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Can Gravitational Collapse Sustain Singularity-Free Trapped Surfaces?

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Abstract

In singularity generating spacetimes both the out-going and in-going expansions of null geodesic congruences θ^+ and θ^- should become increasingly negative without bound, inside the horizon. This behavior leads to geodetic incompleteness which in turn predicts the existence of a singularity. In this work we inquire on whether, in gravitational collapse, spacetime can sustain singularity-free trapped surfaces, in the sense that such a spacetime remains geodetically complete. As a test case, we consider a well known solution of the Einstien Field Equations which is Schwarzschild-like at large distances and consists of a fluid with a $p = -\rho$ equation of state near $r = 0$. By following both the expansion parameters θ^+ and θ^- across the horizon and into the black hole we find that both θ^+ and $\theta^+\theta^-$ have turning points inside the trapped region. Further, we find that deep inside the black hole there is a region $0 \leq r < r_0$ (that includes the black hole center) which is not trapped. Thus the trapped region is bounded both from outside and inside. The spacetime is geodetically complete, a result which violates a condition for singularity formation. It is inferred that in general if gravitational collapse were to proceed with a $p = -\rho$ fluid formation, the resulting black hole may be singularity-free.

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1   Introduction

The concept of a black hole as a gravitationally collapsed object has effectively been with us since 1916 when Schwarzschild first presented a solution to the Einstein field equations for the gravitational field of a spherically symmetric mass. To date, the physics of these end-products has evolved both in breadth and depth, both theoretically and observationally, so much so that black holes are currently considered virtually discovered. While the horizon of a black hole (a 2-sphere coordinate singularity) provides the classical observational limit with regard to the dynamical evolution of the hole, it is widely believed that during spacetime collapse to form a black hole, the matter itself continues to implode unimpeded beyond the horizon towards a physical singularity. Such perception has been motivated, from a theoretical view point, by the apparent absence of any known force that would otherwise counteract the action of gravity under the circumstances.

Lately, the issue of whether or not spacetime singularities do exist is being revisited [1]. There are two main reasons for this. First there is the need to deal with the information loss paradox in black holes [2]. Further, modern frameworks (like string theory) attempting to formulate a quantum theory of gravity assume the existence of fundamental length scale and thus seek singularity free spacetimes. There have been suggestions and speculations that singularity free spacetimes may actually be generic in nature [3]. Indeed the concept of non-singular spacetimes actually dates back quite a while. For example, Einstein [4] invoked a cosmological constant in his field equations in order keep the universe in equilibrium from collapse against its own matter-generated gravity. Later, [5] Sakharov considered the equation of state for a superdense fluid of the form $p = -\rho$ and Gliner suggested [6] that such a fluid could constitute the final state of gravitational collapse. Currently, researchers continue to investigate the possibility of non-singular gravitational collapse.. For example, Dymnikova (see [7] and refs. therein) has constructed a solution of the Einstein field equations for a non-singular black hole containing at the core a fluid $\Lambda_{\mu\nu}(r)$ with anisotropic pressure $T_0^0 = T_1^1$, $T_2^2 = T_3^3$. Another line of investigation in this area initiated by Markov [8], suggests a limiting curvature approach. This idea has been explored further by several authors, see for example [9] [10] [11] [12] [13] [14]. The notion of non-singular collapse does lead (among other things) to some interesting speculations such as universes generated from interior of black holes [15] [16] [17]. Non-singular collapse has also been extended to consid-

erations of other modified forms of gravitational collapse such as boson stars [18] and gravitational vacuum stars (gravastars) [19].

A common feature in all these treatments is that the geometry of the spacetime in question is Schwarschild at large r and de Sitter-like at small r values. Issues to do with direct matching of an external Schwarzschild vacuum to an interior de Sitter one have previously been discussed [20] from the point of view of the junction conditions [21]. In fact this approach has been employed to suggest modifications in some treatments [22]. If gravitational collapse is to result into a non-singular object with a de Sitter-like core then junction constraints imply the matter field profile between or across the Schwarzschild/de Sitter boundary should have a radial dependent equation of state of the form $1 \lesssim w(r) \lesssim -1$ that smoothly changes the matter-energy from a stiff fluid to a de Sitter-like fluid. In particular, inside the black hole where the radial coordinate becomes timelike such a field could take on a quintessential time dependant character, probably evolving as $0 \lesssim w(r) \lesssim -1$.

According to general relativity the 2-spheres inside of and concentric to the event horizon form a family of trapped surfaces Σ, in that the null geodesics of both ingoing and outgoing families, orthogonal to each such a 2-surface, are converging. This means that the tangent vector field l^α of any hypersurface-orthogonal null geodesic defined on any such 2-surfaces have a negative expansion, $\theta = l^\alpha{}_{;\alpha} < 0$. Thus spacetimes with horizons should contain trapped surfaces. In the 1960s, inquiries in the nature of spacetime singularities by Penrose and Hawking led to a quantitative theoretical description of gravitational collapse through the enunciation and proof of the famous Singularity Theorems [23] [24]. These studies were also extended by Hawking and Ellis [25] towards the concept of an initial singularity in cosmology. What the singularity theorems proved was that, subject to certain energy conditions, (see also Section 2.2) a closed trapped surface will contain a singularity. The question then is whether the geometry resulting from gravitational collapse can sustain trapped surfaces without a singularity. This is a reasonable question especially in view of the recent findings by Ellis about a related phenomenon in cosmology [26], namely that cosmic dynamics can allow closed trapped surfaces without leading to an initial singularity. In this paper we investigate whether it is in any way possible to have a family of trapped surfaces Σ_T, resulting, say, from spherically symmetric gravitational collapse, that contain no future-directed singularity.

The rest of the paper is organized as follows. In section 2 we briefly review

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the ideas of convergence of null geodesics and how they lead to the concept of trapped surfaces. Section 3 introduces the general class of spacetimes under consideration and discusses suitable coordinates. In section 4 we compute the expansion of both the out-going and in-going null geodesics in a well known non-singular spacetime that is an exact solution of Einstein Equations. We use the results to demonstrate an unusual character that inside such a black hole there is a region $0 < r < r_0$ for which $\theta^+\theta > 0$. The result violates conditions for the existence of a physical singularity. In section 5 we provide a summary and conclude the paper.

2   Null geodesics and gravitational collapse

In this section we briefly review two concepts to be used in the forthcoming discussion, namely the concept of geodesic convergence and the concept of trapped surfaces.

2.1   Convergence

The propagation of timelike or null rays in a given geometry is governed by the Raychaudhuri equation,

$$\frac{d\theta}{dv} = \varkappa\theta - (\gamma_c^c)^{-1}\theta^2 - \sigma_{\alpha\beta}\sigma^{\alpha\beta} + \omega_{\alpha\beta}\omega^{\alpha\beta} - R_{\alpha\beta}l^\alpha l^\beta, \tag{1}$$

where θ is the expansion rate, σ is the shear rate and ω is the twist. $R_{\alpha\beta}$ is the Ricii tensor. Further, γ_c^c is the trace of the projection tensor for null geodesics while $\varkappa$ is identified with the surface gravity. For simplicity of treatment, we shall consider a twistless $\omega = 0$, shearless, $\sigma = 0$ spacetime that is type D in the Petrov classification [27]. Then the geodesic tangent vector can be chosen to be in the principal null direction. Use of the Einstein Equations implies

$$\frac{d\theta}{dv} + \frac{1}{2}\theta^2 + \kappa\left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T\right)l^\alpha l^\beta = 0. \tag{2}$$

where $\kappa = 8\pi G$. Eq. 2 shows that the expansion/convergence θ of a geodesic congruence will, among other things, depend on energy conditions. In particular, for convergence of a null geodesic congruence, it is necessary (and

sufficient) that fields influencing the geometry satisfy the weak energy condition, i.e. $R_{\alpha\beta}k^\alpha k^\beta \geq 0$ which leads to $\rho + p \geq 0$. The expansion θ of the congruence (or the evolution of the principal null vector l^α) in a given geometry $g_{\alpha\beta}$ is defined by

$$\theta = l^\alpha_{;\alpha} = \frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}l^\alpha\right). \tag{3}$$

Thus with the knowledge of the metric, one can construct θ.

2.2   Trapped surfaces and singularities

Consider a spacelike 3-hyperface Σ on which one has defined a metric γ_{ij} and an extrinsic curvature tensor K_{ij}. The space Σ can be foliated with a family of 2-surfaces, S on which one can define hyperface-othorgonal null vectors l^α. There are two such families of vectors defined on a surface S, namely in-going and out-going rays. The expansion of the in-going congruence is always converging, while that of the out-going one can either diverge or converge depending on the circumstances. In the case that the expansion of both families of these tangent vectors are negative, $\theta < 0$, then the surface S is said to be a trapped surface S_T. There will be a marginally trapped surface(s) S_{Ma} in Σ for which the expansion of the outgoing null geodesics defined on each point on S_{Ma} are vanishing, $\theta = 0$. For black holes in the process of formation, this surface is the apparent horizon. It defines the outer boundary of the family of closed trapped surfaces S_T in this geometry. The existence of trapped surfaces in black holes (along with other conditions) was used by Penrose and Hawking to predict that such spacetimes should have singularities. Briefly, the main argument goes something like this [26] [28]. A given spacetime will contain at least one incomplete geodesic, provided:

(1) The fields satisfy the WEC, $R_{\alpha\beta}l^\alpha l^\beta \geq 0 \Rightarrow \rho + p \geq 0$;

(2) there are no closed timelike loops and

(3) there exists at least one trapped surface S_T,

Geodetic incompleteness then leads to the existence of a singularity.

In our treatment we show that for the spacetime in consideration the trapped region is also bounded from inside so that for $0 < r < r_0$ one finds $\theta^+\theta^- < 0$. This makes such a spacetime geodetically complete (no cusping of the in-going and out-going geodesics) which violates some of the above conditions for the existence of singularity. This is a key result of the paper.

3   The spacetime geometry

3.1   The Metric

In this work we investigate whether the spacetime geometry consequent to gravitational collapse can support trapped surfaces without creating singularities. We assume that the final collapse has resulted in the formation of an event horizon that envelopes the matter fields. Then the real and apparent horizons coincide to locate the *outer* marginally trapped surface. Further, for simplicity, we assume the end-product to admit static, spherically symmetric solutions. In Schwarzschild coordinates the line element for the spacetime can then take the form

$$ds^2 = -A(r)\,dt^2 + A(r)^{-1}\,dr^2 + r^2 d\Omega^2, \tag{4}$$

where $A(r) = 1 - \frac{2m(r)}{r}$ and here the mass $m(r) = 4\pi \int_0^r \rho(r')\, r'^2 dr'$ is a function of the radial coordinate and is distributed in some region $r < 2m(r)$. Such a line element is general enough to contain our spacetimes of interest. There are two boundary requirements for the spacetime of our interest:

(i) The first is that for large radial distance $r_M < r \leq \infty$ the spacetime be asymptotically Schwarzschild

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{5}$$

Here $M = 4\pi \int_0^\infty \rho(r')\, r'^2 dr'$ is the total mass and r_M gives the surface of the matter fields, with $r_M < 2M$.

(ii) The second requirement is that the spacetime be asymptotically de Sitter

$$ds^2 = -\left(1 - \frac{r^2}{r_0}\right) dt^2 + \left(1 - \frac{r^2}{r_0}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{6}$$

for $0 \leq r \leq (r_0 < r_M)$. Here r_0 signals the onset of de Sitter behavior and is given by $r_0 = \sqrt{3/\Lambda} = \sqrt{\frac{3}{\kappa \rho_0}}$, where $\rho_0 = \rho \mid_{r \longrightarrow 0}$ is the upper-bound on the density of the fields and $\kappa = 8\pi G$.

The two boundary requirements in Eqs. (5) and (6) imply that there is an interior region given by $r_0 \leq r \leq r_M$ for which $m = m(r)$. The entire spacetime must therefore satisfy regularity conditions at the two interfaces $r = r_M$ and $r = r_0$. These conditions guarantee (i) continuity of the mass

function and (ii) continuity of the pressure across the interfacing hyperfaces. Thus at each interface we must have for $A(r)$ that

$$\left[A^+ - A^-\right]_{|r=r_i} = 0, \tag{7}$$

and

$$\left[\partial_r A^+ - \partial_r A^-\right]_{|r=r_i} = 0, \tag{8}$$

where $r_i = \{r_m, r_0\}$ and $^+$, $^-$ refer to the exterior and interior values respectively.

It will be convenient to transform (Eq. 4) into coordinates suitable for discussing null geodesics and which cover all the regions of interest[1]. Note that in this a spacetime, null fields propagate along geodesics given by

$$g_{\alpha\beta}k^\alpha k^\beta = 0 = -(1 - \frac{2m(r)}{r})\left(\frac{dt}{d\lambda}\right)^2 + (1 - \frac{2m(r)}{r})^{-1}\left(\frac{dr}{d\lambda}\right)^2$$

where λ is an affine parameter. The radial geodesics of a massless particle are therefore given by

$$dt = \pm \left(1 - \frac{2m(r)}{r}\right)^{-1} dr = dr^*, \quad \text{where} \quad r^* = \pm \int \frac{dr}{1 - \frac{2m(r)}{r}} \tag{9}$$

is the a Regge-Wheeler-like tortoise coordinate in this spacetime and $dt \pm dr^* = 0$. One can set up an Eddington-Finkelstein-like double null coordinate system $\{u, v\}$ given by $u = t - r^*$, $-\infty < u < \infty$ and $v = t + r^*$, $-\infty < v < \infty$. Note that towards the future, the radial coordinate increases along constant u and decreases along v so that u and v are out-going and in-going null coordinates, respectively. The coordinates cover the exterior $r > 2m(r)$ region and the metric (Eq. 4) now takes the form

$$ds^2 = -A(r)\,du dv + r^2 d\Omega^2. \tag{10}$$

In order to regularize the metric in Eq. (6 at the points $A(r) = 0$ it is necessary to choose new coordinates $U(u)$ and $V(v)$ ($v \mapsto U$ and $v \to V$ which keep the metric invariant), so that

$$ds^2 = -\Phi^2(u, v)\,dU dV + r^2(U, V)\,d\Omega^2, \tag{11}$$

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[1]The approach is generalization of the regularization of the Schwarzschild coordinates for a coordinate dependent mass function.

where the function $\Phi^2(u, v)$ can be determined and will, in general, depend on r both explicitly and implicitly through the mass function $m(r)$. In comparing Eq. (11) with Eq. (4) one notes the transformation from r, t to U, V coordinates implies that

$$(\partial_t U)(\partial_t V)\Phi^2 = A(r),$$
$$(\partial_r U)(\partial_r V)\Phi^2 = -A(r)^{-1} \qquad (12)$$
$$[(\partial_r U)(\partial_t V) + (\partial_r V)(\partial_t U)]\Phi^2 = 0$$

The three equations in Eq. (12) are identically satisfied by the choice $U(u) = -Be^{-\frac{1}{2}\gamma u}$ and $V(v) = Be^{\frac{1}{2}\gamma v}$, where B is some scale factor which we can set to unity, and the γs (for the different regions in the spacetime) are chosen so that Φ^2 is regular at all zeros of $A(r)$. With this choice one finds with use of Eq. (12) that

$$\Phi^2 = \frac{A(r)}{4B^2\gamma^2}e^{-\frac{\gamma}{2}(v-u)} = \frac{A(r)}{4B^2\gamma^2}e^{-\gamma r^*}, \qquad (13)$$

where the parameter γ may in general depend on the mass $m(r)$ enclosed at the coordinate r.

3.2 Focusing

Consider now the spacelike subspace Σ of the above spacetime. One can foliate Σ with a family of 2-surfaces S of constant U and V, which are null hyperfaces (U and V are null coordinates of Eq. 11). Note that r is constant on each such surface S. One can then define radial ($d\theta = d\varphi = 0$) vectors $l^\pm$ tangent to the null geodesics orthogonal to S and which are future pointing. The ingoing and outgoing vectors are respectively given by $^+l = \Phi^{-2}\partial_V$, and $^-l = \Phi^{-2}\partial_U$ and satisfy the orthogonality condition $g_{\alpha\beta}l^\alpha l^\beta = 0$. We now consider the divergence θ of the null geodesics of the spacetime given by Eq. (11). Their general evolution is depicted in Eq. 2. Let $^-l^\alpha = (0, a, 0, 0)$ be a tangent vector to an outgoing geodesic. By definition, the null geodesics $x^\alpha(\lambda)$ with affine parameter λ, for which the tangent vectors $\frac{dx^\alpha}{d\lambda}$ are normal to a null hyperface S_N, are the generators of S_N (see e.g. [29]. Then $l^\alpha{}_{;\beta}l^\beta = 0 \Rightarrow a = \Phi^{-2}$ and the expansion $\theta = l^\alpha{}_{;\alpha}$ is, for in-going null geodesics given by

$$\theta^+ = \frac{2}{r\Phi(r)^2}\frac{\partial r}{\partial V}, \qquad (14)$$

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and for out-going null geodesics, by

$$\theta^- = \frac{2}{r\Phi(r)^2} \frac{\partial r}{\partial U}.$$ (15)

4   A Non-Singular Spacetime

The Dymnikova solution [30] depicts a plausible scenario for a spacetime that could result from gravitational collapse. In this solution the fields $T_{\mu\nu}$ satisfy the conditions

$$T^t_t = T^r_r; \ T^\theta_\theta = T^\varphi_\varphi$$ (16)

The radial pressure p_r satisfies a cosmological constant-like equation of state $p_r = -\rho$, while the tangential pressure satisfies $p_\theta = p_\varphi = -\rho - \frac{1}{2} r \frac{\partial \rho}{\partial r}$. The analysis [30] gives an exact solution to the Einstein equations which takes the form of Eqs. 4 - 6 with a mass function of the form

$$m(r) = 4\pi \int_0^x \rho(x) x^2 dx = M \left[1 - e^{-\left(\frac{r^3}{r_0^2 r_g}\right)} \right],$$ (17)

where $m(r)$ is the mass enclosed within a radius r. The radial density profile $\rho(r)$ is given by $\rho(r) = \rho_0 e^{-\left(\frac{r^3}{r_0^2 r_g}\right)}$; with $r_0 = \frac{3}{\Lambda}$ and $r_g = 2MG$.

4.1   Regularized metric

Notice that the Dymnikova solution contains no matter fields (in the usual sense) since the only field existing has a vacuum-like equation of state in the radial direction. This allows one to seek a Kruskal-like (Eq. 11) extension of this spacetime. We are only interested in regularizing the coordinate singularities in Eqs. 4 - 6 in order to follow the expansion θ. In the region $0 \leq r \leq \infty$ this spacetime can be divided into three sub-regions.

Region 1: $r_M < r \leq \infty$. This is the Schwarzschild vacuum region. Here $r_M < 2M$ describes the surface of matter fields and is located deep inside the Schwarzschild horizon. It follows that the the problem of satisfying the junction conditions (for matter fields) across the horizon (see e.g. [1]) does not arise. The regularization of the coordinates at $r = 2M$ can then be effected. Thus Region 1 includes two sectors of the Schwarzschild vacuum: (1a) the exterior $2M < r \leq \infty$ and (1b) the interior $r_M < r \leq 2M$. In region

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1a) we choose [31] $U < 0, V > 0, \gamma_1 = \frac{1}{4M}$ and $A = \frac{1}{2}$. In general for Region 1 we have that the metric is same as in Eq.11 with

$$\Phi^2 = \frac{32M^3}{r}e^{-\frac{r}{2M}}, \ r_M < r \leq \infty. \tag{18}$$

In the present work we shall squash region 1b so that $r_M < r \leq 2M$ consists of only the inner 2-surface r_M^-, and region 2 containing the anisotropic fluid of Eq. starts here[2].

Region 2: $r_0 < r < r_M$. This is the intermediate interior region containing containing a field with a radial dependence $m(r)$ as in Eq. 16 and a vacuum-like equation of state $p_r(r) = -\rho(r)$. Here (as in region 1b) we have $U > 0$, $V > 0$ and the metric is still given by Eq. 10. Since however r^* does not admit a simple closed form integral and one cannot determine γ explicitly. As it turns out this is not a serious problem since the region $r_0 < r < r_M$ has no coordinate or physical singularities. Thus in following θ in this region, we only require to match region to regions 1 and 3 using the matching conditions already mentioned above.

$$\Phi^2 = \frac{A(r)}{4B^2(\gamma_2)^2}e^{-\gamma_2 r^*\left[=-\frac{\gamma}{2}(v-u)\right]}, \ r_0 < r \leq r_M \tag{19}$$

.

Region 3: $0 \leq r \leq r_0$ contains an (almost) constant density ρ_0 vacuum-like fluid of density Λ with a length scale given by $r_0^2 = \frac{3}{\Lambda}$. The solution with the mass function of Eq. (17 has an inner horizon r_- at r_0 (actually at $1.00125\, r_0$) provided the total collapsed mass M is large enough, i.e. $M > m_{crit} \simeq 0.3 m_{Pl}\sqrt{\frac{\rho_{Pl}}{\rho_0}}$. Thus to maintain regularity of the metric across $r = r_0$ we introduce $U > 0$ and $V < 0$. Here the metric becomes asymptotically de Sitter, taking the form of Eq. (6) with $A(r) \sim \left(1 - \frac{r^2}{r_0}\right)$, $0 < r < r_0$. This gives $r^* = \int \left(1 - \frac{r^2}{r_0}\right)^{-1} dr = \frac{r_0}{2} \ln \frac{r_0 + r}{r_0 - r}$. So that now

$$\Phi^2 = \frac{A(r)}{4B^2(\gamma_3)^2}e^{-\gamma_3 r^*} = \left(\frac{1}{r_0}\right)^2 (r_0 + r)^{1-\gamma_3 r_0}(r_0 - r)^{1+\gamma_3 r_0}, \ 0 \leq r \leq r_0. \tag{20}$$

By inspection of Eq. (19), it follows that to make Φ^2 regular in this region one requires that $1 + \gamma_3 r_0 = 0$ giving $\gamma_3 = -\frac{1}{r_0}$.

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[2]As is shown from the form of the mass function Eq. 17, the boundary at r_M^- is a soft boundary which easily satisfies the conditions in Eqs. 7 and 8.
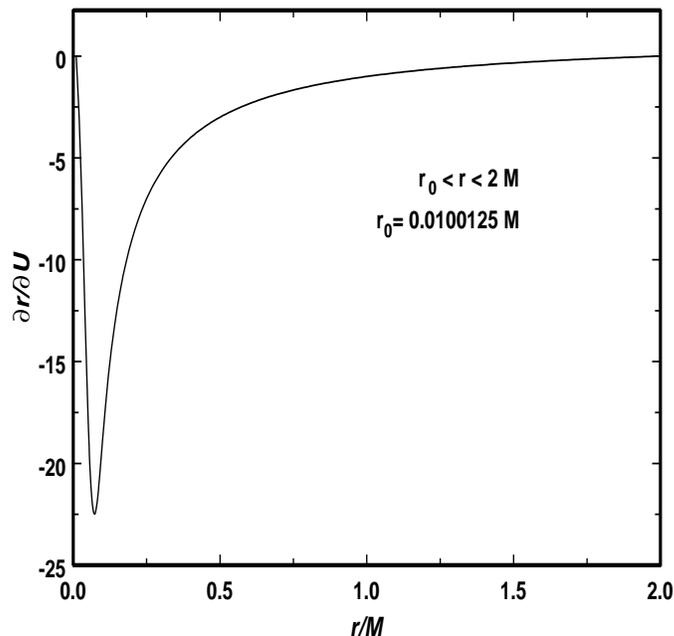
Figure 1: The $\partial r/\partial U$ derivative (to which the expansion θ is proportional) for the region between the outer and inner horizons i.e. $r_0 < r < 2M$ with $r_0 = 0.00100125\,M$ the position of the inner horizon. Instead of reaching $-\infty$ as r approaches zero, the expansion θ begins increasing for $r > (r_0^2 r_g)^{1/3}$ approaching zero at the horizon $r = r_0$.

4.2 Null geodesic convergence in Dymnikova space-time

Using the results in Eqs. 17-19 we can now compute the expansions of the in-going $\theta^+ = \frac{2}{r\Phi(r)^2}\frac{\partial r}{\partial V}$ and out-going $\theta^- = \frac{2}{r\Phi(r)^2}\frac{\partial r}{\partial U}$ null geodesics in the above spacetime for the entire region $0 \le r < \infty$. Fig. 1 is a plot of $\partial r/\partial U$ (to which θ is proportional) for the outgoing mode in this spacetime in the region $r_0 < r < 2M$.

In the region $r > 2M$, the geodesic congruences expand in a similar way as they do in the Schwarzschild spacetime. Thus here $\theta^+ > 0$ and $\theta^- < 0$, so
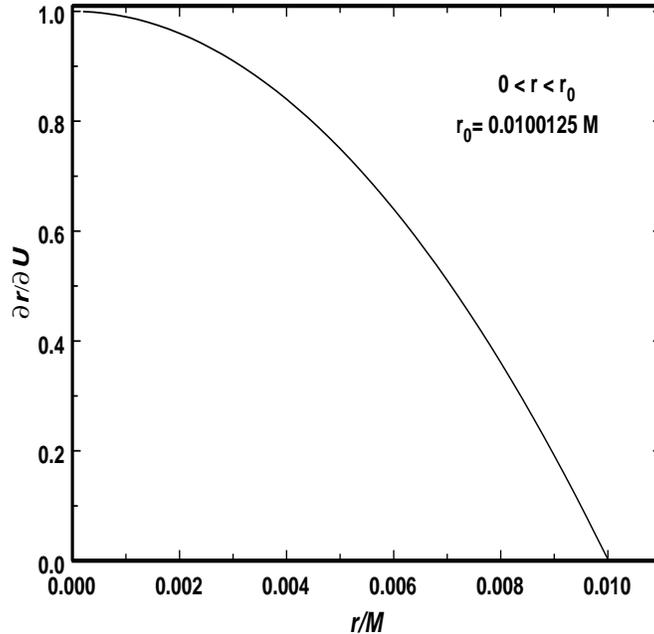
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Figure 2: Same as figure 1 but for the region interior to the inner horizon $0 < r < r_0$. The expansion vanishes on the horizon while it reaches its maximum value at $r = 0$. When plotted in a single figure this curves joins smoothly that of figure 1 at the inner horizon.

that $\theta^+\theta^- < 0$. However as one traverses the region $r < 2M$, the dependence of θ on r evolves very differently. In the Schwarzschild spacetime the out-going null congruences are given by $\theta^+ < 0$ and evolve such that $\theta^+ \rightarrow -\infty$ as $r \rightarrow 0$. But in this region we still have for the in-going congruences that $\theta^- < 0$. Thus it is expected that eventually the two congruences would intersect or cusp leading to an incomplete spacetime. This signifies the existence of a physical singularity [24] [26]. One can compare this situation with that for the spacetime under consideration. One notices that in the region $r < 2M$ of the Dymnikova spacetime (see Fig. 1) the expansion of the outgoing mode θ^+ starts off decreasing as in the Schwarzschild case. However, near $r \simeq (r_0^2 r_g)^{1/3}$ it reaches a minimum and and thereafter, for smaller radii, θ^+ begins increasing as the metric begins deviating significantly from that

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of Schwarzschild. Eventually, at $r = r_0$ (in reality at $r = 1.00125\, r_0$), θ^+ vanishes, to signify the presence an inner marginally trapped surface (and hence horizon) at this point. In figure 2 we show the function $\partial r/\partial U$ for points in the region $r < r_0$. One notices that θ is positive for outgoing modes in this region, as indeed is the case in the $r > 2M$ region, but vanishes at $r = r_0$, which is a marginally trapped surface i.e. a horizon.

In analyzing spacetimes based on the behavior of its null congruences θ^+ we have relied mostly on the behavior of the out-going geodesics. A more recent approach [32] (which we shall also apply) is to consider the evolution of $\theta^+\theta^-$ with the radial coordinate r. This is a more general (and in our case more straight forward) approach. When θ^- remains converging (as in the Schwarzschild case) then it is easy to verify that the sign of $\theta^+\theta^-$ follows the sign of θ^+. Thus $\theta^+\theta^- < 0$ signifies a regular spacetime, $\theta^+\theta^- = 0$ signifies a marginally trapped surface and $\theta^+\theta^- > 0$ signifies a trapped region. In a maximally extended spacetime, Eqs. 14 and 15 imply that the product of the expansions is given by $\theta^+\theta^- = \frac{4}{r^2\Phi^2(r)}\frac{\partial r}{\partial U}\frac{\partial r}{\partial V}$. On using our definitions of $U(u)$ and $V(v)$ to Eqs. 12 we find

$$\theta^+\theta^- = -\frac{4}{r^2}\left[1 - \frac{2m(r)}{r}\right] \tag{21}$$

with $m(r)$ given by Eq. (17). The expression $-(r^2/4)\theta^+\theta^-$ is plotted against r in Figure 3. The plot shows (as expected) that $\theta^+\theta^-$ first changes sign at the outer horizon where it turns from negative to positive as it enters the trapped region $r < 2M$. Inside the trapped region $\theta^+\theta^-$ approaches some maximum and then decreases to zero at some value of r, signifying an exit out of the trapped region. Thereafter, $\theta^+\theta^-$ grows without bound in the negative direction. It is notable that $\theta^+\theta^-$ remains bounded in the trapped region $r_0 < r < 2M$ and does not tend to $+\infty$. Further, it is notable that $\theta^+\theta^- < 0$ in the region $0 \leq r < r_0$, just like in the region $2M < r < +\infty$. It follows that the entire spacetime under consideration is geodetically complete. This violates a condition (in the singularity theorems) for the formation of a matter-generated future-directed singularity. We therefore have an example of a trapped region with no singularity. This finding was the purpose of our inquiry.
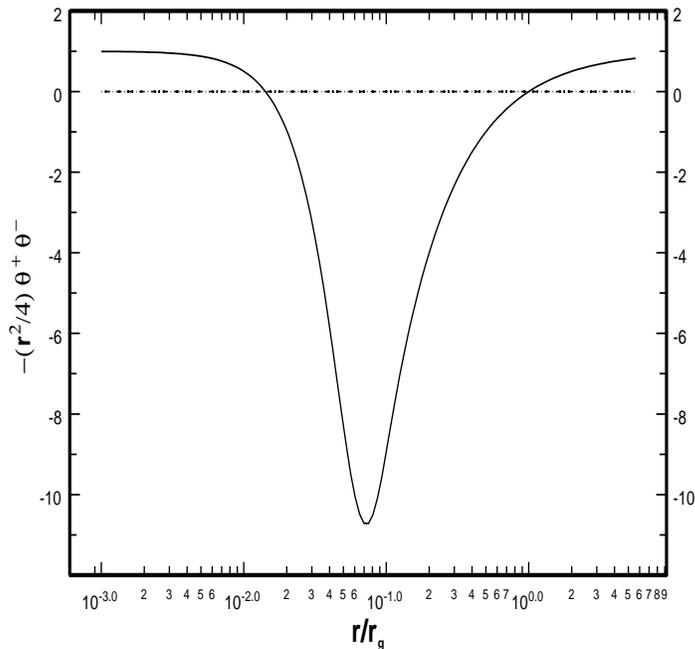
Figure 3: The expression $-(r^2/4)\theta^+\theta^-$ as a function of the normalized radial coordinate r/r_g. It is apparent that the quantity $\theta^+\theta^-$ changes sign at the inner horizon becoming negative for $r < r_0$.

5 Conclusion

In this work we have investigated the question of whether in gravitational collapse spacetime can sustain trapped surfaces without a singularity. As a concrete example we have considered the Dymnikova spacetime, which is a static spherically symmetric solution of the Einstein Field Equations with a de Sitter-like fluid at the core. The investigations performed by following the expansion θ of the null geodesic congruences of the spacetime as a function of the radial coordinate. It is found that for large values of r, i.e. $\infty < r \leq 2M$ the behavior of out-going null θ^+ and that of the product $\theta^+\theta^-$ are similar to those of the Schwarzschild space, in that $\theta^+ \geq 0$ and $\theta^+\theta^- \leq 0$ and vanishing at the horizon boundary, $r \leq 2M$. As one traverses the region $r < M$ of the Dymnikova spacetime one find that, initially $\theta^+ < 0$ and is

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growing more negative and $\theta^+\theta^- < 0$. In the Schwarzschild spacetime this behavior leads to geodetic incompleteness and the occurrence of a singularity. In the Dymnikova spacetime, however, both θ^+ and $\theta^+\theta^-$ have turning points (minimum and maximum, respectively) in this region and eventually both vanish around $r = r_0$. For $0 \leq r < r_0$ one finds that $\theta^+ > 0$ and $\theta^+\theta^- < 0$.

The main results are that in this spacetime, there is a trapped region with an outer and inner boundary. In this trapped region, null congruences do not converge without bound and both θ^+ and $\theta^+\theta^-$ have a minimum. Further, there is a region $0 \leq r < r_0$ including the black hole center $r = 0$, in which the spacetime is regular, with converging in-going congruences $\theta^- < 0$ and diverging out-going congruences, $\theta^+ > 0$ so that $\theta^+\theta^- < 0$. These results suggest that the trapped region Σ_T: (a) does not enclose a singularity and, (b) may not necessarily include the origin.

In [33] we have constructed a solution to the Einstein Field Equations for a static spherically symmetric space-time with an equation of state for a gravitating fluid that transits smoothly between that of a matter-like fluid ($P = w\rho$, $w > 0$) and evolves to vacuum-like state, ($w = -1$). While a maximal extension of this spacetime is still to be made one expects, based on the general arguments above and pending verification, that its trapped surfaces will too be non-singular. Generally, these arguments suggest that if gravitational collapse were to proceed with some phase transition that leads to the creation of some de Sitter-like fluid ($p = -\rho$) at the collapse core deep inside the Schwarzschild surface then it is plausible that a trapped region with no singularity would result inside the black hole.

References

[1] M. Visser and D. L. Wiltshire, Class. Quant. Grav. 21 (2004) 1135-1152.

[2] L. Susskind, Sc. Am. (13) **1** 18(2003).

[3] M. R. Mbonye, Mod. Phys. Lett. A**19** (2004) 117-134.

[4] Goldsmith, D. *Einstein's Greatest Blunder? The Cosmological Constant and Other Fudge Factors in the Physics of the Universe*. Harvard University Press, 1997.

[5] A. D. Sakharov, Hz. Eksp. Teor. Fiz. **49** 542(1965); Sov. Phys.. JETP **22** 378(1966).

[6] E. B. Gliner, gr-qc/9808042.

[7] I. Dymnikova, Int. J. Mod. Phys. D12 (2003) 1015-1034.

[8] Markov [M. A. JEPP Lett. 36 265(1982); Ann. Phys. 155 333(1984).

[9] Makhov, Mukhanov, and Frolov, Phys. Lett. B216 272 (1987).

[10] Makhov, Mukhanov, and Frolov, Phys. Rev D 41, 383(1990).

[11] V. Mukhanov, R. Brandenberger, Phys. Rev. Lett. **68**, 1969(1992).

[12] Brandenberger, V. Mukhanov and A. Sornorger and M. Trodden (in-prep).

[13] M. Trodden, V. F. Mukhanov, Brandenberger, Phys. Lett. B**316** (1993) 483-487.

[14] Easson and Branden berger , JHEP **0106**, 024 (2001).

[15] A. Guth, Phys. Rev. D **23**, 347(1981).

[16] K.M. Larsen and R. L. Mallett, Phys. Rev D **44**, 333(1991).

[17] I. Dymnikova, A. Dobosz, M. L. Fil'chenkov, A. Gromov, Phys. Letts. B **506** 351(2001).

[18] P.O.Mazur and E. Mottola, gr-qc/0109035.

[19] E.W. Mieke and F. E. Schuck, Nucl. Phys. B. **564** 185(2000).

[20] Poisson and Israel, Class. and Quant. Grav. **5**, L201(1988)

[21] O. Grϕn, Lett. Cimento, **44**, 177(1985).

[22] M. Visser and D. Wiltshire, Class. Quant. Grav. **21** 1135(2004).

[23] R. Penrose, Phys. Rev. Lett. **14**, 57(1965).

[24] S. W. Hawking and R. Penrose, Proc. Roy. Soc. Lond. A **314**, 529(1970).

[25] S. W. Hawking and G. F. R. Ellis, ApJ **152**, 25(1968).

[26] G. F. R. Ellis, Gen.Rel.Grav. **35** 1309(2003).

[27] M. Carmeli, *Group Theory and General Relativity*; McGraw-Hill International Inc.; 1977.

[28] S. W. Hawking and G. F. R. Ellis, *The large scale structure of spacetime;* Cambridge Press, 1973.

[29] D. Gerber, Thesis: *Trapped Submanifods in Spacetimes with Symmetries*, ETH Zürich.

[30] I. Dmnikova, Gen. Rel. Grav. **24** 235(1992).

[31] Robert Wald, *General Relativity;* University of Chicago Press,1984.

[32] Hayward, S. A., gr-qc/0504037.

[33] M. R. Mbonye and D. Kazanas Phys. Rev. D**72** (2005) 024016; gr-qc/0506111.