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1. INTRODUCTION

There are many known connections between determinants of tridiagonal matrices and the Fibonacci and Lucas numbers. For example, Strang [5, 6] presents a family of tridiagonal matrices given by:

$$\mathbf{M}(n)=\begin{pmatrix} 3 & 1 & & & \\ 1 & 3 & 1 & & \\ & 1 & 3 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 3 \end{pmatrix}, \tag{1}$$

where $\mathbf{M}(n)$ is $n\times n$. It is easy to show by induction that the determinants $|\mathbf{M}(k)|$ are the Fibonacci numbers F_{2k+2}. Another example is the family of tridiagonal matrices given by:

$$\mathbf{H}(n)=\begin{pmatrix} 1 & i & & & \\ i & 1 & i & & \\ & i & 1 & \ddots & \\ & & \ddots & \ddots & i \\ & & & i & 1 \end{pmatrix}, \tag{2}$$

described in [2] and [3] (also in [5], but with 1 and –1 on the off-diagonals, instead of i). The determinants $|\mathbf{H}(k)|$ are all the Fibonacci numbers F_k, starting with $k=2$. In a similar family of matrices [1], the $(1,1)$ element of $\mathbf{H}(n)$ is replaced with a 3. The determinants now generate the Lucas sequence L_k, starting with $k=2$ (the Lucas sequence is defined by the second order recurrence $L_1=1$, $L_2=3$, $L_{k+1}=L_k+L_{k-1}$, $k\geq 2$).

In this article, we extend these results to construct families of tridiagonal matrices whose determinants generate any arbitrary linear subsequence $F_{\alpha k+\beta}$ or $L_{\alpha k+\beta}$, $k=1,2,\ldots$ of the Fibonacci or Lucas numbers. We then choose a specific linear subsequence of the Fibonacci numbers and use it to derive the following factorization:

$$F_{2mn}=F_{2m}\prod_{k=1}^{n-1}\left(L_{2m}-2\cos\frac{\pi k}{n}\right). \tag{3}$$

This factorization is a generalization of one of the factorizations presented in [3]:

$$F_{2n}=\prod_{k=1}^{n-1}\left(3-2\cos\tfrac{\pi k}{n}\right).$$

In order to develop these results, we must first present a theorem describing the sequence of determinants for a general tridiagonal matrix. Let $A(k)$ be a family of tridiagonal matrices, where

$$A(k)=\begin{pmatrix} a_{1,1} & a_{1,2} & & & \\ a_{2,1} & a_{2,2} & a_{2,3} & & \\ & a_{3,2} & a_{3,3} & \ddots & \\ & & \ddots & \ddots & a_{k-1,k} \\ & & & a_{k,k-1} & a_{k,k} \end{pmatrix}.$$

<u>**Theorem 1**</u>: The determinants $|A(k)|$ can be described by the following recurrence relation:

$$|A(1)| = a_{1,1}$$

$$|A(2)| = a_{2,2}a_{1,1} - a_{2,1}a_{1,2}$$

$$|A(k)| = a_{k,k}\,|A(k-1)| - a_{k,k-1}a_{k-1,k}\,|A(k-2)|, \qquad k \geq 3.$$

Proof: The cases $k=1$ and $k=2$ are clear. Now

$$|A(k)| = \det\begin{pmatrix} a_{1,1} & a_{1,2} & & & & \\ a_{2,1} & a_{2,2} & \ddots & & & \\ & \ddots & \ddots & a_{k-3,k-2} & & \\ & & a_{k-2,k-3} & a_{k-2,k-2} & a_{k-2,k-1} & \\ & & & a_{k-1,k-2} & a_{k-1,k-1} & a_{k-1,k} \\ & & & & a_{k,k-1} & a_{k,k} \end{pmatrix}.$$

By cofactor expansion on the last column and then the last row,

$$|A(k)| = a_{k,k}\,|A(k-1)| - a_{k-1,k}\det\begin{pmatrix} a_{1,1} & a_{1,2} & & & \\ a_{2,1} & a_{2,2} & \ddots & & \\ & \ddots & \ddots & a_{k-3,k-2} & \\ & & a_{k-2,k-3} & a_{k-2,k-2} & a_{k-2,k-1} \\ & & & 0 & a_{k,k-1} \end{pmatrix}$$

$$= a_{k,k}\,|A(k-1)| - a_{k-1,k}a_{k,k-1}\,|A(k-2)|. \qquad \blacksquare$$

2. FIBONACCI SUBSEQUENCES

Using Theorem 1, we can generalize the families of tridiagonal matrices given by (1) and (2) to construct, for every linear subsequence of Fibonacci numbers, a family of tridiagonal matrices whose successive determinants are given by that subsequence.

Theorem 2: The symmetric tridiagonal family of matrices $M_{\alpha,\beta}(k)$, $k = 1,2,\ldots$ whose elements are given by:

$$m_{1,1} = F_{\alpha+\beta}\,,\;\; m_{2,2} = \left\lceil \tfrac{F_{2\alpha+\beta}}{F_{\alpha+\beta}} \right\rceil$$

$$m_{j,j} = L_\alpha\,,\;\; 3 \le j \le k\,,$$

$$m_{1,2} = m_{2,1} = \sqrt{m_{2,2}F_{\alpha+\beta} - F_{2\alpha+\beta}}$$

$$m_{j,j+1} = m_{j+1,j} = \sqrt{(-1)^{\alpha}}\,,\;\; 2 \le j < k\,,$$

with $\alpha \in Z^+$ and $\beta \in N$, has successive determinants $\left|M_{\alpha,\beta}(k)\right| = F_{\alpha k+\beta}$.

In order to prove Theorem 2, we must first present the following lemma:

<u>Lemma 1</u>: $F_{k+n} = L_n F_k + (-1)^{n+1} F_{k-n}$ for $n \ge 1$.

<u>Proof</u>: We use the second principle of finite induction on *n* to prove this lemma:

Let $n = 1$. Then the lemma yields $F_{k+1} = F_k + F_{k-1}$, which defines the Fibonacci sequence. Now assume that $F_{k+n} = L_n F_k + (-1)^{n+1} F_{k-n}$ for $n \le N$. Then

$$\begin{aligned}
F_{k+N+1} &= F_{k+N} + F_{k+N-1} \\
&= L_N F_k + (-1)^{N+1} F_{k-N} + L_{N-1} F_k + (-1)^{N} F_{k-N+1} \\
&= (L_N + L_{N-1}) F_k + (-1)^{N+2} (F_{k-N+1} - F_{k-N}) \\
&= L_{N+1} F_k + (-1)^{N+2} F_{k-(N+1)} \qquad \blacksquare
\end{aligned}$$

Now, using Theorem 1 and Lemma 1, we can prove Theorem 2.

<u>Proof of Theorem 2</u>: We use the second principle of finite induction on *k* to prove this theorem:

$$\left|M_{\alpha,\beta}(1)\right| = \det F_{\alpha+\beta} = F_{\alpha+\beta}\,.$$

$$\left|M_{\alpha,\beta}(2)\right| = \det\begin{pmatrix} F_{\alpha+\beta} & \sqrt{m_{2,2}F_{\alpha+\beta} - F_{2\alpha+\beta}} \\ \sqrt{m_{2,2}F_{\alpha+\beta} - F_{2\alpha+\beta}} & \left\lceil \tfrac{F_{2\alpha+\beta}}{F_{\alpha+\beta}} \right\rceil \end{pmatrix} = F_{2\alpha+\beta}\,.$$

Now assume that $\left|M_{\alpha,\beta}(k)\right| = F_{\alpha k+\beta}$ for $1 \le k \le N$. Then by Theorem 1,

$$\begin{aligned}
\left|M_{\alpha,\beta}(k+1)\right| &= m_{k,k}\left|M_{\alpha,\beta}(k)\right| - m_{k,k-1}m_{k-1,k}\left|M_{\alpha,\beta}(k-1)\right| \\
&= L_\alpha\left|M_{\alpha,\beta}(k)\right| - (-1)^\alpha\left|M_{\alpha,\beta}(k-1)\right| \\
&= L_\alpha F_{\alpha k+\beta} + (-1)^{\alpha+1}F_{\alpha(k-1)+\beta} \\
&= F_{\alpha+\alpha k+\beta} \qquad \text{(by Lemma 1)} \\
&= F_{\alpha(k+1)+\beta} \qquad \blacksquare
\end{aligned}$$

Another family of matrices that satisfies Theorem 2 can be found by choosing the negative root for all of the super-diagonal and sub-diagonal entries. With Theorem 2, we can now construct a family of tridiagonal matrices whose successive determinants form any linear subsequence of the Fibonacci numbers. For example, the determinants of:

$$\begin{pmatrix} 1 & 0 & & & & \\ 0 & 8 & 1 & & & \\ & 1 & 7 & 1 & & \\ & & 1 & 7 & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & 1 & 7 \end{pmatrix}, \begin{pmatrix} 8 & \sqrt{6} & & & & \\ \sqrt{6} & 5 & i & & & \\ & i & 4 & i & & \\ & & i & 4 & \ddots & \\ & & & \ddots & \ddots & i \\ & & & & i & 4 \end{pmatrix}, \text{ and } \begin{pmatrix} 13 & -\sqrt{5} & & & & \\ -\sqrt{5} & 3 & -1 & & & \\ & -1 & 3 & -1 & & \\ & & -1 & 3 & \ddots & \\ & & & \ddots & \ddots & -1 \\ & & & & -1 & 3 \end{pmatrix}$$

are given by the Fibonacci subsequences F_{4k-2}, F_{3k+3} and F_{2k+5}.

3. LUCAS SUBSEQUENCES

We can also generalize the families of tridiagonal matrices given by (1) and (2) to show a similar result for linear subsequences of Lucas numbers. We state this result as the following theorem:

Theorem 3: The symmetric tridiagonal family of matrices $T_{\alpha,\beta}(k)$, $k = 1,2,\ldots$ whose elements are given by:

$$t_{1,1} = L_{\alpha+\beta}\,,\; t_{2,2} = \left\lceil \frac{L_{2\alpha+\beta}}{L_{\alpha+\beta}} \right\rceil$$

$$t_{j,j} = L_{\alpha}\,,\; 3 \le j \le k\,,$$

$$t_{1,2} = t_{2,1} = \sqrt{t_{2,2}L_{\alpha+\beta} - L_{2\alpha+\beta}}$$

$$t_{j,j+1} = t_{j+1,j} = \sqrt{(-1)^{\alpha}}\,,\; 2 \le j < k\,,$$

with $\alpha \in Z^{+}$ and $\beta \in N$, has successive determinants $\left|T_{\alpha,\beta}(k)\right| = L_{\alpha k+\beta}$.

Again we begin with a lemma; its proof imitates the proof of Lemma 1.

Lemma 2: $L_{k+n} = L_n L_k + (-1)^{n+1} L_{k-n}$ for $n \ge 1$.

Proof of Theorem 3: We use induction:

$$\left|T_{\alpha,\beta}(1)\right| = \det L_{\alpha+\beta} = L_{\alpha+\beta}\,.$$

$$\left|T_{\alpha,\beta}(2)\right| = \det\begin{pmatrix} L_{\alpha+\beta} & \sqrt{m_{2,2}L_{\alpha+\beta} - L_{2\alpha+\beta}} \\ \sqrt{m_{2,2}L_{\alpha+\beta} - L_{2\alpha+\beta}} & \left\lceil \frac{L_{2\alpha+\beta}}{L_{\alpha+\beta}} \right\rceil \end{pmatrix} = L_{2\alpha+\beta}\,.$$

Now assume that $\left|T_{\alpha,\beta}(k)\right| = L_{\alpha k+\beta}$ for $1 \le k \le N$. Then by Theorem 1,

$$\begin{aligned}
\left|T_{\alpha,\beta}(k+1)\right| &= t_{k,k}\left|T_{\alpha,\beta}(k)\right| - t_{k,k-1}t_{k-1,k}\left|T_{\alpha,\beta}(k-1)\right| \\
&= L_{\alpha}\left|T_{\alpha,\beta}(k)\right| - (-1)^{\alpha}\left|T_{\alpha,\beta}(k-1)\right| \\
&= L_{\alpha}L_{\alpha k+\beta} + (-1)^{\alpha+1}L_{\alpha(k-1)+\beta} \\
&= L_{\alpha+\alpha k+\beta} \qquad \text{(by Lemma 2)} \\
&= L_{\alpha(k+1)+\beta} \qquad \blacksquare
\end{aligned}$$

With Theorem 3, we can now construct a family of tridiagonal matrices whose successive determinants form any linear subsequence of the Lucas numbers. For example, the determinants of:

$$\begin{pmatrix} 3 & 0 & & & & \\ 0 & 6 & -1 & & & \\ & -1 & 7 & -1 & & \\ & & -1 & 7 & \ddots & \\ & & & \ddots & \ddots & -1 \\ & & & & -1 & 7 \end{pmatrix}, \begin{pmatrix} 18 & \sqrt{14} & & & & \\ \sqrt{14} & 5 & i & & & \\ & i & 4 & i & & \\ & & i & 4 & \ddots & \\ & & & \ddots & \ddots & i \\ & & & & i & 4 \end{pmatrix}, \text{ and } \begin{pmatrix} 29 & \sqrt{11} & & & & \\ \sqrt{11} & 3 & 1 & & & \\ & 1 & 3 & 1 & & \\ & & 1 & 3 & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & 1 & 3 \end{pmatrix}$$

are given by the Lucas subsequences L_{4k-2}, L_{3k+3} and L_{2k+5}.

4. A FACTORIZATION OF THE FIBONACCI NUMBERS

In order to derive the factorization (3) given by $F_{2mn} = F_{2m}\prod_{k=1}^{n-1}\left(L_{2m} - 2\cos\frac{\pi k}{n}\right)$, we consider the symmetric tridiagonal matrices:

$$B_m(n) = \begin{pmatrix} L_{2m}F_{2m} & \sqrt{F_{2m}} & & & & \\ \sqrt{F_{2m}} & L_{2m} & 1 & & & \\ & 1 & L_{2m} & 1 & & \\ & & 1 & L_{2m} & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & 1 & L_{2m} \end{pmatrix}.$$

By Lemma 1, $F_{4m} = L_{2m}F_{2m}$, and $\lceil F_{6m}/F_{4m} \rceil = \lceil L_{2m} - (F_{2m}/F_{4m}) \rceil = L_{2m}$. Furthermore, $\sqrt{\lceil F_{6m}/F_{4m} \rceil F_{4m} - F_{6m}} = \sqrt{L_{2m}F_{4m} - F_{6m}} = \sqrt{F_{2m}}$, so $B_m(n) = M_{2m,2m}(n)$ is a specific instance of the tridiagonal family of matrices described in Theorem 2. Therefore, by Theorem 2, $|B_m(n)| = F_{2m(n+1)}$.

By using the property of determinants that $|AB| = |A||B|$, and by defining $\mathbf{e}_j$ to be the j^{th} column of the $n \times n$ identity matrix $\mathbf{I}$, we have $|B_m(n)| = F_{2m}|C_m(n)|$, where:

$$C_m(n) = \left(\mathbf{I} + \left(\frac{1}{F_{2m}} - 1\right)\mathbf{e}_1\mathbf{e}_1^{\mathbf{T}}\right)B_m(n).$$

The determinant is the product of the eigenvalues. Therefore, let λ_k, $k = 1,2,\ldots n$ be the eigenvalues of $C_m(n)$ (with associated eigenvectors $\mathbf{x}_k$), so $|C_m(n)| = \prod_{k=1}^{n} \lambda_k$. Letting $G_m(n) = C_m(n) - L_{2m}\mathbf{I}$, we see that $G_m(n)\mathbf{x}_k = C_m(n)\mathbf{x}_k - L_{2m}\mathbf{I}\mathbf{x}_k = \lambda_k\mathbf{x}_k - L_{2m}\mathbf{x}_k = (\lambda_k - L_{2m})\mathbf{x}_k$. Then $\gamma_k = \lambda_k - L_{2m}$ are the eigenvalues of $G_m(n)$.

An eigenvalue γ of $G_m(n)$ is a root of the characteristic polynomial $|G_m(n) - \gamma\mathbf{I}| = 0$. Note that $|G_m(n) - \gamma\mathbf{I}| = \left|\left(\mathbf{I} + \left(\sqrt{F_{2m}} - 1\right)\mathbf{e}_1\mathbf{e}_1^{\mathbf{T}}\right)(G_m(n) - \gamma\mathbf{I})\left(\mathbf{I} + \left(1/\sqrt{F_{2m}} - 1\right)\mathbf{e}_1\mathbf{e}_1^{\mathbf{T}}\right)\right|$, so γ is also a root of the polynomial:

$$\begin{vmatrix} -\gamma & 1 & & & & \\ 1 & -\gamma & 1 & & & \\ & 1 & -\gamma & 1 & & \\ & & 1 & -\gamma & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & 1 & -\gamma \end{vmatrix} = 0.$$

This polynomial is a transformed Chebyshev polynomial of the second kind [4], with roots $\gamma_k = -2\cos\frac{\pi k}{n+1}$. Therefore,

$$F_{2m(n+1)} = |B_m(n)| = F_{2m}|C_m(n)| = F_{2m}\prod_{k=1}^{n}\lambda_k = F_{2m}\prod_{k=1}^{n}\left(L_{2m} - 2\cos\tfrac{\pi k}{n+1}\right).$$

(3) follows by a simple change of variables.

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