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Minimal k-rankings and the a-rank number of a path

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Abstract

Given a graph $G$, a function $f : V(G) \to \{1, 2, ..., k\}$ is a $k$-ranking of $G$ if $f(u) = f(v)$ implies every $u - v$ path contains a vertex $w$ such that $f(w) > f(u)$. A $k$-ranking is minimal if the reduction of any label greater than 1 violates the described ranking property. The a-rank number of $G$, denoted $\psi_a(G)$, equals the largest $k$ such that $G$ has a minimal $k$-ranking. We establish new results involving minimal rankings of paths and in particular we determine $\psi_a(P_n)$, a problem suggested by Laskar and Pillone in 2000. We show $\psi_a(P_n) = \left\lceil \log_2 (n+1) \right\rceil + \left\lceil \log_2 \left(n+1 - \left(2^{\left\lceil \log_2 n - 1 \right\rceil}\right)\right) \right\rceil$.

1 Introduction

A labeling $f : V(G) \to \{1, 2, ..., k\}$ is a $k$-ranking of a graph $G$ if and only if $f(u) = f(v)$ implies that every $u - v$ path contains a vertex $w$ such that $f(w) > f(u)$. A $k$-ranking $f$ is minimal if for all $v_i \in V(G)$, a function $g$ satisfying $g(v) = f(v)$ when $v \neq v_i$ and $g(v_i) < f(v_i)$, is not a ranking. That is, if any label in a minimal ranking is replaced with a smaller label the new labeling is not a ranking. Note that for any ranking $f$ there exists a minimal ranking $h$ such that $h(v) \leq f(v)$ for every $v \in V(G)$. The rank number of a graph denoted $\chi_r(G)$, is defined to be the smallest $k$ such that $G$ has a minimal $k$-ranking, and the arank number of a graph denoted $\psi_a(G)$ is defined to be the largest $k$ such that $G$ has a minimal $k$-ranking. When the value of $k$ is unimportant, we will refer to a $k$-ranking as simply a ranking.

The rank number of a graph has been well studied, partially due to its applications to VLSI (Very Large Scale Integration) Layouts and scheduling problems for manufacturing systems [1], [5], [8]. While the rank number has been determined for various families of graphs, the arank number is only known for a few classes of graphs, such as stars and split graphs. An important property of the arank number is that it implies a necessary condition for a given ranking to be minimal. That is, if a ranking contains a label greater than $\psi_a(G)$ it cannot be a minimal ranking.

The problem of determining the arank number of a path was suggested by Laskar and Pillone [7]. In Theorem 13 we provide a complete solution to this problem. In addition, we provide a general result involving necessary conditions for a ranking of a path to be minimal. In Theorem 7 we prove that more than half of the vertices in a minimal ranking of $P_n$ must be labeled 1 or 2.

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2 Background

We will use $P_n$ to denote the Hamiltonian path $v_1, v_2, ..., v_n$ and $(f(v_1), f(v_2), ..., f(v_n))$ to explicitly describe the labels in a ranking $f$. For a given ranking let $S_i$ represent the independent set of all vertices labeled $i$. Given a graph $G$ and a set $S \subseteq V(G)$ the reduction of $G$ is a graph $G^*$ such that $V(G^*) = V(G) - S$ and for vertices $u$ and $v$, $(u, v) \in E(G^*)$ if and only if there exists a $u - v$ path in $G$. Note that if $G$ is a path, $G^*$ is also a path. An example of a reduction is given in Figure 1.

![Figure 1: A reduction with $G = P_7$ and $S = S_1$.](image)

For a ranking $f$ of a graph $G$, $f^*_G$ will represent the ranking of $G^*$ where $f^*_G(v) = f(v) - 1$ for all $v \in V(G)$ with $f(v) > 1$. For any other undefined notation, see the graph theory text by D. B. West [9].

We continue with a series of lemmas involving the frequency and locations of small labels that must appear in a minimal ranking. We restate the following two lemmas from [2].

**Lemma 1** Let $G$ be a graph and $f$ be a minimal ranking of $G$. If $x \in V(G)$ and $f(x) = 2$, then there exists a vertex $u$ adjacent to $x$ such that $f(u) = 1$.

**Lemma 2** If $x$ is a pendant vertex of a graph $G$ and $y$ is adjacent to $x$, then in any minimal ranking $f$ of $G$, either $f(x) = 1$ or $f(y) = 1$.

In the context of paths, this last lemma states that for any minimal ranking one of the first two vertices (or last two) must be labeled 1. If $n \geq 4$, we can use operation of reduction to show that one of the first four (or last four) vertices must be labelled 2. This is presented in our next lemma.

**Lemma 3** Let $f$ be a minimum ranking of a path $P_n = v_1, v_2, ..., v_n$ with $n \geq 4$. Then $f(v_i) = 2$ for some $1 \leq i \leq 4$. Furthermore, if $f(v_i) \neq 2$ for $1 \leq i \leq 3$, then $f(v_1) = f(v_3) = 1$.

**Proof.** Assume the smallest $i$ such that $f(v_i) = 2$ is greater than 4. Then at least two of the first four vertices in the path are labeled with integers greater than 2. It follows that in $f_{P_n^*}$ an end vertex and its neighbor will both have labels greater than 1, contradicting Lemma 2. For the second part, assume $f(v_i) \neq 2$ for $1 \leq i \leq 3$ and $f(v_4) = 2$. Suppose that either $f(v_1) \neq 1$ or $f(v_3) \neq 1$. Then two of the vertices $v_1, v_2$ and $v_3$ will have labels greater than 2. Then again, the pendant vertex and its neighbor will be mapped to a value greater then 1 by $f_{P_n^*}$, contradicting Lemma 2. ■

We next give a bound on the maximum size of a subpath with end vertices labeled $w$ and all internal vertices labelled $z \neq w$. 

2
Lemma 4 If \( f \) is a minimal ranking of \( P_n \) then any subpath of order \( 2m+1 \) has a vertex \( v \) such that \( f(v) = m \).

Proof. The proof is by induction on \( m \). The case where \( m = 1 \) was shown in [7]. The inductive step follows using reduction. ■

It is not difficult to show that if \( P' \) is an induced subpath of a path \( P \), then \( \psi_r(P') \leq \psi_r(P) \). We restate a lemma from [4] which shows that this monotonicity property holds in general.

Lemma 5 Let \( H \) be an induced subgraph of graph \( G \). Then \( \psi_r(H) \leq \psi_r(G) \).

Proof. An alternate proof is found in [4]. Let \( f \) be a minimal \( k \)-ranking of \( H \). We construct a labeling of \( g \) where \( g(v) = f(v) \) for all \( v \in H \) and labeling all other vertices arbitrarily \( k+1, k+2, \ldots, k+|V(G)|-|V(H)| \).

To see that \( g \) is a ranking note that if two vertices in \( G \) have identical labels then both vertices must be in \( H \), and use the fact that \( f \) is a ranking. Although \( g \) may not be a minimal ranking, no label of a vertex in \( H \) may be replaced with a smaller label since \( f \) is a minimal ranking. Replacing labels in \( V(G)-V(H) \) with smaller labels, if needed, will result in a minimal ranking of \( G \) that uses at least \( k \) labels. ■

We conclude this section by restating a lemma from [2] that will play a central role later in our proof of Theorem 7.

Lemma 6 Let \( G \) be a graph and let \( f \) be a minimal \( \psi_r \)-ranking of \( G \). If \( S_1 = \{ x : f(x) = 1 \} \) then \( \psi_r(G_{S_1}^*) = \psi_r(G) - 1 \).

3 Minimal \( k \)-rankings of paths

In our last section we noted many necessary conditions for a given ranking of a path to be minimal in lemmas 2, 3, 4, and 6. All of these lemmas involve the proximity of vertices labeled 1 or 2 in a minimal ranking. This leads to our main result, which states that in any minimal ranking of a path, more than half of the vertices must be labeled 1 or 2.

Theorem 7 If \( f \) is a minimal ranking of \( P_n \) then \( |S_1 \cup S_2| > \frac{n}{2} \).

Proof. Let \( V(P_n) = v_1, v_2, \ldots, v_n \). The vertices in \( S_2 \) partition \( P_n \) into parts \( F_1, F_2, \ldots, F_M \) where each \( x \in S_2 \) is the last vertex in some part \( F_i \), \( 1 \leq i \leq M-1 \) and \( F_M \) consists of the remaining vertices. We illustrate this in Figure 2.

![Figure 2. Partitioning of \( P_{12} \).](image-url)
We note that by Lemma 3, \( |V(F_i)| \leq 4 \) and by Lemma 4 \( |V(F_i)| \leq 8 \) for all \( i = 2, 3, \ldots, M \). Our strategy will be as follows: we will prove that \( |F_i \cap (S_1 \cup S_2)| > \frac{|V(F_i)|}{2} \) and \( |F_i \cap (S_1 \cup S_2)| \geq \frac{|V(F_i)|}{2} \) for all \( i = 2, 3, \ldots, M \). Combining these inequalities will yield \( |V(P_n) \cap (S_1 \cup S_2)| = |S_1 \cup S_2| > \frac{M}{2} \).

First we establish the inequality \( |F_i \cap (S_1 \cup S_2)| > \frac{|V(F_i)|}{2} \). By Lemma 3 the first 2 must appear somewhere among the first four vertices. We consider four cases and show the inequality holds in each one.

- \( (f(v_1) = 2) \) Then \( F_i = v_1 \) and \( |V(F_i) \cap (S_1 \cup S_2)| > \frac{|V(F_i)|}{2} \).
- \( (f(v_2) = 2) \) By Lemma 2 \( f(v_1) = 1 \) and \( |V(F_i) \cap (S_1 \cup S_2)| > 1 = \frac{|V(F_i)|}{2} \).
- \( (f(v_3) = 2) \) By Lemma 2, either \( f(v_1) = 1 \) or \( f(v_2) = 1 \). Hence \( |V(F_i) \cap (S_1 \cup S_2)| > \frac{|V(F_i)|}{2} \).
- \( (f(v_4) = 2) \) By Lemma 3, \( f(v_1) = 1 \) and \( f(v_3) = 1 \). Hence \( |V(F_i) \cap (S_1 \cup S_2)| > \frac{|V(F_i)|}{2} \).

We use a similar argument for \( F_M \) to show \( |V(F_M) \cap (S_1 \cup S_2)| \geq \frac{|V(F_M)|}{2} \). Next we show \( |V(F_i) \cap (S_1 \cup S_2)| \geq \frac{|V(F_i)|}{2} \) for all \( i = 2, 3, \ldots, M - 1 \). Consider \( F_i \) for some \( i, 2 \leq i \leq M \). Let \( v_{i1}, v_{i2}, \ldots, v_{i|V(F_i)|} \) be the vertices of \( F_i \) keeping the same ordering as in \( P_n \). The inequality is clear when \( |V(F_i)| = 2 \). By Lemma 4, \( |V(F_i)| \leq 8 \). We consider cases for the various possible lengths of \( F_i \). For completeness we include the details.

- \( 6 \leq |V(F_i)| \leq 8 \). If \( |F_i \cap S_1| < |V(F_i)| - 4 \) then \( F_i \) contains at least four vertices with labels higher than 2. Then \( f_{P_n}^i \) contains labels for four consecutive vertices that are all greater than 1. By Lemma 4 \( f_{P_n}^i \) cannot be a minimal ranking, a contradiction. Hence \( |V(F_i) \cap S_1| \geq |V(F_i)| - 4 \) and \( |V(F_i) \cap (S_1 \cup S_2)| \geq |V(F_i)| - 3 \geq \frac{|V(F_i)|}{2} \).

- \( |V(F_i)| = 5 \). By Lemma 4 \( |V(F_i) \cap S_1| \geq 1 \) and the vertex labeled 1 can not be the first or fourth vertex of \( F_i \). Assume, without loss of generality, the second vertex is labeled 1. We use \( a, b, \) and \( c \) to denote the first, third and fourth vertices of \( F_i \) respectively. If \( f(c) > f(b) \), then \( f(b) \) can be set to 2 and \( f \) still is a ranking; thus \( f(c) < f(b) \), which implies \( f(c) \) can only equal 1 if the ranking \( f \) is minimal. Hence \( |V(F_i) \cap (S_1 \cup S_2)| \geq 3 \geq \frac{|V(F_i)|}{2} \).

- \( |V(F_i)| = 3 \) or 4. By Lemma 4, \( |V(F_i) \cap S_1| \geq 1 \Rightarrow |V(F_i) \cap (S_1 \cup S_2)| \geq 2 \geq \frac{|V(F_i)|}{2} \).

In our next section we use this result to completely determine the arank number of a path.
4 The $a$-rank number of a path

The $a$-rank number of a path denoted $\psi_r(P_n)$ has been determined for small values of $n$ [2]. These values are given in Table 1.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\psi_r(P_n)$</th>
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<tr>
<td>1</td>
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<td>10</td>
<td>5</td>
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<tr>
<td>11</td>
<td>6</td>
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</tbody>
</table>

Table 1: $a$-rank numbers for small paths

A recursive construction was given in [7] for creating a minimal $(2m - 1)$-ranking of path with $2^m - 1$ vertices and a minimal $(2m - 2)$-ranking of path with $2^m - 2^{m-2} - 1$ vertices. The same construction was used for both families of paths and it was conjectured that the rankings produced by this construction were $\psi_r$-rankings.

The case $m = 1$ is trivial and when $m = 3$, a minimal 3-ranking of a $P_3$ can be constructed simply by labeling the vertices $(3, 1, 2)$. Starting with a $k$-ranking of a path on $w$ vertices, first delete the two end vertices. We next join two copies of the resulting path with a $P_3$ with labels, $(k - 1, k, k - 1)$. Finally add one vertex to each end of the path and label one of these vertices $k + 1$ and the other $k + 2$. An example showing the construction of a minimal 6-ranking of $P_{11}$ is shown in Figure 2.

Figure 3. Construction of a minimal 6-ranking from a minimal 4-ranking.

A direct application of Lemma 6 can be used to show that the rankings produced by the construction are in fact $\psi_r$-rankings. We prove this in the following two lemmas.
Lemma 8 $\psi_r(P_{2^m-1}) = 2m - 1$ for all integers $m \geq 2$.

Proof. We proceed by induction on $m$. As seen in Table 1, $\psi_r(P_{2^2-1}) = 2(2) - 1 = 3$.
Assume the equality holds for $m$. Given a path on $2^{m+1} - 1$ vertices, using the construction from Laskar and Pillone we can produce a $(2m + 1)$-ranking. Hence $\psi_r(P_{2^{m+1}-1}) \geq 2m + 1$. To show the reverse inequality, we assume that $\psi_r(P_{2^{m+1}-1}) \geq 2m + 1$. Then there exists a minimal $2m + 2$-ranking for $P_{2^{m+i}-1}$, in which case reducing $P_{2^{m+i}-1}$ twice produces a path $P$ with a $(2m)$-ranking. By Theorem 7, $P$ must have less than $2^m - 1$ vertices. Then Lemma 5 implies $\psi_r(P_{2^m-1}) \geq 2m$ which contradicts our assumption. ■

Lemma 9 $\psi_r(P_{2^{m-2m-2}-1}) = 2m - 2$ for all integers $m \geq 2$.

Proof. We proceed by induction on $m$. As seen in Table 1, $\psi_r(P_{2^4-22-1}) = \psi_r(P_{11}) = 6 = 2(4) - 2$.
Assume the equality holds for $m$. Given a path on $2^{m+1} - 2^{m-1} - 1$ vertices, we can construct a $2m$-ranking. Hence $\psi_r(P_{2^{m+1}-2m-1}) \geq 2m$. To show the reverse inequality, we assume that $\psi_r(P_{2^{m+1}-2m-1}) = 2m$. Then there exists a minimal $2m + 1$-ranking for $P_{2^{m+1}-2m-1}$. Reducing $P_{2^{m+1}-2m-1}$ twice produces a path $P$ with a $(2m - 1)$-ranking. By Theorem 7, $P$ must have less than or equal to $2m - 2^{m-2} - 1$ vertices. Application of Lemma 5, yields $\psi_r(P_{2^{m-2m-2}-1}) \geq 2m - 1$, which contradicts our assumption. ■

Lemma 10 $\psi_r(P_{2^{m-2m-2}-1}) = 2m - 3$ for all integers $m \geq 2$.

Proof. We proceed by induction on $m$. As seen in Table 1, $\psi_r(P_{2^4-22-2}) = \psi_r(P_{10}) = 5 = 2(4) - 3$.
Assume the equality holds for $m$. Given a path on $2^{m+1} - 2^{m-1} - 2$ vertices, we can construct a $(2m + 1 - 3)$-ranking. Hence $\psi_r(P_{2^{m+1}-2m-1-2}) \geq 2m - 1$. To show the reverse inequality, we assume that $\psi_r(P_{2^{m+1}-2m-1-2}) = 2m$. Then there exists a minimal $2m$-ranking for $P_{2^{m+1}-2m-1-2}$. Reducing $P_{2^{m+1}-2m-1-2}$ twice produces a path $P$ with a $(2m - 2)$-ranking. By Theorem 7, $P$ must have less than or equal to $2m - 2^{m-2} - 2$ vertices. Then by Lemma 5 we have $\psi_r(P_{2^{m-2m-2}-2}) \geq 2m - 2$, a contradiction. ■

Lemma 11 $\psi_r(P_{2^{m-2}}) = 2m - 2$ for all integers $m \geq 2$.

Proof. We proceed by induction on $m$. As seen in Table 1, $\psi_r(P_{2^2-2}) = 2(2) - 2 = 2$.
Assume the equality holds for $m$. Given a path on $2^{m+1} - 2$ vertices, using the construction from Laskar and Pillone we can produce a $2m$-ranking. Hence $\psi_r(P_{2^{m+1}-2}) \geq 2m$. To show the reverse inequality, we assume that $\psi_r(P_{2^{m+1}-2}) \geq 2m + 1$. Then there exists a minimal $(2m + 1)$-ranking for $P_{2^{m+1}-2}$, in which case reducing $P_{2^{m+1}-2}$ twice produces a path $P$ with a minimal $(2m)$-ranking. By Theorem 7, $P$ must have less than or equal to $2m - 2$ vertices. Application of Lemma 5, $\psi_r(P_{2^{m-2}}) \geq 2m$, a contradiction. ■

As mentioned Laskar and Pillone established an upperbound for the arank number of a path. In our next theorem we combine the above four lemmas with Lemma 5 to show that their upper bounds from [7] are in fact tight.
Theorem 12 (arank number of $P_n$)

(i) $\psi_r(P_s) = 2m - 2$ for all integers $s$, $2m - 2^m - 1 \leq s \leq 2m - 2$.

(ii) $\psi_r(P_t) = 2m - 1$ for all integers $t$, $2^{m+1} - 2^{m-1} - 2 \leq t \leq 2m + 1 - 2^m - 2$.

Following algebraic manipulation, the above theorem can be restated as follows to give an explicit formula for the arank number of a path.

**Theorem 13** Let $P_n$ denote a path on $n$ vertices. Then $\psi_r(P_n) = \left\lfloor \log_2 (n + 1) \right\rfloor + \left\lfloor \log_2 (n + 1 - (2^{\left\lfloor \log_2 n \right\rfloor - 1})) \right\rfloor$.

References


