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William Basener
Rochester Institute of Technology

Carl V. Lutzer
Rochester Institute of Technology

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Twisted Solenoids and Maps of $\mathbb{R}^2$ Whose Minimal Sets are Cantor Sets

Bill Basener and Carl V. Lutzer
Rochester Institute of Technology, Rochester, NY, 14623-5603, USA

Abstract. We construct homeomorphisms of $\mathbb{R}^2$ that have a Cantor set which is a minimal set. Some of these homeomorphisms are proven to be C^∞, while the others are conjectured to be nondifferentiable.

1. Introduction

Suppose X is a topological space and $f : X \to X$ is a homeomorphism. A nonempty compact subset $Y \subset X$ is said to be a *minimal set* for f if, for every $y \in Y$, the orbit of y under iterations of f is dense in Y. The set Y is said to be an *exceptional set* if it is both a minimal set and a Cantor set. Denjoy showed that any diffeomorphism of S^1 that has an exceptional set cannot be C^2. In contrast, the return map for a cross section to a solenoid as in [6] is a C^∞ map of the plane with an exceptional set.

In this paper, we define a collection of mappings of the plane which are solenoid-like and have an exceptional set. Each map is determined by a sequence of integers $\{p_i\}$, as in the standard solenoids (see [1], [6]), and we prove in Theorem 2 that the maps are C^∞ when $\sum_{i=1}^\infty \frac{1}{p_i}$ converges. It is conjectured that the map is not $C^\infty$when $\sum_{i=1}^\infty \frac{1}{p_i}$ diverges.

2. The Construction

We begin by introducing a family of disks (depicted in Figure 1) and an associated family of smooth functions. Suppose $p_1, p_2, ...$ is an infinite sequence of positive integers which are pairwise relatively prime, D is the unit disk in $\mathbb{R}^2$, and $f_1 \in C^\infty(\mathbb{R}^2)$ rotates D by $2\pi/p_1$.

Our first step is index a set of p_1 disjoint, closed disks $D_{(i)}$ that are properly contained in D such that f_1 maps $D_{(i)}$ bijectively onto $D_{(i+1 \text{ mod } p_1)}$. For each $i = 0, 1, ..., (p_1 - 1)$ we define a closed disk $E_{(i)}$ such that $D_{(i)} \subset E_{(i)}$ (strict containment) and $E_{(i)} \cap E_{(i')} = \emptyset$ when $i \neq i'$. Lastly, in this step, we choose a smooth function f_2 that rotates each $D_{(i)}$ by $2\pi/p_2$ and that is the identity outside of $\cup_i E_{(i)}$.

The second step begins by indexing a set of p_2 disjoint, closed disks $D_{(i,j)}$ that are properly contained in $D_{(i)}$, for each i, such that f_2 takes $D_{(i,j)}$ to $D_{(i,j+1 \text{ mod } p_2)}$. Hence $f_2 \circ f_1 : D_{(i,j)} \longmapsto D_{(i+1 \text{ mod } p_1, j+1 \text{ mod } p_2)}$. As before, we choose a collection of closed disks $E_{(i,j)}$ that are pairwise

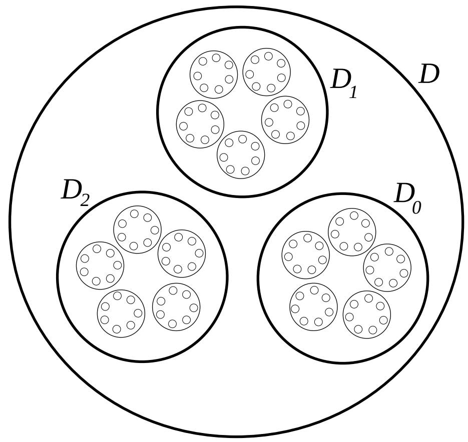

FIGURE 1. The first three levels of nested disks

disjoint such that $D_{(i,j)} \subset E_{(i,j)}$ (strict containment), and we select a smooth function f_3 that rotates $D_{(i,j)}$ by $2\pi/p_3$.

Continuing this pattern, we define closed disjoint disks $D_{(x_1,\ldots,x_i)}$, where $(x_1,\ldots,x_i) \in \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_i-1} \times \mathbb{Z}_{p_i}$, and closed disks $E_{(x_1,\ldots,x_i)} \supset D_{(x_1,\ldots,x_i)}$ (strict containment) such that $E_{(x_1,\ldots,x_i)} \cap E_{(y_1,\ldots,y_i)} = \emptyset$ when $(x_1,\ldots,x_i) \neq (y_1,\ldots,y_i)$. And for each $i \in \mathbb{N}$ we define a homeomorphism $f_i : \mathbb{R}^2 \to \mathbb{R}^2$ that satisfies the following conditions:

(a) The function f_i rotates every $D_{(x_1,\ldots,x_{i-1})}$ by $2\pi/p_i$.
(b) The function f_i is the identity of off the $E_{(x_1,\ldots,x_{i-1})}$.
(c) The composition $f_i \circ f_{i-1} \circ \cdots \circ f_2 \circ f_1$ maps $D_{(x_1,\ldots,x_{i-1},x_i)}$ to $D_{(x_1+1 \text{ mod } (p_1),\ldots,x_{i-1}+1 \text{ mod } (p_{i-1}),x_i+1 \text{ mod } (p_i))}$.

Our objects of study are the limit function F, defined by

$$F = \lim_{i\to\infty} f_i \circ f_{i-1} \circ \cdots \circ f_2 \circ f_1 \tag{1}$$

and its exceptional set.

3. The Exceptional Set of F

Toward describing the exceptional set of F, let

$$C_i = \bigcup_{\{(x_1,x_2,\ldots,x_i)\,|\,x_k \in \{0,1,\ldots,p_k-1\}\}} D_{(x_1,x_2,\ldots,x_i)}.$$

It is easy to verify that the set

$$C = \bigcap_{i=1}^{\infty} C_i$$

is a Cantor set. We will use the group $G = \prod_{i=1}^{\infty} \mathbb{Z}_{p_i}$ to index the points in C via the map $I : G \to C$,

$$I : (x_1, x_2, \ldots) \mapsto D_{(x_1)} \cap D_{(x_1,x_2)} \cap D_{(x_1,x_2,x_3)} \cap \cdots$$

and, henceforth, will identify $(\overline{x})$ and $I(\overline{x})$.

PROPOSITION 1. *The set C is a minimal set for the map F.*

Proof. It follows from our definition that for each $(\overline{x}) \in C$,

$$F(\overline{x}) = (x_1 + 1 \bmod (p_1), x_2 + 1 \bmod (p_2), \ldots) \in C \tag{2}$$

Because $\text{diam} D_{(x_1,\ldots,x_i)} \to 0$ as $i \to \infty$, for any $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for two points $(x_1, x_2, x_3, \ldots)$, $(y_1, y_2, y_3, \ldots) \in C$, $|(x_1, x_2, x_3, \ldots) - (y_1, y_2, y_3, \ldots)| < \varepsilon$ if $x_i = y_i$ for all $i < N$. Thus, the proof will be complete if we can demonstrate that, for any points $(y_1, y_2, y_3, \ldots)$ and $(x_1, x_2, x_3, \ldots) \in C$ and positive integer N, there exists a positive integer k such that the first N entries of $F^k(x_1, x_2, x_3, \ldots)$ agree with the first N entries of $(y_1, y_2, y_3, \ldots)$. This follows easily from Formula (2) because the p_i are pairwise relatively prime. □

We note that the bijection I determines a conjugacy between $F|_C$ and a map on G. To define this conjugacy, we equip G with the cylinder topology, where the basic open sets are of the form

$$B_{b_1,\ldots,b_k} = \{(x_1, x_2, \ldots) \in G, | \, x_i = b_i \text{ for } i = 1, \ldots, k\}.$$

Then the map F restricted to C is conjugate to the map $\alpha : G \to G$ defined by $\alpha(\overline{x}) = (\overline{x}) + (\overline{1})$. Specifically, $I((\overline{x}) + (\overline{1})) = F(I(\overline{x}))$. This follows directly from definition 1 and condition (c) on each f_i.

By comparison, the return map to a solenoid flow is conjugate (on its exceptional set) to an adding machine, or odometer. The adding machine for a sequence of integers $\{p_1, p_2, \ldots\}$ (not necessarily pairwise prime) is the map $\beta : G \to G$, defined by $\beta(x_i, x_2, \ldots) = (y_1, y_2, \ldots)$ where

$$y_i = \begin{cases} x_i + 1 \bmod p_i & \text{if } x_j = p_j - 1 \text{ for all } j < i, \\ x_i & \text{otherwise.} \end{cases}$$

Notice that the map α has more "twisting" than β. That is, the early coordinates of points in G change more under the action of α than under the action of β. In the setting of maps of $\mathbb{R}^2$, it means that the points are rotated more by F then by the return map for the solenoid. For this reason, we refer to the suspension of F as a *twisted solenoid*, and refer to the case where $\sum_{i=1}^{\infty} \frac{1}{p_i}$ diverges as an *over-twisted solenoid*.

4. Analysis of F

The argument that F is smooth relies on Theorem 1, which is an extension of Theorem 7.17 in [7]. Before stating the theorem, we ask the reader to recall the following definition.

DEFINITION 1. *We say an ordered pair α of non-negative integers is a multi-index with length $|\alpha| = \alpha_1 + \alpha_2$, and define the differential operator ∂^α by*

$$\partial^\alpha F = \frac{\partial^{|\alpha|} F}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}.$$

THEOREM 1. *For each $i \in \mathbb{N}$ let $f_i : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth function and suppose the sum*

$$\sum_{i=1}^{\infty} ||f_i - f_{i+1}||_{C^i}$$

converges where, for $F(x) = (F_1(x), F_2(x))$,

$$\|F\|_{C^m} = \sup_{j\in\{1,2\}} \|F_j(x)\|_\infty + \sup_{1\le|\alpha|\le m} \ \sup_{j\in\{1,2\}} \|\partial^\alpha F_j(x)\|_\infty. \tag{3}$$

Then there is a function $f \in C^\infty$ such that $\|f_i - f\|_\infty \to 0$.

For convenience we will identify $\mathbb{R}^2$ with $\mathbb{C}$ and use the notation of the complex plane to state the following technical lemma.

LEMMA 1. *For any positive integers p, k, real numbers $0 < a < b < 1$, and any real number $\varepsilon > 0$ there exists a C^∞ diffeomorphism $\phi : \mathbb{C} \to \mathbb{C}$ that satisfies the following:*

1. *There is a prime number $q > p$ so that $\phi(z) = ze^{2\pi i/q}$ for all $z \in \mathbb{C}$ with $|z| \le a$.*
2. *$\phi(z) = z$ for all $z \in \mathbb{C}$ such that $|z| \ge b$.*
3. *$||\phi - Id||_{C^k} < \varepsilon$, where $Id(z) = z$ is the identity function.*

Proof. Let $\rho : \mathbb{R} \to \mathbb{R}$ be a C^∞ function that is monotonically decreasing on (a, b) such that $\rho(r) = 1$ for $r < a^2$ and $\rho(r) = 0$ for $r > b^2$, and define the matrix valued function $A : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ by

$$A(x) = \begin{bmatrix} \cos\left(\frac{2\pi}{q}\rho(r^2)\right) & -\sin\left(\frac{2\pi}{q}\rho(r^2)\right) \\ \sin\left(\frac{2\pi}{q}\rho(r^2)\right) & \cos\left(\frac{2\pi}{q}\rho(r^2)\right) \end{bmatrix}, \tag{4}$$

where $r^2 = x_1^2 + x_2^2$. Then the function ϕ defined by $\phi(x) = A(x)x$ satisfies (1) and (2) from the theorem. We will show that $\phi(x)$ also satisfies (3) when q is chosen large enough.

From definition (3) we have

$$\|\phi - \mathrm{Id}\|_{C^k} = \tag{5}$$
$$\sup_{j\in\{1,2\}} \|(\phi - \mathrm{Id})_j\|_\infty + \sup_{1\le|\alpha|\le k} \sup_{j\in\{1,2\}} \|\partial^\alpha(\phi - \mathrm{Id})_j\|_\infty$$

We proceed by showing that each of the terms on the right side of this equation can be made less than $\varepsilon/2$ if q is chosen sufficiently large. We begin by considering the first component of $(\phi - \mathrm{Id})(x)$, which is part of the first summand of (5):

$$\begin{aligned} &(\phi - \mathrm{Id})_1(x) \\ &= x_1 \cos\left(\frac{2\pi}{q}\rho(r^2)\right) - x_2 \sin\left(\frac{2\pi}{q}\rho(r^2)\right) - x_1 \\ &= x_1 \left(\cos\left(\frac{2\pi}{q}\rho(r^2)\right) - 1\right) - x_2 \sin\left(\frac{2\pi}{q}\rho(r^2)\right). \end{aligned}$$

Note that $|(\phi - \mathrm{Id})_1(x)| = 0$ when $|x| > 1$ since $\rho(r) = 0$ for $r > 1 > b$. It follows that

$$\|(\phi - \mathrm{Id})_1\|_\infty \le \sup_{\|x\|\le 1} |(\phi - \mathrm{Id})_1(x)|.$$

That is, we may assume that $x_1^2 + x_2^2 \le 1$. This allows us to choose q_1 sufficiently large that, when $q \ge q_1$,

$$\left|\cos\left(\frac{2\pi}{q}\rho(r^2)\right) - 1\right| < \frac{\varepsilon}{4}$$

and

$$\left|\sin\left(\frac{2\pi}{q}\rho(r^2)\right)\right| < \frac{\varepsilon}{4}.$$

Similarly, there is a q_1' associated with the second component of $(\phi - \mathrm{Id})$, and by choosing $q \ge \max\{q_1,\ q_1'\}$ we ensure that the first summand of (5) is bounded above by $\frac{\varepsilon}{2}$.

Next, we address the second summand of equation (5). As before, we need only consider $x_1^2 + x_2^2 \le 1$. The case of $|\alpha| = 1$ is illuminating, so we begin by calculating

$$
\begin{aligned}
\frac{\partial(\phi - \mathrm{Id})_1}{\partial x_1} = \\
&\left[\cos\left(\frac{2\pi}{q}\rho(r^2)\right) - 1\right] \\
&\quad - \sin\left(\frac{2\pi}{q}\rho(r^2)\right)\frac{4\pi x_1^2 \rho'(r^2)}{q} \\
&\qquad - \cos\left(\frac{2\pi}{q}\rho(r^2)\right)\frac{4\pi x_1 x_2 \rho'(r^2)}{q}
\end{aligned} \tag{6}
$$

We can make the first summand on the right-hand side of equation (6) arbitrarily small by choosing q sufficiently large. Further, because ρ is smooth with compact support, its derivatives are bounded. Along with the fact that $x_1^2 + x_2^2 \leq 1$, this allows us to control the magnitude of the second and third summands of (6).

Higher order derivatives of $(\phi - \mathrm{Id})$ result in more factors of ρ' and higher order derivatives of ρ, but each is divided by at least one factor of q. Thus, by choosing q sufficiently large, we can guarantee that $\sup_{1\leq|\alpha|\leq k} \|\partial^\alpha(\phi - \mathrm{Id})_j\| < \frac{\varepsilon}{2}$ for $j \in \{1, 2\}$ and this concludes the proof. □

Let us denote the smooth diffeomorphism of Lemma (1) by $\phi_{(a,b);p,k,\varepsilon}$ and define $f_1 := \phi_{(1,1.1);1,1,0.5}$. We note that f_1 rotates the unit disk by $2\pi/q_1$ for some prime number $q_1 > 1$, and that f_1 is the identity outside of the disk of radius 1.1 centered at the origin. Further,

$$\|f_1 - \mathrm{Id}\|_{C^1} < 0.5.$$

We will use the n^{th} roots of unity to guide our development of f_j, for $j > 1$. Denote these by $u_{n,k} := \left(\cos\left(\frac{2\pi k}{n}\right), \sin\left(\frac{2\pi k}{n}\right)\right)$, $0 \leq k \leq (n-1)$. Toward defining f_2, choose numbers $0 < a_1 < b_1 < 1$ such that $|u_{q_1,i} - u_{q_1,j}| > 4b_1$ whenever $i \neq j$. We define the points

$$c_i = \frac{1}{2}u_{q_1,i} \text{ for } i = 0,\ 1, \ldots,\ (q_1 - 1)$$

and the disks

$$
\begin{aligned}
D_i &= B_{a_1}(c_i) \text{ for } i = 0,\ 1, \ldots,\ (q_1 - 1) \\
E_i &= B_{b_1}(c_i) \text{ for } i = 0,\ 1, \ldots,\ (q_1 - 1)
\end{aligned}
$$

where $B_r(c)$ is the closed ball of radius r centered at the point c. Notice that $f_1(D_i) = D_{i+1 \bmod q_1}$.

Choose some number $p_2 > q_1$ and, continuing to use the notation of $\mathbb{C}$, define

$$\psi_i(z) = \phi_{(a_1,b_1);p_2,2,0.25}(z - c_i) + c_i,\ 0 \le i \le q_1 - 1.$$

Then set $f_2 = \psi_{q_1-1} \circ \psi_{q_1-2} \circ \cdots \circ \psi_0$. The function f_2 rotates each D_i by $2\pi/q_2$, where q_2 is prime and $q_2 > p_2 > q_1$, and it is the identity function outside the disks E_i. Further, $\|f_2 - \mathrm{Id}\|_{C^2} < 0.25$. For each i and j, $0 \le i \le q_1 - 1$ and $0 \le j \le q_2 - 1$, define

$$c_{(i,j)} = c_i + \frac{a_1}{2} u_{q_2,j}.$$

Notice that $f_1(c_{(i,j)}) = c_{(i+1 \bmod q_1,\ j)}$ and $f_2(c_{(i,j)}) = c_{(i,j+1 \bmod q_2)}$, so $f_2 \circ f_1(c_{(i,j)}) = c_{(i+1 \bmod q_1,\ j+1 \bmod q_2)}$.

Choose $0 < a_2 < b_2 < 1$ so that $|c_{(i_1,j_1)} - c_{(i_2,j_2)}| > 4b_2$ whenever $(i_1, j_1) \ne (i_2, j_2)$ and for each i define the disks

$$D_{(i,j)} = B_{a_2}(c_{(i,j)}) \text{ for } j = 0,\ 2, \ldots,\ (q_2 - 1)$$
$$E_{(i,j)} = B_{b_2}(c_{(i,j)}) \text{ for } j = 0,\ 2, \ldots,\ (q_2 - 1).$$

Following the centers $c_{(i,j)}$, we have $f_2 \circ f_1(D_{(i,j)}) = D_{(i+1 \bmod q_1, j+1 \bmod q_2)}$.

We continue by induction. Suppose the maps f_j, $1 \le j \le (n-1)$, and the disks $D_{(x_1,x_2,\ldots,x_j)}$ and $E_{(x_1,x_2,\ldots,x_j)}$ (centered at the point $c_{(x_1,x_2,\ldots,x_j)}$) satisfy the following desiderata:

- For each $j > 1$, the function f_j rotates each $D_{(x_1,x_2,\ldots,x_j)}$ by $2\pi/q_j$ for some prime number $q_j > q_{j-1}$.
- For every j, $\|f_j - \mathrm{Id}\|_{C^j} < 2^{-j}$.
- For every j, the function f_j is the identity outside of the disks $E_{(x_1,x_2,\ldots,x_j)}$.
- The intersection $E_{(x_1,x_2,\ldots,x_j)} \cap E_{(y_1,y_2,\ldots,y_j)} = \emptyset$ when $(x_1, x_2, \ldots, x_j) \ne (y_1, y_2, \ldots, y_j)$.
- For every $(x_1, x_2, \ldots, x_j) \in \mathbb{Z}_{q_1} \times \mathbb{Z}_{q_2} \times \cdots \times \mathbb{Z}_{q_j}$ we have

$$f_j \circ f_{j-1} \circ \cdots \circ f_1(D_{(x_1,x_2,\ldots,x_j)}) = D_{(x_1+1 \bmod q_1, x_2+1 \bmod q_2, \ldots, x_j+1 \bmod q_j)}.$$

- There exist positive numbers $a_{n-1} < b_{n-1} < 1$ such that

$$|c_{(x_1,x_2,\ldots,x_{n-1})} - c_{(y_1,y_2,\ldots,y_{n-1})}| > 2b_{n-1}$$

  for all $(x_1, x_2, \ldots, x_j) \ne (y_1, y_2, \ldots, y_j)$.

We choose some number $p_n > q_{n-1}$ and set $G_n := \mathbb{Z}_{q_1} \times \mathbb{Z}_{q_2} \times \cdots \times \mathbb{Z}_{q_{n-1}}$. Then f or each $\gamma \in G_n$ we define

$$\psi_\gamma(x) = \phi_{(a_{n-1},b_{n-1});p_n,n,2^{-n}}(x - c_\gamma) + c_\gamma.$$

The function f_n is defined as

$$f_n = \circ_{\gamma \in G_n} \psi_\gamma. \tag{7}$$

That is, f_n is the composition of all $\psi_{(x_1,x_2,\ldots,x_{n-1})}$, where $(x_1, x_2, \ldots, x_{n-1}) \in \mathbb{Z}_{q_1} \times \mathbb{Z}_{q_2} \times \cdots \times \mathbb{Z}_{q_{n-1}}$. (The order of composition does not matter because the set of points on which ψ_{γ_1} is not the identity is disjoint from the set of points on which ψ_{γ_2} is not the identity when $\gamma_1,\ \gamma_2$ are distinct elements of G_n.) The function f_n rotates each disk $D_{(x_1,x_2,\ldots,x_{n-1})}$ by $2\pi/q_n$ for some prime number $q_n > p_n > q_{n-1}$, is the identity outside of $\cup_{\gamma \in G_n} E_\gamma$, and $\|f_n - \mathrm{Id}\|_{C^n} < 2^{-n}$.

For each $(x_1, x_2, \ldots, x_n) \in \mathbb{Z}_{q_1} \times \mathbb{Z}_{q_2} \times \cdots \times Z_{q_n}$, define

$$c_{(x_1,x_2,\ldots,x_n)} = c_{(x_1,x_2,\ldots,x_{n-1})} + \frac{a_{n-1}}{2} u_{q_n,x_n}.$$

Then $f_n(c_{(x_1,x_2,\ldots,x_n)}) = c_{(x_1,x_2,\ldots,x_{n+1} \bmod q_n)}$ and

$$f_n \circ f_{n-1} \circ \cdots \circ f_1(c_{(x_1,x_2,\ldots,x_n)}) = c_{(x_1+1 \bmod q_1, x_2+1 \bmod q_2,\ldots,x_n+1 \bmod q_n)}.$$

Choose positive numbers $a_n < b_n < 1$ so that $|c_{(x_1,x_2,\ldots,x_n)} - c_{(y_1,y_2,\ldots,y_n)}| > 4b_n$ whenever $(x_1, x_2, \ldots, x_n) \neq (y_1, y_2, \ldots, y_n)$ and, for each $(x_1, x_2, \ldots, x_n) \in \mathbb{Z}_{q_1} \times \mathbb{Z}_{q_2} \times \cdots \times \mathbb{Z}_{q_n}$ define the disks

$$\begin{aligned} D_{(x_1,x_2,\ldots,x_n)} &= B_{a_n}(c_{(x_1,x_2,\ldots,x_n)}) \\ E_{(x_1,x_2,\ldots,x_n)} &= B_{b_n}(c_{(x_1,x_2,\ldots,x_n)}). \end{aligned}$$

Notice that $f_n(D_{(x_1,x_2,\ldots,x_n)}) = D_{(x_1,x_2,\ldots,x_n+1 \bmod q_n)}$. Hence

$$f_n \circ f_{n-1} \circ \cdots \circ f_1(D_{(x_1,x_2,\ldots,x_n)}) = D_{(x_1+1 \bmod q_1, x_2+1 \bmod q_2,\ldots,x_n+1 \bmod q_n)}.$$

We now define $F_n = f_n \circ f_{n-1} \circ \cdots \circ f_1$. Note that F_n differs from F_{n-1} on only the disks $E_{(x_1,x_2,\ldots,x_n)}$, and then only by the rotation affected by f_n, so $\|F_{n-1} - F_n\|_{C^n} = \|f_n - \mathrm{Id}\|_{C^n} < 2^{-n}$. Thus,

$$\sum_{n=1}^{\infty} \|F_{n-1} - F_n\|_{C^n} < \infty,$$

and Theorem 1 guarantees the existence of a smooth function F such that $F_n \to F$ uniformly. We have thus proved the following theorem.

THEOREM 2. *Suppose $\sum_{i=1}^{\infty} \frac{1}{p_i}$ converges. Then the function F defined by (1) is in C^∞.*

5. Remarks and Conjecture

5.1. **A Homeomorphism on $\mathbb{R}^3$ with Antoine's Necklace as a Minimal Set.** Antoine's Necklace can be defined in the following way (see [7]). Let $q_1, q_2, \dots$ be an infinite sequence of positive integers. Let $T = S^1 \times D^2$ be a solid torus coordinatized by (θ, ϕ, r) where $\theta \in [0, 2\pi]$ mod 2π is the coordinate on S^1 and (ϕ, r) are polar coordinates on D^2. Inside T define a chain of solid tori $T_0, \dots, T_{q_1-1}$ as follows. For each $i \in \{0, 1, \dots, q_1 - 1\}$, let $p_i = (i2\pi/q_1, 0, 0)$, and let γ_i be the circle of radius $3\pi/4q_1$ that is centered at p_i and contained in $\{(\theta, \phi, r) \mid \phi = i2\pi/q_1\}$. (See figure 5.1.) Note that the linking number of γ_i with γ_j is ± 1 if $|i - j \text{ mod } q_1| = 1$ and that γ_i is the image of $\gamma_{i-1 \text{ mod } q_1}$ under the rotation of T by $(\theta, \phi, r) \to (\theta + i2\pi/q_1, \phi + i2\pi/q_1, r)$. From now on we refer to this homeomorphism simply as rotation by $(i2\pi/q_1, i2\pi/q_1)$. For each i define T_i to be a torus neighborhood of γ_i such that all of the T_i are disjoint and a rotation of T by $(i2\pi/q_1, i2\pi/q_1)$ takes T_i to $T_{i+1 \text{ mod } q_1}$. Denote the union of these tori by $C_1 = \bigcup_{i=0}^{q_1-1} T_i$.

In T_0 define a chain of q_2 pairwise disjoint solid tori, $T_{(0,0)}, \dots, T_{(0,q_2-1)}$, such that a rotation of T_0 by $(2\pi/q_2, 2\pi/q_2)$ takes $T_{(0,i)}$ to $T_{(0,i+1 \text{ mod } q_2)}$. Let $T_{(i,j)}$ be the images of $T_{(0,j)}$ under rotations of T by $(2\pi/q_1, 2\pi/q_1)$ and let $C_2 = \bigcup_{j=0}^{q_2-1} \bigcup_{i=0}^{q_1-1} T_{(i,j)}$. Continuing in this manner results in a nested sequence of compact sets $\dots \subset C_2 \subset C_1$ and Antoine's necklace is the set

$$A = \bigcap_{i=1}^{\infty} C_i.$$

The topologically interesting property of Antoine's necklace is that it is a Cantor set and it's complement is not simply connected when imbedded in R^3.

The natural homeomorphism $J : G \to A$ is

$$J(x_1, x_2, \dots) = T \cap T_{x_1} \cap T_{(x_1,x_2)} \cap \cdots .$$

One could define a map that has A as an exceptional set by mimicking the construction outlined in the previous sections.

5.2. **The D-function.** An important topological invariant of a minimal set is the D-function, developed by Ye in [8]. Suppose that $f : X \to X$ is a continuous map of a compact Hausdorff space and that Y is a minimal set for f. The D-function for Y is the function $f_Y : \mathbb{N} \to \mathbb{N}$ which takes the natural number n to the number of distinct minimal sets of f^n which are contained in Y. If $p_1, p_2, ...$ is the sequence of pairwise relatively prime positive integers defining F, then the D-function of F on C is

$$F_C(n) = \prod_{\{p_i : p_i | n\}} p_i.$$

5.3. **A Conjecture.** Theorem 2 begs the question of whether its converse is true. We conjecture that it is and, in fact, over-twisted solenoids are not differentiable.

CONJECTURE 1. *The map F defined by (1) is not C^2 when $\sum_{i=1}^{\infty} p_i$ diverges.*

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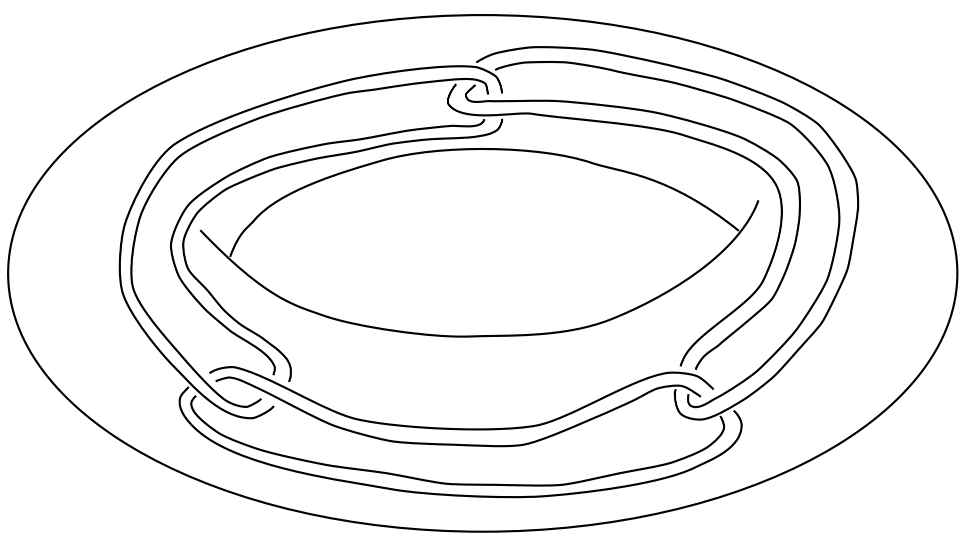

Figure 2. The first two steps in creating Antoine's Necklace.

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