2003

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Recommended Citation

IEEE Transactions on Antennas and Propagation 51N4 (2003) 700-712

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The Spatially Averaged Electric Field in the Near Field and Far Field of a Circular Aperture

Charles J. Daly, Todd W. Nuteson, *Member, IEEE*, and Navalgund A. H. K. Rao

Abstract—This paper presents a theoretical and numerical investigation of the spatially averaged electric field in the beam of a circular aperture. The investigation leads to closed-form analytical expressions, based on scalar diffraction theory, which describe the spatially averaged electric field in the Fresnel region of a circular aperture excited by a spatially uniform, harmonic plane wave. The expressions ultimately permit rapid, practical, and efficient prediction of certain routine electromagnetic measurements. Because the expressions are valid in the Fresnel region, they are also valid in the near field, the far field, and the Fraunhofer region of a circular aperture. In fact, it is shown that the closed-form expressions contain, as special cases, classic on-axis and far-field results associated with a circular aperture. The analytical expressions are based on a generalization of Fresnel diffraction originally developed by Lommel in the late 1800s. Hence, a thorough review of the literature on the Lommel diffraction formulation is presented. Finally, it is shown that results obtained from the closed-form expressions compare quite favorably to results obtained from the exact solution computed via the dyadic Green's function approach.

Index Terms—Aperture antennas, electromagnetic diffraction.

I. INTRODUCTION

THIS paper presents a theoretical and numerical investigation of the spatially averaged electric field as would be measured by a finite receiving aperture centered in the beam of a radiating circular aperture. It complements Bouwkamp's investigation of diffraction from a circular aperture [1] by presenting a relatively simple set of closed-form expressions that, to our knowledge, have not appeared in the literature on electromagnetics. The expressions apply to the practically relevant case of the spatially averaged electric field in the Fresnel region of a circular aperture excited by a spatially uniform, harmonic plane wave with polarization parallel to the plane of the aperture [2].

Because the expressions are valid in the Fresnel region, they are also valid in the near field, the far field, and the Fraunhofer region of a circular aperture. This point is explained in greater detail in Section III. In addition, it is shown that the Lommel diffraction formulation contains, as special cases, the well-known on-axis and far-field results associated with electromagnetic radiation from a circular aperture.

The closed-form expressions are based on a generalization of Fresnel diffraction originally developed by Lommel in the

late 1800s [3]. Hence, a thorough review of the literature on the Lommel diffraction formulation is presented. The expressions ultimately permit rapid, practical, and efficient prediction of certain routine electromagnetic measurements involving circular apertures, dish antennas, and finite receivers. These measurements include antenna calibration and spatial pattern measurements, evaluation of electromagnetic probes for EMI measurements, and safety surveys of radiated electromagnetic fields and power densities [4].

The expressions are derived in sufficient detail so that the reader will be able to understand and program them quite readily. Finally, we show that results obtained from the closed-form spatially averaged expressions compare quite favorably to results obtained from the exact solution computed via the dyadic Green's function approach.

II. LITERATURE REVIEW

In the late 1800s, Lommel investigated optical diffraction [3]. In the course of his investigation, he developed a closed-form description of diffraction from a circular aperture which we call the *Lommel diffraction formulation*. Lommel's work was originally published in German, but excellent accounts of Lommel's work can now be found in several English-language textbooks [5]–[7]. Decades after its publication in 1886, Lommel's work caught the attention of the mathematical community. Watson, for example, investigated Lommel functions in his now classic treatise on Bessel functions [8].

The ultrasound community began investigating the Lommel diffraction formulation in the late 1940s when Huntington and co-workers mentioned it in a paper on ultrasound delay lines and spatially averaged diffraction effects [9]. In the early 1950s, the ultrasound community began developing pulse-echo techniques, and the Lommel diffraction formulation was applied in this new area of ultrasound for a number of years [10]. In 1956, Seki and co-workers published a paper on Lommel functions and diffraction effects in the case of monochromatic excitation of an unfocused piston transducer [11]. Extensive discussion and review of the Lommel diffraction formulation in ultrasound can be found in [12].

Of course, the optical community continued research on the Lommel diffraction formulation. In 1947, for example, Hopkins extended Lommel's original work to include diffraction of waves with radially uniform and nonuniform amplitude [13]. In that same year, Nijboer published a paper on optical aberrations and presented an alternative closed-form expression for the Lommel diffraction formulation; see (23) in [14] or (7) in [15].

Manuscript received November 3, 2000; revised September 21, 2001.

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Digital Object Identifier 10.1109/TAP.2003.811072

In 1951, Wolf extended Lommel's treatment of Fresnel diffraction and derived expressions "for the fraction of the total illumination present within certain regions in receiving planes near focus of spherical waves issuing from a circular aperture ... [16]". In short, Wolf used the Lommel diffraction formulation to spatially integrate the intensity impinging on a disk coaxially located some distance z from the aperture. In the early 1980s, Stamnes and co-workers extended Wolf's work by investigating the encircled energy in systems with centrally obscured circular pupils [17].

Researchers in electromagnetics appear to have become interested in the Lommel diffraction formulation in the early 1970s. In fact, Northover's treatment of the simple microwave lens depends quite heavily on Wolf's treatment of Fresnel diffraction from a circular aperture. We do note, however, that Northover extended Wolf's work by considering the very important case of apodization [2]. Although it did not deal directly with the Lommel diffraction formulation, Bouwkamp's doctoral dissertation on diffraction through a circular aperture is considered seminal. His work was first published in Dutch in 1941, and a good portion of this seminal work was reprised in English by this TRANSACTIONS in 1970 [1].

It must be reiterated that the Lommel diffraction formulation is ultimately a scalar formulation based on the Fresnel approximation. As such, it is mathematically tractable and useful but not general. Indeed, application of the Lommel diffraction may not be suitable for certain problems. For these instances, a text on waves in focal regions by Stamnes [18] is highly recommended. In the text, the author presents rigorous discussion on the subtleties of various diffraction theories such as the Kirchhoff theory, the Debye approximation, and the first and second Rayleigh-Sommerfeld theories. In addition, the text includes a variety of efficient numerical techniques for calculating diffraction integrals.

Perhaps because it is a scalar diffraction theory, the Lommel diffraction formulation appears to have never really caught on in electromagnetics. A sampling of popular texts on electromagnetics corroborates this assertion [19], [20]. Indeed, the Lommel diffraction formulation is not mentioned in Brookner's extensive treatment of circular apertures [21], nor does it appear in Sherman's lengthy review of antenna aperture analysis [4]. A recent exception to this observation is the work of Hongo and co-workers. In 1996, they used generalized Lommel functions to evaluate the radiation fields in the Fresnel regions of circular and elliptical apertures [22].

III. AN IMPORTANT CLARIFICATION

The very useful notion of a near field and a far field is well established in the literature. However, this very useful notion has led to a certain misconception that is both widely held and firmly rooted in electromagnetics. Specifically, the near field is often incorrectly equated, and thus confused, with the Fresnel region [23]. This misconception is generally harmless; it may, however, lead to misinterpretation of our results. Thus, we state explicitly at the outset that the near-field, far-field, and Fraunhofer region are all contained in the Fresnel region.

Fig. 1. Circular aperture geometry for (a) a point receiver and (b) a finite receiver of radius b .

The following observations are made to clarify this statement and prevent misinterpretation of our results. The Rayleigh-Sommerfeld region consists of the entire half-space in front of the aperture [23]. For the circular aperture shown in Fig. 1 (a), the Fresnel region is that portion of the Rayleigh-Sommerfeld region where the Fresnel conditions hold [24, pp. 329–330]:

$$25(a + \rho')^4 < \lambda z^3 = \frac{cz^3}{f}, \quad (1)$$

where a is the radius of aperture, ρ' the cylindrical coordinate depicted in Fig. 1(a), λ is wavelength, c is the speed of light, and f is temporal frequency. Thus, the Fresnel region extends, theoretically, to infinity and is not limited to the near field. The

requirement in (1) is overly stringent, and it can be shown that the Fresnel approximation is in fact valid for points nearer the aperture [25].

The Fraunhofer region is that portion of the Fresnel region where the Fraunhofer approximation holds. Thus, the Fresnel region contains both the Fraunhofer region and the far field [23], [24]. For the purposes of this paper, we follow Robinson and co-workers and define the near field as the region where the electric field oscillates and the far field as the region where the electric field decreases monotonically [26].

The Lommel diffraction formulation and spatially averaged results derived from it are based on the Fresnel diffraction integral. Thus, the Lommel diffraction formulation and expressions based on it are valid over a large portion of the half space in front of a circular aperture.

IV. THE LOMMEL DIFFRACTION FORMULATION

In this section, the diffraction problem depicted in Fig. 1 will be solved using scalar diffraction theory along with the Fresnel approximation. A circular aperture of radius a is shown, and it is excited with a harmonic plane wave. We are interested in describing the electric field as measured in front of the aperture by: i) a point receiver Fig. 1(a) and ii) a finite receiver of radius b centered in the beam of a radiating circular aperture Fig. 1(b).

The Kirchhoff boundary conditions along with an infinite ground plane and spatially uniform, monochromatic excitation are assumed. Only the case where the direction of polarization of the excitation is parallel to the aperture is considered. The aperture is symmetrically positioned at the origin of the source ($z = 0$) plane. Other assumptions will be explained as the need arises.

To define terms and establish notation, we outline Papoulis's derivation of the Lommel diffraction formulation [24, pp. 320–330]; other derivations can be found in the literature [6], [7], [14], [27]. The claim that the Lommel diffraction formulation holds in both the near field and far field is proved in Section V.

Under the conditions and assumptions stated above, Fresnel diffraction [24], [25] provides a reasonable scalar estimate of the electric field sensed by a fictitious point receiver P located sufficiently far from the aperture shown in Fig. 1(a). According to Papoulis, the Fresnel diffraction integral in rectangular coordinates is

$$\hat{E}(x, y, z) = \frac{e^{-jk_0 z}}{2\pi z} e^{-jk_0 \frac{(x^2 + y^2)}{2z}} \times \int_{y'} \int_{x'} f(x', y') e^{-jk_0 \frac{(x'^2 + y'^2)}{2z}} e^{-j \frac{k_0}{z} (xx' + yy')} dx' dy'. \quad (2)$$

Throughout this paper, the prime notation ($'$) denotes the source ($z = 0$) plane, and k_0 is the free space wave number. The excitation distribution across the face of the aperture is $f(x', y')$ and is unity due to the assumption of spatially uniform excitation. Per convention, the time-dependence of $\hat{E}(x, y, z)$ on $e^{j\omega t}$ is suppressed. The hat notation signifies estimation or approximation.

Circular symmetry allows (2) to be simplified and written in cylindrical coordinates:

$$\hat{E}(\rho, \phi, z) = \frac{k_0^2}{2z} e^{-jk_0 \left(z + \frac{\rho^2}{2z}\right)} \int_0^a e^{-jk_0 \frac{(\rho')^2}{2z}} J_0 \left(\frac{k_0 \rho}{z} \rho'\right) \rho' d\rho' \quad (3)$$

where $\rho' = \sqrt{(x')^2 + (y')^2}$ is the off-axis distance at the source plane and $\rho = \sqrt{x^2 + y^2}$ is the off-axis distance at the z plane [24, p. 330]. We note that $\hat{E}(\rho, \phi, z)$ is independent of ϕ because of circular symmetry; however, explicit functional notation is used for consistency with equations developed in Section VII.

A prominent and familiar feature of Fresnel diffraction is its interpretation as a convolution involving a quadratic phase term [23], [24]. This feature is obscured in (3). However, if the singularity function

$$p_a(\rho') = \begin{cases} 1, & \rho' \leq a; \\ 0, & \rho' > a \end{cases} \quad (4)$$

is introduced in the integrand of (3) and the upper limit of integration changed to ∞ , then (3) becomes

$$\hat{E}(\rho, \phi, z) = \frac{k_0^2}{2z} e^{-jk_0 \left(z + \frac{\rho^2}{2z}\right)} \times \int_0^\infty p_a(\rho') e^{-jk_0 \frac{(\rho')^2}{2z}} J_0 \left(\frac{k_0 \rho}{z} \rho'\right) \rho' d\rho'. \quad (5)$$

This improper integral may be interpreted as the Hankel transform of the product of the singularity function $p_a(\rho')$ and a quadratic phase term. The convolution theorem for Hankel transforms allows (5) to be rewritten

$$\hat{E}(\rho, \phi, z) = \frac{k_0}{2} e^{-jk_0 \left(z + \frac{\rho^2}{2z}\right)} \left[\frac{a}{\rho} J_1 \left(\frac{k_0 a \rho}{z}\right) * \frac{1}{j} e^{j \frac{k_0 \rho^2}{2z}} \right] \quad (6)$$

where the convolution is with respect to $k_0 \rho / z$. The familiar interpretation of Fresnel diffraction is made explicit in (6).

Equation (3) can be cast in closed form via the Lommel functions of two variables, $U_n(u, v)$, which are defined in Appendix A. The closed-form result in terms of $U_n(u, v)$ is

$$\hat{E}(\rho, \phi, z) = \frac{k_0}{2} \exp \left[-j \left(k_0 z + \frac{v^2}{2u} + \frac{u}{2} \right) \right] \times [U_1(u, v) + jU_2(u, v)] \quad (7)$$

where the substitutions $u = k_0 a^2 / z$ and $v = k_0 a \rho / z$ result in more compact notation.

Equation (7) is the Lommel diffraction formulation; it is easily programmed because of its closed-form and is amenable to either direct or recursive calculation [28]. It can be used to compute the electric field in the near field, the far field, the Fresnel region, and the Fraunhofer region. Finally, we note that the Lommel diffraction formulation can be modified to account for focusing; details can be found in [2], [7], [24], [29].

V. SPECIAL CASES AND COMPUTATIONAL ISSUES

In this section, we examine two special cases of the Lommel diffraction formulation and discuss three issues that arise when computing the Lommel diffraction formulation. The reader is again referred to Appendix A for functional definitions. The first special case of interest is when $\rho = 0$ or, equivalently, when $v = ka\rho/z = 0$. In this case, $\hat{E}(0, \phi, z)$ describes the on-axis fluctuations of the electric field.

Since $U_1(u, 0) = \sin(u/2)$ and $U_2(u, 0) = 1 - \cos(u/2)$ [8, p. 540], we note that

$$\hat{E}(0, \phi, z) = \frac{k_0}{2} e^{-j(k_0 z + \frac{u}{2})} \left(\sin \frac{u}{2} + j \left[1 - \cos \frac{u}{2} \right] \right) \quad (8a)$$

$$= \frac{k_0}{2} e^{-j(k_0 z + \frac{u}{2})} \left(j + \sin \frac{u}{2} - j \cos \frac{u}{2} \right) \quad (8b)$$

$$= \frac{k_0}{2} e^{-j(k_0 z + \frac{u}{2})} (j - j e^{-j \frac{u}{2}}) \quad (8c)$$

$$= \frac{-j k_0}{2} e^{-j(k_0 z + \frac{u}{4})} (e^{j \frac{u}{4}} - e^{-j \frac{u}{4}}) \quad (8d)$$

$$= k_0 e^{-j(k_0 z + \frac{u}{4})} \frac{e^{j \frac{u}{4}} - e^{-j \frac{u}{4}}}{2j} \quad (8e)$$

$$= k_0 e^{-j(k_0 z + \frac{u}{4})} \sin \left(\frac{u}{4} \right). \quad (8f)$$

Thus, the on-axis intensity predicted by the Lommel diffraction formulation is

$$\left| \hat{E}(0, \phi, z) \right|^2 = k_0^2 \sin^2 \frac{u}{4} \quad (9)$$

which is consistent with the classic result used to define the near field and far field of a circular aperture. Equation (9) can also be derived by evaluating the integral in (3) with ρ set to zero and subsequently squaring the magnitude of the result. For completeness, we note that for a circular aperture $z < a^2/\lambda$ is generally considered the near field while $z \geq a^2/\lambda$ is considered the far field.

The second special case of interest is when z is large. When z is large (the far field), both $u = k_0 a^2/z$ and $v = k_0 a\rho/z$ in (7) are small for geometries of practical interest. Because the arguments in the Lommel functions are small, terms involving Bessel functions of order greater than one can be neglected. Thus,

$$\hat{E}(\rho, \phi, z) = \frac{k_0 a}{2\rho} e^{-jk_0 \left(z + \frac{\rho^2}{2z} + \frac{a^2}{2z} \right)} J_1 \left(\frac{k_0 a \rho}{z} \right) \quad (10)$$

for large z . Basic trigonometry allows us to write $\rho^2 + z^2 = r^2$, where r , in this case, is the distance from the center of the aperture to the off-axis point ρ . With z large, $\rho/z \approx \sin \theta$ and

$$\hat{E}(\rho, \phi, z) = \frac{k_0^2 a^2}{2z} e^{-jk_0 \left(z + \frac{\rho^2}{2z} + \frac{a^2}{2z} \right)} \frac{J_1(k_0 a \sin \theta)}{k_0 a \sin \theta}. \quad (11)$$

Equation (11) captures both the far-field directivity pattern of a circular aperture [20], [30] and the celebrated Airy disk pattern [18].

Finally, three computational issues require discussion. First, computation of $U_n(u, v)$, the Lommel function of two variables defined in Appendix A, may fail if $u/v > 1$ because $U_n(u, v)$ converges slowly when $u/v > 1$. Therefore, it is prudent to compute $U_n(u, v)$ in terms of $V_n(u, v)$ via (29) when $u/v > 1$.

Second, the Lommel functions must be computed using a sufficient number of terms to obtain meaningful results; $n + 2s \leq 49$ was used in this work. Calculating a Bessel function with this high an order may cause underflow on some computers [31, p. 57–58]. This problem can be avoided if recursive methods are used. For more details, the reader is referred to Boersma's work on computing Lommel functions [28].

Third, on-axis ($\rho = 0$) values of (7) can be calculated via appropriate algorithmic handling of the Lommel functions when $v = 0$ [8, p. 540], or they may be calculated using (8) directly. The latter method was used in our computations.

VI. THE SPATIALLY AVERAGED ELECTRIC FIELD

ANSI C63.5 describes standards for calibrating antennas used for radiated emissions in EMI control, and IEEE STD 1309–1996 describes standards for calibration of EM field sensors and probes. Both standards describe routine electromagnetic measurements involving circular apertures or dish antennas; these include antenna calibration and spatial pattern measurements, evaluation of electromagnetic probes for EMI measurements, and safety surveys of radiated electromagnetic fields and power densities.

In this section, we present closed-form expressions that will allow for rapid, practical, and efficient prediction of certain types of electromagnetic measurements like those aforementioned. Detailed derivation of the expressions is presented in Appendix B. Indeed, derivation of the closed-form expressions, the expressions themselves, and the discussion of results in Section VIII represent the major contribution of this theoretical and numerical investigation.

Consider Fig.1 (b). The figure depicts a circular transmitting aperture of radius a . A finite receiving aperture or measurement circle of radius b is centered in the beam of the transmitting aperture. We are interested in describing the spatially averaged one-way diffraction effects for this situation. A reasonable approximation is obtained by spatially averaging the Lommel diffraction formulation in (7).

Spatially integrating and averaging the Lommel diffraction formulation yields

$$\langle \hat{E}(z) \rangle_b = \frac{1}{\pi b^2} \int_{\phi=0}^{2\pi} \int_{\rho=0}^b \hat{E}(\rho, \phi, z) \rho d\rho d\phi \quad (12a)$$

$$= \frac{1}{\pi b^2} \left[2\pi \int_0^b \hat{E}(\rho, z) \rho d\rho \right]. \quad (12b)$$

The notation $\langle \rangle_b$ denotes spatial averaging over a receiving aperture or measurement circle of radius b .

The closed-form expressions obtained by performing the operations indicated in (12b) are:

$$\begin{aligned} \langle \hat{E}(z) \rangle_b &= -\frac{z}{b^2} e^{-j(k_0 z + \frac{\pi}{2})} \\ &\times \begin{cases} \left(j \frac{v_b^2}{2u} e^{j\frac{\pi}{2}} + e^{-j\frac{v_b^2}{2u}} [Y_2(u, v_b) - jY_1(u, v_b)] \right), & b < a \\ \left(j \frac{u}{2} e^{j\frac{\pi}{2}} + e^{-j\frac{\pi}{2}} \left[\frac{u}{2} J_1(u) - j \frac{u}{2} J_0(u) \right] \right), & b = a \\ \left(j \frac{u}{2} e^{j\frac{\pi}{2}} + e^{-j\frac{v_b^2}{2u}} [X_2(u, v_b) - jX_1(u, v_b)] \right), & b > a. \end{cases} \quad (13) \end{aligned}$$

The $X_n(u, v)$ and $Y_n(u, v)$ functions are defined in Appendix A, and a detailed derivation of (13) is presented in Appendix B.

Before turning our attention to the dyadic Green's function approach, we must point out an important yet subtle distinction. Because it includes phase information in the integrand, (12) is a coherent average. The incoherent average

$$\langle |\hat{E}(z)|^2 \rangle_b = \frac{1}{\pi b^2} \left[2\pi \int_0^b |\hat{E}(\rho, z)|^2 \rho d\rho \right] \quad (14)$$

and similar incoherent averages have received a great deal of attention in the past, particularly in optics [2], [16], [17]. Stamnes and co-workers note that, in the case of the incoherent average in (14), the so-called *encircled energy* is the spatial average not of the field itself, as in (12), but of its intensity or amplitude squared [17]. More discussion of coherent and incoherent averaging in the beam of a circular aperture can be found in [29].

VII. DYADIC GREEN'S FUNCTION APPROACH

In this section, the formulation of the exact solution of the electric field from a circular aperture derived directly from Maxwell's equations is given. In the next section, we will compare results obtained from this exact solution to results obtained from expressions based on the approximate Lommel diffraction formulation.

Using the dyadic Green's function approach, we write the electric field as

$$\mathbf{E}(\mathbf{r}) = \int_{\Omega'} \bar{\bar{\mathbf{G}}}_E(\mathbf{r}|\mathbf{r}') \cdot \mathbf{J}(\mathbf{r}') dv', \quad (15)$$

where $\bar{\bar{\mathbf{G}}}_E$ is the electric field dyadic Green's function and $\mathbf{J}$ is the electric current density source used to excite the fields, the

primed notation $\mathbf{r}' = \{x', y', z'\}$ denotes the source coordinates and the unprimed notation $\mathbf{r} = \{x, y, z\}$ denotes the observation coordinates.

In the spatial domain, the free space dyadic Green's function is given as [32]

$$\bar{\bar{\mathbf{G}}}_{E0}(\mathbf{r}|\mathbf{r}') = -j\omega\mu_0 \left(\bar{\bar{\mathbf{I}}} - \frac{\nabla\nabla}{k_0^2} \right) G_0(\mathbf{r}|\mathbf{r}'), \quad (16)$$

where μ_0 is the permeability of free space,

$$G_0(\mathbf{r}|\mathbf{r}') = \frac{e^{-jk_0|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}, \quad (17)$$

and

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}. \quad (18)$$

$\bar{\bar{\mathbf{I}}}$ is the unit dyad defined as

$$\bar{\bar{\mathbf{I}}} = \hat{\mathbf{a}}_x \hat{\mathbf{a}}_x + \hat{\mathbf{a}}_y \hat{\mathbf{a}}_y + \hat{\mathbf{a}}_z \hat{\mathbf{a}}_z. \quad (19)$$

The current density in (15) is related to the electric field as follows:

$$\mathbf{J}(\mathbf{r}') = \frac{j}{\omega\mu_0} \left(\nabla\nabla \cdot \mathbf{E}(\mathbf{r}') - \nabla^2 \mathbf{E}(\mathbf{r}') - k_0^2 \mathbf{E}(\mathbf{r}') \right). \quad (20)$$

Note that the exact EM solution given in (15) contains all electric field components ($\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, and $\hat{\mathbf{a}}_z$). To validate the closed-form expressions (7) and (13), the vector electric field in (15) must be reduced to a scalar field quantity.

First, we assume a circular aperture with an incident plane wave excitation that has a parallel polarization with respect to the aperture radius for a given ϕ angle. Such an excitation field can mathematically be represented by

$$\mathbf{E}(\mathbf{r}') = E_0 \hat{\mathbf{a}}_\rho = E_0 (\cos \phi'_i \hat{\mathbf{a}}_x + \sin \phi'_i \hat{\mathbf{a}}_y), \quad (21)$$

where E_0 is a constant and ϕ'_i is the plane wave excitation angle on the circular aperture.

Since the electric field excitation has only one component, $\hat{\mathbf{a}}_\rho$, the electric field in (15) for a circular aperture with radius a , reduces to (see equations at bottom page) E_0 is the electric field magnitude source, usually represented by 1 V/m, and Z_0 is the free space wave impedance ($\approx 377 \Omega$). The δ_z in (22) is an artifact of the volume integration from (15). The source excitation is assumed to be infinitesimal in the z dimension so

$$E(\rho, \phi, z) = \frac{-jk_0 E_0 \delta_z}{Z_0} \int_0^{2\pi} \int_0^a G_{E0}(R) \rho' d\rho' d\phi' \quad (22)$$

where

$$\begin{aligned} G_{E0}(R) = & -\frac{jZ_0}{k_0} G_0(R) \left[k_0^2 - \frac{jk_0}{R} - \frac{1 + k_0^2(\rho \cos \phi - \rho' \cos \phi')^2}{R^2} \right. \\ & \left. + \frac{j3k_0(\rho \cos \phi - \rho' \cos \phi')^2}{R^3} + \frac{3(\rho \cos \phi - \rho' \cos \phi')^2}{R^4} \right] \end{aligned} \quad (23)$$

and

$$R = \sqrt{(\rho \cos \phi - \rho' \cos \phi')^2 + (\rho \sin \phi - \rho' \sin \phi')^2 + (z - z')^2}. \quad (24)$$

Fig. 2. Electric field magnitude along the z -axis for $\rho = 0$: approximate results from (7) and exact results from (22).

the δ_z is kept in (22) to keep the units (V/m) consistent. It is important to note that (22), unlike (3) which is based on the Fresnel approximation, is exact.

The exact spatially averaged electric field is found by integrating the point receiver electric field (22) over a finite circular aperture of radius b :

$$\langle E(z) \rangle_b = \frac{1}{\pi b^2} \int_{\phi=0}^{2\pi} \int_{\rho=0}^b E(\rho, \phi, z) \rho d\rho d\phi \quad (25a)$$

$$= \frac{1}{\pi b^2} \left[2\pi \int_0^b E(\rho, z) \rho d\rho \right]. \quad (25b)$$

Compare (25) and (12); they are virtually identical except that (12) contains the Fresnel approximation $\hat{E}(\rho, \phi, z)$ to the exact scalar electric field $E(\rho, \phi, z)$ in (25). Thus, (25), unlike (12) which is based on the Fresnel approximation, is exact. Expressions (22) and (25) are used to validate the closed-form expressions (7) and (13), respectively, for a point receiver and finite receiver.

VIII. RESULTS AND DISCUSSION

In this section, we validate (7) and (13), both of which are based on the Fresnel approximation, by comparing results obtained from these expressions with results obtained from the exact expressions in (22) and (25). All expressions were evaluated at a frequency of 300 MHz ($\lambda_0 = 1$ m) for a circular aperture having a radius of 5 m or 5 wavelengths. Note that for these parameters, the traditionally defined demarcation between the near field and far field is $z = a^2/\lambda = 25$ m. Figs. 2–5 show results for a point receiver which were computed using (7) and

(22), while Figs. 6–8 show results for a finite receiver which were computed using (13) and (25).

The electric field magnitude along the longitudinal axis (z -axis) with $\rho = 0$ is shown in Fig. 2. The closed-form solution given by (7) compares favorably with the EM solution given by (22) for $z \geq 10$ m. Fig. 3(a) and (b) shows a transverse cut of the electric field, magnitude and phase respectively, at $z = 20$ m in the near field of the circular aperture.

The approximate and exact results shown in Fig. 3 compare quite favorably out to $\rho \approx 4.5$ m. For $\rho > 4.5$ m, the Lommel diffraction formulation predicts local magnitude minima when in fact the exact EM solution indicates local magnitude maxima. In addition, the phase predicted by the two methods disagrees when $\rho > 4.5$ m. This disagreement in magnitude and phase is not surprising because the Lommel diffraction formulation is based on the Fresnel approximation and, as a result, is not valid for large values of ρ .

The approximate and exact results shown in Fig. 4 were computed with $z = 100$ m which is in the Fraunhofer region of the circular aperture being considered. The approximate and exact results shown in the figure compare quite favorably in both magnitude and phase out to $\rho \approx 20$ m; however, the agreement breaks down for $\rho > 20$ m. Again, this disagreement is a result of the Fresnel approximation. A far-field antenna pattern of the electric field in spherical coordinates is shown in Fig. 5 for $r = 10000$ m. This pattern and the transverse cuts shown in Figs. 3, 4 show good agreement out to the second sidelobe.

Spatially averaged results are shown Figs. 6–8. Fig. 6 shows results for $b = 1$ m $< a$. Results obtained from the approximate and exact expressions compare quite favorably for $z > 12$ m. Compare the results shown in Figs. 2 and 5. Spatial averaging smooths out oscillations in the near field. Fig. 7, 8 shows results for $b = 5$ m $= a$ and for $b = 9$ m $> a$, respectively. Once again, the approximate and exact results show very good agreement and differ by, at most, only a fraction of a decibel. The closed-form expression (13) for $b < a$ was also observed to converge to the point receiver solution (7) as b approached zero; a mathematical proof that confirms this observation is outlined in Appendix B.

This section is closed with a brief discussion of computational efficiency. Both the Lommel-based expressions and the dyadic Greens' function approach were simulated using Mathcad version 8. Practical simulations usually involve 200 or more data points to make a useful plot, and this was the case in obtaining the plots shown in Figs. 2–8. Computation of the closed-form expressions was very efficient whereas explicit computation of the EM solution was slow and cumbersome. For example, the closed-form expression (13) for the spatially averaged example took only a couple of minutes to run whereas the EM solution (25) took several hours.

It is important to note that computation of the EM solution for the spatially averaged case in (25b) ultimately requires numerical evaluation of three integrals. There are several numerical techniques to speed up the evaluation of diffraction integrals such as (25), but discussion and comparison of these methods is beyond the scope of this paper. It is sufficient to note that the more important of these methods employ linear or parabolic approximations to the phase and amplitude of the integrand in

Fig. 3. Electric field profile. (a) Magnitude. (b) Phase, along the ρ -axis for $z = 20$ m: approximate results from (7) and exact results from (22).

Fig. 4. Electric field profile. (a) Magnitude. (b) Phase, along the ρ -axis for $z = 100$ m: approximate results from (7) and exact results from (22).

a diffraction integral [33]–[35]. Stamnes provides an excellent discussion and a detailed comparison of these methods in his text on waves in focal regions [18pp. 60–85].

IX. CONCLUSION

This paper presented a theoretical and numerical investigation of the spatially averaged electric field in the beam of a circular aperture. The investigation was based on scalar diffraction

theory and, more specifically, the Lommel diffraction formulation. It led to closed-form analytical expressions (13) which describe the spatially averaged electric field in the Fresnel region of a circular aperture excited by a spatially uniform, harmonic plane wave with polarization parallel the aperture. The expressions ultimately permit rapid, practical, and efficient prediction of certain routine electromagnetic measurements. Because the expressions are valid in the Fresnel region, they are also valid in the near field, the far field, and the Fraunhofer region of a

Fig. 5. Far-field electric field pattern for $r = 10\,000$ m: approximate results from (7) and exact results from (22).

Fig. 7. Magnitude of spatially averaged electric field for a finite receiver of radius $b = 5$ m: approximate results from (13) and exact results from (25).

Fig. 6. Magnitude of spatially averaged electric field for a finite receiver of radius $b = 1$ m: approximate results from (13) and exact results from (25).

Fig. 8. Magnitude of spatially averaged electric field for a finite receiver of radius $b = 9$ m: approximate results from (13) and exact results from (25).

circular aperture. In fact, it was shown that the Lommel diffraction formulation contains, as special cases, classic on-axis and far-field results for a circular aperture. Finally, it was shown that results obtained from the closed-form expressions in (7) and (13) compared quite favorably to results obtained from exact expressions in (22) and (25) which were derived via the dyadic Green's function approach. This agreement was not all encompassing; regions where agreement broke down were commented

upon and differences were explained by the Fresnel approximation.

APPENDIX A MATHEMATICAL DEFINITIONS

Per convention, $J_n(v)$ denotes a Bessel function of the first kind of order n , where n is a nonnegative integer and v is a real variable. $U_n(u, v)$ and $V_n(u, v)$ are the Lommel functions

of two variables—where u like v is a real variable. We follow various authors and define the Lommel functions as

$$U_n(u, v) \stackrel{\text{def}}{=} \sum_{s=0}^{\infty} (-1)^s \left(\frac{u}{v}\right)^{n+2s} J_{n+2s}(v) \quad (26)$$

$$V_n(u, v) \stackrel{\text{def}}{=} \sum_{s=0}^{\infty} (-1)^s \left(\frac{v}{u}\right)^{n+2s} J_{n+2s}(v). \quad (27)$$

It is imperative to note that the definition of $V_n(u, v)$ invoked here is consistent with that of Gray and Mathews [6], Born and Wolf [7], and Lommel himself [3]; it differs from the definition used by other authors by the factor $(-1)^n$ [8], [36].

The Lommel functions are infinite summations and can be computed only approximately. These approximations can be computed in a do-loop or recursively [6], [16]. The following recursion relations [37] will prove useful:

$$U_n(u, v) + U_{n+2}(u, v) = \left(\frac{u}{v}\right)^n J_n(v) \quad (28a)$$

$$V_n(u, v) + V_{n+2}(u, v) = \left(\frac{v}{u}\right)^n J_n(v) \quad (28b)$$

The reader is referred to Boersma's paper and the references contained therein for more details on computing Lommel functions [28].

$U_n(u, v)$ converges rapidly when $u < v$. On the other hand, it converges slowly when $u > v$. Thus, the following formulae from Gray and Mathews [6p. 185, Eq. 20] are provided for completeness.

$$U_{2n+1}(u, v) + V_{-2n+1}(u, v) = (-1)^n \sin\left(\frac{1}{2} \left[u + \frac{v^2}{u}\right]\right) \quad (29a)$$

$$-U_{2n+2}(u, v) + V_{-2n}(u, v) = (-1)^n \cos\left(\frac{1}{2} \left[u + \frac{v^2}{u}\right]\right) \quad (29b)$$

Computers and modern computer languages make it possible to calculate $U_n(u, v)$ without resort to (29); nonetheless, (29) offers computational-efficiency. Various simplifications for the case when $u = v$ can be found in [6], [8], [24].

A group of functions related to the Lommel functions will also be encountered, and they are defined below. First, Wolf [16] defined

$$W_n(u, v) \stackrel{\text{def}}{=} \sum_{s=0}^{\infty} (-1)^s (s+1) \left(\frac{v}{u}\right)^{n+2s} J_{n+2s}(v) \quad (30)$$

and noted that

$$\sum_{s=0}^{\infty} (-1)^s V_{n+2s}(u, v) = W_n(u, v). \quad (31)$$

Wolf's assertion can be proved by writing

$$\begin{aligned} & \sum_{s=0}^{\infty} (-1)^s V_{n+2s}(u, v) \\ &= \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} (-1)^{s+p} \left(\frac{v}{u}\right)^{n+2s+2p} J_{n+2s+2p}(v) \end{aligned} \quad (32)$$

and then collecting terms of the same order in J and arranging the series in ascending order. The grouping of terms is justified since the series is absolutely convergent.

Similarly, we define

$$Z_n(u, v) \stackrel{\text{def}}{=} \sum_{s=0}^{\infty} (-1)^s (s+1) \left(\frac{u}{v}\right)^{n+2s} J_{n+2s}(v) \quad (33)$$

and note that

$$\sum_{s=0}^{\infty} (-1)^s U_{n+2s}(u, v) = Z_n(u, v). \quad (34)$$

This assertion can be proved by writing

$$\begin{aligned} & \sum_{s=0}^{\infty} (-1)^s U_{n+2s}(u, v) \\ &= \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} (-1)^{s+p} \left(\frac{u}{v}\right)^{n+2s+2p} J_{n+2s+2p}(v) \end{aligned} \quad (35)$$

and then collecting terms of the same order in J and arranging the series in ascending order. The grouping of terms is justified since the series is absolutely convergent [12], [16].

Finally, we have

$$X_n(u, v) = \frac{1}{2} \left[\frac{v^2}{u} U_{n+1}(u, v) + u U_{n-1}(u, v) \right] \quad (36a)$$

$$= \sum_{s=0}^{\infty} (-1)^s (n+2s) \left(\frac{u}{v}\right)^{n+2s} J_{n+2s}(v) \quad (36b)$$

and

$$Y_n(u, v) = \frac{1}{2} \left[\frac{v^2}{u} V_{n-1}(u, v) + u V_{n+1}(u, v) \right] \quad (37a)$$

$$= \sum_{s=0}^{\infty} (-1)^s (n+2s) \left(\frac{v}{u}\right)^{n+2s} J_{n+2s}(v). \quad (37b)$$

Details on the origins and applications of the functions defined in this section can be found in [12], [16], [37].

APPENDIX B DERIVATION OF (13)

Writing (12) in terms of Lommel functions, we have

$$\begin{aligned} \langle \hat{E}(z) \rangle_b &= \frac{z}{ub^2} e^{-j(k_0 z + \frac{u}{2})} \int_0^{v_b} [U_1(u, v) + jU_2(u, v)] e^{-j\frac{v^2}{2u}} v dv \\ &\quad (38) \end{aligned}$$

where, as before, $u = k_0 a^2/z$, $v = k_0 a \rho/z$, and $v_b = k_0 a b/z$. This integral will be solved for the three case of b : i) $b < a$; ii) $b = a$; and iii) $b > a$.

A. Case I: $b < a$

When $b < a$, $u/v_b > 1$. Thus, we use (29) in conjunction with Euler's formula to rewrite (38) in terms of $V_n(u, v)$:

$$\begin{aligned} \langle \hat{E}(z) \rangle_{b < a} &= \frac{z}{ub^2} e^{-j(k_0 z + \frac{u}{2})} \\ &\times \int_0^{v_b} \left[-j e^{j\frac{1}{2}(u + \frac{v^2}{u})} - V_1(u, v) + jV_0(u, v) \right] e^{-j\frac{v^2}{2u}} v dv. \end{aligned} \quad (39)$$

Now it is simply a matter of integrating each term in the integrand. Integration of the exponential term, is relatively straightforward and left to the reader. Integration of the $V_n(u, v)$ functions is, however, another matter.

Wolf has proven elsewhere [16 Lemma 9] that

$$\begin{aligned} & \int_0^{v_0} V_n(u, v) e^{-j\frac{v^2}{2u}} v dv \\ &= u [W_{n+1}(u, v_0) + jW_{n+2}(u, v_0)] e^{-j\frac{v_0^2}{2u}}. \end{aligned} \quad (40)$$

Thus,

$$\begin{aligned} \langle \hat{E}(z) \rangle_{b < a} &= \frac{z}{b^2} e^{-j(k_0 z + \frac{\pi}{4})} \\ &\times \left\{ -j \frac{v_b^2}{2u} e^{j\frac{\pi}{2}} - 2W_2(u, v_b) e^{-j\frac{v_b^2}{2u}} \right. \\ &\left. + j[W_1(u, v_b) - W_3(u, v_b)] e^{-j\frac{v_b^2}{2u}} \right\}. \end{aligned} \quad (41)$$

This intermediate result is simplified by noting that $W_1(u, v) - W_3(u, v) = Y_1(u, v)$ and $2W_2(u, v) = Y_2(u, v)$. This simplification results in the first spatially averaged expression

$$\begin{aligned} \langle \hat{E}(z) \rangle_{b < a} &= -\frac{z}{b^2} e^{-j(k_0 z + \frac{\pi}{4})} \\ &\times \left\{ j \frac{v_b^2}{2u} e^{j\frac{\pi}{2}} + e^{-j\frac{v_b^2}{2u}} [Y_2(u, v_b) - jY_1(u, v_b)] \right\}. \end{aligned} \quad (42)$$

The Y_n functions are highly convergent and fairly easy to compute in terms of coding and processor time.

It is gratifying to note that

$$\lim_{b \rightarrow 0} \langle \hat{E}(z) \rangle_b = k_0 e^{-j(k_0 z + \frac{\pi}{4})} \sin\left(\frac{u}{4}\right) = \hat{E}(0, \phi, z). \quad (43)$$

The results of (43) and (8f) are identical, and (43) can be easily proved by considering the following substitutions into (42). First, ku/v_b^2 can be substituted for z/b^2 in (42). Second, (37b) indicates that

$$\lim_{b \rightarrow 0} \frac{u}{v_b^2} Y_1(u, v_b) = \frac{1}{2} \quad (44)$$

and

$$\lim_{b \rightarrow 0} \frac{u}{v_b^2} Y_2(u, v_b) = 0. \quad (45)$$

These last two equations are based on the fact that $\lim_{x \rightarrow 0} J_1(x)/x = 1/2$ and that $\lim_{x \rightarrow 0} J_n(x) = 0$ when $n \neq 0$.

Making the substitutions into and subsequent algebraic manipulation of (42) along the lines of (8) results in (43). Thus, the spatially averaged Lommel diffraction formulation for $b < a$ in (42) incorporates the on-axis, point receiver description of the Lommel diffraction formulation in (8) as a special case.

B. Case II: $b = a$

In the second case, $b = a$, $u = v_b$, and (38) succumbs to a welcome simplification that is a special case of (42). Specifically (37a) and (28b) indicate that $Y_1(u, u) = 0.5 u J_0(u)$ and $Y_2(u, u) = 0.5 u J_1(u)$; thus the expression for the spatially averaged one-way diffraction effects when $b = a$ is

$$\begin{aligned} \langle \hat{E}(z) \rangle_{b=a} &= -\frac{z}{b^2} e^{-j(k_0 z + \frac{\pi}{4})} \\ &\times \left\{ j \frac{u}{2} e^{j\frac{\pi}{2}} + e^{-j\frac{u^2}{2}} \left[\frac{u}{2} J_1(u) - j \frac{u}{2} J_0(u) \right] \right\}. \end{aligned} \quad (46)$$

Equation (46) is a special case of (42), is very easy to compute, and was first reported decades ago by Torikai [38] and Rogers and Van Buren [39] as part of their research in ultrasound. Their derivations, however, are significantly different than the one presented here.

C. Case I: $b > a$

The closed-form expression for $\langle \hat{E}(z) \rangle_{b > a}$ is derived by splitting the range of integration in (38) into two intervals: $0 \leq \rho \leq a$ and $a \leq \rho \leq b$. The problem then becomes one of solving

$$\begin{aligned} \langle \hat{E}(z) \rangle_{b > a} &= \frac{1}{\pi b^2} \left(\int_{\phi=0}^{2\pi} \int_{\rho=0}^a \hat{E}(\rho, \phi, z) \rho d\rho d\phi \right. \\ &\left. + \int_{\phi=0}^{2\pi} \int_{\rho=a}^b \hat{E}(\rho, \phi, z) \rho d\rho d\phi \right) \end{aligned} \quad (47a)$$

$$= \langle \hat{E}(z) \rangle_a + \hat{E}_{ab}(z) \quad (47b)$$

where $\hat{E}_{ab}(z)$ is simply a notational convenience. The closed-form solution for $\langle \hat{E}(z) \rangle_a$ was derived in Section B.2 and is given by (46); thus, (47b) becomes

$$\begin{aligned} \langle \hat{E}(z) \rangle_{b > a} &= -\frac{z}{b^2} e^{-j(k_0 z + \frac{\pi}{4})} \\ &\times \left\{ j \frac{u}{2} e^{j\frac{\pi}{2}} + e^{-j\frac{u^2}{2}} \left[\frac{u}{2} J_1(u) - j \frac{u}{2} J_0(u) \right] \right\} + \hat{E}_{ab}(z). \end{aligned} \quad (48)$$

The integral represented by $\hat{E}_{ab}(z)$ written in terms of $U_n(u, v)$ functions is

$$\begin{aligned} \hat{E}_{ab}(z) &= \frac{z}{ub^2} e^{-j(k_0 z + \frac{\pi}{4})} \\ &\times \int_u^{v_b} [U_1(u, v) + jU_2(u, v)] e^{-j\frac{v^2}{2u}} v dv. \end{aligned} \quad (49)$$

Equation (49) can be solved by consideration of the more general integral

$$\frac{1}{u} \int_u^{v_b} U_n(u, v) e^{-j\frac{v^2}{2u}} v dv. \quad (50)$$

Writing the $U_n(u, v)$ in terms of Bessel functions yields

$$\begin{aligned} \frac{1}{u} \int_u^{v_b} U_n(u, v) e^{-j\frac{v^2}{2u}} v dv \\ = \sum_{s=0}^{\infty} (-1)^s \int_u^{v_b} \left(\frac{v}{u} \right)^{1-(n+2s)} J_{n+2s}(v) e^{-j\frac{v^2}{2u}} dv. \end{aligned} \quad (51)$$

Advantageous manipulation of integration and summation operations is permitted because the Lommel function in the integrand on the right-hand side (RHS) of (51) is uniformly convergent over the interval of integration [16].

Lommel's reciprocity formulae [8p. 543] will help us solve the RHS of (51). Close inspection of Lommel's reciprocity formulae in conjunction with Euler's formula reveals that

$$\begin{aligned} y \left(\frac{wt}{y} \right)^{1-n} J_n(yt) e^{-j\frac{1}{2}wt^2} \\ = \frac{d}{dt} \left[j e^{-j\frac{1}{2}wt^2} U_n\left(\frac{y^2}{w}, yt\right) - e^{-j\frac{1}{2}wt^2} U_{n+1}\left(\frac{y^2}{w}, yt\right) \right] \end{aligned} \quad (52)$$

where t , w , and y are dummy variables. Setting $t = v$, $w = 1/u$, and $y = 1$ yields

$$\left(\frac{v}{u}\right)^{1-n} J_n(v) e^{-j\frac{v^2}{2u}} = \frac{d}{dv} \left[j e^{-j\frac{v^2}{2u}} U_n(u, v) - e^{-j\frac{v^2}{2u}} U_{n+1}(u, v) \right]. \quad (53)$$

Observe that the left-hand side (LHS) of (53) and the integrand on the RHS of (51) are in the same form.

These manipulations and observations along with the fundamental theorem of integral calculus allow us to conclude that

$$\begin{aligned} \frac{1}{u} \int_u^{v_b} v U_n(u, v) e^{-j\frac{v^2}{2u}} dv &= \left\{ e^{-j\frac{u}{2}} \left[\sum_{s=0}^{\infty} (-1)^s U_{n+1+2s}(u, u) \right. \right. \\ &\quad \left. \left. - j \sum_{s=0}^{\infty} (-1)^s U_{n+2s}(u, u) \right] \right. \\ &\quad \left. + e^{-j\frac{v_b^2}{2u}} \left[j \sum_{s=0}^{\infty} (-1)^s U_{n+2s}(u, v_b) \right. \right. \\ &\quad \left. \left. - \sum_{s=0}^{\infty} (-1)^s U_{n+1+2s}(u, v_b) \right] \right\}. \quad (54) \end{aligned}$$

Application of (34) to (54) yields the desired intermediate result

$$\begin{aligned} \frac{1}{u} \int_u^{v_b} v U_n(u, v) e^{-j\frac{v^2}{2u}} dv &= \left\{ e^{-j\frac{u}{2}} [Z_{n+1}(u, u) - j Z_n(u, u)] \right. \\ &\quad \left. + e^{-j\frac{v_b^2}{2u}} [j Z_n(u, v_b) - Z_{n+1}(u, v_b)] \right\} \quad (55) \end{aligned}$$

Straightforward application of this intermediate result to the real and imaginary parts of (49) yields

$$\begin{aligned} \hat{E}_{ab}(z) &= \frac{z}{b^2} e^{-j(k_0 z + \frac{u}{2})} \\ &\times \left\{ e^{-j\frac{u}{2}} [2Z_2(u, u) - j Z_1(u, u) + j Z_3(u, u)] \right. \\ &\quad \left. + e^{-j\frac{v_b^2}{2u}} [j Z_1(u, v_b) - j Z_3(u, v_b) \right. \\ &\quad \left. - 2Z_2(u, v_b)] \right\}. \quad (56) \end{aligned}$$

Now (56) may be simplified because $Z_1(u, v) - Z_3(u, v) = X_1(u, v)$ and $2Z_2(u, v) = X_2(u, v)$. Thus, (56) becomes

$$\begin{aligned} \hat{E}_{ab}(z) &= \frac{z}{b^2} e^{-j(k_0 z + \frac{u}{2})} \\ &\times \left\{ e^{-j\frac{u}{2}} [X_2(u, u) - j X_1(u, u)] \right. \\ &\quad \left. + e^{-j\frac{v_b^2}{2u}} [j X_1(u, v_b) - X_2(u, v_b)] \right\}. \quad (57) \end{aligned}$$

But, (36a) in conjunction with (28a) reveals that $X_1(u, u) = 0.5 u J_0(u)$ and $X_2(u, u) = 0.5 u J_1(u)$, so

$$\begin{aligned} \hat{E}_{ab}(z) &= \frac{z}{b^2} e^{-j(k_0 z + \frac{u}{2})} \\ &\times \left\{ e^{-j\frac{u}{2}} \left[\frac{u}{2} J_1(u) - j \frac{u}{2} J_0(u) \right] \right. \\ &\quad \left. + e^{-j\frac{v_b^2}{2u}} [j X_1(u, v_b) - X_2(u, v_b)] \right\}. \quad (58) \end{aligned}$$

Substitution of (58) into (48) and simple addition and subtraction yield the final result

$$\begin{aligned} \langle \hat{E}(z) \rangle_{b>a} &= -\frac{z}{b^2} e^{-j(k_0 z + \frac{u}{2})} \\ &\times \left\{ j \frac{u}{2} e^{j\frac{u}{2}} + e^{-j\frac{v_b^2}{2u}} [X_2(u, v_b) - j X_1(u, v_b)] \right\}. \quad (59) \end{aligned}$$

The X_n functions, defined in Appendix A, are highly convergent and fairly easy to compute in terms of coding and processor time. Finally, we note that (36a) and (28a) indicate that $X_1(u, u) = 0.5 u J_0(u)$ and $X_2(u, u) = 0.5 u J_1(u)$; thus, (59) reduces to (46) when $b = a$.

ACKNOWLEDGMENT

The authors express their deep appreciation to the anonymous referees. Their thorough review contained insightful comments, helpful suggestions, and numerous references, all of which improved our submission significantly.

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