 since Curve #5 is a 2 car multiple of the simulation done for Curve #4. Notice that the two car "heavier train" Curve #5, accelerates slightly faster than the single car train, Curve #4 because of the greater tractive effort demand initially generated. The peak acceleration of the two car trains
is greater than that of the single car train. In Curve #2, the "heavier train" again accelerates faster than the lighter train of Curve #1.

The relationship between train weight and the loop time constant is as follows:

\[ \omega_n = \sqrt{51K_{10}K_2} = \sqrt{\frac{(NC_1g)(K_6)K_2}{W'}} \quad \text{EQ. 75} \]

\[ \tau = \frac{1}{\omega_n} \quad \text{. EQ. 76} \]

but \( \omega_n = 1 \)

\[ \tau = \frac{1}{\omega_n} = \frac{1}{\sqrt{\frac{(NC_1g)(K_6)K_2}{W'}}} \quad \text{EQ. 77} \]

Thus, as the weight, \( W' \), increases, the time constant, \( \tau \), will increase. Finally, note that the steady state error is zero as predicted for all 5 curves.

Because there is no acceleration limit imposed, the vehicle acceleration exceeds the desired maximum acceleration of 3 mph/sec, in Curves #1 - #5. In addition, the jerk limit of 1.5 mph/sec^2 is also exceeded. This can be verified intuitively by noting the sharp transition in the acceleration curves of Response Curves 1-5 or by "drawing" a tangent to the acceleration curve at various points and computing the slope per unit time. This transient response would probably be adequate if one were controlling the velocity of a machine tool. However, for a rapid transit vehicle, the acceleration and jerk ride comfort constraints are essential for a smooth ride.
THE EFFECT OF CONSTRAINING VEHICLE ACCELERATION

The vehicle transient response examined so far has exceeded both the 3 mph/sec acceleration and 1.5 mph/sec² jerk constraint. In an actual system, the motor controller on the vehicle has circuitry to limit the motor and ultimately the vehicle acceleration. Circuits that integrate the velocity step command input can to some extent control the acceleration and jerk. The larger the step velocity command at the motor input, the greater the velocity ramp output (acceleration) slope. Thus, a small velocity step input (which could result because of only a 5 mph desired speed increase) severely penalizes the system with a loss in headway due to the very gentle acceleration. To minimize headway loss, the system should generate the maximum allowable acceleration until the tractive effort demand generated by the speed regulator is satisfied. One method of obtaining this end is a torque limiting circuit. By limiting the motor torque to some prescribed value the vehicle acceleration is also limited. However, just limiting torque is still inadequate since a constant maximum torque for a lightly loaded vehicle will cause a greater acceleration than the same maximum torque applied to a heavily loaded vehicle. It is clear that some sort of vehicle "load weighing" scheme must be utilized to increase the maximum torque limit for heavily
loaded vehicles and decrease it for lightly loaded vehicles. The constant weighing of a vehicle in order to establish an acceleration limit is a scheme of acceleration control employed by some propulsion manufacturers. It is particularly useful in the braking or deceleration mode by allowing more brake cylinder air pressure for a heavier vehicle and less pressure for a light one. The maximum braking effort can thus remain fairly constant from vehicle to vehicle.

Acceleration limiting was easily simulated on the digital computer. Figure 13 is the flow chart followed to accomplish the acceleration limiting. This flow chart is identical to that of Figure 12 up to the determination of $T_{E_{net}}$. The decision diamond immediately following the determination of $T_{E_{net}}$ tests to determine if the tractive effort to weight ratio exceeds a predetermined dimensionless constant (0.1365). This constant was derived using the acceleration formula:

$$A \text{ (mph/sec)} = \frac{T_{E_{net}} \text{ (lbs)} x g \text{ (ft/sec}^2) x K_6 \text{ (Hr) (Ft)}}{W' \text{ (lbs)}} \text{ EQ. 75}$$

In the above formula $g = 32.2 \text{ ft/sec}^2$, $K_6 = 0.6818 (\text{Mile})(\text{Sec}) \text{ (Hr)(Ft)}$

If one lets $A = 3 \text{ mph/sec}$ and solves for $T_{E_{net}}/W'$, one obtains:

$$\frac{T_{E_{net}}}{W'} = \frac{A}{gK_6} = 0.1365 \text{ EQ. 76}$$

In EQ. 76 it is evident that if either $T_{E_{net}}$ is increased
without a corresponding increase in $W'$ or if $W'$ is decreased without a corresponding decrease in $T_{E\text{net}}$ that $A$ will exceed 3 mph/sec. The digital simulation tests the ratio of $T_{E\text{net}}$ to $W'$. If the ratio exceeds 0.1365, $A$ is set equal to 3.0 mph/sec, clamping the acceleration. The ratio of $|T_{E\text{net}}|$ to $T_{E\text{net}}$ determines the sign of the acceleration. The "clamped" value of $A$ is used in succeeding calculations in the flow chart until the ratio of $T_{E\text{net}}/W'$ is less than 0.1365. If the ratio of $T_{E\text{net}}/W'$ is less than 0.1365, acceleration is determined in the same manner as in Figure 12, Page B.

$$A = \frac{T_{E\text{net}} \times g \times K_6}{W'} \quad \text{EQ. 77}$$

The effect of constraining vehicle acceleration to 3 mph/sec is shown in response curves #6-#7. As expected, the loop delay introduced by the constraint caused a velocity overshoot. Since the constraint introduced no additional poles or zeros into the loop, the linearized loop transfer function is unchanged and the steady state step input error remains at zero. Comparing curve #3 with #6 and curves #5 with #7, one can see that acceleration limiting produces approximately 13% of overshoot. The loop is still trying to regulate to a 40 mph step velocity input, but the imposed acceleration limit has reduced the system's bandwidth and hence, increased its response time. Notice that none of the regulator gain constants have at this point been changed.
Several corrective measures may be used to produce the desired zero percent overshoot with the acceleration limit, some of which are not suitable. The system transfer function is:

\[ \frac{V_{act}}{V_{ref}} = \frac{K_1 K_2}{S^2 + K_3 K_1 S + K_1 K_2} = \frac{\omega_n^2}{S^2 + 2\omega_n S + \omega_n^2} \quad \text{EQ. 78} \]

where \( \zeta \), the damping ratio, equals \( \frac{K_3 K_10}{2\omega_n} \).

Since overshoot has occurred, \( \zeta \) must be readjusted so that it is 1.0 (for critical damping). Since \( \zeta = \frac{K_3 K_10}{2\omega_n} \), one can increase either \( K_3 \) or \( K_10 \), or decrease \( \omega_n \) in order to restore \( \zeta \) to 1.0. \( K_10 \), however, is equal to \( \frac{C_1 g K_6}{W} \)

and is considered fixed. \( \omega_n \), which is equal to \( \sqrt{K_10 K_2} \), can be decreased by reducing \( K_2 \). However, if \( \omega_n \) decreases so does the system's bandwidth. This is due to the increase in the system time constant, \( T = 1/\omega_n \). Thus, for a constant \( \zeta = 1.0 \), the system response time to a desired step input is slower. Therefore, \( \omega_n \) should remain at least equal to its present value. This fixes \( K_2 \). The only suitable means left for increasing \( \zeta \) is increasing \( K_3 \). Since the average overshoot in response curves #6 and #7 is 13%, the value of \( K_3 \) was increased by 13%, from 0.204 to 0.230.

The effect of the 13% increase of \( K_3 \) is shown in response curves #8 and #9. Note that an overshoot still occurs but that it is reduced to approximately a 5% overshoot.
Notice that the acceleration curve dips negative as it did in curves #6 and #7, but that the magnitude of the dip is reduced in curves #7 and #8. Increasing the feedback gain, $K_3$, by the same amount as the overshoot does not produce critical damping probably because of the nonlinearities of the control loop, the greatest contributor being the acceleration limiting.

Since an increase of $K_3$ by 13% resulted in reducing the overshoot by 60%, an additional 8% increase in $K_3$ should by pure guess result in a zero percent overshoot. Curves #10 and #11 demonstrate the effect of increasing $K_3$ to a value of 0.250. Notice that the overshoot is reduced to essentially zero.

Up until this point there has been no jerk limiting constraint placed upon the loop response. You will note that the acceleration and velocity curves discussed so far are not projected toward the origin (0 mph/sec and 0 mph respectively). The reason is that at 0 seconds, both acceleration and velocity are zero, but at $0^+$ seconds, the acceleration jumps to 3 mph/sec. The 3 mph/sec jump over the very small time interval (1 sample interval equals $1/50^\text{th}$ of a second) produces a jerk or rate of change of acceleration of at least $(3 \text{ mph/sec}/.02) 150 \text{ mph/sec}^2$. The first data point recorded occurs at 1 second of elapsed time, well after the initial jerk has subsided. For an average jerk of $1.5 \text{ mph/sec}^2$, two seconds must elapse between
an acceleration of 0 mph/sec and 3 mph/sec.

Jerk limiting devices on the motor consist basically of some type of integrating network that limits the rate of change of motor field current which is proportional to the motor's developed torque. The digital simulation of jerk limiting is illustrated in the Flow Chart of Figure 14, Pages A-C. Notice that acceleration limiting is also included in this flow chart. Page A of Figure 14 is identical to Page A of Figures 12 and 13. Notice that the acceleration limiting blocks following the determination of $T_{E_{\text{net}}}$ differ slightly from the method used in Figure 13. The "sign" of the net tractive effort is determined by dividing the absolute value of $T_{E_{\text{net}}}$ by $T_{E_{\text{net}}}$. Next the ratio of $|T_{E_{\text{net}}}|$ divided by $W'$ is examined. If the ratio exceeds 0.1365, the acceleration is clamped at 3 mph/sec. If the ratio does not exceed 0.1365, the acceleration is determined by the formula:

$$A_n = \frac{|T_{E_{\text{net}}}|}{W'} \times \text{"sign"} \times g \times K_6 \text{ (mph/sec)} \quad \text{EQ. 79}$$

This relationship yields identical results to the relationship used in the flow chart of Figure 12:

$$A = T_{E_{\text{net}}} \times g \times K_6 \frac{1}{W'} \quad \text{EQ. 80}$$

where the sign of $A$ was determined by the sign of $T_{E_{\text{net}}}$. The acceleration determined in Figure 14 is subscripted with an $n$. This is to denote that it represents the acceleration of the $n^{\text{th}}$ time interval and is to be stored for later calculations.
The acceleration determined in the \((n-1)\)th time interval is then subtracted from that determined in the present \(n\)th interval. A decision is made as to whether the magnitude of this acceleration difference is greater or less than the constant 0.03. This constant was determined using the relationship:

\[ J = \frac{\text{An} - \text{An-1}}{T} \quad \text{EQ. 81} \]

where \(J = \text{jerk (mph/sec}^2\)\)

\(T = \text{time interval of interest (seconds)}\).

In this instance \(J\) is made equal to our desired jerk limit, 1.5 mph/sec\(^2\), and the time interval, \(T\), is that of our sample interval 0.02 seconds. Thus, \(\frac{\text{An} - \text{An-1}}{T}\) becomes \(J \times T\) or \((1.5 \text{ mph/sec}^2) \times (0.02 \text{ sec}) = 0.03 \text{ mph/sec}\). If the acceleration difference is greater than 0.03 mph/sec in the 0.02 second interval then the jerk of 1.5 mph/sec\(^2\) has been exceeded and must be limited. Limiting the jerk consists of controlling the maximum rate of change of acceleration over the 0.02 second interval. The relationship \(\text{An} = \text{An-1} + (0.02) \times (\text{Desired Jerk}) \times (\text{sign})\), effectively increases or decreases the previous acceleration, \(\text{An-1}\), by 0.03 mph/sec, if the desired Jerk is to be 1.5 mph/sec\(^2\). The next decision diamond checks to see if the new \(\text{An}\) exceeds the acceleration limit of 3.0 mph/sec again. If it does, the new acceleration, \(\text{An}\), is clamped to 3.0 mph/sec times the sign of the net tractive effort. If \(\text{An}\) does not exceed 3.0 mph/sec, then \(\text{An}\) becomes the new acceleration.
for the $n$th time interval. Finally, if $|An - An-1|$ does not exceed 0.03 mph/sec over the time interval 0.02 seconds, the acceleration, $An$, determined in the decision blocks following determination of $TE_{net}$ is used.

The effect of jerk limiting in addition to acceleration limiting may be observed in Response Curves #12, and #13. Notice the acceleration curve. For the step velocity input, the acceleration increases from 0 mph/sec to 3 mph/sec over the time interval of 0 to 2 seconds. This yields an average jerk of 1.5 mph/sec$^2$. Note also that the acceleration limits at 3 mph/sec. The velocity response overshoots by approximately 15 percent for both the 2 car heavily loaded and the single car average loaded train. This overshoot can again be attributed to the loop delay this time caused by the jerk constraint. Again note that the steady state error is unaffected. The values of the gain constants $K_2$ and $K_3$ used in generating curves #12 and #13 are those that produce a critically damped system when only acceleration limiting occurred.

Using the same reasoning as was earlier employed for the case of acceleration limiting, $K_3$ is increased to reduce the overshoot. The value of $K_3$ was increased from 0.250 to 0.284. Response Curves #14, #15, and #16 show the effect of this increase on the system transient response. Notice
that while little or no overshoot occurs in response curves #14 and #16, that a slight "ring" or droop of 2 mph occurs just after the desired velocity has been achieved. The response settling time (time to achieve and maintain the desired velocity) has been increased considerably over that observed in earlier response curves. The settling time in these 2 response curves is approximately 34 seconds whereas in curve #10 it is only 20 seconds. However, note the smooth seemingly overdamped response of curve #15. The only variable change in all 3 curves is vehicle weight. Since one and only one value of $K_2$ and $K_3$ is selected for all vehicle weights it is expected that response variations will occur with various vehicle weights, as observed. The selection of the constant, $K_3$, was accomplished iteratively on the computer increasing it until no overshoot occurred. Since the constant selection was based on a 72,000 lb. vehicle, the effect of additional weight acts to increase system loop response time shown in response curve #15. Mathematically this is shown in Equations #75 through #77. As weight $W'$, increases, $T$, the loop time constant, increases. In addition $\omega n$, which is equal to $\sqrt{K_{10} K_2}$, decreases, decreasing the system bandwidth. The net result is a smoother response curve as shown in curve #15.

With acceleration and jerk constraints imposed, step decreases in velocity were examined for an 87,000 lb.
vehicle. The controller gain constants $K_3$ and $K_2$ used are those just selected to produce a critically damped response. A 40 mph to 20 mph step decrease is exhibited in response curve #17 and a 40 mph to a 5 mph step decrease is exhibited in curve #18. Notice that the transitions are smooth, constrained by both acceleration and jerk limiting. Also notice the absence of undershoot or ringing and the zero steady state error. In curve #17, the velocity step was small enough so that the acceleration did not reach the 3 mph/sec limit. In curves #1-#16, the 40 mph step input was chosen because it would require the maximum 3 mph/sec acceleration rate.

The transient response to a step loop disturbance is next examined. The linearized block diagram analysis of Figure 11 discussed earlier resulted in a velocity output:

$$ Va = \frac{-K_{10}K_d/MC_1}{S^2 + K_{10}K_3 S + K_2K_{10}} \text{ mph} \quad \text{EQ. 82} $$

for a step disturbance $K_d/S$. If $\lambda = 1.0$, the output becomes:

$$ Va = \frac{(-K_{10}K_d)t\exp(-K_{10}K_3t/2)}{(2\pi^2 C_1)} \text{ mph} \quad \text{EQ. 83} $$

A plot of this expression, $Va$ versus time, is shown in Curve #20 for $K_3 = 0.204$ and $K_3 = 0.284$ with a disturbance amplitude, $K_d$, equal to 6090 lbs. The 6090 lb. step disturbance used is of the same magnitude as that encountered by a vehicle suddenly climbing a 7% grade as discussed earlier. Comparing Curves #19 and #20, one can see that
the predicted loop response, curve #20, and the actual loop response, curve #19, are very similar in their general appearance. Notice the absence of undershoot or ringing and the zero steady state error. A good estimation of loop delay can be obtained by examining curve #19. The peak of the velocity droop occurs after about 4 seconds of elapsed time. This compares with a system time constant of 1.45 seconds determined from

\[ \tau = \frac{1}{2\pi \omega_n} \] for a \( K_3 = 0.204 \), using the linearized mathematical model. It appears, therefore, that acceleration and jerk limiting have added about 2.5 seconds to the loop delay.
CONCLUSION

It appears from the results of this paper that the overall closed loop velocity control system exhibits a response that meets the desired design objectives of a jerk and acceleration limited response having no overshoot and zero steady state error, for $K_2 = 0.0349 \text{ sec}^{-1}$, $K_3 = 0.284$ and a weight range from 72,000 lbs. to 108,000 lbs. The speed regulator controller is simple in design and lends itself to easy hardware implementation. In addition to solving the design objectives this paper has presented the background and a systematic approach that can be used as a guide in solving future speed control problems. The combination of classical control theory analysis and digital computer simulation has shown itself to be a very useful method of practical problem solution.
FIGURE 1. THE BASIC ESSENTIALS OF A CLOSED LOOP RAPID TRANSIT SPEED REGULATING CONTROL SYSTEM
FIGURE 2. THE BASIC ESSENTIALS OF AN OPEN LOOP RAPID TRANSIT SPEED REGULATING CONTROL SYSTEM
**Figure 3. The Speed Regulating Controller Used**
FIGURE 6. BASIC CLOSED LOOP REGULATOR - ACCELERATION/DECELERATION MODES
FOR A RAPID TRANSIT VEHICLE WITH SYNCHRONOUS MOTOR PROPULSION
FIGURE 8. THE SPEED REGULATING CONTROLLER - DIFFERENTIATOR FORM
FIGURE 9. LINEARIZED CLOSED LOOP SYSTEM
THE OPEN LOOP TRANSFER FUNCTION $G(s)$ IS:

$$G(s) = \frac{K_1 K_2}{s(s+K_3 K_1)}$$

**Figure 10.** Rearrangement of transfer function

$$\frac{K_1 K_2}{s^2 + (K_3 K_1) s + K_10 K_2}$$

To include unity feedback.
**FIGURE 12 A**

**FLOW CHART FOR SIMULATION OF CLOSED LOOP REGULATOR**

1. **READ**
   - $V_{REF}$, $V_{AN}$, $V_H$, $P_n$, $N$, $W$, $K_3$, $K_2$,
   - $t_{m1}$, $t_{m2}$

2. $V = V_{REF} - V_{AN}$

3. $P_n = P_{n-1} + 0.02(K_2)(V)$

4. $TE_1 = P_n - K_3 V_A$

5. **DIAGRAM**
   - $V_A > 24$ **YES**
     - $TE_2 = \frac{TE_1(N)(C_1)(24)}{V_A}$
   - $V_A > 24$ **NO**
   - $TE_2 = TE_1(N)C_1$

6. **RETURN TO 2**
\[ T_{E_{\text{NET}}} = T_E - \frac{(1.2)W}{2000} - 116N - \frac{0.045W_A}{2500} - \frac{0.03(N)[V_A + V_B]^2}{1} \]

\[ A = \frac{T_{E_{\text{NET}}} - 0(K_b)}{W} \]

\[ V_{A_n} = V_{A_{n-1}} + 0.02A \]

\[ \text{time}_n = \text{time}_{n-1} + 0.02 \]

**IS**

- time, AN INTEGERS
  - **NO**
  - **YES**

**PRINT**
FIGURE 13 A
FLOW CHART FOR SIMULATION OF CLOSED LOOP REGULATOR - INCLUDING ACCELERATION LIMITING

READ

\[ V = V_{REF} - V_A \]

\[ P_n = P_{n-1} + 0.02K_2V \]

\[ TB_1 = P_n - K_3V_A \]

\[ V_A > 24 \]

YES

\[ TB_2 = \frac{TB_1(N)(C_1)(24)}{V_A} \]

NO

\[ TB_2 = TB_1(N)C_1 \]

2

V_{REF}, V_A, V_H, N, P_n, W, K_3, K_2, time
\[ \frac{TR_{NET}}{2} = TB_2 - \frac{(1.3)W}{2000} - 116N - \frac{0.045Wv_A}{2000} - \frac{0.2+.03(N+1)[v_A+v_H]^2}{1} \]

\[ A = \frac{TR_{NET}}{g(K_6)} \]

\[ v_A = v_A + .02A \]

\[ \text{time} = \text{time} + .02 \]

CONTINUE

\[ \text{time} = \text{time} \]

NO \rightarrow 3

YES

\[ \text{time, } A, v_A \]

PRINT

3
FIGURE 14 A
FLOW CHART SIMULATING CLOSED LOOP REGULATOR
(JERK PLUS ACCELERATION LIMITED)

V Feinstein, Vn, Vn, N, Pn
W, K3, K2, time, An, JERK

Vn = V Feinstein = Vn

Pn = Pn-1 + 0.02K2V

TB1 = Pn - K3Vn

Vn > 24 YEB

TB2 = TB1(N)(C1)(24) / Vn

TB2 = TB1(N)C1

2
FIGURE 14 B

\[ \text{TB}_{\text{NET}} = \text{TB}_2 - \frac{1.3(W)}{2000} - \frac{0.045W_A}{2000} - \frac{(N-1)[V_A+V_H]^2}{1} \]

\[ \left| \frac{\text{TB}_{\text{NET}}}{\text{TR}_{\text{NET}}} \right| = \text{SIGN} \]

\[ \left| \frac{\text{TB}_{\text{NET}}}{W} \right| > 1.3 \rightarrow \text{NO} \]

\[ \left| \frac{\text{TB}_{\text{NET}}}{W} \right| 

\[ A_n = \left| \frac{\text{TB}_{\text{NET}}}{W} \right| \text{SIGN}(g)(K_6) \]

\[ A_n = (3)\text{SIGN} \]

\[ \left| A_n - A_{n-1} \right| > 0.03 \rightarrow \text{YES} \]

\[ A_n = A_{n-1} + (0.02)(\text{JERK})(\text{SIGN}) \]

\[ \left| A_n \right| > 3 \rightarrow \text{NO} \]

\[ \left| A_n \right| > 3 \rightarrow \text{YES} \]

5

4
TRANSIENT RESPONSE OF A 72,000 LB CAR TO A 40 MPH STEP INPUT

CONSTRAINTS:
K3 = 0.204
K2 = 0.0349
Vh = 15 mph.

No Jerk Limiting
No Acceleration Limiting

Vehicle Acceleration

Vehicle Velocity Response

TIME - SECONDS

ACCELERATION - MPH/SEC.

VELOCITY - MPH

CURVE 1
TRANSIENT RESPONSE OF A 2 CAR 14,000 LB TRAIN TO A 40 MPH STEP INPUT

CONTRAINTS:
K3 = 0.204
K2 = 0.0349
Vh = 15 mph.

No Jerk Limiting
No Acceleration Limiting

Vehicle Acceleration
Vehicle Velocity Response

CURVE 2
TRANSIENT RESPONSE OF A 87,000 CAR TO A 40 MPH STEP INPUT

CONTRAINTS:
K3 = 0.204
K2 = 0.0349
Vh = 15 mph.
No Jerk Limiting
No Acceleration Limiting

Vehicle Acceleration

Vehicle Velocity Response

CURVE 3
TRANIENT RESPONSE OF A 102,000 LB CAR TO A 40 MPH STEP INPUT

CONSTRAINS:
K3 = 0.204
K2 = 0.0349
Vh = 15 mph.

No Jerk Limiting
No Acceleration Limiting

Vehicle Acceleration

Vehicle Velocity Response

CURVE 4
TRANSIENT RESPONSE OF A 2 CAR TRAIN (216,000 LB TOTAL) TO A 60 MPH STEP INPUT

CONTRAINTS:
K3 = 0.204
K2 = 0.0349
Vh = 15 mph.

No Jerk Limiting
No Acceleration Limiting
TRANIENT RESPONSE OF A 87,000 LB CAR TO A 40 MPH STEP INPUT

CONstrains:
K3 = 0.204
K2 = 0.0349
Vh = 15 mph.

No Jerk limiting
Motor is Acceleration Limited

Vehicle Velocity Response

Vehicle Acceleration

TIME - SECONDS

VELOCITY - MPH

ACCELERATION - MPH/SEC

CURVE 6
TRANSIENT RESPONSE OF A 2 CAR TRAIN (216,000 LR) TO A 60 MPH VELOCITY STEP

CONTRAINTS:
K3 = 0.230
K2 = 0.0349
Vh = 15 mph.

No Jerk Limiting
Motor is Acceleration Limited

Vehicle Velocity Response

Vehicle Acceleration

TIME - SECONDS

VELOCITY - MPH

ACCELERATION - MPH/SEC
TRANSIENT RESPONSE OF A 37,000 LB VEHICLE TO A 40 MPH VELOCITY STEP

CONTRAINTS:
K3 = 0.230
K2 = 0.0349
Vh = 15 mph.

No Jerk Limiting
Motor is acceleration Limited

Vehicle Velocity Response

Vehicle Acceleration

TIME - SECONDS

ACCELERATION - MPH/SEC

VELOCITY - MPH

CURVE 0
TRANSIENT RESPONSE OF A 97,000 LB VEHICLE TO A 40 LBF VELOCITY STEP

CONTRAINTS:

\[ \begin{align*}
K_3 &= 0.250 \\
K_2 &= 0.0349 \\
V_h &= 15 \text{ mph.}
\end{align*} \]

No Jerk Limiting
Motor is Acceleration Limited

Vehicle Velocity Response

Vehicle Acceleration

TIME - SECONDS

ACCELERATION - MPH/SEC

VELOCITY - MPH

CURVE 10
TRANSIENT RESPONSE OF A 2 CAR TRAIN (216,000 LB) TO A 40 MPH STEP INPUT

CONTRAINTS:
K3 = 0.250
K2 = 0.0340
Vh = 15 mph.
No Jerk Limiting
Motor is Acceleration Limited

Vehicle Velocity Response

Vehicle Acceleration

CURVE 1

TIME - SECONDS

ACCELERATION - MPH/SEC

VELOCITY - MPH
TRANSIENT RESPONSE OF A 2 CAR TRAIN (216,000)
TO A 40 MPH VELOCITY STEP INPUT

CONTRAINTS:
K3 = 0.250
K2 = 0.0349
Vh = 15 mph.

Motor is Jerk Limited
Motor is Acceleration Limited

Vehicle Acceleration

Vehicle Velocity Response

VELOCITY - MPH
ACCELERATION - MPH/SEC
TIME - SECONDS

CURVE 12
TRANSIENT RESPONSE OF A 87,000 LB VEHICLE TO A 40 MPH VELOCITY STEP INPUT

CONSTRANTS:
K3 = 0.250  
K2 = 0.0349  
Vh = 15 mph.

Motor is Jerk Limited  
Motor is Acceleration Limited

Vehicle Acceleration  
Vehicle Velocity Response

CURVE 13
TRANSIENT RESPONSE OF A 87,000 LB VEHICLE TO A 40 MPH VELOCITY STEP INPUT

CONTRAINTS:
K3 = 0.250
K2 = 0.0349
Vh = 15 mph.

Motor is Jerk Limited
Motor is Acceleration Limited

Vehicle Velocity Response
Vehicle Acceleration

CURVE 14
TRANSIENT RESPONSE OF A 2 CAR TRAIN (216,000)
TO A 40 MPH VELOCITY STEP INPUT

CONSTRAINTS:
K3 = 0.284
K2 = 0.0349
Vh = 15 mph.

Motor is Jerk Limited
Motor is Acceleration Limited

Vehicle Acceleration

Vehicle Velocity Response
TRANSIENT RESPONSE OF A 1 CAR (72,000 LB) TRAIN
TO A 40 MPH VELOCITY STEP INPUT

CONTRASTS:
K3 = 0.284
K2 = 0.0349
Vb = 15 mph.

Motor is Jerk Limited
Motor is Acceleration Limited

Vehicle Acceleration
Vehicle Velocity Response

CURVE 16
Transient response of a 87,000 lb vehicle to a step velocity command from 40 mph to 30 mph.

Constraints:
- \( K_3 = 0.284 \)
- \( K_2 = 0.0349 \)
- \( V_h = 15 \text{ mph} \)

Motor is Jerk Limited
Motor is Acceleration Limited

Vehicle Acceleration
Vehicle Velocity Response

Time - Seconds
TRANSIENT RESPONSE OF A 87,000 LB CAR TO
A STEP VELOCITY COMMAND FROM 40 MPH TO 5 MPH

CONSTRAINTS:
K3 = 0.284
K2 = 0.0349
Vh = 15 mph.

Motor is Jerk Limited
Motor is Acceleration Limited

Vehicle Acceleration

Vehicle Velocity Response

CURVE 18
TRANSIENT RESPONSE OF A 87,000 LB CAR TO A 7% GRADE DISTURBANCE

VELOCITY - MPH

TIME - SECONDS

ACCELERATION - MPH/SEC

Vehicle Acceleration

Vehicle Velocity Response

CONSTRAINTS:

A = 0.0349

B = 0.00349

V = 15 mph

Motor is Jack Limited

Motor is Acceleration Limited

CURVE 19
ANALYSIS OF DISTURBANCE Kd/S ON LOOP RESPONSE

A Plot of the Response:

\[ V_a = \frac{(k_1)(k_d)}{2(c_1)T} \exp \left( -\frac{k_1(k_3)}{T} \right) \]

Where:

- \( K_d = 6090 \) lbs.
- \( K_1 = 3.36 \) sec.\(^{-1} \)

For various \( K_3 \) values.
REFERENCES


4. Dorf, R. C., Modern Control Systems, Addison-Wesley, Massachusetts, 1967