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William Basener

Carl Lutzer

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A $C^\infty$ diffeomorphism of $\mathbb{R}^2$ that has a Cantor set that is a minimal set. - DRAFT

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BILL BASENER AND CARL LUTZER

ABSTRACT. We present a $C^\infty$ diffeomorphism of $\mathbb{R}^2$ that has a Cantor Set that is a minimal Set. The Cantor Set is contained inside an annulus.

1. Introduction

For a homeomorphism $f : X \rightarrow X$ of a topological space $X$, a nonempty compact subset $Y \subset X$ is a minimal set if for every $y \in Y$ the orbit of $y$ is dense in $Y$. Denjoy showed (see [4]) that any diffeomorphism of $S^1$ that has a Cantor set which is a minimal set cannot be $C^2$. Our example shows that this restriction does not hold for a diffeomorphism of the annulus.

This raises the question of whether diffeomorphisms of other manifolds can be smoother than $C^2$ and have a Cantor set as a minimal set. We answer this in the affirmative by constructing a $C^\infty$ diffeomorphism of $\mathbb{R}^2$ that has a Cantor set which is a minimal set. We will refer to a Cantor Set that is a minimal set as a Cantor minimal set.

We need the following definition

DEFINITION 1. For $F : \mathbb{R} \rightarrow \mathbb{R}$ any $j$-times differentiable map we define

$$||F||_{C^j} = \sup_{x \in \mathbb{R}, 1 \leq i \leq j} \left| \frac{d^i F}{dx^i}(x) \right| + \sup_{x \in \mathbb{R}} |F(x)|,$$

and we need the following theorem.

THEOREM 1. Let $f_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $i = 1, 2, \ldots$ be a sequence of functions such that:

1. For every $i$, $f_i$ is $C^\infty$.
2. The sum $\sum_{i=1}^\infty ||f_i - f_{i+1}||_{C^j}$ converges.

Then $f_i \rightarrow f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $f$ is $C^\infty$.

The map can be loosely described as follows. Let $p_1, p_2, \ldots$ be an infinite sequence of positive integers which are pairwise relatively prime. Let $D$ be the unit disk in $\mathbb{R}^2$. Let $f_1$ be a rotation of $D$ by $2\pi/p_1$. For each $i = 0, 1, \ldots, p_1 - 1$ let $D(i)$ be a closed disk contained in $D$ such that $f_1$ takes $D(i)$ to $D(i+1 \mod p_1)$ and such that $D(i) \cap D(j) = \emptyset$ for
$i \neq i'$. (See Figure 1.) For each $i = 0, 1, \ldots, p_1 - 1$ define a closed disk $D^{(i)}$ such that $D^{(i)} \subseteq \text{intt} D^{(i)}$ and $D^{(i)} \cap D^{(i')} = \emptyset$ for $i \neq i'$.

Let $f_2$ be a function that rotates each $D^{(i)}$ by $2\pi/p_2$ and is the identity outside of the $D^{(i)}$. For each $i = 0, 1, \ldots, p_1 - 1, j = 0, 1, \ldots, p_2 - 1$ define a closed disk $D^{(i,j)}$ such that such that $f_2$ takes $D^{(i,j)}$ to $D^{(i,j+1 \mod p_2)}$. Hence $f_2 \circ f_1$ takes $D^{(i,j)}$ to $D^{(i+1 \mod (p_1), j+1 \mod (p_2))}$.

Continuing by induction, for every $i \in \mathbb{N}$, we define a homeomorphism $f_i : \mathbb{R}^2 \to \mathbb{R}^2$ such that:

- $f_1$ rotates every $D^{(x_1,\ldots,x_{i-1})}$ by $2\pi/p_1$, where $(x_1,\ldots,x_{i-1}) \in \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_{i-1}}$.
- $f_i$ is the identity off the $D^{(x_1,\ldots,x_{i-1})}$.

Then define disjoint disks $D^{(x_1,\ldots,x_{i-1},x_i)}$, where $(x_1,\ldots,x_{i-1},x_i) \in \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_{i-1}} \times \mathbb{Z}_{p_i}$, and disks $D^{(x_1,\ldots,x_{i-1},x_i)}$ with $D^{(x_1,\ldots,x_{i-1},x_i)} \subset \text{intt} D^{(x_1,\ldots,x_{i-1},x_i)}$, such that

- $D^{(x_1,\ldots,x_{i-1},x_i)} \cap D^{(y_1,\ldots,y_{i-1},y_i)} = \emptyset$ for $(x_1,\ldots,x_{i-1},x_i) \neq (y_1,\ldots,y_{i-1},y_i)$
- $f_i \circ f_{i-1} \circ \cdots \circ f_2 \circ f_1$ takes $D^{(x_1,\ldots,x_{i-1},x_i)}$ to $D^{(x_1+1 \mod (p_1),\ldots,x_{i-1}+1 \mod (p_{i-1}),x_i+1 \mod (p_i))}$
- $f_i$ is the identity off the $D^{(x_1,\ldots,x_{i-1},x_i)}$.

We show that the map $f = \cdots \circ f_i \circ f_{i-1} \circ \cdots \circ f_2 \circ f_1$ is continuous in Section 2.

**Figure 1.** The first three steps in creating the Cantor set $C$ for $p_1 = 3$, $p_2 = 5$, and $p_3 = 7$.

The points in the Cantor set $C = \bigcap_{i=1}^{\infty} \left( \bigcup_{(x_1,x_2,\ldots,x_i) \in \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_i}} D^{(x_1,x_2,\ldots,x_i)} \right)$ can be indexed by the group

$$G = \times_{i=1}^{\infty} \mathbb{Z}_{p_i}$$
using the map

\[(x_1, x_2, x_3, \ldots) \mapsto D_{(x_1)} \cap D_{(x_2)} \cap D_{(x_1, x_2, x_3)} \cap \cdots.\]

We can now prove that \(C\) is a minimal set for \(f\).

**Proposition 1.** The set \(C\) is a minimal set for the map \(f\).

**Proof.** It follows from our definition that for \((x_1, x_2, x_3, \ldots) \in C\),

\[f(x_1, x_2, \ldots) = (x_1 + 1 \mod (p_1), x_2 + 1 \mod (p_2), \ldots) \in C\]

For any \(\epsilon > 0\) there exists an \(N\) such that for two points \((x_1, x_2, x_3, \ldots), (y_1, y_2, y_3, \ldots) \in C\), \(|(x_1, x_2, x_3, \ldots) - (y_1, y_2, y_3, \ldots)| < \epsilon\) if \(x_i = y_i\) for all \(i < N\). This follows because \(\text{diam} D_{(x_1, \ldots, x_i)} \to 0\) as \(i \to \infty\). So to show that \(C\) is a minimal set for \(f\) it suffices to show that for any \((x_1, x_2, x_3, \ldots), (y_1, y_2, y_3, \ldots) \in C\) and positive integer \(N\) there exists a positive integer \(k\) such that the first \(N\) entries of \(f^k(x_1, x_2, x_3, \ldots)\) agree with the first \(N\) entries of \((y_1, y_2, y_3, \ldots)\). This follows easily from Formula 1 because the \(p_i\) are pairwise relatively prime. \(\square\)

2. The Map \(f\) Can Be \(C^\infty\)

For convenience we will use \(\mathbb{C}\) instead of \(\mathbb{R}^2\). We begin with a technical but useful lemma.

**Lemma 1.** For any positive integers \(p, k\), real numbers \(0 < a < b < 1\), and any real number \(\epsilon > 0\) there exists a \(C^\infty\) diffeomorphism \(\phi : \mathbb{C} \to \mathbb{C}\) such that:

1. \(\phi(z) = ze^{2\pi i/p}\) for all \(z \in \mathbb{C}\) such that \(|z| \leq a\) and for some prime number \(p' > p\).
2. \(\phi(z) = z\) for all \(z \in \mathbb{C}\) such that \(|z| \geq b\).
3. \(||\phi(z) - z||_{C^k} < \epsilon\).

**Proof.** Let \(\rho : \mathbb{R} \to \mathbb{R}\) be a \(C^\infty\) function such that \(\rho(r) = 1\) for \(r < a\), \(\rho\) is monotonically decreasing on \((a, b)\), and \(\rho(r) = 0\) for \(r > b\). For any prime number \(p' > p\) the function

\[\phi(z) = ze^{2\pi i\rho(|z|)/p'}\]

satisfies (1) and (2) from the theorem. We will show that if \(p'\) is chosen large enough then \(\phi(z)\) from Equation 3 also satisfies (3).

Using Definition 1,

\[||\phi(z) - z||_{C^k} = \sup_{z \in \mathbb{C}, |z| \leq 1} \left| \frac{d^i[\phi(z) - z]}{dz^i} \right| + \sup_{x \in \mathbb{C}} |ze^{2\pi i\rho(|z|)/p'} - z|\]

We will show that each of the terms on the right side of this equation can be made less than \(\epsilon/2\) if \(p'\) is chosen small enough.

We first show this for the term \(\sup_{x \in \mathbb{C}} |ze^{2\pi i\rho(|z|)/p'} - z|\). Since \(\rho(r) = 0\) for \(r > 1\), \(|ze^{2\pi i\rho(|z|)/p'} - z| = 0\) if \(|z| > 1\). So

\[\sup_{z \in \mathbb{C}} |ze^{2\pi i\rho(|z|)/p'} - z| = \sup_{|z| \leq 1} |ze^{2\pi i\rho(|z|)/p'} - z| \leq \sup_{|z| \leq 1} |e^{2\pi i\rho(|z|)/p'} - 1|.
\]
Since $|e^{2\pi i \rho(|z|)/p'} - 1| \to 0$ as $p' \to \infty$, we can choose $p'$ so that

\[(4) \sup_{z \in \mathbb{C}} |ze^{2\pi i \rho(|z|)/p'} - z| \leq \sup_{|z| \leq 1} |e^{2\pi i \rho(|z|)/p'} - 1| < \frac{\epsilon}{4}.
\]

Note that we bound this term by $\epsilon/4$.

Now we show that the term $\sup_{z \in \mathbb{C}, 1 \leq i \leq j} \left| \frac{d^i \phi(z) - z}{dx^i} \right| < \epsilon/2$ if $p'$ is chosen small enough. As before, $\phi(z) - z = 0$ if $|z| > 1$ so it suffices to prove that $\sup_{|z| \leq 1, 1 \leq i \leq j} \left| \frac{d^i \phi(z) - z}{dx^i} \right| < \epsilon/2$ if $p'$ is chosen large enough. We demonstrate this by showing that if $p'$ is large enough then $\sup_{|z| \leq 1} \left| \frac{d^i \phi(z) - z}{dx^i} \right|$ for every $1 \leq i \leq j$. For the case $i = 1$, (using the triangle inequality and Equation 4.)

\[
\sup_{|z| \leq 1} \left| \frac{d^i \phi(z) - z}{dx^i} \right| = \sup_{|z| \leq 1} \left| e^{2\pi i \rho(|z|)/p'} + z \left( \frac{2\pi i d\rho(|z|)}{p'} \right) e^{2\pi i \rho(|z|)/p'} - 1 \right|
\]

\[
< \sup_{|z| \leq 1} \left| e^{2\pi i \rho(|z|)/p'} - 1 \right| + \sup_{|z| \leq 1} \left| z \left( \frac{2\pi i d\rho(|z|)}{p'} \right) e^{2\pi i \rho(|z|)/p'} \right|
\]

\[
< \frac{\epsilon}{4} + \frac{1}{p'} \sup_{|z| \leq 1} \left| z \left( \frac{2\pi i d\rho(|z|)}{d^i} \right) e^{2\pi i \rho(|z|)/p'} \right|
\]

The function $\left| z \left( \frac{2\pi i d\rho(|z|)}{d^i} \right) e^{2\pi i \rho(|z|)/p'} \right|$ is continuous on $|z| \leq 1$ so it achieves its max $M = \sup_{|z| \leq 1} \left| z \left( \frac{2\pi i d\rho(|z|)}{d^i} \right) e^{2\pi i \rho(|z|)/p'} \right|$ on this set. Hence choosing $p' > \frac{4M}{\epsilon}$ gives

\[
\sup_{|z| \leq 1} \left| \frac{d^i \phi(z) - z}{dx^i} \right| < \frac{\epsilon}{4} + \frac{1}{p'} \sup_{|z| \leq 1} \left| z \left( \frac{2\pi i d\rho(|z|)}{d^i} \right) e^{2\pi i \rho(|z|)/p'} \right|
\]

\[
< \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.
\]

For $i > 1$, we can write

\[
\sup_{|z| \leq 1} \left| \frac{d^i \phi(z) - z}{dx^i} \right| = \frac{1}{p'} \sup_{|z| \leq 1} |F(z)|
\]

where $F : \mathbb{Z} \to \mathbb{Z}$ is a continuous function. Hence $|F(z)|$ achieves its max on $|z| \leq 1$ and if $p'$ is large enough,

\[
\sup_{|z| \leq 1} \left| \frac{d^i \phi(z) - z}{dx^i} \right| = \frac{\epsilon}{2}.
\]
This proves that if \( p' \) is large enough then both of the terms on the right hand side of Equation 4 are less than \( \epsilon /2 \), which finishes the proof of (3).

Denote the function \( \phi : \mathbb{C} \rightarrow \mathbb{C} \) associate with positive integers \( p, k \), real numbers \( 0 < a < b < 1 \), and \( \epsilon > 0 \) by

\[
\phi_{p,k,(a,b),\epsilon} : \mathbb{C} \rightarrow \mathbb{C}.
\]

Denote the \( n \)th roots of unity by

\[
\{u_k^n = e^{k2\pi i/n}\}_{k=1}^n
\]

Define

\[
f_1(z) = \phi_{1,1,(1,1),1/2}.
\]

So \( f_1 \) rotates the unit disk by \( 2\pi/p'_1 \) for some prime number \( p'_1 > 1 \), \( f_1 \) is the identity outside of the disk of radius 1.1 centered at the origin, and \( ||f_1(z) - z||_{C1} < 1/2 \). Choose \( 0 < a_1 < b_1 < 1 \) such that \( |1/2 u_i^{p'_1} - 1/2 u_j^{p'_1}| > 2b_1 \) for all \( i \neq j \). Define the points

\[
c_i = \frac{1}{2} u_i^{p'_1}, \text{ for } i = 0, ..., p'_1 - 1,
\]

and the disks

\[
D_i = B_{a_1} (c_i), \text{ for } i = 0, ..., p'_1 - 1,
\]

\[
\overline{D_i} = B_{b_1} (c_i), \text{ for } i = 0, ..., p'_1 - 1,
\]

where \( B_r(c) \) is the ball of radius \( r \) centered at the point \( c \). Notice that

\[
f_1(D_i) = D_{i+1 \mod (p'_1)}.
\]

Define

\[
\psi_i(z) = \phi_{p'_1,2,(a_1,b_1),1/4}(z - c_i) + c_i.
\]

for some prime number \( p'_2 > p'_1 \), and let

\[
f_2(z) = \psi_{p'_1} \circ \cdots \circ \psi_0(z).
\]

So \( f_2 \) rotates each disk \( D_i \) by \( 2\pi/p'_2 \) for some prime number \( p'_2 > p'_1 \), \( f_2 \) is the identity outside of the disks \( \overline{D_i} \), and \( ||f_2(z) - z||_{C2} < 1/4 \). For each \( i = 0, 1, ..., p'_1 \) and \( j = 0, 1, ..., p'_2 \), define

\[
c_{(i,j)} = c_i + \frac{a_2}{2} u_j^{p'_2}
\]

Notice that \( f_1(c_{(i,j)}) = c_{(i+1 \mod (p'_1),j)} \) and \( f_2(c_{(i,j)}) = c_{(i,j+1 \mod (p'_2))} \). Hence, \( f_2 \circ f_1(c_{(i,j)}) = c_{(i+1 \mod (p'_1),j+1 \mod (p'_2))} \). Choose \( 0 < a_2 < b_2 < 1 \) such that \( |c_{(i_1,j_1)} - c_{(i_2,j_2)}| > 2b_2 \) for all \( (i_1,j_1) \neq (i_2,j_2) \). Define the disks

\[
D_{(i,j)} = B_{a_2} (c_{(i,j)}), \text{ for } i = 0, ..., p'_1 - 1,
\]

\[
\overline{D_{(i,j)}} = B_{b_2} (c_{(i,j)}), \text{ for } i = 0, ..., p'_1 - 1.
\]

Notice that \( f_2(D_{(i,j)}) = D_{(i,j+1 \mod (p'_2))} \) and hence,

\[
f_2 \circ f_1(D_{(i,j)}) = D_{(i+1 \mod (p'_1),j+1 \mod (p'_2))}.
\]
We continue by induction as described in Section 1. Suppose maps $f_1, f_2, \ldots, f_{i-1}$ are given with disks $D_{(x_1, x_2, \ldots, x_{i-1})}$ and $D_{(x_1, x_2, \ldots, x_{i-1})}$ centered at $C_{(x_1, x_2, \ldots, x_{i-1})}$, with $D_{(x_1, x_2, \ldots, x_{i-1})}$ centered at $C_{(x_1, x_2, \ldots, x_{i-1})}$, such that

- Each $f_j$ rotates each $D_{(x_1, x_2, \ldots, x_{j})}$ by $2\pi/p'_j$ for some prime number $p'_j > p'_{j-1}$.
- For every $f_j$, $||f_j(z) - z||_C < 1/2^j$.
- Each $f_j$ is the identity outside of the disks $\overline{D_{(x_1, x_2, \ldots, x_{j})}}$.
- For $(x_1, x_2, \ldots, x_{j}) \neq (y_1, y_2, \ldots, y_{j})$, $D_{(x_1, x_2, \ldots, x_{j})} \cap \overline{D_{(y_1, y_2, \ldots, y_{j})}} = \emptyset$.
- For every $(x_1, \ldots, x_{j}) \in \mathbb{Z}_{p'_1} \times \cdots \times \mathbb{Z}_{p'_j}$, $f_j \circ \cdots \circ f_1(D_{(x_1, \ldots, x_{j})}) = D_{(x_1+1 \pmod{p'_1}, \ldots, x_{j}+1 \pmod{p'_j})}$.

- There exist $0 < a_{i-1} < b_{i-1} < 1$ such that $|c_{(x_1, x_2, \ldots, x_{i-1})} - c_{(y_1, y_2, \ldots, y_{i-1})}| > 2b_{i-1}$ for all $(x_1, x_2, \ldots, x_{i-1}) \neq (y_1, y_2, \ldots, y_{i-1})$.

For each $(x_1, \ldots, x_{i-1}) \in \mathbb{Z}_{p'_1} \times \cdots \times \mathbb{Z}_{p'_{i-1}}$ define

$$\psi_{(x_1, \ldots, x_{i-1})}(z) = \phi_{p'_1, i-1}(a_{i-1}, b_{i-1}) \cdot 1/2^i |z - c_{(x_1, \ldots, x_{i-1})}| + c_{(x_1, x_2, \ldots, x_{i-1})}.$$ 

for some prime number $p'_i > p'_{i-1}$, and let

$$f_i(z) = \circ_{(x_1, \ldots, x_{i-1})} \in \mathbb{Z}_{p'_1} \times \cdots \times \mathbb{Z}_{p'_{i-1}} \psi_{(x_1, \ldots, x_{i-1})} \in \mathbb{Z}_{p'_i}(z).$$

That is, $f_i$ is the composition of all of the $\psi_{(x_1, \ldots, x_{i-1})}$, where $(x_1, \ldots, x_{i-1}) \in \mathbb{Z}_{p'_1} \times \cdots \times \mathbb{Z}_{p'_{i-1}}$, and the order of composition does not matter because for any $(x_1, \ldots, x_{i-1}) \neq (y_1, y_2, \ldots, y_{i-1})$, the set of points for which $\psi_{(x_1, \ldots, x_{i-1})}$ is not the identity is disjoint from the set of points for which $\psi_{(y_1, \ldots, y_{i-1})}$ is not the identity. So $f_i$ rotates each disk $D_{(x_1, \ldots, x_{i-1})}$ by $2\pi/p'_i$ for some prime number $p'_i > p'_{i-1}$, $f_i$ is the identity outside of the disks $(x_1, \ldots, x_{i-1})$, and $||f_i(z) - z||_C < 1/2^{i+1}$. For each $(x_1, \ldots, x_i) \in \mathbb{Z}_{p'_1} \times \cdots \times \mathbb{Z}_{p'_i}$ and define

$$c_{(x_1, \ldots, x_i, x_{i+1})} = c_{(x_1, \ldots, x_{i+1}, x_{i})} + \frac{a_{i-1}}{2} u_{x_i}^{p'_i}.$$ 

Notice that $f_i(c_{(x_1, \ldots, x_i, x_{i+1})}) = c_{(x_1, \ldots, x_{i+1}, x_{i+1})} \pmod{(p'_i)}$. Hence,

$$f_i \circ f_{i-1} \circ \cdots \circ f_1(c_{(x_1, \ldots, x_{i+1}, x_{i+1})}) = c_{x_1+1} \pmod{(p'_1), \ldots, x_{i+1}+1 \pmod{(p'_i)}}.$$ 

Choose $0 < a_i < b_i < 1$ such that $|c_{(x_1, \ldots, x_{i+1})} - c_{(y_1, y_2, \ldots, y_{i+1})}| > 2b_i$ for all $(x_1, \ldots, x_{i+1}) \neq (y_1, y_2, \ldots, y_{i+1})$. Define the disks

$$D_{(x_1, \ldots, x_{i+1}, x_{i+1})} = B_{a_i}(c_{(x_1, \ldots, x_{i+1}, x_{i+1})}), \text{ for each } (x_1, \ldots, x_i) \in \mathbb{Z}_{p'_1} \times \cdots \times \mathbb{Z}_{p'_i},$$

$$D_{(x_1, x_2, \ldots, x_{i+1}, x_{i+1})} = B_{b_i}(c_{(x_1, x_2, \ldots, x_{i+1}, x_{i+1})}), \text{ for each } (x_1, \ldots, x_i) \in \mathbb{Z}_{p'_1} \times \cdots \times \mathbb{Z}_{p'_i},$$

Notice that $f_i(D_{(x_1, \ldots, x_{i+1}, x_{i+1})}) = D_{(x_1, \ldots, x_{i+1}, x_{i+1})} \pmod{(p'_i)}$. Hence,

$$f_i \circ f_{i-1} \circ \cdots \circ f_1(D_{(x_1, \ldots, x_{i+1}, x_{i+1})}) = D_{(x_1+1 \pmod{(p'_1)}, \ldots, x_{i+1}+1 \pmod{(p'_i)})}.$$
REFERENCES


