Iterative morphological filters and application in document restoration

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Iterative Morphological Filters
and Application in Document Restoration

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Submitted to the Center for Imaging Science
in partial fulfillment of the requirements for the degree of

Master of Science

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December 1995

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CERTIFICATE OF APPROVAL

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MASTER THESIS  
CERTIFICATE OF APPROVAL

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December 12, 1995  
Date
Iterative Morphological Filters
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Abstract
The binary nature of document degradation decides the suitability of morphological methods for restoration. Although the computational burden in morphological filter design can be mitigated by imposing constraints on the filter and employing the morphological filter MAE theorem in an efficient search strategy, the design constraints on the filter limit the performance of single-pass filter. It has been shown that iterative morphological filters can outperform single-pass filters. The investigation of iterative morphological filter design for image restoration is the main contribution of the present thesis. The study of iterative morphological filter design provides the understanding in depth of how filters achieve a better restoration in an iterative way. Various image-noise processes have been used to examine the effect of iteration on window constraint. Through iteration we have increased the class of filters from which an increasing estimator may be designed, so that the window constraint can be compensated by employing iterative morphological filter. Practically, we arrive at the conclusion that smaller size observation windows can achieve very similar restoration result in a MAE sense as large size windows by employing iterative design. It provides us a better practical design of increasing operators for document restoration compared to the single-pass filter using large size window. Theoretically, we arrive at the conclusion that it is not important if two operators are quite different in logical structure, and they can achieve very similar restoration effect as long as they are statistically similar.
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Dedication

This thesis is dedicated to my husband

Yongyi Ye

and

my parents
Chapter 1  Introduction

1.1 Statement of Problem and Objective

In order to extract information about a prescribed quantity of interest from an image, a statistical estimation rule is often employed to filter out noise, or to smooth information by previously obtained data or to predict the underlying process of the information. In the case of linear filters, where the filter output is a linear function of the observations, one can design a filter that minimizes the mean square error (MSE) of the output. For stationary input, the optimal solution is the Wiener filter, while for the nonstationary case, a common solution is the Kalman filter.

However, a digital image does not always possess linear properties due to digitization, quantization and sharp transitions (edges)\(^1\)\(^2\). Take the digitized document as an example. They often suffer from degradations that render them aesthetically unpleasing, machine unreadable, or human illegible. The degradations may have been introduced by the scanning process or may have been present on the hard copy original. For instance, in the scanning process, text degradation may have occurred due to system noise in the scanner, uneven illumination, or improper choice of threshold. The original hard copy may also have been degraded prior to digitization due to aging, weathering, or multi-generation reproduction processes. Or the original may be in a form that is difficult to digitize, such as transparency material or print on colored or textured paper stock. In each of these cases, the image setting is essentially binary, and therefore restoration by traditional linear method is generally inappropriate. The intrinsic nonlinearity of digital images determines the superiority of nonlinear filters in performance compared to its linear counterpart. Among the class of nonlinear operators used in signal processing and image processing
applications, morphological operators are known for their outstanding performance, parallelism and other attractive features. Restoration typically involves background cleaning, hole filling, smoothing ragged edges, or correcting stroke width. Binary morphological filters are well suited for such tasks.

An analysis of discrete morphological filters based on a finite number of observations has been given by Dougherty\(^3\)[4][5][6] in the context of the mean-square-error (MSE) criterion. There are two key aspects to the analysis. First, the entire methodology is based on the Matheron representation for translation-invariant, monotonically increasing filters; namely, that a binary (or gray-scale) filter is represented by an union (maximum in the gray-scale case) of erosions over structuring elements in the filter basis. Second, the analysis describes the optimal filter by a system of nonlinear inequalities with no known method of solution, and thus reduces filter design to the implementation of minimal search strategies over subsets of N-dimensional discrete Cartesian space, and as noted in the original work, that search analysis applies equally well to mean-absolute-error (MAE) optimization. Although the search analysis is definitive, in the sense that it derives minimal search spaces for optimal filter design, the search space can be prohibitively large if it is not mitigated in some way.

Loce extended from Dougherty's starting point to the MAE analysis for the various filter settings\(^7\). In each case, a theorem is derived that expresses overall filter MAE as a sum of MAE values of individual structuring element filters and MAE of combinations of unions (maxim) of those elements. Recursive forms of the theorems can be employed in a computer algorithm to rapidly evaluate combinations of structuring elements and search for an optimal filter basis. Although the MAE theorems provide a rapid means for examining the filter design space, the combinatorial nature of this space is, in general, too large for an exhaustive search. To render the search tractable, Loce and Dougherty developed a
constraint methodology that is employed in conjunction with the computer search\textsuperscript{[8]}. The design methodology involves constraining the class of filters from which the optimal morphological filter is to be chosen; hence, from the perspective of the original optimization paradigm of Dougherty, this approach produces a suboptimal filter. The following three constraints have been imposed to render the filter design process tractable: (1) Limiting structuring element size and shape, which is referred to as window constraint; (2) Limiting the number of structuring elements forming the filter, termed basis-size constraint; (3) Limiting the search to structuring elements derived from some library that has been chosen in an expert manner or by means of some preliminary statistics, which is called library constraint.

However, there are some drawbacks to the constrained-design approach for filter design. Chen first presented this problem\textsuperscript{[9]}. For instance, small window size limits its ability to filter out large noise components. Small basis size limits the number of geometric shapes treated by the structuring elements. Also, single-pass filters may over-filter the image in order to achieve minimum MAE. Iterative morphological filtering is an effective way to compensate for those drawbacks, thereby designing a filter from a larger class, hence producing a better statistical estimator. The investigation of iterative morphological filters for image restoration is the main contribution of the present thesis.

Image restoration involves application of a filter to an observed image, assumed to be degraded, with the intent being to restore the observed image to some ideal form. The methodology is based on statistical analysis of image and noise processes, and the consequent development of a filter (statistical estimator) to take the observed image as an input and yield an estimate of the uncorrupted image as an output. In terms of the model, the image, noise, and noisy image are random processes, so that the estimator is a function of random inputs, but in actual practice the filter is applied to a realization of the image-
noise process. In this thesis, several image models and noise models, such as maximum noise, maximum plus minimum noise, edge noise, have been developed to simulate realistic degradations that occur in binary images such as digital documents. In the filter design process, these image and image-noise models are implemented to generate realizations from which we extract single-erosion MAE statistics. These realization-based statistics are utilized in the search for the optimal combination of structuring elements.

1.2 Overview of Thesis

In Chapter 2, the general theory of binary morphological filters, binary morphological filter MAE theorems and the constraint methodology for filter design is reviewed. In Chapter 3, three types of image-noise processes are tested for restoration using iterative filtering: squares with maximum noise, squares with maximum and minimum noise, and text with edge noise. First, various image-noise processes with different initial MAE values are used in a statistical experiment to show the restoration effect of iterative design. Then, an intensive study is applied on a 5-element 3x3 observation window versus a 13-element 5x5 observation window. It is this study that is the central to the thesis. We calculate MAE values through the iterative restoration process for degraded images using the 5-element 3x3 window and the 13-element 5x5 window, respectively. We also perform Boolean logic calculation and simplification to compare two operators from two different windows as mentioned above. In practice, we try to prove whether we can get a similar restoration result through iterative design using a smaller size window instead of a larger size window. In theory, we try to analyze the underlying relationship between the two filter operators (5-element 3x3 window, 13-element 5x5 window). Chapter 4 is the conclusion and brief summary of the thesis.
Chapter 2  Optimal Binary Morphological Filters

This section presents the theory and design methodology of optimal (suboptimal) morphological filters. We first review some basic binary morphological filter theory and the optimal estimation paradigm of Dougherty. Then we review the recursive form of the binary morphological filtering MAE theorem of Loce. Last, the constrained design methodology that facilitates computationally tractable design is described in detail.

2.1 Binary Morphological Filtering

We first describe the fundamental morphological operations, then proceed onto general morphological filters and filter optimization. The definitions of the morphological operations and of general morphological filters are independent of dimension and they apply to both 2-dimensional images and 1-dimensional signals. For simplicity, we at times employ the signal notation; however, we apply the results to images later on.

There are four elementary morphological operations: erosion, dilation, opening, and closing. In designing and analyzing morphological filters, the operation of erosion plays a key role, and erosion is the operation employed in the present thesis. In the binary setting, erosion is defined by

\[ S \ominus B = \{ z : B + z \subseteq S \} \]  

(2.1)

where \( B + z = \{ b + z : b \in B \} \), and \( B \) is called a structuring element.

More generally, we consider a binary morphological filter to be a set mapping \( \Psi \) that is \textit{increasing} \( [S \subseteq T \text{ implies } \Psi(S) \subseteq \Psi(T)] \) and \textit{translation invariant} \( [\Psi(S+z) = \Psi(S) + z] \).
The kernel of an increasing, translation-invariant mapping \( \Psi \), \( \text{Ker}[\Psi] \), is the class of all images \( S \) such that \( \Psi(S) \) contains the origin.

A fundamental proposition of mathematical morphology is the Matheron representation\(^{[10]}\): a filter is translation invariant and increasing if and only if it can be expressed as a union of erosions by its kernel elements. As noticed by Dougherty and Giardina\(^{[11]}\) and Maragos and Schafer\(^{[12][13]}\), excluding certain pathological cases, a morphological filter \( \Psi \) has a basis, \( \text{Bas}[\Psi] \), of structuring elements such that the Matheron expansion can be taken over \( \text{Bas}[\Psi] \) instead of \( \text{Ker}[\Psi] \):

\[
\Psi(S) = \bigcup \{ S \ominus B : B \in \text{Bas}[\Psi] \}
\] (2.2)

The basis is a minimal (nonredundant) class of structuring elements within the kernel: for any \( B \in \text{Ker}[\Psi] \), there exists \( B' \in \text{Bas}[\Psi] \) such that \( B' \subset B \); and there does not exist a pair of structuring elements in \( \text{Bas}[\Psi] \) properly related by set inclusion. If \( B \) is the basis for a filter, we will sometimes denote the filter by \( \Psi_B \).

### 2.2 Optimal Estimation in the Binary Morphological Setting

To adapt the erosion and morphological filter definition to a digital environment and to statistical estimation, consider \( N \) binary observation random variables \( X_1, X_2, ..., X_N \). Each realization of the random vector \( \mathbf{X} = (X_1,X_2,\ldots,X_N) \) is a 0-1 \( N \)-tuple denoted by \( \mathbf{x} = (x_1,x_2,\ldots,x_N) \). If we let 1 and 0 denote points of the domain of \( \mathbf{X} \) that lie within or outside the domain \( \{1, 2, \ldots, N\} \), then each realization \( \mathbf{x} \) of \( \mathbf{X} \) constitutes a subset of \( \{1, 2, \ldots, \} \).
N\}, and we can erode \( x \) by a deterministic structuring element \( b = (b_1, b_2, \ldots, b_N) \), \( b_j \) being 0 or 1. The erosion \( x \ominus b \) is a binary functional, its value being either 0 or 1:

\[
 x \ominus b = \min_j \{ x_j; b_j = 1 \}
\]  

(2.3)

Using the Matheron representation Eq.(2.2) as a guide, in Ref.[5] an N-observation digital morphological filter is defined to be a functional of the form

\[
\Psi(x) = \max_i \{ x \ominus b_i \} = \max_i \{ \min_j \{ x_j; b_{ij} = 1 \} \}
\]  

(2.4)

where \( x \) and \( b_i \) are deterministic binary N-vectors, and assuming nonredundancy, \( \{b_i\} \) is called the basis of \( \Psi \).

Given \( N \) observation random variables \( X_1, X_2, \ldots, X_N \) and a random variable \( Y \) to be estimated, the optimal mean-absolute-error(MAE) estimator is the function \( g(X_1, X_2, \ldots, X_N) \) that minimizes the expected value,

\[
\text{MAE} = E \left[ |Y - g(X_1, X_2, \ldots, X_N)| \right]
\]  

(2.5)

Mean-square-error estimators for optimal digital morphological filters have been characterized by Dougherty\[5\][6]. In the binary setting, mean-absolute error equals the mean-square error, and therefore the analysis of Dougherty applies directly to the present study. Furthermore, it is well known that the optimal MSE estimator is given by the conditional expectation; however, rather than find the best estimator, it is common practice to look for the best estimator among a class of estimators, thereby restricting the nature of the estimation rule \( g \). Here we require \( g \) to be morphological.

For a random vector \( X \) and fixed structuring element \( b \), erosion defines an estimator in Eq.(2.4) that can be used to estimate another random variable \( Y \). The optimal mean-absolute-error erosion filter is the one defined by the structuring element \( b \) minimizing
Extension of optimality to general morphological filters involves filters minimizing $\text{MAE}(\Psi)$ over all possible choices of N-observation morphological filters $\Psi$. The optimal mean-absolute-error N-observation morphological filter is given by

$$\text{MAE}(\Psi) = E[|Y - \Psi(X)|] = E[|Y - \max_i \{X \ominus b_i\}|]$$

(2.7)

Since $\Psi$ is fully determined by its basis, finding the optimal N-observation filter reduces to selecting the subset of the $2^N$ structuring elements that yields minimum $\text{MAE}(\Psi)$.

A striking feature of the optimal MAE erosion filter is that, relative to the entire image, it is not a morphological filter. Because filtering is considered as point-wise estimation, each pixel possesses its own optimal basis. Thus, it is not spatially invariant, and therefore not translation invariant. In the case of optimizing a linear estimator, wide-sense stationarity yields spatial invariance; however, morphological estimators require strict-sense stationarity for spatial invariance. Even though such an assumption is often not fully warranted, it is typical of the kind of modeling assumption one must often apply in practice. So for the filter design in this thesis, the term of optimal morphological filters actually refers to optimal increasing filters.

2.3 Binary Mean-Absolute-Error Theorem

Loce presented a binary morphological filter MAE theorem\textsuperscript{[8]} that can be employed in a computer algorithm to search for an optimal basis. The theorem states that the MAE of a morphological filter can be expressed as a linear combination of the MAE's of its individual basis elements and their unions. Under the assumption of stationarity, a computer search
approach could directly employ Eqs.(2.6) and (2.7), which give the numerical MAE expressions for single-erosion and multiple-erosion filters, respectively, to estimate MAE from image realizations by comparing filtered noise-corrupted realizations to corresponding uncorrupted realizations. However, rather than actually filtering images to determine estimation efficiency and find the optimal filter basis, it is computationally more efficient to employ the general theorem regarding MAE for morphological filters. Employed over a range of basis sizes (basis size refers to number of structuring elements in the basis), to an allowed limit, an algorithm employing the theorem can provide the basis that yields minimum MAE.

The theorem depends upon certain structuring-element "fit" or "subset" statistics based upon an image model and image degradation model, or realizations of such models. We first examine the single-erosion case. Let S and S' respectively denote the uncorrupted and corrupted image, and B denotes the structuring element. Under the assumption of strict-sense stationarity, we can speak of the MAE for a filter as being an image error, since it is pixel independent. For a single erosion by B,

\[
\text{MAE}(B) = E \left[ | S(z) - (S' \Theta B)(z) | \right] \\
= P \left[ S(z) \neq (S' \Theta B)(z) \right] \\
= P[ (S(z)=1) \cap ( (S' \Theta B)(z)=0 ) ] + P[ (S(z)=0) \cap ( (S' \Theta B)(z)=1 ) ]
\] (2.8)

When estimating \( S(z) \) by \( (S' \Theta B)(z) \), one of two types of estimation error can occur. Type-0 error as that which occur when \( S(z) = 1 \) but \( (S' \Theta B)(z) = 0 \); eroding the observed image results in a zero at a location where the ideal value is one. Type-1 error occurs when \( S(z) = 0 \) but \( (S' \Theta B)(z) = 1 \); the erosion estimate is one where the ideal state of the pixel is zero.
The mean-absolute-error of estimation at $z$ is the sum of these two mutually exclusive error probabilities:

$$\text{MAE}(B) = p_0[B] + p_1[B]$$

(2.9)

where $p_0[B]$ and $p_1[B]$ are the probabilities of type-0 and type-1 estimation error, respectively, when erosion by $B$ is the estimation rule:

$$p_0[B] = P[ (S(z)=1) \cap (S'\Theta B)(z)=0 ) ]$$

(2.10)

$$p_1[B] = P[ (S(z)=0) \cap (S'\Theta B)(z)=1 ) ]$$

(2.11)

The assumption of strict-sense stationarity results in the MAE relative to the entire image being equal to the point-wise MAE. Provided with a suitable image model and image degradation model we may extract the probabilities of type-0 and type-1 errors, and thus calculate MAE for a given single-erosion filter.

In the general $n$-erosion case, we have the binary morphological filter MAE theorem. An $n$-erosion binary morphological filter $\Psi$ possessing basis $\text{Bas}[\Psi] = \{ B_1, B_2, \ldots, B_n \}$ will provide a point estimation with mean-absolute-error given by

$$\text{MAE}(\Psi) = \sum_{j=1}^{n} (-1)^{j+1} \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq n} \text{MAE}( \bigcup_{k=1}^{j} B_{i_k} )$$

(2.12)

The theorem states that the MAE of a morphological filter can be expressed as a linear combination of the MAE's of its individual basis elements and their unions.

For computer search efficiency, it is desirable to rewrite an expression for MAE in recursive form. A recursive form of the $n$-erosion MAE theorem can best be understood in terms of a single-erosion filter $B_n$ and two $(n-1)$-erosion filters, $\Psi_{n-1}$ and $\Phi_{n-1}$ in the following way:
An n-erosion binary morphological filter $\Psi$ possessing basis $\text{Bas}[\Psi] = \{B_1,B_2,...,B_n\}$ provides a point estimate with mean-absolute-error given by

$$\text{MAE}(\Psi_n) = \text{MAE}(\Psi_{n-1}) + \text{MAE}(B_n) - \text{MAE}(\Phi_{n-1})$$  \hspace{1cm} (2.13)

where

$$\text{Bas}[\Psi_{n-1}] = \{B_1,B_2,...,B_{n-1}\}$$  \hspace{1cm} (2.14)

$$\text{Bas}[\Psi_n] = \text{Bas}[\Psi_{n-1}] \cup \{B_n\} = \{B_1,B_2,...,B_n\}$$  \hspace{1cm} (2.15)

$$\text{Bas}[\Phi_{n-1}] = \{B_1 \cup B_n, B_2 \cup B_n,...,B_{n-1} \cup B_n\}$$  \hspace{1cm} (2.16)

To more clearly see the form of the binary morphological filter $\text{MAE}$ theorem and its recursive formulation, we write the expressions in expanded form for a specific basis size. The $\text{MAE}$ of a 3-erosion filter is given by

$$\text{MAE}(B_1,B_2,B_3) = \text{MAE}(B_1) + \text{MAE}(B_2) + \text{MAE}(B_3) - \text{MAE}(B_1 \cup B_2)$$

$$\text{MAE}(B_1 \cup B_3) - \text{MAE}(B_2 \cup B_3) + \text{MAE}(B_1 \cup B_2 \cup B_3)$$  \hspace{1cm} (2.17)

or, in recursive form,

$$\text{MAE}(\Psi_3) = \text{MAE}(B_1,B_2) + \text{MAE}(B_3) - \text{MAE}(B_1 \cup B_3, B_2 \cup B_3)$$  \hspace{1cm} (2.18)

Single-erosion-filter $\text{MAE}$ for structuring elements $B_1$, $B_2$, $B_3$ and their unions may be obtained through image models and image degradation models. $\text{MAE}$ of the corresponding 3-erosion filter may be calculated through the linear operation indicated in the $\text{MAE}$ theorem. An optimal 3-erosion filter may be designed by searching the combination which yields minimum $\text{MAE}$.
2.4 Constrained Optimality

Mean-absolute-error optimal binary morphological filters have been characterized previously in terms of the Matheron erosion representation and the binary morphological filter MAE theorem. Included in the characterization is the minimal search strategy for the optimal filter basis. However, without prior statistical information or an adequate image-noise model, even in the binary setting design is computationally intractable for moderately sized observation windows. It has been shown that the computational burden can be mitigated by imposing constraints on the filter\cite{14}. Although the resulting constrained filter will be suboptimal, if the constraints are imposed in a suitable manner, there is little loss of filter performance in return for design tractability. In this thesis, we are concerned mainly with three constraint paradigms. Although we describe each of the three individually, tractable design often involves using all three constraints in conjunction.

2.4.1 Size Constraint

First, we might wish to constrain the number of basis elements to some pre-fixed limit m to obtain the optimal m-erosion filter. By fixing a limit m we do not require that there be exactly m erosions in the filter, only that the number be bounded by m. This convention is crucial because if $\mathbf{B}$ and $\mathbf{E}$ are two structuring-element classes such that $\mathbf{B} \subseteq \mathbf{E}$, then $\Psi_\mathbf{B} \subseteq \Psi_\mathbf{E}$. Thus, forcing too large a basis can cause over-estimation: there will be too many erosions in the Matheron expansion. We call this type of constraint size constraint.

Rather than fix the maximum number of basis elements at the outset, we often choose to limit the number dynamically by plotting MAE against the number of structuring elements. When this size-MAE curve begins to flatten, we recognize that it is highly unlikely that
larger bases will have a significant effect. We could, of course, wait for a definitive increase in the size-MAE curve; however, computation time increases combinatorically with increasing basis size. Therefore it is pragmatic to choose some heuristic decision mechanism to judge when the basis is sufficiently large to preclude further significant reduction in MAE.

2.4.2 Window Constraint

A second constraint arising from our desire to reduce computation in filter design involves observation restriction. Rather than consider all possible structuring elements in a window \( W \), we might restrict our attention to structuring elements in some subwindow \( W' \) in \( W \). Strictly speaking, optimization relative to \( W' \) without constraint on the basis is actually unconstrained optimization over \( W' \); however, assuming that we would actually like to optimize over \( W \), we can view restriction to \( W' \) as a constraint that will allow us to consider only a subclass of all \( W \)-bases, thereby likely yielding a filter with increased MAE. In effect, this type of constraint is a special case of the general constraint problem in which basis restriction is relative to window geometry. We term it *window constraint*.

Suppose we wish to reduce the computational burden by employing structuring elements only in a subwindow \( W' \) of the window \( W \). Heuristically, we will delete pixels that we believe to form structuring elements that do not play a major role in reducing MAE. If our assumptions are reasonable, we will obtain a suboptimal filter that is only marginally less efficient than the optimal filter over \( W \). For instance, we might desire a 5x5 square window; however, such a window possesses \( 2^{25} = 33,554,432 \) structuring elements, and therefore there are \( 33,554,432!/m!(33,554,432-m)! \) m-element structuring element combinations. Of course, a multitude of these are redundant due to basis minimality; nevertheless, even if we limit ourselves to small \( m \), a search through all possible
structuring-element combinations is still computationally burdensome. Rather than use the 5x5 window, we can greatly decrease the search space by eliminating 8 pixels to form the subwindow W' shown in Figure 1. The cost is a constraint that impinges ultimately on filter properties, as well as on MAE.

![Figure 1](image)

**Figure 1.** (a) Unconstrained 5x5 window W. (b) Constrained 5x5 window W'.

### 2.4.3 Library Constraint

A key method of achieving design tractability over a given window W is to limit the set of potential basis elements; rather than optimize over all nonredundant collections of structuring elements within W, we restrict ourselves to some predetermined subset, or subsets, of structuring elements in the window. Specifically, we might postulate m collections of structuring elements, L₁, L₂, ..., Lₘ, each of which is suited for accomplishing a certain type of filtering task. Letting \( L = \bigcup L_i \), we constrain our basis selection process to L, and we say that the optimal filter from among those whose bases are formed from L is L-optimal. Library optimization constraint is characterized by the limitations of L. In selecting L, two conditions must be met: (1) bases formed from L must produce a class of filters that provides good suboptimality over the image range of interest, and (2) the size of L must be sufficiently small to yield design tractability.
While one might approach the construction of a library in a number of ways, here we will describe two methods. One will be based on knowledge of important filter bases. Because knowledge of filter behavior will be contained in the final library, we call it an expert library. For example, we have one class of filters. This class of filters will fill holes created by min noise; and we may call it hole-filler library. Another method of library construction is based on first-order statistical properties of structuring elements, and we call it first-order library. A first-order library is used through this thesis, so we will discuss it in detail.

A library or sublibrary may be constructed on the basis of statistics of the individual structuring elements. One key statistic of a structuring element, relative to forming filter basis, is the MAE incurred by single-erosion filtering with that element. Limiting a library to structuring elements individually possessing low MAE is a first-order-statistics approach. Various sublibraries can be constructed using this method. These first-order sublibraries may be used in conjunction with one another, or with sublibraries formed by other methods, such as expert approach. In designing a first-order library, the number of elements employed will typically be limited so that the design procedure is tractable and computationally feasible using the available computing tools.

The simplest methodology for obtaining a first-order library is to begin with a class $C$ of structuring elements (perhaps all possible structuring elements in the window), find MAE for each single-erosion filter arising from a structuring element in $C$, and let the library $L$ be composed of the $q$ structuring elements possessing the least MAE. The $L$-optimal or $m$-erosion $L$-optimal filter can then be found in the usual manner.
Chapter 3  Iterative Morphological Filters

It has been discussed in the previous chapter how the computational burden in morphological filter design can be mitigated by imposing constraints on the filter, and employing the morphological filter MAE theorem in an efficient search strategy. On the other hand, the design constraints on the filter result in suboptimal performance of the single-pass filter. For instance, small window size limits a single-pass filter's ability to filter large noise components; and the constrained optimal filter may over-filter the image in order to achieve minimum MAE. This chapter is mainly concerned with compensating for the observation window constraint by employing iterative morphological filters.

3.1 Iterative Filter Design Methodology

Consider a sequential operator $\Psi^n$ of the form

$$\Psi^n = \Psi_1 \Psi_2 \Psi_3 \ldots \Psi_n$$

where the superscript denotes operation of the complete sequence of $n$ operators and the subscript denotes the operator at a particular stage of the sequence. For a sequence of translation-invariant increasing operators, $\Psi^n$ is also an increasing translation-invariant operator, and thus possesses a basis in the Matheron representation[Eq.(2.2)]. The goal of employing iteration in filter design is to obtain an increasing operator $\Psi^n$ that approximates the ideal optimal increasing operator $\Psi$, where design of $\Psi$ by search techniques is generally an intractable problem, but design of constrained operators $\Psi_j$ of the sequence is tractable. The constraints utilized in Ref.[14] restrict the class from which filters may be designed, thus limiting the performance of the filter. Filters designed from a larger class
have improved performance. The effect of iteration on observation window size is that the class from which filters are designed is increased. That is the reason why we can use $\Psi^n$ to approximate the ideal optimal operator $\Psi$.

For $j > 1$, $\Psi_j$ operates on $N_j$ windowed variables, which themselves are each a result of $\Psi_{j-1}$ operating on $N_{j-1}$ windowed variables. For $\Psi_1$ and $\Psi_2$ we have

$$
\Psi^2 = \Psi_2 [\Psi_1(x_{1,1}, x_{1,2}, \ldots, x_{1,M}), \Psi_1(x_{2,1}, x_{2,2}, \ldots, x_{2,M}), \ldots, \Psi_1(x_{N,1}, x_{N,2}, \ldots, x_{N,M})]
$$

(3.2)

where $x_{i,j}$ denotes the $j$th observation taken at the window position $i$. For a sequential operator $\Psi^n$, the observation window on the unprocessed random variables is, in effect, the dilation of the $n$ observation windows in the sequence:

$$
W^n = W_1 \oplus W_2 \oplus \ldots \oplus W_n
$$

(3.3)

If we directly employ larger observation window $W^n$ in filter design, a search through all possible structuring-element combinations based on $W^n$ is computationally burdensome. Instead, if we iteratively employ smaller size window $W_1$, the search space can be greatly decreased. An effect of iteration is to allow for larger structuring elements in Bas[$\Psi^n$]; the class of structuring elements from which we design filters is increased. Consequently, for ideal filter $\Psi$ and sequential operator $\Psi^i$

$$
\text{MAE} \langle \Psi \rangle \leq \text{MAE} \langle \Psi^n \rangle \leq \text{MAE} \langle \Psi^{n-1} \rangle \leq \ldots \leq \text{MAE} \langle \Psi^2 \rangle \leq \text{MAE} \langle \Psi^1 \rangle
$$

(3.4)
Note that iteration of optimal filters where the window contains the origin can never increase MAE because the constrained approach to design can produce the identity filter for \( \Psi^j \).

It is important to realize that although \( W^n = \bigoplus_{i=1}^{m} W_i \), due to the nature of the sequential operations, \( \Psi^n \) is not designed from the class of all possible structuring elements generated by \( W^n \).

In this thesis, we use the following iterative filter design methodology: (1) Given ideal image \( S \) and degraded image \( S' \), search the optimal filter based on single-erosion MAE, i.e., first-order library. (2) Apply the optimal filter obtained in (1) on degraded image \( S' \) to obtain \( S'' = \Psi_1(S') \). (3) Using \( S'' \) as degraded image, repeat steps (1) and (2). Note that a new library is used for each design iteration.

To illustrate iterative morphological filter performance, we examine several image restoration examples involving different image types and image degradation types. Images in their ideal state, degraded, and filtered are shown with the corresponding filter bases and mean-absolute-error.

### 3.2 Restoration of Squares with Maximum Noise

Consider image processes that consist of squares Poisson randomly distributed in the grid with two different densities \( D_{s_1} = 0.002 \) and \( D_{s_2} = 0.006 \), and each square is \( Z \) by \( Z \) in dimension, \( Z \) being Gaussian distributed with fixed variance \( \sigma^2 = 4 \) and six different mean values \( U_z = 8, 9, 10, 11, 12 \) and 13 respectively. Overlap of squares is allowed in these
image processes. Also consider noise processes which consist of 3x3 supports Poisson randomly distributed in the grid with two different densities $D_{n1} = 0.02$ and $D_{n2} = 0.08$, and within each 3x3 noise support, the quantity of noise pixels are randomly uniformly distributed over [1,5]. There are totally 24 combinations of image processes and noise processes. The corrupted image is the set addition of the noise and the uncorrupted image (maximum noise). Of the 24 image-noise processes, the minimum MAE = 0.022291 with maximum area coverage 60.6% (when $U_s = 13$, $D_s = 0.006$ and $D_n = 0.02$); the maximum MAE = 0.191813 with minimum area coverage 11.1% (when $U_s = 8$, $D_s = 0.002$ and $D_n = 0.08$).

The sizes of realizations of image-noise processes need to be large enough in order for us to extract enough statistics from those realizations for a robust filter design. A 1024x1024 pixel $^2$ realization of each process is used for filter design in this thesis. A 9-element 3x3 observation window is utilized, as shown in Figure 2. In this thesis, the same window shape is used for iterations. However, we need to point out that the experiment could also be done using different window shapes for iterations. We limit the number of structuring elements forming each filter basis to 6. Iterative filters have been designed using both expert library and first-order library methods, with the library size being 100. In the first iteration, because only maximum noise exists, the first-order library method employs a 100-element max type library, which consists of 100 structuring elements possessing lowest MAE and having an activated pixel at the origin (except the single-pixel origin element). It is termed a "max library" because the elements have a restorative effect in the presence of max degradation (where the noise process consists of set additions). After the first iteration, it is quite possible that the degraded image is over-filtered, which means both maximum noise and minimum noise exist in the first-filtered image. So the following iterations employ a 100-element general library.
We organize the 24 image-noise processes into 4 groups. Within each group, 6 image-noise processes possess the same ideal image density and noise density, but different ideal image component size mean values. The MAE vs. iteration number for those 4 groups of processes are plotted in Figure 3 to Figure 6, respectively. From these plots, we can conclude: (1) Because of the maximum noise process, the initial MAE for each image-noise process is quite different, even though some of them possess the same noise density. The initial MAE is large if the process has small image density, small ideal image component size and large noise density. (2) When the noise density is small (with low noise density $D_n = 0.02$), after the first iteration, there is almost no improvement on MAE for more iterations. (3) When noise is relatively large (with high noise density $D_n = 0.08$), after three iterations, there is almost no improvement on MAE for more iterations. Comparing with conclusion (2), it is clear that the iterative filters have a better performance than a single-pass filter when the initial MAE of the degraded image is relatively high. (4) The larger the initial MAE in the degraded image, the larger the final MAE in the finally restored image.

Take one image-noise process as an example, which possesses image component size mean value $U_s = 11$, image density $D_s = 0.002$, and noise density $D_n = 0.08$. The ideal image, degraded image, restored images and filter bases at each iteration are shown in Figure 7 to Figure 13, respectively.

In order to see how degraded images are filtered at each iteration in more detail, we observe the area of an image enclosed in a rectangle. After the first iteration, most maximum noise
is filtered out. But, at the same time, some of the corners of image squares are missing too, which means the image is over-filtered. After the second iteration, less maximum noise is left while more corners are missing. After the third iteration, most of missing corners are put back again. This observation shows that the over-filtering of a single-pass filter can be compensated by using iterative filters. In other words, iterative filters can perform a better job than a single-pass filter to remove noise and also not destroy the image itself too much.

We can also analyze the behavior of iterative filters from the point of view of filter bases. Figure 13 shows filter bases at each iteration. In iteration #1, the first three structuring elements perform a function similar to opening. They tend to smooth horizontal and vertical edges within an image, and eliminate thin protrusions and isolated point noises. It is quite possible that the image is over-filtered (e.g. missing corners) by those opening-like operations. In the filter bases of iteration #3, the first four structuring elements perform corner-filling in order to put those missing corners back. The process of corner-filling is illustrated in Figure 14: $S'$ is a square with the right-bottom corner missing. After applying the first structuring element in the filter basis for iteration #3, we get the square $S$ with the missing corner coming back.
Figure 3. Mean-absolute-error vs. iteration number

( $D_s = 0.002, D_n = 0.02$ )
**Figure 4.** Mean-absolute-error vs. iteration number  
\( (D_s = 0.002, D_n = 0.08) \)
Figure 5. Mean-absolute-error vs. iteration number

\((D_s = 0.006, D_a = 0.02)\)
Figure 6. Mean-absolute-error vs. iteration number

\[ D_s = 0.006, \quad D_n = 0.08 \]
Figure 7. Original image (area coverage=20%)

Figure 8. Degraded image (max noise, MAE=0.172179)
Figure 9. After first iteration restoration (MAE=0.016712)

Figure 10. After second iteration restoration (MAE=0.009536)
Figure 11. After third iteration restoration (MAE=0.006770)

Figure 12. Restored image (MAE=0.005548)
Figure 13. Optimal bases for each iteration

Figure 14. Illustration of corner-filling
3.3 Restoration of Squares with Maximum and Minimum Noise

Now consider the image processes and noise processes of the previous section. But this time the corrupted image is both set addition and set subtraction of the noise and the uncorrupted image, which means both maximum and minimum noise are applied to the ideal image. A 1024x1024 pixel\(^2\) realization of each image-noise process is used in filter design with the 9-element 3x3 observation window shown in Figure 2. We limit the number of structuring elements forming the filter basis to 6. Iterative filters have been designed using a 100-element general first-order library.

The MAE vs. iteration number for those 4 groups of image-noise processes are plotted in Figure 15 to Figure 18, respectively. In each case, the curve stops when MAE value does not change at all or only changes very little. From these plots, we can conclude: (1) Because image processes are degraded by both maximum noise and minimum noise, the ideal image-noise processes with same noise density possess very similar initial MAE values. The initial MAE is around 0.058 when the noise density is low \(\mathbf{D}_n = 0.02\); the initial MAE is around 0.215 when the noise density is relatively high \(\mathbf{D}_n = 0.08\). (2) When noise density is low, after the fourth iteration, there is not much more improvement on MAE for more iterations. (3) When noise density is high, after the eighth iteration, there is not much more improvement on MAE for more iterations. (4) In the case of low image density, the smaller the image component size, the smaller the final MAE value. The reason is that there are more image edges in the realizations when image component size becomes larger. Hence, there is more opportunities that maximum noise and minimum noise degrade edge parts of the image, i.e., more edge noise is introduced into the image process. Edge noise is more difficult to filter out compared to those isolated maximum noise and minimum noise. So the image-noise process with larger image component size and more edge noise ends up with larger final MAE value. (5) In the case of high image density, we can get the
same conclusion as (4) when image component size is small. But when image component size becomes larger, there is more overlap between image squares, which leads to fewer image edges when the density of the image is very high. So, when image component size becomes larger, we get the opposite conclusion that the larger the image component size, the smaller the final MAE value. (6) Comparing Figure 15 and Figure 17, image-noise processes in Figure 15 and those in Figure 17 possess the same noise density and similar initial MAE value, but different image densities. Final MAE is larger in the case of high image density, because there is more edge noise in the corresponding realizations. We can get the same conclusion when comparing Figure 16 and Figure 18.

Take one image-noise process as an example, which possesses image size mean value $U_s = 11$, image density $D_s = 0.002$, and noise density $D_n = 0.08$. The ideal image, degraded image, restored images and filter bases at each iteration are shown in Figure 19 to Figure 26, respectively.

Because both maximum and minimum noise are applied to the ideal image, the restoration process becomes more complicated than the case of Section 3.2, in which only maximum noise was used. From the filter bases shown in Figure 26, we see that smoothing and corner-filling alternately happened during the iterative restoration of the degraded image.
Figure 15. Mean-absolute-error vs. iteration number

( $D_s = 0.002$, $D_a = 0.02$ )
Figure 16. Mean-absolute-error vs. iteration number

\( D_s = 0.002, D_n = 0.08 \)
Figure 17. Mean-absolute-error vs. iteration number

(\( D_i = 0.006, D_n = 0.02 \))
Figure 18. Mean-absolute-error vs. iteration number

\( D_s = 0.006, \quad D_e = 0.08 \)
Figure 19. Original image (area coverage=20%)

Figure 20. Degraded image (max and min noise, MAE=0.215855)
Figure 21. After first iteration restoration (MAE=0.099984)

Figure 22. After second iteration restoration (MAE=0.072196)
Figure 23. After third iteration restoration (MAE=0.060059)

Figure 24. After eleventh iteration restoration (MAE=0.045965)
Figure 25. Restored image (MAE=0.045928)

Iteration #1:

Iteration #2:

Iteration #3:

Iteration #4:
Figure 26. Optimal bases for each iteration
3.4 Restoration of Text with Maximum Edge Noise

Consider a text image with 19% area coverage shown in Figure 27. And consider a realistic maximum edge-noise process in which the maximum noise has been randomly adjoined to the edge pixels of the text image. Figure 28 shows a degraded realization with MAE = 0.023113. By changing thresholds in the generating edge-noise process, we can get image-noise processes with different MAE values. We generated 4 image-noise processes with MAE = 0.023113, 0.020845, 0.018738 and 0.016590, respectively. A 1024x1024 pixel\(^2\) realization of each image-noise process is used in filter design with the 9-element 3x3 observation window shown in Figure 2. Basis size is limited to 6 and the iterative filters have been designed using a 100-element first-order library (max type).

The MAE vs. iteration number for those 4 image-noise processes are plotted in Figure 29. The plot illustrates: (1) The MAE value has little improvement beyond the second or third iteration. (2) The larger the initial MAE in the degraded image, the larger the final MAE in the finally restored image.

The restored images of Figure 28 and the corresponding filter bases at each iteration are shown in Figure 30 to Figure 34, respectively. In this case, the filter bases slowly restore the degraded image until no further improvement.
Figure 27. Original text image (area coverage=19%)

Figure 28. Degraded image (max edge noise, MAE=0.023113)
Figure 29. Mean-absolute-error vs. iteration number

Figure 30. After first iteration of restoration of Figure 28 (MAE=0.012922)
Figure 31. After second iteration of restoration of Figure 28 (MAE=0.010771)

Figure 32. After third iteration of restoration of Figure 28 (MAE=0.010319)
Figure 33. Restored image of Figure 28 (MAE=0.010213)

Figure 34. Optimal bases for each iteration
3.5 3x3 vs. 5x5 Window for Restoration of Squares with Maximum and Minimum Noise

Consider a 5-element 3x3 observation window \( W_1 \) as shown in Figure 35 and a 13-element 5x5 observation window \( W_2 \) as shown in Figure 36. In effect, observation window \( W_2 \) is the dilation of two observation windows, \( W_1 \):

\[
W_2 = W_1 \oplus W_1
\]  

(3.5)

![Figure 35. 5-element 3x3 window \( W_1 \)](image)

![Figure 36. 13-element 5x5 window \( W_2 \)](image)

We experimentally show that, if we perform two iteration restoration on a degraded image using observation window \( W_1 \), we get a similar restoration effect as if we perform only one iteration restoration using observation window \( W_2 \). In Section 3.3, we tested 4 groups of image-noise processes. Here we choose one image-noise process from each group and perform iterative restoration using observation windows \( W_1 \) and \( W_2 \), respectively. Basis size is constrained to be 6. The MAE vs. iteration number for those 4 image-noise processes are plotted in Figure 37 to Figure 40, respectively. We see that, in each case, two iteration restoration using observation window \( W_1 \) gives us very similar MAE value as the one iteration restoration using observation window \( W_2 \). And also we can see that,
Figure 37. MAE vs. iteration number using windows $W_1$ and $W_2$
($U_s = 8, D_s = 0.002, D_n = 0.02$)

Figure 38. MAE vs. iteration number using windows $W_1$ and $W_2$
($U_s = 8, D_s = 0.002, D_n = 0.08$)
Figure 39. MAE vs. iteration number using windows $W_1$ and $W_2$
\( U_s = 9, D_s = 0.006, D_n = 0.02 \)

Figure 40. MAE vs. iteration number using windows $W_1$ and $W_2$
\( U_s = 8, D_s = 0.006, D_n = 0.08 \)
with the iteration number increasing, the curve of window $W_1$ can finally get very close to or merge with the curve of window $W_2$. These observations mean that the observation window constraint can be compensated by employing iterative morphological filters.

Now we take the image-noise process of Figure 37 as an example to do more detailed analysis for iterative filters. Two iteration filter bases from $W_1$ are plotted in Figure 41, and the one iteration filter basis from $W_2$ in Figure 42.

**Figure 41.** Filter bases $\Psi_1$ and $\Psi_2$ for window $W_1$

**Figure 42.** Filter basis $\Psi^1$ for window $W_2$
In the case of Figure 41, two operators $\Psi_1$ and $\Psi_2$ sequentially operate on the degraded image $S'$ in the form of:

$$\Psi^2 = \Psi_2 (\Psi_1 (S'))$$

(3.6)

We perform logical calculation of Eq.(3.6) and logical simplification using mathematical software MAPLE. The resulting operator $\Psi^2$ consists of 54 structuring elements, as shown in Figure 43.
Now we know operator $\Psi^2$ and operator $\Psi^1$ give us almost the same MAE value ($\Delta\text{MAE} = 3 \times 10^{-5}$). So what is the underlying relationship between $\Psi^2$ and $\Psi^1$? Next we try to answer this question using the error analysis method presented by Dougherty and Loce in their paper about the precision of filter estimation via realizations\(^{15}\).

For error analysis in binary-filter design we treat the N observations in the window as an N-vector $\mathbf{x}$. These vectors are $\{0,1\}$-valued, each consists of a string of N 0's and 1's, and they form a lattice, $L_N$, under the ordering $\mathbf{w} \geq \mathbf{x}$ if and only if every $\{0,1\}$-element in the string defining $\mathbf{w}$ is greater than or equal to the corresponding element in the string defining $\mathbf{x}$. The upper set $U[\mathbf{x}]$ of a vector $\mathbf{x}$ is the collection of all vectors $\mathbf{w} \geq \mathbf{x}$ in the lattice. Figure 44 shows the upper set of 0011 in $L_4$. The element 0011 and its upper set are shown highlighted in black.
A binary filter $\Psi$ defined on $N$ windowed binary variables can be considered to be a binary mapping on $L_N$. The requirement that $\Psi$ be a translation-invariant windowed mapping means that it is only dependent on $L_N$, not on the window position in the grid. Given $\Psi$ is defined on $L_N$, we call it morphological if it is increasing on $L_N$: $w \geq x$ implies $\Psi(w) \geq \Psi(x)$.

Any binary mapping on $L_N$ is determined by $S_1(\Psi) = \{ x: \Psi(x) = 1 \}$ or, equivalently, by $S_0(\Psi) = \{ x: \Psi(x) = 0 \}$. Indeed, this is just a special instance of the fact that any mapping is determined by its inverses over its range. We call $S_1(\Psi)$ and $S_0(\Psi)$ the 1-set and 0-set of $\Psi$, respectively. If the optimal filter $\Psi$ has basis $B = \{ x_1, x_2, ..., x_m \}$, the 1-set of the filter $\Psi$ is the union of the upper sets of the basis elements:

**Figure 44.** Upper set of 0011 in lattice $L_4$
\[ S_1(\Psi) = \cup_{x \in B} U[x] \tag{3.7} \]

Since a binary filter \( \Psi \) is determined by its 1-set or 0-set, we can compare two filters \( \Psi^2 \) and \( \Psi^1 \) by examining the difference between their 1-sets and 0-sets.

Table 1 lists the number of correct and incorrect elements found in the 1-set and 0-set of \( \Psi^2 \) relative to \( \Psi^1 \). From Table 1, the number of error elements of \( \Psi^2 \) relative to \( \Psi^1 \) is \( 1205 + 870 = 2075 \), and it is \( 2075/2^{13} = 25.33\% \) the size of the lattice. Such a high percentage implies that, from the logical structure point of view, \( \Psi^2 \) and \( \Psi^1 \) are quite different. We calculate the probability that those error elements can be seen in the degraded image, and the resulting probability is only 1.4%. It means it is not important that \( \Psi^2 \) and \( \Psi^1 \) are structurally different, and as long as we hardly see those error elements in the degraded image, \( \Psi^2 \) and \( \Psi^1 \) can still be quite similar from the statistics point of view. Actually, that is the reason why the two filters \( \Psi^2 \) and \( \Psi^1 \) can produce very similar MAE values.

The original image, the degraded image and the restored images filtered by \( \Psi^1 \) and \( \Psi^2 \) are shown in Figure 45 to Figure 48, respectively.

<table>
<thead>
<tr>
<th>Filter</th>
<th>Number of elements in 1-set of ( \Psi^1 ) and 1-set of ( \Psi^2 ) (correct elements)</th>
<th>Number of elements in 1-set of ( \Psi^1 ) and 0-set of ( \Psi^2 ) (error elements)</th>
<th>Number of elements in 0-set of ( \Psi^1 ) and 0-set of ( \Psi^2 ) (correct elements)</th>
<th>Number of elements in 0-set of ( \Psi^1 ) and 1-set of ( \Psi^2 ) (error elements)</th>
</tr>
</thead>
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<td>( \Psi^1 )</td>
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<td>1552</td>
<td>1205</td>
<td>4565</td>
<td>870</td>
</tr>
</tbody>
</table>

**Table 1.** Effect of iterative filter on the 1-set and 0-set
Figure 45. Original image (area coverage=11%)

Figure 46. Degraded image (max and min noise, MAE=0.058942)
Figure 47. Restored image by $\Psi^i$ (MAE=0.01291)

Figure 48. Restored image by $\Psi^2$ (MAE=0.01294)
3.6 3x3 vs. 5x5 Window for Restoration of Text with Maximum Edge Noise

In Section 3.4, we tested 4 text-edgenoise processes. Here we perform iterative restoration on one of those processes (case 1 in Figure 29) using observation window $W_1$ in Figure 35 and observation window $W_2$ in Figure 36, respectively. As above, we constrain basis size to be 6. The MAE vs. iteration number for this process is plotted in Figure 49. Again we see that two iteration restoration using $W_1$ yields a very similar MAE value as the one iteration restoration using $W_2$, and after several iteration, the curve of window $W_1$ merges with the curve of window $W_2$. Two iteration filter bases from $W_1$ are plotted in Figure 50, and one iteration filter basis from $W_2$ in Figure 51. We perform logical calculation and logical simplification on $\Psi_1$ and $\Psi_2$ in Figure 50 according to Eq.(3.6). The resulting operator $\Psi^2$ consists of 86 structuring elements, as shown in Figure 52.

![Figure 49](image.png)  
*Figure 49.* MAE vs. iteration number using windows $W_1$ and $W_2$
Iteration #1:
\[ \Psi_1 \]

Iteration #2:
\[ \Psi_2 \]

Figure 50. Filter bases \( \Psi_1 \) and \( \Psi_2 \) for window \( W_1 \)

Figure 51. Filter basis \( \Psi^1 \) for window \( W_2 \)
Now we compare operator $\Psi^2$ in Figure 52 and operator $\Psi^1$ in Figure 51. Table 2 lists the number of correct and incorrect elements found in the 1-set and 0-set of $\Psi^2$ relative to $\Psi^1$.

From Table 2, the number of error elements of $\Psi^2$ relative to $\Psi^1$ is $1495 + 336 = 1831$, and it is $1831/2^{13} = 22.35\%$ of the size of the lattice. We calculate the probability that we can see those error elements in the degraded image, and the resulting probability is only $0.4\%$. So for text-edgenoise model, we arrive at the same conclusion as in Section 3.5: even though $\Psi^2$ and $\Psi^1$ are quite different from the logical structure point of view, as long as the probability we see those error elements in the degraded image is very small, $\Psi^2$ and $\Psi^1$ can be statistically similar, and therefore they can produce very close MAE values.
The original image and the degraded image are showed in Figure 27 and Figure 28, respectively. The restored images filtered by $\Psi^1$ and $\Psi^2$ are shown in Figure 53 and Figure 54, respectively.

<table>
<thead>
<tr>
<th>Filter</th>
<th>Number of elements in 1-set of $\Psi^1$ and 1-set of $\Psi^2$ (correct elements)</th>
<th>Number of elements in 1-set of $\Psi^1$ and 0-set of $\Psi^2$ (error elements)</th>
<th>Number of elements in 0-set of $\Psi^1$ and 0-set of $\Psi^2$ (correct elements)</th>
<th>Number of elements in 0-set of $\Psi^1$ and 1-set of $\Psi^2$ (error elements)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Psi^1$</td>
<td>2896</td>
<td>0</td>
<td>5296</td>
<td>0</td>
</tr>
<tr>
<td>$\Psi^2$</td>
<td>1401</td>
<td>1495</td>
<td>4960</td>
<td>336</td>
</tr>
</tbody>
</table>

Table 2. Effect of iterative filter on the 1-set and 0-set

Figure 53. Restored image by $\Psi^1$ (MAE=0.012933)
Figure 54. Restored image by $\psi^2$ (MAE=0.012933)
Chapter 4  Conclusion

The present thesis has described binary iterative optimal morphological filters and has applied this class of filters to restoration of digital documents. The optimal morphological filter paradigm, MAE theorems and the constraint methodology that render filter design practical has been reviewed. Although the computational burden in morphological filter design can be mitigated by imposing constraints on the filter and employing the morphological filter MAE theorem in an efficient search strategy, the design constraints on the filter limit the performance of single-pass filters. It has been shown that iterative morphological filters can outperform single-pass filters. The investigation of iterative morphological filter design for image restoration is the main contribution of the present thesis.

The study of iterative morphological filter design provides the understanding in depth of how filters achieve a better restoration in an iterative way. Various image-noise processes have been used to examine the effect of iteration on window constraint. Through iteration we have increased the class of filters from which an increasing estimator may be designed, so that the window constraint can be compensated by employing iterative morphological filters. Practically, we arrive at the conclusion that smaller size observation windows can achieve very similar restoration result in a MAE sense as large size windows by employing iterative design. It provides us a better practical design of increasing operators for document restoration for certain models compared to the single-pass filter using large size window. Theoretically, we arrive at the conclusion that it is not important if two operators are quite different in logical structure, and they can achieve very similar restoration effect as long as they are statistically similar.
References


