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A spatially averaged impulse response for an unfocused piston transducer

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Fourier–Bessel theory is used to derive a closed-form solution for the spatially averaged velocity-potential impulse response associated with one-way diffraction from an unfocused piston transducer of radius a . The derivation provides additional insight into the problem of diffraction from an unfocused piston transducer. © 1999 Acoustical Society of America.

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INTRODUCTION

Computation of a spatially integrated or spatially averaged impulse response is the first step when correcting for diffraction effects in attenuation measurements with ultrasound.¹ Williams² makes the case for spatial integration, while Harris³ makes the case for spatial averaging. The two cases differ by a multiplicative factor only, and we do not dwell on the difference here. The spatially averaged impulse response for an unfocused piston transducer is amenable to closed-form solution. Hence, piston transducers have been researched for over fifty years.¹ Indeed, Huntington *et al.*,⁴ Williams,^{2,5} Seki *et al.*,⁶ Bass,⁷ Rhyne,⁸ and Rogers and Van Buren⁹ all derived closed-form spatially averaged one-way diffraction corrections. Khimunin¹⁰ and Benson and Kiyohara¹¹ tabulated extensive numerical results, and more recently, Harris,³ Cassereau *et al.*,¹² and Chen *et al.*¹³ have continued the work of the early researchers.

In this paper, Fourier–Bessel theory¹⁴ is used to present a novel derivation of the spatially averaged impulse response for an unfocused piston transducer. The derivation leads to additional insight into the problem of diffraction from an unfocused piston transducer.

I. DERIVATION AND DISCUSSION

With c and $J_n(x)$ denoting the speed of sound and an n th-order Bessel function of the first kind, respectively, the arccos diffraction formulation in *integral form* is

$$h_1(\rho, z, t) = \begin{cases} ac \int_0^\infty J_0(\tau\rho) J_1(\tau a) J_0(\tau\sqrt{(ct)^2 - z^2}) d\tau, & ct > z, \\ 0, & ct < z, \end{cases} \quad (1)$$

and in *closed form* involves eponymous arccos terms. The integral and closed-form results were derived analytically by Oberhettinger¹⁵ in 1961. A decade later, Stepanishen derived the closed-form solution geometrically and interpreted it as

an impulse response.¹⁶ Hence, the arccos diffraction formulation is also known as the velocity-potential impulse response.

Equation 1 assumes a piston transmitter of radius a and a *point* receiver. The spatially averaged impulse response in the case of a *finite* piston receiver of radius $b \leq a$ is

$$\langle h_1(z, t) \rangle_b = \frac{1}{\pi b^2} \left[2\pi \int_0^b h_1(\rho, z, t) \rho d\rho \right]. \quad (2)$$

Rhyne⁸ derived a closed-form expression for $\langle h_1(z, t) \rangle_b$ with $b = a$ by integrating the closed-form arccos diffraction formulation directly, and Cassereau *et al.*¹² generalized Rhyne's result by doing the same for $0 < b < \infty$.

These results can be placed in new perspective by spatially averaging the integral form of the arccos diffraction formulation [Eq. (1)] and interpreting the result as a Fourier–Bessel or Hankel transform

$$\begin{aligned} \langle h_1(z, t) \rangle_b &= \frac{1}{\pi b^2} \left[2\pi ac \int_{\rho=0}^b \int_{\tau=0}^\infty J_0(\tau\rho) J_1(\tau a) \right. \\ &\quad \times J_0(\tau\sqrt{(ct)^2 - z^2}) \rho d\rho d\tau \Big], \\ &= \frac{1}{\pi b^2} \left[2\pi abc \int_{\tau=0}^\infty \tau^{-1} J_1(\tau b) J_1(\tau a) \right. \\ &\quad \times J_0(\tau\sqrt{(ct)^2 - z^2}) d\tau \Big], \\ &= \frac{1}{\pi b^2} \left[2\pi abc \int_{\tau=0}^\infty \tau^{-2} J_1(\tau b) \right. \\ &\quad \times J_1(\tau a) J_0(\gamma\tau) \tau d\tau \Big], \\ &= \frac{c}{\pi b^2} \mathcal{H}\{2\pi\tau^{-1} b J_1(\tau b) \tau^{-1} a J_1(\tau a)\}, \end{aligned} \quad (3)$$

where $\gamma = \sqrt{(ct)^2 - z^2}$, and $\mathcal{H}$ denotes the Hankel transform with conjugate variables γ and τ . The convolution theorem for Hankel transforms allows Eq. (3) to be written

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$$\langle h_1(z, t) \rangle_b = \frac{c}{\pi b^2} \text{cyl}\left(\frac{\gamma}{2b}\right) * \text{cyl}\left(\frac{\gamma}{2a}\right), \quad (4)$$

where $\text{cyl}(r)$ is defined in Ref. 17. Gaskill derived a closed-form solution to Eq. (4) via graphical convolution. With $\alpha = (\gamma^2 + a^2 - b^2)/(2\gamma a)$, and $\beta = (\gamma^2 + b^2 - a^2)/(2\gamma b)$, the solution is

$$\langle h_1(z, t) \rangle_b = \begin{cases} c, & \gamma < a - b, \\ \frac{ca^2}{\pi b^2} [\cos^{-1}(\alpha) - \alpha \sqrt{1 - \alpha^2}] \\ + \frac{c}{\pi} [\cos^{-1}(\beta) - \beta \sqrt{1 - \beta^2}], & a - b \leq \gamma \leq a + b, \\ 0, & \gamma > a + b. \end{cases} \quad (5)$$

Equation (5) reduces to Rhyne's result when $b = a$, and, with the exception of a multiplicative constant, Eq. (5) appears to be the same as the result derived by Cassereau *et al.*

Even though it was derived under the assumption $b \leq a$, Eq. (5) is completely general. That is, a and b can

simply be interchanged in Eq. (5) if $b > a$. Reasons for this will be discussed in the next section. Algebraic manipulation of γ reveals that, for a fixed b , the spatially averaged velocity-potential impulse response $\langle h_1(z, t) \rangle_b$ is compressed in time as z increases.^{12,1}

Equation (5) was used to compute $\langle h_1(z, t) \rangle_b$ for six values of b at two depths: $z = 3$ cm and $z = 9$ cm. The speed of sound was set at $c = 1540$ m/s, and the diameter of the piston transmitter was set at $2a = 13$ mm; the radius of the receiver b is annotated in each of the plots. The transmitter was assumed to have an infinitely broadband or Dirac response;¹² the excitation was assumed to be an impulse; and the sampling frequency was set at $f_s = 36$ MHz. The results are shown in Figs. 1 and 2.

Spatially averaged velocity-potential impulse responses for the case $b = a/1000$ are shown in Fig. 1(a) and (b) and are consistent with point-receiver theory.¹⁶ In particular, the case $b = a/1000$ approximates an on-axis *point receiver*, and point-receiver theory predicts that the velocity-potential impulse response for an on-axis point receiver is a rectangular pulse with a pulse width that is compressed in time with increasing depth. The spatially averaged results obtained using Eq. (5) with $b = a/1000$ are consistent with this predic-

FIG. 1. Spatially averaged velocity-potential impulse responses.

FIG. 2. More spatially averaged velocity-potential impulse response.

tion. Figure 1(c) and (d) reveal that the pulse-like nature of the spatially averaged impulse response decays as b is increased.

Figure 1(e) and (f) show spatially averaged impulse responses for the case $b = a$, and the results are consistent with those presented by Rhyne.⁸ Figure 2(a)–(f) reveal that a pulse-like behavior begins to re-emerge in the spatially averaged impulse responses as b becomes larger than a . This behavior will be discussed in the next section. Finally, the graphs in both Fig. 1 and Fig. 2 confirm the prediction that, for a fixed b , the spatially averaged velocity-potential impulse response is compressed in time as z increases.

II. FURTHER INSIGHTS

Additional insight can be gained by considering Eqs. (4) and (5). First, the generality of Eq. (5) is due to the commutativity of convolution in Eq. (4) which, in this context, may be interpreted as a mathematical manifestation of Helmholtz's reciprocity theorem.¹⁸ Consequently, a and b can simply be interchanged in Eq. (5) if $b > a$.

Furthermore, point-receiver theory complemented with Helmholtz's reciprocity theorem can be used to predict that the spatially averaged velocity-potential impulse response will resemble a rectangular pulse when $b \gg a$. That is, the

impulse response will be a rectangular pulse when the transmitter, relative to the receiver, approximates an on-axis *point transmitter*. The graphs in Fig. 2 show that the spatially averaged impulse response becomes more pulse-like as b is increased. Thus, the prediction is theoretically confirmed and the behavior of the impulse responses for $b > a$ explained.

Second, Eq. (5) is well known in optics. Gaskill calls it the *cylinder-function cross correlation* (Ref. 17, pp. 302–304), while Bracewell gives the name *chat* function (Ref. 19, pp. 187–192) to the special case that results when $b = a$. Bracewell coined this term because the shape of the graph of Eq. (5) as a function of γ for $b = a$ resembles a Chinese farmer's hat. Thus, insights and results developed in optics for Eq. (5) may benefit researchers in ultrasound.

Finally, the graph that results from plotting Eq. (5) as a function of time t for $b = a$ is strikingly similar to the celebrated *brachistochrone*,^{20–22} compare Fig. 1(e) and (f) and Fig. 3. Indeed, the form of Eq. (5) for the case $b = a$ is similar to the equation for the brachistochrone; Eq. (5) for the case $b = a$ is

$$\langle h_1(z, t) \rangle_a = \frac{2c}{\pi} \left[\arccos\left(\frac{\gamma}{2a}\right) - \frac{\gamma}{2a} \sqrt{1 - \frac{\gamma^2}{4a^2}} \right], \quad (6)$$

and, with the parameterization

FIG. 3. A brachistochrone.

$$x = a(\theta - \sin \theta) \quad (7a)$$

and

$$y = a(1 + \cos \theta), \quad (7b)$$

where $0 \leq \theta \leq \pi$, the equation for the brachistochrone is

$$x = a \arccos\left(\frac{y-a}{a}\right) - a \sqrt{1 - \left(\frac{y-a}{a}\right)^2}. \quad (8)$$

Note the similarity between Eq. (6) and Eq. (8). The graphical and functional similarities just noted are not surprising when one considers the physical origins of and mathematical solutions to the brachistochrone and diffraction problems.

Specifically, both problems can be formulated in terms of Hamilton's physical principle of least action.^{23,21,24} Thus, both are mathematically amenable to solution via the calculus of variations. More rigorous comparison of these two problems may lead to deeper understanding of diffraction from piston transducers and transducers involving other geometries.

III. CONCLUSION

Fourier-Bessel theory was used to present a novel derivation of the spatially averaged velocity-potential impulse response associated with one-way diffraction from an unfocused piston transducer of radius a . The derivation also led to insights which may be of theoretical and practical interest to researchers in ultrasound.

- ¹G. Harris, "Review of transient field theory for a baffled planar piston," *J. Acoust. Soc. Am.* **70**, 10–19 (1981).
- ²A. Williams, Jr., "Integrated signal on circular piston receiver centered in a piston beam," *J. Acoust. Soc. Am.* **48**, 285–289 (1970).
- ³G. Harris, "Transient field of a baffled planar piston transducer having an arbitrary vibration amplitude distribution," *J. Acoust. Soc. Am.* **70**, 186–204 (1981).
- ⁴H. Huntington, A. Emslie, and V. Hughes, "Ultrasonic delay lines. I," *J. Franklin Inst.* **245**(1), 16–23 (1948).
- ⁵A. Williams, Jr., "The piston source at high frequencies," *J. Acoust. Soc. Am.* **23**, 1–6 (1951).
- ⁶H. Seki, A. Granato, and R. Truell, "Diffraction effects in the ultrasonic field of a piston source and their importance in the accurate measurement of attenuation," *J. Acoust. Soc. Am.* **28**, 230–238 (1956).
- ⁷R. Bass, "Diffraction effects in the ultrasonic field of a piston source," *J. Acoust. Soc. Am.* **30**, 602–605 (1958).
- ⁸T. Rhyne, "Radiation coupling of a disk to a plane and back or a disk to disk: An exact solution," *J. Acoust. Soc. Am.* **61**, 318–324 (1977).
- ⁹P. Rogers and A. Van Buren, "An exact expression for the Lommel diffraction correction integral," *J. Acoust. Soc. Am.* **55**, 728–728 (1974).
- ¹⁰A. Khimunin, "Numerical calculation of the diffraction corrections for the precise measurement of ultrasound absorption," *Acustica* **27**, 173–181 (1972).
- ¹¹G. Benson and O. Kiyohara, "Tabulation of some integral functions describing diffraction effects in the ultrasonic field of a circular piston source," *J. Acoust. Soc. Am.* **55**, 184–185 (1974).
- ¹²D. Cassereau, D. Guyomar, and M. Fink, "Time deconvolution of diffraction effects—Application to calibration and prediction of transducer waveforms," *J. Acoust. Soc. Am.* **84**, 1073–1085 (1988).
- ¹³X. Chen, K. Schwarz, and K. Parker, "Acoustic coupling from a focused transducer to a flat plate and back to the transducer," *J. Acoust. Soc. Am.* **95**, 3049–3054 (1994).
- ¹⁴A. Papoulis, *Systems and Transforms with Applications in Optics* (McGraw-Hill, New York, 1968), Chaps. 5 and 9.
- ¹⁵F. Oberhettinger, "On transient solutions of the 'baffled piston' problem," *J. Res. Natl. Bur. Stand., Sect. B* **65B**(1), 1–5 (1961).
- ¹⁶P. Stepanishen, "Transient radiation from pistons in an infinite planar baffle," *J. Acoust. Soc. Am.* **49**, 1627–1638 (1971).
- ¹⁷J. Gaskill, *Linear Systems, Fourier Transforms, and Optics* (Wiley, New York, 1978).
- ¹⁸L. Kinsler and A. Frey, *Fundamentals of Acoustics* (Wiley, New York, 1982), 3rd ed.
- ¹⁹R. Bracewell, *Two-Dimensional Imaging* (Prentice-Hall, Englewood Cliffs, NJ, 1995).
- ²⁰R. Courant and F. John, *Introduction to Calculus and Analysis*, Vol. I (Interscience, New York, 1965).
- ²¹H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, MA, 1950).
- ²²G. Thomas, Jr., *Calculus* (Addison-Wesley, Reading, MA, 1969).
- ²³M. Born and E. Wolf, *Principles of Optics* (Pergamon, New York, 1980), 6th ed.
- ²⁴P. Morse and K. Ingard, *Theoretical Acoustics* (McGraw-Hill, New York, 1968).