Optimal linearization of anharmonic oscillators

Jungkun Lee

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Optimal Linearization of Anharmonic Oscillators

by

Jungkun Lee

A Thesis Submitted
in
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of the
Requirements for the Degree of
MASTER OF SCIENCE
in
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Rochester, New York
1991
Optimal Linearization of Anharmonic Oscillators

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Abstract

This investigation is based on the geometric analysis of phase trajectories and incurred vector fields associated with nonlinear oscillators. Optimal curve fitting techniques are applied in the phase plane, in an effort to generate a so-called “geometric averaging”. The results are then compared with those generated by classical techniques such as harmonic balance and equivalent linearization, as well as by numerical integration. The investigation is extended to nonlinear multiple-degree-of-freedom systems. Frequencies of oscillations and mode shapes are derived based on the optimal equivalent linearization process. The results are also compared with numerical integration for justification. It is shown that the proposed linearization methods are simple to implement and provide an efficient methodology for the analysis of nonlinear oscillations.
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A  amplitude of displacement
A₀, A₁, Aₙ  Fourier coefficients
A₁, A₂, Aₙ  amplitude coefficients of cosine functions
Aⱼ  amplitude of the mode j
A  semi-axis of an ellipse
Aₖ  Fourier coefficient
aᵢ  coefficient of a differential equation
B  modal matrix
Bᵢ  element (i) of B
B₁, Bₙ  Fourier coefficients
→B  vector form of B
→B⁽ʲ⁾  vector form of B for the mode j
B  semi-axis of an ellipse
Bₖ  Fourier coefficient
C₁  Fourier coefficient
C₁, C₂, Cₙ  amplitude coefficients of sine functions
c  constant of integration
c₀, c₁, c₂, c₃  coefficients of a frequency polynomial
ĉ  equivalent damping coefficient
C  constant of integration; curve of a limit cycle
D₀, D₁, Dₙ  Fourier coefficients
Ω  total distance between points and the optimal ellipse
dᵢ  each distance between a point(i) and the optimal ellipse
Eₙ  Fourier coefficient
e  exponential function
F  nonlinear function of x and y; column matrix of nonlinear terms
F̂  nonlinearity matrix
List of Symbols

(continued)

f  nonlinear function
f^i  element (i) of F matrix
\vec{G}  vector function of state variables
g  gravitational acceleration
h  general nonlinear function
I  unit matrix
i  complex number
\vec{i}, \vec{j}  unit vectors
K  transformed stiffness matrix
k  stiffness constant
\bar{k}  equivalent stiffness coefficient
\bar{K}  stiffness matrix
k_{ij}  element (ij) of K
k_{ii}  element (ii) of K
\hat{K}  equivalent stiffness matrix
\hat{K}_{ij}  element (ij) of \hat{K}
\hat{K}_{ii}  element (ii) of \hat{K}
\bar{M}  mass matrix
m  mass
N  number of points on a phase plane; number of modes
O  higher-order term
q  normal coordinate
\vec{q}  vector form of q
q_{i}  normal coordinate of mode i
\ddot{q}_{i}  acceleration of q_{i}
r  amplitude of a linearized system
\dot{r}  velocity of r
r_{0}  initial value of r
T  period of oscillation
t  time
List of Symbols

(continued)

\[ U \]
unit modal matrix

\[ \vec{u}_i, \vec{u}_j \]
normalized eigenvectors of mode \( i, j \)

\[ u_{nj} \]
element \( n \) of modal eigenvector \( j \)

\[ \nabla \]
vector field of a nonlinear system

\[ \nabla_i \]
vector function on a point \((i)\) derived from \( \nabla \)

\[ \nabla_l \]
vector field of an equivalent linear system

\[ (\nabla_i)_i \]
vector function on a point \((i)\) derived from \( \nabla_i \)

\[ X \]
column vector of generalized coordinates

\[ \dot{X} \]
acceleration vector of \( X \)

\[ \dot{x} \]
rectangular coordinate

\[ \dot{x} \]
velocity of \( x \)

\[ \ddot{x} \]
acceleration of \( x \)

\[ x^{(i)} \]
ith order derivative of \( x \)

\[ x_i \]
rectangular coordinate of a point \((i)\) on a phase plane; element \((i)\) of \( \dot{X} \)

\[ \dot{Y} \]
column vector of generalized coordinates

\[ \ddot{Y} \]
acceleration vector of \( Y \)

\[ y \]
rectangular coordinate

\[ y_i \]
rectangular coordinate of a point \((i)\) on a phase plane; element \((i)\) of \( \dot{Y} \)

\[ \alpha \]
coefficient of a linear term

\[ \Gamma \]
phase curves

\[ \epsilon \]
nonlinearity coefficient

\[ \vec{\epsilon} \]
vector form of \( \epsilon \)

\[ \theta \]
angular displacement

\[ \theta_i \]
orientation of a point \((i)\) on an ellipse

\[ \lambda_j \]
eigenvalue of mode \( j \)

\[ \mu \]
nonlinearity coefficient

\[ \Phi \]
function of phase curves in terms of \( x, y \)
List of Symbols

(continued)

\( \phi, \phi_k \) phase shift

\( \phi_i \) orientation of a point \((i)\) on a phase plane

\( \varphi \) phase angle of a linearized system

\( \dot{\varphi} \) velocity of \( \varphi \)

\( \psi \) phase of a linearized system

\( \dot{\psi} \) velocity of \( \psi \)

\( \omega \) frequency of a nonlinear system

\( \omega_0 \) frequency of a linear system

\( \omega_i \) frequency of mode \(i\) of a multi-degree-of-freedom system

\( \text{sgn} \) signum function

\( \Delta \) difference between two systems

\( \overrightarrow{\Delta} \) vector form of \( \Delta \)

\( \Delta_i \) difference between two systems with respect to mode \(i\)
I Introduction

The study of oscillating motion is of great importance in physics, particularly in mechanics and engineering. Analysis of oscillations is also the essential part of mechanical vibration and is an important design consideration in most structural systems. Therefore, the effects of oscillation must be widely and carefully examined by designers and engineers.

Linear vibration analysis has been adequate for most applications. However, due to higher operating speeds and increased demand for system performance, the applicability of linear analysis becomes questionable. A large number of failures are a result of nonlinear effects in systems that were essentially designed under the assumption of linear behavior. In physical terms, a nonlinear system is one for which the output is no longer proportional to the input. Failures in machinery and structures result from unpredicted phenomena not encountered in linear systems.

One of the distinguishing characteristics of nonlinear oscillations is the dependence of the frequency of oscillations on the amplitudes of motion. For linear systems, the natural frequencies are independent of initial conditions. This well-known fact, coupled with the principle of superposition, forms the basis of standard linear analysis. The objective of this investigation is to develop geometric averaging techniques in an effort to replace a nonlinear system by an equivalent linear system, while retaining the nonlinear characteristics of the dynamics.

Differential equations are used to model the oscillatory
motion of physical systems. Based on the analytic structure of the differential equations, oscillatory motion can be generally classified as linear or nonlinear. It is recognized that a differential equation of the form

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \ldots + a_1 x^{(1)} + a_0 x = f(t) \quad (1.1)$$

where $x^{(i)} = \frac{d^i x}{d t^i}$, $i = 1, 2, \ldots n.$

is called linear if its coefficients $a_i(t)$ are continuous functions of only the independent variable $t$ (usually time). In typical cases, these functions reduce to constants. On the other hand, a differential equation such as Eq. (1.1) is nonlinear if its coefficients $a_i$ are also continuous functions of the dependent variable $x$ or its derivatives [1].

There are many characteristics which distinguish between the nature of linear and nonlinear differential equations. For example, the fundamental system of solutions exists only for linear differential equations. This implies that if certain basic solutions are known, the general solution will be a linear combination of these fundamental solutions. Moreover, the principle of superposition can only apply to linear differential equations. These fundamental properties also help in the analysis of linear oscillations. Obviously, there are many available methods and techniques to solve linear differential equations, including both analytical and numerical approaches [2]. However, it is more often than not very difficult to solve nonlinear differential equations. There is no general way (except in certain special cases) to obtain analytic solutions of nonlinear differential equations. Consequently, both approximate methods and qualitative analyses of the solutions become significantly important in
studying the nature of nonlinear oscillations.

The purpose of qualitative analysis of nonlinear oscillations is to acquire important information about the solution of differential equations without actually solving the equations. An indispensable tool for qualitative analysis is the phase plane. By investigating the geometric nature of trajectories in the phase plane, many properties such as equilibrium, periodicity and stability of the oscillations are obtained. A nonlinear oscillatory system can generally be expressed by the following differential equation:

$$\dot{x} + f(x, \dot{x}, t) = 0$$  \hspace{1cm} (1.2)

where \( f(x, \dot{x}, t) \) is a nonlinear function of \( x, \dot{x} \) and \( t \). If the independent variable \( t \) does not appear explicitly, Eq. (1.2) is written as

$$\dot{x} + f(x, \dot{x}) = 0$$  \hspace{1cm} (1.3)

A system described by Eq. (1.3) is designated as an autonomous system since all the properties of the system do not change with the time. For the sake of simplicity, we will restrict this investigation to autonomous systems. Eq. (1.3) can be transformed into two simultaneous differential equations of the form

$$\frac{dx}{dt} = y$$  \hspace{1cm} (1.4)

$$\frac{dy}{dt} = -f(x, y)$$

If \( x \) and \( y \) are Cartesian coordinates, the plane of the state variables (\( x \) and \( y \)) is defined as the phase plane. Therefore, solutions \( x(t) \) and \( y(t) \) of Eq. (1.4) are
represented by curves on the phase plane. Since the solution curves are graphically represented on the phase plane, topological methods of analysis are utilized to investigate the oscillations of nonlinear systems. By this method, solutions of Eq. (1.4) are not explicitly expressed by the functions of time \( t \), but rather from the resultant flow on the phase plane. The qualitative features and some quantitative information can be acquired through the investigation of the trajectories in the phase plane. However, the applicability of the graphical method is unfortunately confined to autonomous systems of lower orders [3].

On the phase plane, each point corresponds to a state of the system. In mechanics, the state of a system is described by the dependent variables of position and velocity. Figure 1.1 shows the phase plane of a nonlinear system expressed by Eq. (1.4), the direction of motion along the trajectories is indicated by the arrows.

![Diagram of phase plane](image)

Figure 1.1 The phase plane of a nonlinear system.

The state of the system is determined by the values of the coordinates \( x \) and \( y \). In mechanical systems, \( x \) is the
positional coordinate and \( y \) is the velocity of the system as defined in Eq. (1.4). The curves in Figure 1.1 are called the phase curves or trajectories. The complete collection of phase curves is called the phase diagram for the system. The dimension of the phase space is typically twice the dimension of the physical space.

In Eq. (1.4), if \( \dot{y} \) is divided by \( \dot{x} \), we obtain a first order differential equation in the variables \( x \) and \( y \) as

\[
\frac{dy}{dx} = \frac{-f(x,y)}{y} = \Phi(x, y) \tag{1.5}
\]

Upon integration of Eq. (1.5), we obtain an analytic relation between the state variables and thus deduce the equation of the phase curves in terms of \( x \) and \( y \). The phase plane graphically represents the dynamics and qualitatively describes the evolution of all the possible states of a system. Numerous techniques are available for the construction of phase curves. The interested reader should consult references [2, 3, 4, 6].

Note that a closed phase curve represents periodic motion, since the state of a system will eventually return to its original state along the curve. Since the state of the system does not return to its original state along a non-closed phase curve, the motion is verified as non-periodic. In this investigation, we will limit our study to strictly periodic oscillation.

Because the time derivative is applied in Eq. (1.4), the velocity of each point on the phase plane is also defined by Eq. (1.4) and is known as the phase velocity [4]. Note here the distinction between the phase velocity and the actual physical velocity of a system. Phase velocity can be expressed by a plane vector with the components \( \dot{x} \) and \( \dot{y} \).
Thus this unique vector is tangent to the corresponding phase curve and points in the direction of motion along the phase trajectories. In Figure 1.1, a vector \( \vec{V} \) of phase velocity is assigned at the point P. The direction of this vector is also defined by Eq. (1.4). As a result, by assigning a phase velocity vector to each point of the phase space, we acquire the collection of all the vectors which is defined as the vector field associated with the dynamical system. Figure 1.2 shows the vector field representing the dynamics of a nonlinear system.

![Vector Field](image)

**Figure 1.2** The vector field of a nonlinear system.

The vector field is very useful in analyzing the nonlinear system. We will discuss the analysis of the vector field in later sections.

Furthermore, both the equilibrium and related stability conditions can also be investigated through the phase plane. Referring to Eq. (1.4), if \( \dot{x} \) and \( \dot{y} \) are both zero, this indicates that the phase velocity of the system vanishes, hence both the velocity and acceleration of the system are zero. Thus an equilibrium point in a phase plane is defined by the condition
\[
\dot{x} = 0 \tag{1.6}
\]
\[
\dot{y} = 0
\]

This is equivalent (from Eq. (1.4)) to requiring
\[
y = 0 \tag{1.7}
\]
\[
f(x, y) = 0
\]

In mechanical systems, typically governed by Eq. (1.4), the equilibrium points are thus always located on the horizontal \((x)\) axis. This is not generally the case, however since Eq. (1.4) was derived from a single second-order differential equation. In addition, Eq. (1.5) implies that the slopes of the phase curves are uniquely determined everywhere except at the equilibrium points. Thus phase curves can intersect only at these equilibrium points. This is consistent with the uniqueness theorems of differential equations. Figure 1.2 indicates that point 0 is an equilibrium point, since the vector field vanishes there. It is known that a stable equilibrium point refers to a state of minimum potential energy [5]. Moreover, if the potential energy at an equilibrium point is not a relative minimum, then the equilibrium state is unstable. Since we are interested in the geometric nature of the phase curves, the mathematical analysis of stability conditions will not be extensively discussed. However, in terms of the phase plane, by the theory of Poincaré Stability an equilibrium state is called stable whenever a small disturbance remains small [4].

In this investigation, we will consider three examples of typical nonlinear oscillators to explore the features of the phase plane. Furthermore, these examples will be
utilized for explicit illustration of various approximation methods.

Example 1.1 Nonlinear Pendulum.

An undamped pendulum of length $\ell$ is governed by the differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell} \sin \theta = 0 \quad (1.8)$$

where $\theta$ is the inclination of the string to the downward vertical position. Figure 1.3 illustrates the oscillation of a nonlinear pendulum.

![Diagram of nonlinear pendulum](image)

**Figure 1.3** The oscillation of a nonlinear pendulum.

For simplicity, let $x = \theta$ and $\omega_0^2 = g/\ell$. Eq. (1.8) can be rewritten as:

$$\frac{d^2x}{dt^2} + \omega_0^2 \sin x = 0 \quad (1.9)$$

To transform Eq. (1.9) into an autonomous system of two simultaneous equations, we obtain

$$\frac{dx}{dt} = y \quad (1.10)$$

$$\frac{dy}{dt} = -\omega_0^2 \sin x \quad (1.11)$$
If Eq. (1.11) is divided by Eq. (1.10), we obtain a new differential equation which relates $y$ and $x$ instead of $x$ and $t$ as:

$$\frac{dy}{dx} = -\frac{\omega_0^2}{y}\sin x \tag{1.12}$$

Eq. (1.12) describes the analytical character of the phase curves. After integration of Eq. (1.12) with respect to $x$, the equation of the phase curves can be expressed as:

$$\frac{1}{2}y^2 - \omega_0^2\cos x = c \tag{1.13}$$

where $c$ is a constant of integration. In Eq. (1.13), the first term pertains to the kinetic energy and the second term pertains to the potential energy. Thus $c$ is a constant which corresponds to the total energy of the system.

Utilizing Eq. (1.13), various curves with different values of the total energy $c$ are plotted as shown in Figure 1.4.

![Figure 1.4 Phase diagram for the undamped pendulum.](image)

The curves in Figure 1.4 are called the phase curves or trajectories, and the complete Figure 1.4 is called the phase diagram for the system. Each curve uniquely
corresponds to a particular level of energy. The directions of the phase curves are indicated by the arrows on Figure 1.4, in which, when \( y \) is positive, \( x \) is increasing with time, and when \( y \) is negative, \( x \) is decreasing with time. This information is deduced from the system of Eqs. (1.10) and (1.11). Therefore, Figure 1.4 also illustrates the global dynamics of the oscillating system as time is increasing. The points labeled A, B and 0 are called equilibrium points, because both Eq. (1.10) and Eq. (1.11) are zero at these points. Geometrically, if we apply a small displacement at either A or B, the state of the system will be moved far away from these equilibrium points. Thus both points A and B are referred to as unstable equilibria. On the contrary, when we displace slightly from the center 0, the state of the system tends to stay close to the state of center 0, hence point 0 corresponds to a state of stable equilibrium. The stability of equilibrium points can also be verified by the principle of minimum potential energy. Verified by Eq. (1.13), points A and B represent the local maximum points of potential energy. Point 0 represents a relative minimum point of potential energy. Therefore, points A and B are called unstable equilibrium points and point 0 is called the stable equilibrium. In the neighborhood of the center 0, the phase curves are closed, hence the oscillation is periodic. The non-closed curves represent motions with high energy levels. Physically, this can be interpreted as the pendulum spinning about its pivot point.

Example 1.2 Oscillation Induced by a Nonlinear Restoring Spring.

The oscillation of a particle on a spring exerted by an anti-symmetrical restoring force of constant magnitude is governed by the second order differential equation [6]
\[ \ddot{x} + \text{sgn}(x) = 0 \] (1.14)

where the \( \text{sgn} \) (signum) function is defined by

\[
\text{sgn}(x) = \begin{cases} 
1, & x > 0 \\
0, & x = 0 \\
-1, & x < 0 
\end{cases} \] (1.15)

The plot of the function \( \text{sgn}(x) \) is shown in Figure 1.5. Similarly, following the previous example, we obtain two simultaneous differential equations to express the system and derive the differential equation of the phase curves as follows:

\[
\frac{dx}{dt} = y 
\] (1.16)

\[
\frac{dy}{dt} = -\text{sgn}(x) 
\] (1.17)

which results in

\[
\frac{dy}{dx} = -\frac{1}{y}\text{sgn}(x) 
\] (1.18)

Figure 1.5  Plot of the function \( \text{sgn}(x) \).
In order to construct the profile of the phase plane, Eq. (1.18) is integrated and the equations of the phase curves are expressed in the form

\[ y^2 = -2x + c, \quad x \geq 0 \]  

and

\[ y^2 = 2x + c, \quad x < 0 \]  

(1.19) \hspace{1cm} (1.20)

Apparently, the phase curves consist of two families of parabolas as shown in Figure 1.6. The direction along the phase curves and the equilibrium point 0 are also shown in Figure 1.6. In view of the closed phase curves, we accertain that the oscillations are periodic.

![Figure 1.6 Phase diagram for the piecewise constant restoring spring system.](image)

Example 1.3 Van der Pol Oscillator.

The Van der Pol equation has been widely studied, because it displays the existence of a limit cycle on the phase plane [7]. This equation is well known as
\[ \ddot{x} + \mu \dot{x}(x^2 - 1) + x = 0 \]  

(1.21)

where \( \mu \) is a coefficient of the nonlinear damping term. Eq. (1.21) can be written in the form of two simultaneous equations

\[ \dot{x} = y \]  

(1.22)

\[ \dot{y} = \mu y(1 - x^2) - x \]

We can interpret the above equations as a spring-mass system with viscous damping which is nonlinear and depends on both the velocity and the amplitude. If \( x^2 > 1 \), the damping is positive and the system exhibits energy dissipation. However, if \( x^2 < 1 \), the damping is negative and it suggests that energy is being "pumped" into the system. In Figure 1.7, a phase diagram for the Van der Pol Oscillator (\( \mu = 0.1 \)) is shown.

Figure 1.7 Phase diagram for the Van der Pol Oscillator.
In Figure 1.7, we use two phase curves to explain the nature of the limit cycle. On the phase plane, one phase curve starts well outside the limit cycle, while the other begins near the origin. Again, the arrows indicate the direction of the phase flow. Note that the two phase curves approach the same limit cycle as time approaches infinity. Thus, a limit cycle (C) on the phase plane is an isolated closed curve, for which all other nonclosed phase curves (Γ) in the form of spirals wind into or unwind from the limit cycle. Concerning the stability, the winding or unwinding property can be interpreted as stable or unstable, respectively. In addition, if the phase curves wind themselves onto the limit cycle from one side and unwind themselves from the other side or vice versa, the limit cycle is called semistable [7]. Figure 1.8 illustrates the three generic stability conditions of limit cycles.

![Figure 1.8](image)

(i) Stable  (ii) Unstable  (iii) Semistable

Figure 1.8 Stability conditions of limit cycles.

Stable limit cycles are characteristic of self-sustained oscillations, that is, periodic solutions displayed by certain unforced dissipative systems. Unstable limit cycles, on the other hand, have no definite physical meaning. In a nonlinear conservative system, the amplitude (or radii) of closed phase curves is dependent on the
initial conditions. On the other hand, the amplitude of a stable limit cycle is independent of the initial conditions, since the phase curves will reach the limit cycle following a corresponding spiral. Since this investigation is concerned only with periodic oscillations, the analysis of the transient state before a limit cycle will not be discussed.

As previously mentioned, it is always simpler to solve linear equations rather than nonlinear equations. Based on this judgement, it is possible to take advantage of this fact in deriving the approximate solutions of nonlinear oscillations through various methods of linearization. One of the early and simplest linearization methods is the assumption of small oscillation, which utilizes the characteristics of small oscillations and linearizes the terms in the nonlinear equation. For instance, in Eq (1.8) for a nonlinear pendulum, \( \sin \theta \approx \theta \) if the oscillation \( \theta \) is small. Then Eq. (1.9) can be approximated as

\[
\frac{d^2x}{dt^2} + \omega_0^2 x = 0 \tag{1.23}
\]

Thus the nonlinear term of Eq. (1.9) is transformed into a linear one. It is assumed here that for small oscillations, the dynamics will also be approximated by a linear oscillator. Eq. (1.23) corresponds to the well-known differential equation of a simple harmonic oscillator. Consequently, as long as the oscillation is small, not only the derivation of the solution is simplified, but the solution can be very close to an exact one for the nonlinear case. Unfortunately, direct linearization of a governing equation results in the loss of an important characteristic of nonlinear oscillations, namely, the dependence of frequency on the amplitude of motion.
Nevertheless, the same nonlinear system can be linearized and solved by different methodologies. For example, we can linearize either the nonlinear characteristics of the dynamics or the nonlinear equation itself. The former entails the averaging of dynamical properties. Conventional linearization methods include harmonic balance [6] and equivalent linearization [8], which are based on analytical measures. Both methods are commonly applied in the analysis of nonlinear oscillations. We will discuss them in detail in the subsequent sections. Recently, computers and numerical methods have provided an effective and efficient tool for performing tedious arithmetic computations. These tools also provide powerful new alternatives in the analysis of nonlinear oscillations. Assisted by computers, it becomes more feasible to investigate the phase curves and vector fields on the phase plane, as well as to acquire solutions by optimization techniques. Computers enable us to optimize the difference, in various contexts, between the nonlinear system and an assumed equivalent linear system. Furthermore, tedious computations associated with nonlinear multiple-degree-of-freedom systems can be readily implemented.

The application of optimization techniques in the analysis of nonlinear oscillations is the central theme of this investigation. In the later sections, we will propose and illustrate the new linearization methods. Finally, the comparison between conventional and new linearization methods will also be presented in this investigation.
II Conventional Linearization Methods

I Harmonic Balance Method

The Harmonic Balance Method is one of the simplest methods for approximating periodic solutions of nonlinear systems. Consider the general second order nonlinear differential equation

\[ \ddot{x} + \epsilon h(x, \dot{x}) + x = 0 \quad (2.1) \]

in which \( \epsilon \) is a small parameter. Suppose that Eq. (2.1) has a periodic solution \( x(t) \). By expanding \( x(t) \) in a Fourier series, we assume a solution developed as

\[ x(t) = \sum_{k=0}^{\infty} (A_k \cos k\omega t + B_k \sin k\omega t) \quad (2.2) \]

or

\[ x(t) = \sum_{k=0}^{\infty} A_k \cos (k\omega t + \phi_k) \quad (2.3) \]

Recall that a periodic solution of Eq. (2.1) corresponds to a simple closed curve in the associated phase plane. The shape of this closed curve is essentially determined by the fundamental harmonic in Eq. (2.3). Thus, for the first approximation we assume a solution of the form

\[ x(t) = A \cos (\omega t + \phi) \quad (2.4) \]

in which \( A \) is the amplitude of oscillation and \( \phi \) is the associated phase shift. Since the system is autonomous, i.e. time-invariant, we set \( \phi = 0 \) and thus

\[ x(t) = A \cos \omega t \quad (2.5) \]
This is possible, since time-invariance implies that the initial time may be taken as arbitrary. Given this form of the solution, the nonlinear terms in Eq. (2.1) become periodic and can also be expanded as a Fourier series. It follows that

\[ h(x, \dot{x}) \approx h(A \cos \omega t, -A \omega \sin \omega t) \]
\[ = C_1(A) \cos \omega t + D_1(A) \sin \omega t \]
\[ + \text{higher harmonics} \quad (2.6) \]

Note here the dependence of the primary Fourier coefficients on the amplitude \( A \) of the assumed solution (2.5). The Harmonic Balance Method is based on the analysis of the fundamental harmonic. Therefore, the higher harmonics in the series expansions will be neglected.

Substitution Eq. (2.6) into Eq. (2.1) results in

\[ (1 - \omega^2)A \cos \omega t + \epsilon C_1(A) \cos \omega t + \epsilon D_1(A) \sin \omega t \]
\[ + \text{higher harmonics} = 0 \quad (2.7) \]

Higher harmonics are ignored, since the periodicity of the solution depends essentially on the fundamental component. Based on the balance of harmonics and equating coefficients of the trigonometric functions of the fundamental frequency, Eq. (2.7) will hold for all \( t \) only if

\[ (1 - \omega^2)A + \epsilon C_1(A) = 0 \quad (2.8) \]
\[ D_1(A) = 0 \]

which gives a relation between \( A \) and \( \omega \). The coefficient \( C_1 \) depends on the particular form of the function \( h(x, \dot{x}) \). Such a relation is a characteristic of nonlinear systems. That is, the frequency of oscillation depends on
amplitudes. However, if it is not possible to equate and match the terms in the nonlinear equation as in the form of Eq. (2.7), we still can apply Fourier series to convert \( h(x, \dot{x}) \) into a trigonometric series and matching could then be completed. The method of harmonic balance can also be adapted to convert the nonlinear equation to a linear one without losing the characteristics of nonlinearity, such as the dependent feature of the frequency on the amplitude. We illustrate both methods in the following examples.

Example 2.1 Nonlinear Pendulum.

By Taylor series expansion and truncation, for convenience we substitute \( \sin x \approx x - \frac{1}{6}x^3 \) into Eq. (1.9), so the equation of motion becomes

\[
\ddot{x} + \omega_0^2(x - \frac{1}{6}x^3) = 0 \tag{2.9}
\]

Assume an approximate solution of the form \( x = A\cos \omega t \) and substitute into Eq. (2.9) to obtain

\[
-\omega^2A\cos \omega t + \omega_0^2[A\cos \omega t - \frac{1}{6}A^3(\frac{3}{4}\cos \omega t - \frac{1}{4}\cos 3\omega t)] = 0
\]

(2.10)

Ignoring the higher harmonic and collecting the coefficients of \( \cos \omega t \) yields

\[
\omega^2 = \omega_0^2(1 - \frac{1}{8}A^2) \tag{2.11}
\]

Therefore, the frequency-amplitude relation will be

\[
\omega = \omega_0\sqrt{1 - \frac{1}{8}A^2} \approx \omega_0(1 - \frac{1}{16}A^2) \tag{2.12}
\]

Eq. (2.12) is valid when the amplitude \( A \) is not too
large. Further, Eq. (2.9) can also be solved by equivalent linearization based on harmonic balance. We assume \( x = A \cos \omega t \) and again examine the nonlinear term in the equation to obtain

\[
-\frac{1}{6}x^3 = -\frac{1}{6}A^3\left(\frac{3}{4}\cos \omega t + \frac{1}{4}\cos 3\omega t\right)
\]  

(2.13)

But since \( x = A \cos \omega t \), it follows that

\[
-\frac{1}{6}x^2 A \cos \omega t = -\frac{1}{8}A^3 \cos \omega t - \frac{1}{24}A^3 \cos 3\omega t
\]  

(2.14)

Matching the terms of \( \cos \omega t \) results in

\[
-\frac{1}{6}x^2 = -\frac{1}{8}A^2
\]  

(2.15)

Now substitute Eq. (2.15) into Eq. (2.9), which will replace Eq. (2.9) by a linear equation given as

\[
\ddot{x} + \omega_0^2 (x - \frac{1}{8}A^2 x) = 0
\]  

(2.16)

or

\[
\ddot{x} + \omega_0^2 (1 - \frac{1}{8}A^2) x = 0
\]  

(2.17)

Since Eq. (2.17) is linear, the frequency is deduced as

\[
\omega = \omega_0 \sqrt{1 - \frac{1}{8}A^2} \approx \omega_0 (1 - \frac{1}{16}A^2)
\]  

(2.18)

Eq. (2.18) is consistent with the relation in Eq. (2.12).

Example 2.2 Oscillation Induced by a Nonlinear Restoring Spring.

Substituting the assumed solution (2.5) into Eq. (1.14), the nonlinear term is now expressed as

\[
\text{sgn}(x) = \text{sgn}(A \cos \omega t)
\]  

(2.19)
The graph of $\text{sgn}(x)$ and $\text{sgn}(\cos \omega t)$ are shown in Figure 2.1.

![Graphs of sgn(x) and sgn(Acos\omega t)](image)

Figure 2.1 (a) graph of $\text{sgn}(x)$ (b) graph of $\text{sgn}(\cos \omega t)$

The period of $\text{sgn}(\cos \omega t)$ is $2\pi/\omega$, so we can approximate the function $\text{sgn}(\cos \omega t)$ by the first term of its Fourier series on the interval $(0, 2\pi/\omega)$, thus

$$\text{sgn}(\cos \omega t) = A_1 \cos \omega t + B_1 \sin \omega t + \text{higher harmonics}$$

(2.20)

Since $\text{sgn}(\cos \omega t)$ is even, there is no sine term, so $B_1 = 0$ and

$$A_1 = \frac{\omega}{\pi} \int_{0}^{\frac{2\pi}{\omega}} \text{sgn}(\cos \omega t) \cos \omega t \, dt$$

$$= \frac{1}{\pi} \int_{0}^{2\pi} \text{sgn}(\cos \theta) \cos \theta \, d\theta$$

$$= \frac{4}{\pi}$$

(2.21)
Therefore, \( \text{sgn}(x) \) is converted to \((4/\pi)\cos\omega t\) or approximated to first order by \(4x/\pi A\). As a result, the equivalent linear equation is given by

\[
\ddot{x} + \frac{4}{\pi A}x = 0
\]  

(2.22)

Obviously, the solution of Eq. (2.22) is

\[
x(t) = A\cos\left\{\left(\frac{4}{\pi A}\right)^{\frac{1}{2}}t\right\}
\]  

(2.23)

with

\[\omega = \frac{2}{\sqrt{\pi A}}\]

Again, the nonlinear character of the oscillation is preserved by this frequency-amplitude relation (2.23).

Example 2.3 Van der Pol Oscillator.

Rewriting Eq. (1.21) in the form

\[
\ddot{x} + x = -\mu(x^2 - 1)x
\]  

(2.24)

we assume an approximate solution \( x = A\cos\omega t \) and substitute it into Eq. (2.24). This results in

\[
(-\omega^2 + 1)A\cos\omega t = -\mu(A^2\cos^2\omega t - 1)(-A\omega\sin\omega t)
\]

\[= \mu\omega(A^2 - 1)\sin\omega t + \mu A^3\omega\sin3\omega t\]

(2.25)

Equating the terms of \( \cos\omega t \) and \( \sin\omega t \), and simultaneously ignoring higher harmonics, two equations are generated:
\[ 1 - \omega^2 = 0 \]  
\[ \frac{1}{4}A^2 - 1 = 0 \]

From Eqs. (2.26) and (2.27), solutions are obtained as

\[ \omega = \pm 1 \]

and

\[ A = \pm 2 \]

Similarly, we can approximate a solution of Eq. (1.21) from the linearization of the nonlinear term. We note that

\[ \mu(x^2 - 1)\dot{x} = -\mu(A^2\cos^2\omega t - 1)A\omega\sin\omega t \]

\[ = -\mu A\omega(\frac{1}{4}A^2 - 1)\sin\omega t - \frac{1}{4}\mu A^3\omega\sin3\omega t \]

Neglecting the effect of the higher harmonic and assuming \( \dot{x} = -A\omega\sin\omega t \) for the periodic solution, Eq. (2.28) is rewritten as

\[ \mu(x^2 - 1)\dot{x} \approx \mu(\frac{1}{4}A^2 - 1)\dot{x} \]

After substitution into Eq. (1.19), we obtain the linear equation

\[ \ddot{x} + \mu(\frac{1}{4}A^2 - 1)\dot{x} + x \approx 0 \]

In Eq. (2.30), if \( A = \pm 2 \), the damping term vanishes and the solution has a characteristic frequency \( \omega = 1 \). Moreover, the amplitude \( (A) \) can be interpreted as the average radius of the limit cycle, which equals \( +2 \).
Although the harmonic balance method is easy to implement, there exists a certain deficiency. In order to obtain a consistent solution, we need to examine the order of the coefficients of all neglected harmonics. In addition, the error of solution must be carefully examined. Otherwise, we might have an inaccurate approximation. For instance, in Example 2.1, if the amplitude (A) is too large, the solution from Eq. (2.18) will be unfavorable, since the expression inside the radical may become negative. In reference to the example of the Van der Pol Oscillator, if $\mu$ is very large, then the higher harmonic cannot be neglected, hence the harmonic balance method is recommended only for $\mu \ll 1$.

II Equivalent Linearization Method

This method was developed by Krylov and Bogoliubov and improved by Mitropolsky and is known as the KBM Method [8]. We shall review the essential points and show some examples analyzed by this method. Consider a differential equation of the form

$$\ddot{x} + a^2 x + \epsilon f(x, \dot{x}) = 0 \quad (2.31)$$

where $f(x, \dot{x})$ is a nonlinear function of $x$ and $\dot{x}$, and $\epsilon$ is a small parameter. Such an equation is known as weakly nonlinear. If $\epsilon = 0$, Eq. (2.31) reduces to a linear differential equation whose solution is

$$x = r \sin(at + \varphi) \quad (2.32)$$

with

$$\dot{x} = a \cos(at + \varphi) \quad (2.33)$$
in which, \( r \) and \( \varphi \) are constants determined by the initial conditions.

The idea of this method is to incorporate Eqs. (2.32) and (2.33) as the solution of Eq. (2.31) when \( \epsilon \neq 0 \), but essentially small (\( \epsilon \ll 1 \)). Hence, \( r \) and \( \varphi \) must be considered as certain functions of \( t \). Due to the assumption \( \epsilon \ll 1 \), \( r \) and \( \varphi \) may be considered as "almost constant", i.e. slowly-varying functions of \( t \). Since \( r = r(t) \) and \( \varphi = \varphi(t) \), we differentiate Eq. (2.32) with respect to time and obtain

\[
\dot{x} = \dot{r}\sin(at + \varphi) + r\dot{\varphi}\cos(at + \varphi) + r\dot{c}\cos(at + \varphi)
\]  

(2.34)

Imposing Eq. (2.33), we have

\[
\dot{r}\sin(at + \varphi) + r\dot{\varphi}\cos(at + \varphi) = 0
\]  

(2.35)

and thus

\[
\ddot{x} = \ddot{r}\cos(at + \varphi) - r\dot{c}^2\sin(at + \varphi) - r\dot{\varphi}\sin(at + \varphi)
\]  

(2.36)

Substituting Eqs. (2.33) and (2.35) into Eq. (2.31) results in

\[
\dot{r}\cos(at + \varphi) - r\dot{\varphi}\sin(at + \varphi) + \epsilon f[r\sin(at + \varphi)] = 0
\]  

(2.37)

Simultaneously solving Eqs. (2.35) and (2.37) leads to
\[ \dot{\phi} = \frac{f}{r_0} \sin(\alpha t + \phi) f[r \sin(\alpha t + \phi), \arccos(\alpha t + \phi)] \quad (2.38) \]

\[ \dot{r} = -\frac{f}{\alpha} \cos(\alpha t + \phi) f[r \sin(\alpha t + \phi), \arccos(\alpha t + \phi)] \quad (2.39) \]

Therefore, the original differential Eq. (2.31) is now replaced by Eqs. (2.38) and (2.39) with \( r(t) \) and \( \phi(t) \) as variables. In the above, \( \epsilon \) is a small parameter and so if \( f(x, \dot{x}) \) is bounded function, Eqs. (2.38) and (2.39) show that \( r(t) \) and \( \phi(t) \) are slowly-varying functions of \( t \).

Setting \( \psi = \alpha t + \phi \) and expanding the right hand sides of Eq. (2.38) and Eq. (2.39) into Fourier series

\[ f(r \sin \psi, \arccos \psi) \sin \psi = A_0(r) + \sum_{n=1}^{\infty} \left[ A_n(r) \sin(n \psi) + B_n(r) \cos(n \psi) \right] \quad (2.40) \]

and

\[ f(r \sin \psi, \arccos \psi) \cos \psi = D_0(r) + \sum_{n=1}^{\infty} \left[ D_n(r) \sin(n \psi) + E_n(r) \cos(n \psi) \right] \quad (2.41) \]

where

\[ A_0(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r \sin \psi, \arccos \psi) \sin \psi d\psi \quad (2.42) \]

\[ D_0(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r \sin \psi, \arccos \psi) \cos \psi d\psi \quad (2.43) \]

\[ A_n(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r \sin \psi, \arccos \psi) \sin \psi \sin(n \psi) d\psi \quad (2.44) \]

\[ B_n(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r \sin \psi, \arccos \psi) \sin \psi \cos(n \psi) d\psi \quad (2.45) \]
\[
D_n(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r\sin\psi, \arccos\psi) \cos\psi \sin(n\psi) d\psi 
\quad (2.46)
\]

\[
E_n(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r\sin\psi, \arccos\psi) \cos\psi \cos(n\psi) d\psi 
\quad (2.47)
\]

As previously mentioned, \( r \) and \( \varphi \) are assumed to be slowly-varying functions of time, thus we exclude the terms of higher frequencies and retain only two terms \( A_0 \) and \( D_0 \) of the sums. Under these assumptions, Eqs. (2.38) and (2.39) can be written as

\[
\dot{\varphi} = \frac{\epsilon}{\alpha} A_0(r) 
\quad (2.48)
\]

\[
\dot{r} = -\frac{\epsilon}{\alpha} D_0(r) 
\quad (2.49)
\]

Since \( \dot{\psi} = \alpha + \dot{\varphi} \), Eq. (2.48) can be rewritten as

\[
\dot{\psi} = \alpha + \frac{\epsilon}{\alpha} A_0(r) 
\quad (2.50)
\]

Hence, from Eqs. (2.49) and (2.50), we obtain the well-known Krylov-Bogoliubov formulas for the first approximation as

\[
\dot{r} = -\frac{\epsilon}{\alpha} \frac{1}{2\pi} \int_0^{2\pi} f(r\sin\psi, \arccos\psi) \cos\psi d\psi 
\quad (2.51)
\]

\[
\dot{\psi} = \alpha + \frac{\epsilon}{\alpha} \frac{1}{2\pi} \int_0^{2\pi} f(r\sin\psi, \arccos\psi) \sin\psi d\psi 
\quad (2.52)
\]

As a result, the amplitude and the frequency for a solution of a nonlinear differential equation can be approximated by Eqs. (2.51) and (2.52). Meanwhile, in Eq. (2.52), since \( \alpha \) is the linear frequency corresponding to \( \epsilon = 0 \), the second term on the right hand side of Eq. (2.52) represents the nonlinear frequency correction.
It is interesting to note that the Method of Equivalent Linearization based on KBM Method can be applied to transform a nonlinear differential equation into an equivalent linear one. Consider the differential equation

\[ m\ddot{x} + kx + \epsilon f(x, \dot{x}) = 0 \]  
(2.53)

The first approximation is given as

\[ x = r \cos \psi \]  
(2.54)

\[ \psi = \alpha t + \phi \]  
(2.55)

in which \( a^2 = \frac{k}{m} \). Following the steps outlined above, it can be shown that \( r \) and \( \psi \) satisfy

\[ \ddot{r} = \frac{\epsilon}{2\pi m} \int_0^{2\pi} f(r \cos \psi, -r \sin \psi) \sin \psi d\psi \]  
(2.56)

\[ \dot{\psi} = \omega(r) = \alpha + \frac{\epsilon}{2\pi m r^2} \int_0^{2\pi} f(r \cos \psi, -r \sin \psi) \cos \psi \sin \psi d\psi \]  
(2.57)

Two functions of the amplitude, \( \bar{k}(r) \) and \( \bar{c}(r) \), defined by the equations

\[ \bar{c}(r) = -\frac{\epsilon}{\pi r^2} \int_0^{2\pi} f(r \cos \psi, -r \sin \psi) \sin \psi d\psi \]  
(2.58a)

\[ \bar{k}(r) = k + \frac{\epsilon}{\pi r^2} \int_0^{2\pi} f(r \cos \psi, -r \sin \psi) \cos \psi d\psi \]  
(2.58b)

may be introduced. Squaring Eq. (2.57),

\[ \omega^2 = a^2 + \frac{\epsilon}{\pi m r^2} \int_0^{2\pi} f(r \cos \psi, -r \sin \psi) \cos \psi d\psi + O(\epsilon^2) \]  
(2.59)
So to first order,

\[
\omega^2 = \frac{k}{m} + \frac{\epsilon}{\pi m r^2} \int_0^{2\pi} f(r \cos \psi, -\arcsin(\psi)) \cos \psi d\psi
\]

\[
= \frac{\tilde{k}}{m}
\]

(2.60)

In terms of these functions, \( \bar{c}(r) \) and \( \bar{k}(r) \), the Eqs. (2.56) and (2.57) become

\[
\dot{r} = \frac{\bar{c}}{2m} r
\]

(2.61a)

\[
\dot{\psi} = \omega = \sqrt{\frac{\bar{k}}{m}}
\]

(2.61b)

Differentiating Eq. (2.54) twice and taking into account Eqs. (2.58) to (2.61), as well as ignoring terms of order higher than \( \epsilon \), Eq. (2.53) can now be expressed as the linear differential equation

\[
m\ddot{x} + c\dot{x} + \tilde{k}x \approx 0
\]

(2.62)

The analysis above shows that if the parameters \( \bar{c} \) and \( \bar{k} \) are obtained, a nonlinear equation such as Eq. (2.53) can be replaced by an equivalent linear equation as Eq. (2.62) with an accuracy of the order of \( \epsilon^2 \). Hence, \( \bar{c} \) is called an equivalent coefficient of damping and \( \bar{k} \) is called an equivalent stiffness coefficient [8]. Eq. (2.62) is important in the linearization of certain nonlinear systems. For instance, in control problems, linearization of nonlinear links is necessary in converting the original system to a linear system. The derivation and solution of Eq. (2.62) is no simpler than the analysis of system of Eqs. (2.56) and (2.57), since integration is required in both sets of equations. Therefore, we will apply only the approximation method given by Eqs. (2.51) and (2.52) to
illustrate the following examples of nonlinear oscillators.

Example 2.4 Nonlinear Pendulum.

From Eq. (2.9), the equation of motion can be approximated as

$$\ddot{x} + \omega_0^2 x - \frac{1}{6} \omega_0^2 x^3 = 0$$  \hspace{1cm} (2.63)

Applying Eqs. (2.51) and (2.52), with

$$\alpha = \omega_0$$  \hspace{1cm} (2.64)

$$\epsilon = -\frac{1}{6} \omega_0^2$$  \hspace{1cm} (2.65)

and

$$f(x, \dot{x}) = x^3$$  \hspace{1cm} (2.66)

We have

$$\dot{r} = \frac{\omega_0}{12\pi} \int_0^{2\pi} r^3 \sin^3 \psi \cos \psi \, d\psi$$

$$= 0$$  \hspace{1cm} (2.67)

and

$$\dot{\psi} = \omega_0 - \frac{\omega_0}{12\pi r} \int_0^{2\pi} r^3 \sin^3 \psi \sin \psi \, d\psi$$

$$= \omega_0 \left(1 - \frac{r^2}{16}\right)$$  \hspace{1cm} (2.68)

Comparing Eq. (2.68) to Eq. (2.12), we have deduced the same approximation as given by the Harmonic Balance Method.
Example 2.5 Van der Pol Oscillator.

Referring to Eq. (1.21), the parameters and nonlinearity are taken as

\[ a = 1.0; \quad \epsilon = \mu \]  \hspace{1cm} (2.69)

\[ f(x, \dot{x}) = \dot{x}(x^2 - 1) \]  \hspace{1cm} (2.70)

From Eqs. (2.51) and (2.52), we obtain

\[ \dot{r} = -\frac{\mu}{2\pi} \int_{0}^{2\pi} \text{arccos}(r^2 \sin^2 \psi - 1) \cos \psi \, d\psi \]

\[ = -\frac{\mu r^3}{8} + \frac{\mu r}{2} \]

\[ = \frac{\mu r}{2} \left( 1 - \frac{r^2}{4} \right) \]  \hspace{1cm} (2.71)

and

\[ \dot{\psi} = 1.0 + \frac{\mu}{2\pi r} \int_{0}^{2\pi} \text{arccos}(r^2 \sin^2 \psi - 1) \sin \psi \, d\psi \]

\[ = 1.0 \]  \hspace{1cm} (2.72)

Again, the result is consistent with prior harmonic balance results, which implies that if \( r = 2 \), then Eq. (2.71) has a zero right hand side and the damping effect is negated. This will result in periodic oscillation. Also from Eq. (2.72), the frequency of the oscillator is 1.0.

Furthermore, Equations (2.71) and (2.72) imply that an approximate solution for the Van der Pol oscillator is given by

\[ x(t) = r(t) \cos(\omega t) \]
in which

\[ r(t) = \frac{2r_0}{\sqrt{r_0^2 + (4 - r_0^2)e^{-\mu t}}} \]

where \( r_0 \) is the initial value of \( r(t) \).
III Proposed Methods of Equivalent Linearization

I Method of Optimal Ellipse.

In this section, we develop an elliptical phase path of an equivalent linear system to approximate the solution of a nonlinear oscillator. We consider the equation of motion of a simple harmonic oscillator in the form:

\[ m\ddot{x} + kx = 0 \]  \hspace{1cm} (3.1)

where \( m \) is the mass of the oscillator and \( k \) is the stiffness constant. We rewrite Eq. (3.1) as

\[ \dot{x} + \omega_0^2 x = 0 \]  \hspace{1cm} (3.2)

in which

\[ \omega_0 = \sqrt{\frac{k}{m}} \]  \hspace{1cm} (3.3)

where \( \omega_0 \) is referred to as the natural circular frequency of the linear system. Multiplying Eq. (3.2) by \( \dot{x} \) and integrating, we obtain

\[ \dot{x}^2 + \omega_0^2 x^2 = 2C \]  \hspace{1cm} (3.4)

Here \( C \) is an arbitrary constant which also represents the energy level of the system as in Eq. (1.13). If we divide Eq. (3.4) by \( 2C \) and set

\[ \dot{x} = y \]

\[ 2C = 3^2 \]
and

\[ \frac{2c}{\omega_0^2} = A^2 \]

we will get the equation of an ellipse with the semi-axes \( A \) and \( B \) as

\[ \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1 \]  \hspace{1cm} (3.5)

which characterizes the phase curves on the phase plane as in Figure 3.1.

![Figure 3.1 Phase curves of a harmonic oscillator.](image)

Thus, by Eq. (3.5), we acquire a family of ellipses corresponding to different values of the energy constant \( c \). From the identifications above, the frequency of the linear system can be derived from the semi-axes as follows:

\[ \omega_0^2 = \frac{B^2}{A^2} \]  \hspace{1cm} (3.6)

or

\[ \omega_0 = \frac{B}{A} \]  \hspace{1cm} (3.7)

Both Eqs. (3.5) and (3.7) provide an easy measure of
approximation to the solution of a nonlinear oscillator without damping. The objective is to find the semi-axes of a "best-fit" or an optimal ellipse for a closed phase path of a nonlinear oscillator, and by applying Eq. (3.7), to derive an estimate for the frequency. Hence, if a phase curve of a nonlinear oscillator is obtained, we are able to approximate the solution without solving or even knowing the equation of motion of the nonlinear system. For example, the ellipse could be fit to data points generated by an experiment.

Formulation of the optimal ellipse is based on minimizing the difference between two phase curves. One is generated from the nonlinear oscillator and the other is associated with the elliptical path of an equivalent harmonic oscillator. The coordinates of each point on the phase curve of the nonlinear oscillator are given as \((x_i, y_i)\), \(i = 1, 2, \ldots, N\). The difference is expressed by the total distance between the two phase curves. This distance is discretely computed as the sum of distances between the data points \((x_i, y_i)\) and corresponding nearest points on the ellipse. Each unique nearest point is characterized by an associated angle \(\theta_i\). Illustrated by Figure 3.2, the optimal ellipse is obtained by finding the angles \(\theta_i\), \(i = 1, 2, \ldots, N\), as well as the optimal semi-axes \(A\) and \(B\). The angles \(\theta_i\) are the orientations of projections of the data points onto the ellipse. These projections define the distance \(d_i\), \(i = 1, 2, \ldots, N\). Therefore, we determine the optimal ellipse by minimizing the total distance between the phase points and the optimal ellipse. Summing the square of each distance, we have

\[
\Phi = \sum_{i=1}^{N} d_i^2
\]

\[
= \sum_{i=1}^{n} \left[ (x_i - A\cos\theta_i)^2 + (y_i - B\sin\theta_i)^2 \right]
\]

(3.8)
Figure 3.2 Optimal ellipse of the nonlinear oscillator.

Eq. (3.8) is an objective function with \( N + 2 \) variables, which are \( A, B \) and \( \theta_i, i = 1, 2, \ldots, N \). Meanwhile, there is no constraint for Eq. (3.8), since the quantities \( A\cos\theta_i \) and \( B\sin\theta_i \) automatically represent the coordinates of points on the optimal ellipse. In order to find the minimum of Eq. (3.8), we set derivatives with respect to each variable equal to zero. We obtain

\[
\frac{\partial \mathcal{F}}{\partial A} = 0: \quad \sum_{i=1}^{n} 2(x_i - A\cos\theta_i)(-\cos\theta_i) = 0 \quad (3.9)
\]

\[
\frac{\partial \mathcal{F}}{\partial B} = 0: \quad \sum_{i=1}^{n} 2(y_i - B\sin\theta_i)(-\sin\theta_i) = 0 \quad (3.10)
\]

and for each \( i = 1, 2, \ldots, N \)

\[
\frac{\partial \mathcal{F}}{\partial \theta_i} = 0: \quad 2(x_i - A\cos\theta_i)(A\sin\theta_i) + 2(y_i - B\sin\theta_i)(-B\cos\theta_i) = 0
\]

(3.11)
We end up with $N + 2$ equations as

$$\sum_{i=1}^{n} x_i \cos \theta_i = \mathcal{A} \sum_{i=1}^{n} \cos^2 \theta_i \quad (3.12)$$

$$\sum_{i=1}^{n} y_i \sin \theta_i = \mathcal{B} \sum_{i=1}^{n} \sin^2 \theta_i \quad (3.13)$$

and for each $i$,

$$\frac{\mathcal{A}(x_i - \mathcal{A}\cos \theta_i)}{\cos \theta_i} = \frac{\mathcal{B}(y_i - \mathcal{B}\sin \theta_i)}{\sin \theta_i} \quad (3.14)$$

The semi-axes $\mathcal{A}$ and $\mathcal{B}$ are approximated from Eqs. (3.12), (3.13) and (3.14), which results in the approximation of the frequency of the nonlinear oscillator by Eq. (3.7). The computer program used to solve the system of equations is listed in Appendix A.

II Method of Optimal Vector Field.

Finite dimensional continuous dynamical systems are defined by a system of first order ordinary differential equations given in vector form by

$$\dot{\vec{x}} = \vec{G}(\vec{x}) \quad (3.15)$$

in which $\vec{x} = (x_1, x_2, \ldots, x_n)$ represents a point in an $n$-dimensional phase space. The function $\vec{G}$ is a vector function of the state variables. Eq. (3.15) represents a vector field in the phase space and thereby determines the phase flow (or evolution) of a dynamical system [6]. Such a vector field associated with a dynamical system provides a useful tool in the analysis of nonlinear oscillators.

Recall that the Method of Optimal Ellipse pertains to the
construction of a best-fit ellipse to a set of data points in the phase plane. The generated ellipse becomes the linear approximation to the phase curve of a nonlinear oscillator. By its very nature, this method is restricted to a space of two state variables, that is, single degree of freedom systems.

An alternate approach to optimal linearization is the approximation of the actual vector field instead of the solution curve. It is important to note that the phase velocities (given by the vector field) are tangent to the trajectories at each point in the phase space. So the idea is to optimally match the vector field of a nonlinear system with the vector field associated with a linear oscillator. This method has two distinct advantages. First, the vector field is given explicitly by the equation of motion (3.15). Thus the governing differential equations need not be solved. Second, this method is applicable in any dimension. However, to illustrate these concepts, we first examine a single degree of freedom system.

Consider a nonlinear oscillator described by a system of differential equations as

\[ \dot{x} = y \]

\[ y = F(x, y) \]  

(3.16)

where \( F(x, y) \) represents a nonlinear function of \( x \) and \( y \). With respect to the periodic oscillation, we assume \( \Gamma \) to be a closed phase curve on the phase plane. There is one and only one vector assigned to express Eq. (3.16) at each point on \( \Gamma \). If we introduce Cartesian coordinates \( x \) and \( y \), then for all these points on \( \Gamma \), a vector field is generated and expressed by the following vector function as
This vector field represents the phase velocity at each point on the phase plane. Both the phase curve \( \Gamma \) and the resulting vector field are shown in Figure 3.3.

\[
\vec{V}(x, y) = y \vec{i} + F(x, y) \vec{j}
\]  

(3.17)

Figure 3.3 Phase curve \( \Gamma \) and the resulting vector field of a nonlinear oscillator.

For the first approximation of a nonlinear system, we assume an equivalent linear system (with \( x = A \cos \omega t \)) whose associated vector field is in the form

\[
\vec{V}_l(x, y) = \dot{x} \vec{i} + \dot{y} \vec{j}
\]

\[
= -A \omega \sin \omega t \vec{i} - A \omega^2 \cos \omega t \vec{j}
\]  

(3.18)

The vector field of the equivalent linear system is shown in Figure 3.4. Consequently, from the linear system, if we can derive an optimal vector field which is the "best fit" vector field to the vector field of the nonlinear system, then the approximation of the solution is acquired.
Illustrated by Figure 3.5, the implementation of this method based on first picking \( N \) points on an arbitrary ellipse whose coordinates are defined by \((x_i, y_i), i = 1, 2, \ldots, N\). Thus vector functions are expressed as

\[
y_i \hat{i} + F(x_i, y_i) \hat{j}
\]

on the closed phase curve of a linear oscillator.

![Figure 3.4 Vector field of the equivalent linear system.](image)

![Figure 3.5 Selected points and vector function on the closed phase curve of a nonlinear oscillator.](image)
Setting $\phi_i = \omega t_i$, and incorporating the vector field of the linear system, we obtain two vectors at each point on the closed phase curve as shown in Figure 3.6.

\[ (\vec{V}_i)_1 = -A\omega \sin \phi_i \hat{i} - A\omega^2 \cos \phi_i \hat{j} \]

\[ \vec{V}_i = y_i \hat{i} + F(x_i, y_i) \hat{j} \]

**Figure 3.6** Vector functions of the linear and nonlinear systems.

For the purpose of optimization, we sum up the difference between vector $(\vec{V}_i)_1$ and vector $\vec{V}_i$ at each point to obtain

\[ \mathcal{F} = \sum_{i=1}^{n} | \vec{V}_i - (\vec{V}_i)_1 |^2 \]

\[ = \sum_{i=1}^{n} (y_i + A\omega \sin \phi_i)^2 + [F(x_i, y_i) + A\omega^2 \cos \phi_i]^2 \quad (3.19) \]

Further, by the first approximation from Eq. (2.5), we set $x_i = A \cos \phi_i$ and $y_i = -A \omega \sin \phi_i$. Thus Eq. (3.19) is simplified and transformed as

\[ \mathcal{F} = \sum_{i=1}^{n} [F(A \cos \phi_i, -A \omega \sin \phi_i) + A\omega^2 \cos \phi_i]^2 \quad (3.20) \]
Eq. (3.20) is the objective function for the minimizing procedure, in order to acquire the approximation of the frequency \( \omega \). Taking the derivative of Eq. (3.20) with respect to the variable \( \omega \), we obtain

\[
\frac{d\Phi}{d\omega} = 2 \sum_{i=1}^{n} \left[ F(\cos\phi_i, -\omega\sin\phi_i) + A\omega^2 \cos\phi_i \right] \cdot \left[ \frac{\partial F}{\partial y}(-\sin\phi_i) + 2A\omega \cos\phi_i \right]
\]

(3.21)

So for a minimum, we need

\[
0 = \sum_{i=1}^{n} \left[ F(\cos\phi_i, -\omega\sin\phi_i) + A\omega^2 \cos\phi_i \right] \cdot \left[ \frac{\partial F}{\partial y}(-\sin\phi_i) + 2A\omega \cos\phi_i \right]
\]

(3.22)

Let \( F_i = F(\cos\phi_i, -\omega\sin\phi_i) \), then Eq.(3.22) is rewritten as

\[
0 = \sum_{i=1}^{n} \left[ F_i + A\omega^2 \cos\phi_i \right] \cdot \left[ \frac{\partial F}{\partial y}(-\sin\phi_i) + 2A\omega \cos\phi_i \right]
\]

\[
= \sum_{i=1}^{n} \left\{ (-\sin\phi_i)F_i \frac{\partial F}{\partial y} + 2A\omega \cos\phi_i F_i - A^2 \omega^2 \sin\phi_i \cos\phi_i \frac{\partial F}{\partial y} + 2A^2 \omega^3 \cos^2\phi_i \right\}
\]

(3.23)

Note here that

\[
\frac{\partial F}{\partial y} = \frac{\partial F}{\partial y} \bigg|_{(x = \cos\phi_i, \ y = -\omega\sin\phi_i)}
\]

Eq. (3.23) leads to a polynomial in \( \omega \), i.e.

\[
C_3 \omega^3 + C_2 \omega^2 + C_1 \omega + C_0 = 0
\]

(3.24)

in which the coefficients are given by
\[ C_0 = - \sum_{i=1}^{n} (Asin\phi_i)F_i \frac{\partial F}{\partial y} \] (3.25)

\[ C_1 = 2A \sum_{i=1}^{n} F_i \cos\phi_i \] (3.26)

\[ C_2 = -A^2 \sum_{i=1}^{n} \sin\phi_i \cos\phi_i \frac{\partial F}{\partial y} \] (3.27)

\[ C_3 = 2A^2 \sum_{i=1}^{n} \cos^2\phi_i \] (3.28)

Since iterative computation is applied in solving Eq. (3.24), an initial approximation of \( \omega \) is required. The initial approximation of \( \omega \) can be easily obtained by setting the nonlinear term in the governing equation to be zero. Therefore, the frequency of the nonlinear oscillator is approximated by solving Eq. (3.24). Moreover, in the case of no damping effects, Eq. (3.24) can be further simplified. Without damping, we have \( \frac{\partial F}{\partial y} = 0 \), so

\[ F = F(x) \]

and

\[ F_i = F(A\cos\phi_i) \]

Thus both \( C_0 \) and \( C_2 \) are equal to zero in Eqs. (3.25) and (3.27). Rewriting Eq. (3.24), we have

\[ 0 = \left(2A^2 \sum_{i=1}^{n} \cos^2\phi_i\right)\omega^3 + \left\{2A \sum_{i=1}^{n} F(A\cos\phi_i)\cos\phi_i\right\}\omega \] (3.29)

which gives

\[ \omega^2 = \frac{- \sum_{i=1}^{n} F(A\cos\phi_i)\cos\phi_i}{A \sum_{i=1}^{n} \cos^2\phi_i} \] (3.30)

as the non-trivial solution. In an effort to obtain a meaningful result, we discard \( \omega = 0 \) from the solution set of Eq. (3.29) and adapt Eq. (3.30) as the approximate solution.
expression for the frequency of the nonlinear oscillator. The computer programs which are used to solve Eqs. (3.24) and (3.30) are listed in Appendix B.

III Method of Optimal Equivalent Linearization for Multiple-Degree-of-Freedom Systems.

In contrast with a single-degree-of-freedom (SDOF) system, which has only one natural frequency and a single mode of motion, a linear multiple (N)-degree-of-freedom (MDOF) system has N natural frequencies and N corresponding modes. For a MDOF system, the equations of motion can be formulated through matrix techniques, such as stiffness matrix or flexibility matrix. Hence, with respect to a linear MDOF system, eigenvalues and eigenvectors which are derived from these matrices provide the solution for the natural frequencies and normal mode shapes of the system [9]. For a nonlinear MDOF system, since superposition principles do not hold, it becomes invalid to obtain the solution from the eigenvalues and eigenvectors of the system due to the existence of nonlinear terms. Therefore, it is generally much more complicated and tedious to develop the solution of a nonlinear MDOF system rather than a linear MDOF system. In fact, the lack of a solution in terms of elementary functions is the rule rather than the exception.

Nevertheless, it is possible to approximate the solution of a nonlinear MDOF system by replacing it with an equivalent linear MDOF system. Therefore, we develop the Method of Optimal Equivalent Linearization for MDOF systems, in which, the approximate solution of a nonlinear MDOF system is acquired through an equivalent linear MDOF system. The
objective of the proposed method is to minimize the difference between the original nonlinear system and the equivalent linear system through the technique of least square minimization. Equivalent Linearization through least square minimization of a SDOF system has been effectively analyzed by Ivan and Patula [10]. Accordingly, we take the following nonlinear SDOF system to demonstrate this method.

Example 3.1 Nonlinear Pendulum.

The equation of motion for a nonlinear pendulum is given as

$$\frac{d^2x}{dt^2} + \omega_0^2 \sin x = 0$$  \hspace{1cm} (3.31)

and the proposed linear equation is of the form

$$\ddot{x} + kx = 0$$  \hspace{1cm} (3.32)

which possesses periodic solutions of the form

$$x = A \cos \omega t$$ \hspace{1cm} (3.33)

The difference between Eqs. (3.31) and (3.32) will be

$$\Delta = \omega_0^2 \sin x - kx$$ \hspace{1cm} (3.34)

Taking the square of this difference and implementing the assumed solution expressed by Eq. (3.33), we have

$$\Delta^2 = [\omega_0^2 \sin (A \cos \omega t) - kA \cos \omega t]^2$$ \hspace{1cm} (3.35)

The idea here is to minimize the error over an entire period. Applying the least square minimization procedure results in
\[ 0 = \int_0^{2\pi} \frac{\partial A^2}{\partial k} \, dt \]

or more explicitly

\[ 0 = \int_0^{2\pi} [\omega_0^2 \sin(A \cos \omega t) - kA \cos \omega t] \cos \omega t \, dt \quad (3.36) \]

From the integration in Eq. (3.36), we deduce the approximation of \( k \) as

\[ k = \omega_0^2 \left(1 - \frac{A^2}{8} + \frac{A^4}{192}\right) \quad (3.37) \]

If \( A \) is not too large, we ignore the higher-order term and substitute Eq. (3.37) into Eq. (3.32) to formulate the equivalent linear equation

\[ \ddot{x} + \omega_0^2 \left(1 - \frac{A^2}{8}\right)x = 0 \quad (3.38) \]

Comparing Eq. (3.38) to Eq. (2.12) of the Harmonic Balance Linearization, we have arrived at a familiar equation.

Similarly, based on the preceding methodology, we can extend the procedure to nonlinear MDOF systems. For the sake of simplicity, we will confine our study to undamped nonlinear MDOF systems. Detailed formulation of the proposed Optimal Linearization Method is developed as follows.

For an undamped nonlinear MDOF system, we consider a vector differential equation of motion of the form

\[ \ddot{\mathbf{X}} + \mathbf{KX} + \mathbf{f}(\mathbf{X}, \dot{\mathbf{X}}) = \mathbf{0} \quad (3.39) \]
with $\vec{X} = [x_i]$ (column vector of generalized coordinates, $i = 1, 2, \ldots, N$)

In Eq. (3.39), $\tilde{M}$, $\tilde{K}$ and $\tilde{F}$ represent the mass matrix, stiffness matrix and nonlinearity matrix, respectively. The vector $\vec{c}$ is in a sense a measure of the nonlinearities of the system. Thus, we further assume that $\tilde{F}(\vec{X}, \vec{0}) = \vec{0}$. For brevity, set $\vec{Y} = \tilde{M} \vec{X}$, then Eq. (3.39) can be transformed into a more convenient form as

$$\vec{Y} + K \vec{Y} + F(\vec{Y}, \vec{c}) = \vec{0}$$  \hspace{1cm} (3.40)

where $\vec{Y} = [y_i]$ (column vector of generalized coordinates)

$K = [k_{ij}]$ (stiffness matrix)

$F = [f_i]$ (matrix of nonlinear terms)

with $i, j = 1, 2, \ldots, N$.

Since Eq. (3.40) is nonlinear, the natural frequencies will depend on the corresponding amplitudes of motion. This is an inherent characteristic of all nonlinear systems possessing periodic solutions. Next, we assume an equation of an equivalent linear MDOF system, which is of the form

$$\vec{Y} + \hat{K} \vec{Y} = \vec{0}$$  \hspace{1cm} (3.41)

with $\hat{K} = [\hat{k}_{ij}]$ (Equivalent stiffness matrix). Since Eq. (3.41) is linear, we are able to uncouple the system by setting $y_i = Bi e^{i\omega t}$ (i is a complex number), thus arriving at the familiar equation

$$[\hat{K} - \omega^2 I][\vec{B}] = \vec{0}$$  \hspace{1cm} (3.42)

The linear system has $N$ eigenvalues $\omega_i^2$, $i = 1, 2, \ldots, N$ and corresponding $N$ eigenvectors $\vec{u_i}$, $i = 1, 2, \ldots, N$, which satisfy the equation.
\[ [\hat{K}] [\vec{u}_i] = \omega_i^2 [\vec{u}_i] \]  

(3.43)

\( \vec{u}_i \) are taken to be the normalized eigenvectors so that

\[ |\vec{u}_i| = 1. \]

with respect to the usual Euclidean norm. To uncouple Eq. (3.41), we apply the linear transformation

\[ [y_i] = [U] [q_i] \]

(3.44)

where the columns of the modal matrix \([U]\) consist of the unit eigenvectors, \(\vec{u}_i\). The \([q_i]\) are also referred to as normal coordinates [9]. Uncoupling Eq. (3.41), in terms of \(\vec{q}\), the individual equations are expressed as

\[ q_i + \omega_i^2 q_i = 0 \]

(3.45)

each with general solution

\[ q_i = A_i \cos \omega_i t + C_i \sin \omega_i t \]

(3.46)

From Eqs. (3.43) to (3.45), we obtain

\[
\begin{pmatrix}
\vec{y}_1 \\
\vec{y}_2 \\
\vdots \\
\vec{y}_n
\end{pmatrix} = [U]
\begin{pmatrix}
A_1 \cos \omega_1 t + C_1 \sin \omega_1 t \\
A_2 \cos \omega_2 t + C_2 \sin \omega_2 t \\
\vdots \\
A_n \cos \omega_n t + C_n \sin \omega_n t
\end{pmatrix}
\]

\[
= [\vec{B}]
\begin{pmatrix}
\cos \omega_1 t \\
\cos \omega_2 t \\
\vdots \\
\cos \omega_n t
\end{pmatrix}
\]

(3.47)
The final form on the right hand side of Eq. (3.47) is acceptable, since Eq. (3.45) is autonomous. Calculating with each mode separately, we obtain the intermediate result which is expressed in the form

\[ \vec{B}^{(j)} = A_j \begin{bmatrix} u_{1j} \\ u_{2j} \\ \vdots \\ u_{nj} \end{bmatrix} \]

\[ = A_j \vec{u}_j \quad (j = 1, 2, \ldots, N) \quad (3.48) \]

in which, \( j \) is assigned as the mode number and \( A_j \) is its amplitude.

Since the \( \vec{u}_i \) are normalized (unit) eigenvectors, the magnitude relationship is

\[ A_j^2 = A_j^2 |\vec{u}_j|^2 \]

\[ = |\vec{B}^{(j)}|^2 \quad (3.49) \]

where \( A_j \) is the amplitude of \( \cos \omega_j t \).

We proceed to minimize the difference between Eqs. (3.40) and (3.41). The difference is expressed as

\[ \vec{\Delta} = (K - \hat{K}) \vec{Y} + \vec{F}(\vec{Y}, \vec{\tau}) \quad (3.50) \]

which contains \( N \) difference terms, for each \( \Delta_i \). Explicitly,

\[ \Delta_i = (k_{i1} - \hat{k}_{i1})y_1 + (k_{i2} - \hat{k}_{i2})y_2 + \ldots + \]

\[ + (k_{in} - \hat{k}_{in})y_n + F_i(\vec{Y}, \vec{\tau}) \quad (3.51) \]
From Eq. (3.51), the total sum of squared differences is computed:

\[ \Delta^2 = \Delta_1^2 + \Delta_2^2 + \ldots + \Delta_n^2 \]  

(3.52)

where

\[ \Delta_1 = \Delta_1(\hat{k}_{i1}, \hat{k}_{i2}, \ldots, \hat{k}_{in}) \]

\[ \Delta_i = \Delta_i(\hat{k}_{i1}, \hat{k}_{i2}, \ldots, \hat{k}_{in}) \]

\[ \vdots \]

\[ \Delta_n = \Delta_n(\hat{k}_{n1}, \hat{k}_{n2}, \ldots, \hat{k}_{nn}) \]

As before, the difference is to be minimized over the entire period, T. Applying the mean square minimization, we need

\[ 0 = \int_0^T \frac{\partial \Delta^2}{\partial k_{ij}} dt \]

which reduces to

\[ 0 = \int_0^T \Delta_i y_{ij} dt \quad (i, j = 1, 2, \ldots, N) \]  

(3.53)

From Eq. (3.47), setting \( \vec{y} = \vec{B} \cos \omega t \) and using a Fourier series to expand \( \vec{F}(\vec{y}, \vec{c}) \), we transform Eq. (3.53) into a more acceptable form. For convenience, the mode number \( (j) \) is dropped to allow the derivation of a general frequency-amplitude relation.

For each difference \( \Delta_i \),

\[ 0 = \int_0^T \left[ (k_{i1} - \hat{k}_{i1})B_1 \cos^2 \omega t + (k_{i2} - \hat{k}_{i2})B_2 \cos^2 \omega t + (k_{in} - \hat{k}_{in})B_n \cos^2 \omega t + F_i(B \cos \omega t, \vec{c}) \cos \omega t \right] dt \]  

(3.54)
or

\[
0 = (k_{i1} - \hat{k}_{i1})B_1 \frac{T}{2} + (k_{i2} - \hat{k}_{i2})B_2 \frac{T}{2} + (k_{in} - \\
\hat{k}_{in})B_n \frac{T}{2} + \int_0^T F_i(B \cos \omega t, \tilde{\tau}) \cos \omega t dt
\]  

(3.55)

Thus, for each \( i = 1, 2, \ldots, N \).

\[
0 = \sum_{i=1}^n (k_{ii} - \hat{k}_{ii})B_i + \frac{2}{T} \int_0^T F(B \cos \omega t, \tilde{\tau}) \cos \omega t dt
\]  

(3.56)

From Eq. (3.42), we rewrite Eq. (3.56) as

\[
0 = \sum_{i=1}^n k_{ii}B_i - \omega^2 B_i + \frac{2}{T} \int_0^T F(B \cos \omega t, \tilde{\tau}) \cos \omega t dt
\]  

(3.57)

Expressing Eq. (3.57) in vector form,

\[
\vec{\tau} = [K]B - \omega^2 B + \frac{2}{T} \int_0^T F(B \cos \omega t, \tilde{\tau}) \cos \omega t dt
\]  

(3.58)

which is subject to the constraint from Eq. (3.49), that is

\[
A^2 = B_1^2 + B_2^2 + \ldots + B_n^2
\]  

(3.59)

Recall that \( A \) is now the assumed amplitude of a particular mode. Therefore, if \( A \) is given, we can construct \( N + 1 \) equations which provide the approximation of the equivalent linear MDOF system. Moreover, since Eqs. (3.58) and (3.59) are nonlinear equations, they must be solved iteratively. Thus, initial approximations are required for the computation. By setting \( \tilde{\tau} = 0 \) in Eq. (3.58), we obtain

\[
[K]B = \omega^2 B
\]  

(3.60)

Hence appropriate initial approximations will be given from Eq. (3.60), which are the eigenvalues \( \lambda_j = \omega_j^2 \) and the corresponding eigenvectors \( \vec{B}^{(j)} \) for the linear system given
by \( \bar{\tau} = \bar{0} \). After solving Eqs. (3.58) and (3.59) for the approximate frequencies \( \omega_j \), the equivalent stiffness matrix \([\bar{k}]\) can also be acquired from the vectors \( \bar{B}^{(j)} \). The normalized eigenvectors are given by

\[
\bar{u}_j = \frac{1}{\bar{A}} \bar{B}^{(j)}
\]

and the modal matrix of Eq. (3.40) is constructed as

\[
[U] = [\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n]
\]

Thus, given the modal matrix, \([U]\), the linearized stiffness matrix \([\bar{K}]\) is determined by the orthogonality relationship [11], which is

\[
[\bar{K}] = [U] \begin{bmatrix}
\omega_1^2 & 0 & \cdots & 0 \\
0 & \omega_2^2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \omega_n^2
\end{bmatrix} [U]^{-1}
\]

As a result, using the above methodology, we are able to avoid the complexity associated with obtaining general solutions of nonlinear MDOF systems. In addition, an equivalent linear MDOF system is also established for further reference and analysis.
IV Results and Recommendations

In this section, proposed methods of optimal equivalent linearization for SDOF systems are demonstrated through previously discussed physical examples. With respect to MDOF systems, nonlinear spring-mass systems are utilized to illustrate the proposed method. Furthermore, we will compare these results to the solutions obtained from conventional linearization methods. Both the computer programs and output results which are needed to analyze these examples are given in the Appendices.

I Method of Optimal Ellipse

In order to implement this method, it is necessary to provide the coordinates of discrete points on the phase plane. Instead of securing them from experimental results, the coordinates of points are simulated and obtained by the computer program PHASE in Appendix A, which implements the Runge-Kutta algorithm [12]. In addition, the program PHASE also computes the time series for graphically approximating the frequency. Also in Appendix A, a computer program OPTELP based on Newton’s Method [12] is used to solve Eqs. (3.12), (3.13) and (3.14), which yields values for “best-fit” semi-axes and solutions of the system. To initially examine the validity of this proposed method, the following example of a linear oscillator is illustrated.

Example 4.1 Harmonic Oscillator.

From previous sections, the differential equation that governs a harmonic oscillator is given as
\[ \ddot{x} + \omega_0^2 x = 0 \]  \hspace{1cm} (3.2)

where \( \omega_0 \) clearly represents the natural circular frequency of the system. Moreover, the energy equation that defines the trajectories on the phase plane is in the form

\[ \dot{x}^2 + \omega_0^2 x^2 = 2C \]  \hspace{1cm} (3.4)

where \( C \) represents the energy level of the system. Thus the discrete data points on a phase plane can be obtained by either numerical integration of Eq. (3.2) (program PHASE) or the implicit equation of the phase curves Eq. (3.4). Figure 4.1 shows the discrete data points on the phase plane of a harmonic oscillator.

![Phase plane plot](image)

Figure 4.1 Simulated points on the phase plane of a harmonic oscillator (Amplitude = 2.0, \( \omega_0 = 0.7071 \)).
Using these data points as input, the computer program OPTELP is applied to approximate the frequency of the harmonic oscillator. The results of approximation from the program OPTELP for different amplitudes and frequencies are shown in Table 4.1.

Table 4.1 Exact and approximate frequencies of harmonic oscillators.

<table>
<thead>
<tr>
<th>Amplitude</th>
<th>Exact Frequency</th>
<th>Optimal Ellipse</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\omega_0$</td>
<td>$A$</td>
</tr>
<tr>
<td>1.0</td>
<td>0.7071</td>
<td>1.0000</td>
</tr>
<tr>
<td>2.0</td>
<td>0.7071</td>
<td>2.0000</td>
</tr>
<tr>
<td>1.5</td>
<td>1.4142</td>
<td>1.5000</td>
</tr>
<tr>
<td>3.0</td>
<td>1.4142</td>
<td>3.0000</td>
</tr>
</tbody>
</table>

Table 4.1 demonstrates that highly accurate approximations are generated by the proposed method. Therefore, it is feasible to approximate the frequency of an anharmonic oscillator through the geometric nature of its phase portrait.

Incidentally, from the results of program OPTELP in Table 4.1, different amplitudes result in different semi-axes, but with the same value of the combination ($B/A$), which represents the frequency of the system. This demonstrates the well-known fact that the natural frequency of a linear oscillator is independent of its amplitude. However, this independence of amplitude and frequency is generally not true for a nonlinear oscillator. The relationship between frequency and amplitude is a primary distinction between linear and nonlinear oscillators.
With respect to nonlinear oscillators, application of this proposed method in analyzing previous examples is illustrated as follows.

Example 4.2 Nonlinear Pendulum.

To facilitate the computation, we let $\omega_0 = 1.0$ as in Eq. (1.9). By using the program PHASE with a step size 0.1 and an amplitude of 1.0, 200 points are generated and shown in Figure 4.2. Meanwhile, the time series for $x(t)$ is also given in Figure 4.3, from which the frequency of the system is approximated as 0.952. Utilizing program OPTELP, the semi-axes are generated and results in the approximate frequency of the system as 0.9586. The iteration history and the solutions generated from the program are also given in Appendix A. From the Method of Harmonic Balance, the frequency of the system is obtained by Eq. (2.11) to be 0.9354. Comparing the preceding two approximations, there is only about a 2% difference between them, thus a favorable result is acquired by the proposed method. The comparison of solutions generated from the Harmonic Balance, the program PHASE and the program OPTELP for various amplitudes is listed in Table 4.2.

<table>
<thead>
<tr>
<th>Amplitude</th>
<th>Frequency (Runge-Kutta)</th>
<th>Frequency (Harmonic Balance)</th>
<th>Frequency (Optimal Ellipse)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.9973</td>
<td>0.9843</td>
<td>0.9896</td>
</tr>
<tr>
<td>1.0</td>
<td>0.9520</td>
<td>0.9354</td>
<td>0.9586</td>
</tr>
<tr>
<td>1.5</td>
<td>0.8607</td>
<td>0.8478</td>
<td>0.9073</td>
</tr>
<tr>
<td>2.0</td>
<td>0.7570</td>
<td>0.7071</td>
<td>0.8373</td>
</tr>
<tr>
<td>2.5</td>
<td>0.6100</td>
<td>0.4677</td>
<td>0.7470</td>
</tr>
</tbody>
</table>
Figure 4.2 Simulated points on the phase plane of a nonlinear pendulum ($A = 1.0$).

Figure 4.3 Time series for $x(t)$ of a nonlinear pendulum ($A = 1.0$).
As shown in Table 4.2, when the amplitude is small, the Method of Optimal Ellipse has resulted in better accuracy than the Method of Harmonic Balance. However it can be seen that when the amplitude is rather large, a significant error is generated for both methods. Thus, the same limitation is imposed to both methods in this example, which is, the amplitude can not be too large. However, the Optimal Ellipse Method has a significant advantage, in that it can be applied to experimental data, for which a governing equation may not be available.

Example 4.3 Oscillation Induced by a Nonlinear Restoring Spring.

Equation of motion for this oscillator is given in Eq. (1.14). Applying same procedures as Example 4.1, 200 points are generated by a step size 0.1 and an amplitude 1.0, which are plotted on the phase plane as in Figure 4.4. Figure 4.5 shows the time series plot which yields the approximate frequency to be 1.1220. Again, the frequency obtained from the program OPTELP is 1.2948, while the approximate frequency from Eq. (2.23) using Harmonic Balance, is approximated as 1.1284. The output for this Example generated from program OPTELP is also shown in Appendix A. Table 4.3 lists solutions computed from the discussed approaches for various amplitudes.

Examining Table 4.3, we find a significant error existing in the Optimal Ellipse Method. Observing the points on the phase plane in Figure 4.4, two sharp corners are found on top and bottom of the closed phase curve if we connect these points. The sharp corners cause large deviation of accuracy in deriving the optimal ellipse. These corners are a result of the discontinuity in the forcing function.
Figure 4.4 Simulated points on the phase plane of a nonlinear restoring spring ($A = 1.0$).

Figure 4.5 Time series for $x(t)$ of a nonlinear restoring spring ($A = 1.0$).
Table 4.3 Approximate solutions of a nonlinear restoring spring oscillator for various amplitudes.

<table>
<thead>
<tr>
<th>Amplitude</th>
<th>Frequency (Runge-Kutta)</th>
<th>Frequency (Harmonic Balance)</th>
<th>Frequency (Optimal Ellipse)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>1.6111</td>
<td>1.5958</td>
<td>1.9144</td>
</tr>
<tr>
<td>1.0</td>
<td>1.1220</td>
<td>1.1284</td>
<td>1.2948</td>
</tr>
<tr>
<td>1.5</td>
<td>0.8976</td>
<td>0.9213</td>
<td>1.0575</td>
</tr>
<tr>
<td>2.0</td>
<td>0.7854</td>
<td>0.7979</td>
<td>0.9040</td>
</tr>
<tr>
<td>2.5</td>
<td>0.6943</td>
<td>0.7136</td>
<td>0.8028</td>
</tr>
</tbody>
</table>

Example 4.4 Van der Pol Oscillator.

As verified in a previous chapter, periodic oscillation occurs only when the amplitude is approximately 2.0 for this unique oscillator. Hence, from Eq. (1.21), we use a step size 0.1 and an amplitude 2.0 for creating the points on phase plane, which are shown in Figure 4.6. Once again, the numerically generated response is illustrated in Figure 4.7, which indicates an approximate frequency of 1.0. Computation from the computer program yields the approximate frequency of 1.0087, which is consistent with the solution of Eq. (2.26) using the Harmonic Balance Method. The output results for this Example are given in Appendix A. With respect to various values of the nonlinearity coefficient (μ) in Eq. (1.21), we obtain the approximations and present comparisons in Table 4.4.
Figure 4.6 Simulated points on the phase plane of a Van der Pol Oscillator ($\mu = 0.2$).

Figure 4.7 Time series for $x(t)$ of a Van der Pol Oscillator ($\mu = 0.2$).
Table 4.4 Approximate solutions of the Van der Pol Oscillator with various nonlinearity coefficients ($\mu$).

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Frequency</th>
<th>Frequency</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>(Runge-Kutta)</td>
<td>(Harmonic Balance)</td>
<td>(Optimal Ellipse)</td>
</tr>
<tr>
<td>0.1</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0018</td>
</tr>
<tr>
<td>0.2</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0087</td>
</tr>
<tr>
<td>0.3</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0208</td>
</tr>
<tr>
<td>0.4</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0379</td>
</tr>
<tr>
<td>0.5</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0596</td>
</tr>
</tbody>
</table>

In summary, from the preceding three examples, it is evident that a more accurate solution requires the contour of a closed phase curve constituted by the data points to have more resemblance to an ellipse or a circle. In Example 4.2, when $x$ equals to $\pm \pi$, which represents two unstable equilibrium points as points A and B (Figure 1.4) whereon sharp corners are formed on the phase plane. Hence, when $x$ is close to $\pm \pi$, the more inaccurate approximation is generated as in Table 4.2. Further, in Example 4.3, since sharp corners are existing in all the phase curves as shown in Figure 1.6, unfavorable approximations are generated.

With respect to Example 4.4, higher nonlinearity coefficients can cause certain quasi-corners on the phase curve and thus generate errors in the curve fitting procedure. These can be seen from Figures 4.6 and 4.8, in which the values of $\mu$ are 0.2 and 0.5, respectively. Consequently, inspecting the contour of a phase curve is recommended before applying the Method of Optimal Ellipse. A possible rectification of this problem is to fit curves with appropriate large curvatures near these singularity points.
II. Method of Optimal Vector Field.

In the previous chapter, we have shown that two equations, Eqs. (3.24) and (3.30), can be used to approximate the dynamical characteristics of a weakly nonlinear system by the Method of Optimal Vector Field. Again, previous examples are utilized to demonstrate the proposed method. Concerning an undamped nonlinear system in previous Examples 4.2 and 4.3, a computer program OPTVF in Appendix B is used to solve Eq. (3.30). Moreover, for a damped nonlinear system, the computer program OPTVFD in Appendix B is applied to solve Eq. (3.24). Additionally, the generated solutions will be compared to the results obtained by Runge-Kutta numerical integration and the Harmonic Balance Method.
Example 4.5 Nonlinear Pendulum.

The equation of motion of a nonlinear pendulum is given by Eq. (1.9). Since this is an undamped system, we initially need to define the nonlinear part $F(x)$ of the equation in order to use Eq. (3.30) for obtaining the solution. From Eqs. (1.9) and (3.16), as well as letting $\omega_0 = 1.0$, we obtain

$$F(x) = -\sin x$$  \hspace{1cm} (4.1)

and thus

$$F(A\cos\phi_i) = -[\sin(A\cos\phi_i)]$$  \hspace{1cm} (4.2)

From Eq. (3.30), we have

$$\omega^2 = \frac{\sum_{i=1}^{n} \cos\phi_i [\sin(A\cos\phi_i)]}{A \sum_{i=1}^{n} \cos^2\phi_i}$$  \hspace{1cm} (4.3)

Setting the number of points equal to 200 and the amplitude equal to 1.0, we use program OPTVF to solve Eq. (4.3). As a result, the frequency is approximated as 0.9381. The output from program OPTVF is given in Appendix B. In addition, the solutions and comparison of different methods with various amplitudes are given in Table 4.5.

Examining Table 4.5, we attain quite favorable approximations from the Method of Optimal Vector Field. Table 4.5 also shows that for this example the proposed method is more accurate than the Harmonic Balance Method, particularly when the amplitude is not small.
Table 4.5 Approximate solutions of a nonlinear pendulum for various amplitudes.

<table>
<thead>
<tr>
<th>Amplitude</th>
<th>Frequency (Runge-Kutta)</th>
<th>Frequency (Harmonic Balance)</th>
<th>Frequency (Optimal V. Field)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.9973</td>
<td>0.9843</td>
<td>0.9844</td>
</tr>
<tr>
<td>1.0</td>
<td>0.9520</td>
<td>0.9354</td>
<td>0.9381</td>
</tr>
<tr>
<td>1.5</td>
<td>0.8607</td>
<td>0.8478</td>
<td>0.8625</td>
</tr>
<tr>
<td>2.0</td>
<td>0.7570</td>
<td>0.7071</td>
<td>0.7594</td>
</tr>
<tr>
<td>2.5</td>
<td>0.6100</td>
<td>0.4677</td>
<td>0.6306</td>
</tr>
</tbody>
</table>

Example 4.6 Oscillation Induced by a Nonlinear Restoring Spring.

For the case of no damping, we apply the same procedures as in Example 4.5. Recall Eq. (1.14), in which we have

\[ F(x) = -\text{sgn}(x) \quad (4.4) \]

and so

\[ F(A\cos\phi_i) = -\text{sgn}(A\cos\phi_i) \]

\[ = -\text{sgn}(\cos\phi_i) \quad (4.5) \]

By Eq. (3.30),

\[ \omega^2 = \frac{\sum_{i=1}^{n} \cos\phi_i \text{sgn}(\cos\phi_i)}{A \sum_{i=1}^{n} \cos^2\phi_i} \]

\[ = \frac{\sum_{i=1}^{n} |\cos\phi_i|}{A \sum_{i=1}^{n} \cos^2\phi_i} \quad (4.6) \]
In Eq. (4.6), we set the number of points equal to 200 and the amplitude equal to 1.0. The frequency (Ω) is approximated as 1.1283 by the program OPTVF. Results from program OPTVF are shown in Appendix B. Table 4.6 shows solutions for various amplitudes approximated by several methods. Comparing the results in Table 4.6, solutions from the proposed method are almost identical to those generated by the Method of Harmonic Balance.

Table 4.6 Approximate solutions of a nonlinear restoring spring oscillator for various amplitudes.

<table>
<thead>
<tr>
<th>Amplitude</th>
<th>Frequency (Runge-Kutta)</th>
<th>Frequency (Harmonic Balance)</th>
<th>Frequency (Optimal V. Field)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>1.6110</td>
<td>1.5958</td>
<td>1.5957</td>
</tr>
<tr>
<td>1.0</td>
<td>1.1855</td>
<td>1.1284</td>
<td>1.1283</td>
</tr>
<tr>
<td>1.5</td>
<td>0.9240</td>
<td>0.9213</td>
<td>0.9213</td>
</tr>
<tr>
<td>2.0</td>
<td>0.8160</td>
<td>0.7979</td>
<td>0.7979</td>
</tr>
<tr>
<td>2.5</td>
<td>0.7060</td>
<td>0.7136</td>
<td>0.7136</td>
</tr>
</tbody>
</table>

Example 4.7 Van der Pol Oscillator.

The equation of motion for this oscillator is given by Eq. (1.21). Since this is a damped system, we need to implement Eqs. (3.24) to (3.28) and use computer program OPTVFD to approximate the solution of the system. The computation process is as follows:

\[
\begin{align*}
F_i &= F(x_i, y_i) \\
&= -A\cos\phi_i + \mu(A^2\cos^2\phi_i - 1)A\omega\sin\phi_i \\
\frac{\partial F}{\partial y_i} &= -\mu(A^2\cos^2\phi_i - 1)
\end{align*}
\]
Using Eqs. (4.7) and (4.8) to acquire each coefficient of the polynomial (3.24), we have

from Eq. (3.25),

\[
C_0 = - \sum_{i=1}^{n} (A \sin \phi_i) \frac{\partial F_i}{\partial y_i} \\
= - \sum_{i=1}^{n} (A \sin \phi_i) [\cos \phi_i + \mu (A^2 \cos^2 \phi_i - 1) A \omega \sin \phi_i] \\
[-\mu (A^2 \cos^2 \phi_i - 1)] \\
= - \sum_{i=1}^{n} (\mu A^4 \sin \phi_i \cos^3 \phi_i - \mu A^2 \sin \phi_i \cos^2 \phi_i) + \omega \sum_{i=1}^{n} (\mu^2 A^6 \cos^4 \phi_i \sin^2 \phi_i \\
- 2\mu A^4 \cos^2 \phi_i \sin^2 \phi_i + \mu^2 A^2 \sin^2 \phi_i) \\
(4.9)
\]

From Eq. (3.26),

\[
C_1 = 2A \sum_{i=1}^{n} \cos \phi_i F_i \\
= 2A \sum_{i=1}^{n} \cos \phi_i [\cos \phi_i + \mu (A^2 \cos^2 \phi_i - 1) A \omega \sin \phi_i] \\
= \sum_{i=1}^{n} (-2A^2 \cos^2 \phi_i) + \omega \sum_{i=1}^{n} (2\mu A^4 \cos^3 \phi_i \sin \phi_i - \\
2\mu A^2 \cos \phi_i \sin \phi_i) \\
(4.10)
\]

From Eq. (3.27),

\[
C_2 = -A^2 \sum_{i=1}^{n} \sin \phi_i \cos \phi_i \frac{\partial F_i}{\partial y_i} \\
= -A^2 \sum_{i=1}^{n} \sin \phi_i \cos \phi_i [\mu (A^2 \cos^2 \phi_i - 1)] \\
(4.11)
\]
From Eq. (3.28),

\[ C_3 = 2A^2 \sum_{i=1}^{n} \cos^2 \phi_i \]  \hspace{1cm} (4.12)

By substituting Eqs. (4.9) to (4.12) into Eq. (3.24), we obtain a polynomial of the form

\[ C_3' \omega^3 + C_2' \omega^2 + C_1' \omega + C_0' = 0 \]  \hspace{1cm} (4.13)

for which

\[ C_0' = -\mu \sum_{i=1}^{n} \sin \phi_i \cos \phi_i (A^2 \cos^2 \phi_i - 1) \]  \hspace{1cm} (4.14)

\[ C_1' = \mu^2 \sum_{i=1}^{n} \sin^2 \phi_i (A^4 \cos^4 \phi_i - 2A^2 \cos^2 \phi_i + 1) 
\] 
\[ - 2 \sum_{i=1}^{n} \cos^2 \phi_i \]  \hspace{1cm} (4.15)

\[ C_2' = 3\mu \sum_{i=1}^{n} \sin \phi_i \cos \phi_i (A^2 \cos^2 \phi_i - 1) \]  \hspace{1cm} (4.16)

\[ C_3' = 2 \sum_{i=1}^{n} \cos^2 \phi_i \]  \hspace{1cm} (4.17)

In computer program OPTVFD, we use a subroutine VANDP to calculate values of the coefficients (4.14) to (4.17) and then apply Newton's Method [12] in the subroutine NEWTON2 to solve the polynomial (4.13). This yields the solution as shown in Appendix B. Solutions of the system are shown in Table 4.7 for different nonlinearity coefficients. In Table 4.7, larger nonlinearity coefficients tend to produce higher inaccuracy of solutions, which results from calculation errors and the fact that the limit cycle becomes highly distorted.
Table 4.7 Approximate solutions of the Van der Pol Oscillator with various nonlinearity coefficients ($\mu$).

<table>
<thead>
<tr>
<th>Coefficient $\mu$</th>
<th>Frequency (Runge-Kutta)</th>
<th>Frequency (Harmonic Balance)</th>
<th>Frequency (Optimal V. Field)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.9975</td>
</tr>
<tr>
<td>0.2</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.9900</td>
</tr>
<tr>
<td>0.3</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.9772</td>
</tr>
<tr>
<td>0.4</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.9592</td>
</tr>
<tr>
<td>0.5</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.9354</td>
</tr>
</tbody>
</table>

III. Method of Optimal Equivalent Linearization for Undamped MDOF Systems.

Investigations of nonlinear MDOF systems have generally focused on models with cubic spring elements. Restoring forces involving a cubic term are typically used to model elastic systems undergoing large deformations. The assumption of small deformation allows the approximation of the restoring force by a linear function. As displacements get large, higher order terms must be added to the restoring force in order to approximate the nonlinear action of a spring. A spring is said to become "harder" or "softer", depending on the sign of the nonlinear term [7]. Figure 4.9 shows a plot of the restoring force versus displacement of a nonlinear spring element. Huang [13] and Hovanessian [14] considered cubic springs to model nonlinear two and three degree of freedom systems, respectively. Moochhala [15] also used cubic springs to explore the free vibration of nonlinear MDOF systems.
Accordingly, we will utilize cubic spring systems to demonstrate the proposed method with respect to two and three degree of freedom systems.

\[ kx + \xi x^3 \]

**Figure 4.9** Restoring force versus displacement of a nonlinear spring element.

Example 4.8 An Undamped Two-degree-of-Freedom System.

To examine the validity of the proposed method, detailed derivation of the generalized Eqs. (3.58) and (3.59) is also demonstrated in this example. We consider the undamped MDOF system depicted in Figure 4.10.

**Figure 4.10** An undamped two-degree-of-freedom system.
For the sake of convenience, we set the mass (m) as a unit mass. Thus, the differential equations of motion for the system become

\[ \ddot{y}_1 + 2ky_1 - ky_2 + cy_1^3 = 0 \]  
\[ \ddot{y}_2 - ky_1 + ky_2 = 0 \]  

From Eq. (3.41), we assume an equivalent linear MDOF system in the form

\[ \ddot{y}_1 + \hat{k}_{11}y_1 + \hat{k}_{12}y_2 = 0 \]  
\[ \ddot{y}_2 + \hat{k}_{21}y_1 + \hat{k}_{22}y_2 = 0 \]  

or in matrix form

\[
\begin{bmatrix}
\ddot{y}_1 \\
\ddot{y}_2
\end{bmatrix} +
\begin{bmatrix}
\hat{k}_{11} & \hat{k}_{12} \\
\hat{k}_{21} & \hat{k}_{22}
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

Since Eq. (4.20) is linear, we can uncouple the system by setting

\[ y_1 = B_1 e^{i\omega t} \]  
\[ y_2 = B_2 e^{i\omega t} \]

which yields the familiar vector equation

\[
\begin{bmatrix}
\hat{k}_{11} - \omega^2 & \hat{k}_{12} \\
\hat{k}_{21} & \hat{k}_{22} - \omega^2
\end{bmatrix}
\begin{bmatrix}
B_1 \\
B_2
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

Therefore, the linear system has two eigenvalues, \( \omega_1^2 \) and \( \omega_2^2 \),
along with two corresponding eigenvectors given by

\[ \vec{u}_1 = \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} \]  \hspace{1cm} (4.24)

and

\[ \vec{u}_2 = \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix} \]  \hspace{1cm} (4.25)

which will satisfy the equations

\[
\begin{bmatrix}
\hat{k}_{11} & \hat{k}_{12} \\
\hat{k}_{21} & \hat{k}_{22}
\end{bmatrix}
\begin{bmatrix}
u_{11} \\
u_{21}
\end{bmatrix}
= \omega_1^2
\begin{bmatrix}
u_{11} \\
u_{21}
\end{bmatrix}
\]  \hspace{1cm} (4.26)

and

\[
\begin{bmatrix}
\hat{k}_{11} & \hat{k}_{12} \\
\hat{k}_{21} & \hat{k}_{22}
\end{bmatrix}
\begin{bmatrix}
u_{12} \\
u_{22}
\end{bmatrix}
= \omega_2^2
\begin{bmatrix}
u_{12} \\
u_{22}
\end{bmatrix}
\]  \hspace{1cm} (4.27)

Since we assume normalized eigenvectors, we have

\[ u_{11}^2 + u_{21}^2 = 1 \]  \hspace{1cm} (4.28)

\[ u_{12}^2 + u_{22}^2 = 1 \]  \hspace{1cm} (4.29)

To uncouple Eq. (4.20), we apply the linear transformation

\[
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix}
= \begin{bmatrix}
u_{11} & u_{12} \\
u_{21} & u_{22}
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2
\end{bmatrix}
\]  \hspace{1cm} (4.30)
where $q_1$ and $q_2$ are known as normal coordinates. Thus from Eqs. (4.26), (4.27) and (4.30), we uncouple Eq. (4.20) as

$$q_1 + \omega_1^2 q_1 = 0 \hspace{1cm} (4.31)$$

and

$$q_2 + \omega_2^2 q_2 = 0 \hspace{1cm} (4.32)$$

with

$$q_1 = A_1 \cos \omega_1 t + C_1 \sin \omega_1 t \hspace{1cm} (4.33)$$

and

$$q_2 = A_2 \cos \omega_2 t + C_2 \sin \omega_2 t \hspace{1cm} (4.34)$$

Substitute Eqs. (4.33) and (4.34) into Eq. (4.30) and consider each mode separately, then, for the first mode ($A_1 = 1.0; A_2, C_1, C_2 = 0$)

$$y_1 = u_{11} A_1 \cos \omega_1 t$$
$$= B_1 \cos \omega_1 t \hspace{1cm} (4.35)$$

$$y_2 = u_{21} A_1 \cos \omega_1 t$$
$$= B_2 \cos \omega_1 t \hspace{1cm} (4.36)$$

where from Eq. (4.28)

$$B_1^2 + B_2^2 = A_1^2 \hspace{1cm} (4.37)$$

Therefore, substituting Eqs. (4.35) and (4.36) into Eq. (4.26), we have

$$\hat{k}_{11} B_1 + \hat{k}_{12} B_2 = \omega_1^2 B_1 \hspace{1cm} (4.38)$$

$$\hat{k}_{21} B_1 + \hat{k}_{22} B_2 = \omega_1^2 B_2$$
For convenience, we still use $B_1$ and $B_2$ to represent the modal eigenvectors for the second mode. Thus,

for the second mode ($A_2 = 1.0; A_1, C_1, C_2 = 0$)

$$y_1 = u_{12}A_2 \cos \omega_2 t$$
$$= B_1 \cos \omega_2 t \quad (4.39)$$

$$y_2 = u_{22}A_2 \cos \omega_2 t$$
$$= B_2 \cos \omega_2 t \quad (4.40)$$

and from Eq. (4.29)

$$B_1^2 + B_2^2 = A_2^2 \quad (4.41)$$

Substituting Eqs. (4.39) and Eq. (4.40) into Eq. (4.27), we obtain

$$\hat{k}_{11}B_1 + \hat{k}_{12}B_2 = \omega_2^2 B_1 \quad (4.42)$$

$$\hat{k}_{21}B_1 + \hat{k}_{22}B_2 = \omega_2^2 B_2$$

With respect to each mode, we proceed to minimize the difference between Eqs. (4.18) and (4.19). For the first mode:

$$\Delta^2 = \Delta_1^2 + \Delta_2^2$$

$$= \left[(2k - \hat{k}_{11})y_1 - (k + \hat{k}_{12})y_2 + cy_1^3\right]^2$$

$$+ \left[-(k - \hat{k}_{21})y_1 + (k - \hat{k}_{22})y_2\right]^2 \quad (4.43)$$

where $\Delta_1 = \Delta(\hat{k}_{11}, \hat{k}_{12})$

$$\Delta_2 = \Delta(\hat{k}_{21}, \hat{k}_{22})$$
Applying the mean square minimization to Eq. (4.43) over the entire period, $T$, we have for each unknown

\[ 0 = \int_0^T \frac{\partial \Delta^2}{\partial k_{11}} \, dt \]

\[ 0 = \int_0^T \Delta_1 y_1 dt \]

\[ = \int_0^T [(2k - \hat{k}_{11}) y_1 - (k + \hat{k}_{12}) y_2 + \epsilon y_1^3] y_1 dt \quad (4.44) \]

\[ 0 = \int_0^T \frac{\partial \Delta^2}{\partial k_{12}} \, dt \]

\[ 0 = \int_0^T \Delta_1 y_2 dt \]

\[ = \int_0^T [(2k - \hat{k}_{11}) y_1 - (k + \hat{k}_{12}) y_2 + \epsilon y_1^3] y_2 dt \quad (4.45) \]

\[ 0 = \int_0^T \frac{\partial \Delta^2}{\partial k_{21}} \, dt \]

\[ 0 = \int_0^T \Delta_2 y_1 dt \]

\[ = \int_0^T [-(k + \hat{k}_{21}) y_1 + (k - \hat{k}_{22}) y_2] y_1 dt \quad (4.46) \]

and
\[ 0 = \int_0^T \frac{\partial \Delta^2}{\partial k_{22}} \, dt \]
\[ 0 = \int_0^T \Delta_2 y_2 \, dt \]
\[ = \int_0^T \left[ -(k + \dot{k}_{21}) y_1 + (k - \dot{k}_{22}) y_2 \right] y_2 \, dt \quad (4.47) \]

Recall that \( y_1 = B_1 \cos \omega_1 t \) and \( y_2 = B_2 \cos \omega_1 t \), thus Eqs. (4.44) and (4.45) can be simplified as
\[ 0 = \int_0^T \left\{ \left[ (2k - \dot{k}_{11}) B_1 \cos \omega_1 t - (k + \dot{k}_{12}) B_2 \cos \omega_1 t \right. \right. \]
\[ \left. \left. + eB_1^3 \cos^3 \omega_1 t \right] \cos \omega_1 t \right\} \, dt \]
\[ = (2k - \dot{k}_{11}) B_1 - (k + \dot{k}_{12}) B_2 + \frac{3}{4} e B_1^3 \quad (4.48) \]

Using Eq. (4.38)
\[ 0 = 2kB_1 - kB_2 - \omega_1^2 B_1 + \frac{3}{4} e B_1^3 \quad (4.49) \]

Similarly, Eqs. (4.46) and (4.47) are simplified as
\[ 0 = \int_0^T \left\{ \left[ -(k + \dot{k}_{21}) B_1 \cos \omega_1 t + (k - \dot{k}_{22}) B_2 \cos \omega_1 t \right] \cos \omega_1 t \right\} \, dt \]
\[ = -kB_1 + kB_2 - \omega_1^2 B_2 \quad (4.50) \]

In order to approximate the frequency of the first mode, Eqs. (4.49) and (4.50) are utilized as objective functions for optimization. In addition, the objective functions are subject to the constraint of Eq. (4.37). Thus, following the same procedure, we can derive the objective functions for the second mode, which will be of the same form as Eqs. (4.49) and (4.50). Eq. (4.41) will then be used as the
constraint for the optimization process.

However, the same objective functions can also be obtained directly from the generalized Eq. (3.58), which is (for each mode)

\[
\begin{bmatrix}
0 \\
0
\end{bmatrix} = \begin{bmatrix}
2k & -k \\
-k & k
\end{bmatrix} \begin{bmatrix}
B_1 \\
B_2
\end{bmatrix} - \varepsilon^2 \begin{bmatrix}
B_1 \\
B_2
\end{bmatrix}
\]

\[
+ \frac{2}{T} \int_0^T \cos \omega t \begin{bmatrix}
\epsilon B_1^3 \cos^3 \omega t \\
0
\end{bmatrix} dt
\]

(4.51)

where \( \omega \) generically represents the frequency of each mode. After integration of Eq. (4.51), we will attain the same result as Eqs. (4.49) and (4.50).

For brevity, Eqs. (4.49) and (4.50) are divided by \( k \), thus we summarize the equations for optimization as

\[
0 = 2B_1 - B_2 - \Omega^2 B_1 + \frac{3\epsilon B_1^3}{4k}
\]

(4.52)

\[
0 = -B_1 + B_2 - \Omega^2 B_2
\]

where \( \Omega^2 = \frac{\omega^2}{k} \)

Eq. (4.52) is subject to the constraints of Eqs. (4.37) and (4.41) with respect to each separate mode.

As previously mentioned, initial approximations of \( B_1 \), \( B_2 \) and \( \Omega \) are required for solving Eq. (4.52) by an iterative procedure. Therefore, the initial approximations of \( B_1 \), \( B_2 \) and \( \Omega \) are obtained by finding all eigenvectors and eigenvalues of the system defined by setting \( \epsilon = 0 \) in Eq.
The computer program OPTMDS is used to approximate the solution of constrained Eq. (4.52) and is listed in Appendix C. In program OPTMDS, subroutines of both International Mathematical Statistics Library (IMSL) [16] and Automated Design Synthesis (ADS) [17] are invoked to execute matrix operations and numerical optimization procedures [18], respectively. The results of program OPTMDS are also shown in Appendix C.

For instance, by setting $A = 4.0$ for each mode, $k = 5.0$ and $\varepsilon = 1.0$; the approximate solution of the preceding MDOF system is generated by the program OPTMDS as

$$\omega_1 = 1.5735; \quad \omega_2 = 4.7078$$

with

$$[B] = \begin{bmatrix} 1.7947 & -3.7940 \\ 3.5748 & 1.2673 \end{bmatrix}$$

where $[B]$ is the modal matrix of the equivalent linearized system. Meanwhile, by Eq. (3.63) and the results of program OPTMDS, the equivalent linear system is also obtained as

$$\ddot{y}_1 + 19.3360y_1 - 8.4646y_2 = 0$$

$$\ddot{y}_2 - 5.6318y_1 + 5.3032y_2 = 0$$

Verification of the results given by Eq. (4.53) is examined by the program MULD1. The program MULD1 utilizes Advanced Continuous Simulation Language (ACSL) [19] to integrate the nonlinear MDOF system and is listed in Appendix C. In Eq. (4.53), matrix $[B]$ represents the modal matrix (eigenvectors), hence the required initial conditions of program MULD1 are also given by the matrix $[B]$. Moreover, utilizing the same initial conditions as in ACSL, program
MULD2 is used to integrate the equivalent linear system of Eq. (4.54). Program MULD2 is also listed in Appendix C. Applying programs MULD1 and MULD2, the time series of \( y_1(t) \) and \( y_2(t) \) for the nonlinear and equivalent linear MDOF systems are illustrated and compared in Figure 4.11, with respect to different modes. The simulations also yield frequencies of the nonlinear system as \( \omega_1 = 1.5734 \) and \( \omega_2 = 4.7124 \). Comparing approximate solutions from the programs OPTMDS, MULD1 and MULD2, it is verified that a satisfactory approximation is obtained from the proposed method.

With respect to different values of the nonlinearity coefficient (\( \epsilon \)), results of the analysis using the programs OPTMDS and MULD1 are listed in Table 4.8 for evaluation.

<table>
<thead>
<tr>
<th></th>
<th>( \omega )</th>
<th>0.5</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Program</td>
<td>( \omega )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>OPTMDS</td>
<td>( \omega_1 )</td>
<td>1.5005</td>
<td>1.5735</td>
<td>1.6153</td>
<td>1.6446</td>
</tr>
<tr>
<td></td>
<td>( \omega_2 )</td>
<td>4.1393</td>
<td>4.7078</td>
<td>5.2785</td>
<td>5.7987</td>
</tr>
<tr>
<td>MULD1</td>
<td>( \omega_1 )</td>
<td>1.5400</td>
<td>1.5734</td>
<td>1.6391</td>
<td>1.6581</td>
</tr>
<tr>
<td></td>
<td>( \omega_2 )</td>
<td>4.0484</td>
<td>4.7124</td>
<td>5.2785</td>
<td>5.7987</td>
</tr>
</tbody>
</table>

Inspecting the results in Table 4.8, it is apparent that when \( \epsilon \) is not small (\( \epsilon > 1.0 \)), the approximate solutions of the proposed method are still significantly favorable.

Example 4.9 Undamped Three-Degree-of-Freedom System.

To further illustrate the use of the discussed method, we consider a three-degree-of-freedom system, as shown in Figure 4.12.
Figure 4.11 Time series for $y_1(t)$ and $y_2(t)$ of the undamped nonlinear and equivalent linear systems ($\epsilon = 1.0$) with respect to:
(a) the first mode  (b) the second mode.
By considering the mass \( m \) as a unit mass \( (m = 1.0) \), the system is governed by the equations of motion

\[
\begin{align*}
\ddot{y}_1 + 2k_y - ky_2 + \epsilon y_1^3 &= 0 \\
\ddot{y}_2 - ky_1 + 2k_y - ky_3 - \epsilon (y_3 - y_2)^3 &= 0 \\
\ddot{y}_3 - ky_2 + ky_3 + \epsilon (y_3 - y_2)^3 &= 0
\end{align*}
\tag{4.55}
\]

From Eq. (3.41), an equivalent linear MDOF system is assumed. We thus proceed to optimize the parameters of the following linear system:

\[
\begin{align*}
\ddot{y}_1 + \hat{k}_{11}y_1 + \hat{k}_{12}y_2 + \hat{k}_{13}y_3 &= 0 \\
\ddot{y}_2 + \hat{k}_{21}y_1 + \hat{k}_{22}y_2 + \hat{k}_{23}y_3 &= 0 \\
\ddot{y}_3 + \hat{k}_{31}y_1 + \hat{k}_{32}y_2 + \hat{k}_{33}y_3 &= 0
\end{align*}
\tag{4.56}
\]

To demonstrate the ease of application of the proposed method, Eqs. (3.58) and (3.59) are directly applied to approximate the frequencies of the system. Thus, for each mode, we minimize the difference between Eqs. (4.55) and (4.56) over a period \( T \) of its motion. Thus
\[
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
= \begin{bmatrix}
2k & -k & 0 \\
-k & 2k & -k \\
0 & -k & k
\end{bmatrix}
\begin{bmatrix}
B_1 \\
B_2 \\
B_3
\end{bmatrix}
- \omega^2
\begin{bmatrix}
B_1 \\
B_2 \\
B_3
\end{bmatrix}
+ \\
\frac{2}{T} \int_0^T \cos\omega t \begin{bmatrix}
\epsilon B_1^3 \cos^3 \omega t \\
-\epsilon (B_3 - B_2)^3 \cos^3 \omega t \\
\epsilon (B_3 - B_2)^3 \cos^3 \omega t
\end{bmatrix} dt
\]  

(4.57)

Performing the required integration and simplification of Eq. (4.57), we have

\[
0 = 2B_1 - B_2 - \Omega^2 B_1 + \frac{3\epsilon}{4k} B_1^3
\]

\[
0 = -B_1 + 2B_2 - B_3 - \Omega^2 B_2 - \frac{3\epsilon}{4k} (B_3 - B_2)^3
\]  

(4.58)

\[
0 = -B_2 + B_3 - \Omega^2 B_3 + \frac{3\epsilon}{4k} (B_3 - B_2)^3
\]

where \( \Omega^2 = \frac{\omega^2}{k} \)

From Eq. (3.59), the constraint for Eq. (4.58) is

\[
A^2 = B_1^2 + B_2^2 + B_3^2
\]  

(4.59)

Again, the program OPTMDS is used to execute the solution of Eqs. (4.58) and (4.59). As an example, let \( A = 2.0 \) for each mode, with parameters \( k = 5.0 \) and \( \epsilon = 0.5 \). The approximate solution is obtained as follows:

\[
\omega_1 = 1.0080; \quad \omega_2 = 2.9932; \quad \omega_3 = 4.3697;
\]

(4.60)

\[
[B] = \begin{bmatrix}
0.6481 & 1.5710 & 1.0617 \\
1.1811 & 0.7221 & -1.4893 \\
1.4789 & -1.0050 & 0.8095
\end{bmatrix}
\]
The corresponding equivalent linear system is given as

\[
\dot{y}_1 + 11.4210y_1 - 5.5424y_2 - 0.1334y_3 = 0
\]

\[
\dot{y}_2 - 5.3597y_1 + 11.8480y_2 - 6.3025y_3 = 0 \quad (4.61)
\]

\[
\dot{y}_3 + 0.97971y_1 - 6.5278y_2 + 5.8005y_3 = 0
\]

The detailed output for this example is shown in Appendix C. In order to check the results of Eq. (4.60), a program MULD3, which is similar to prior MULD1, is implemented and shown in Appendix C. Time series of \( y_1(t) \), \( y_2(t) \) and \( y_3(t) \), with respect to each corresponding mode, are generated from program MULD3. Individual time plots are also illustrated in Figure 4.13. Figure 4.13 also yields the frequencies of the system as \( \omega_1 = 0.9967 \), \( \omega_2 = 2.9197 \) and \( \omega_3 = 4.3633 \). From the comparison of the preceding results, we have attained a favorable approximation using the proposed method.

Furthermore, with respect to various values of the nonlinearity coefficient(\( \epsilon \)), the frequencies approximated by programs OPTMDS and MULD3 for the discussed system are tabulated below for comparison.

<table>
<thead>
<tr>
<th>Program</th>
<th>(\omega)</th>
<th>(\epsilon)</th>
<th>0.5</th>
<th>1.0</th>
<th>1.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>OPTMDS</td>
<td>(\omega_1)</td>
<td>1.0080</td>
<td>1.0130</td>
<td>1.0170</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(\omega_2)</td>
<td>2.9932</td>
<td>3.3304</td>
<td>3.3352</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(\omega_3)</td>
<td>4.3697</td>
<td>4.4540</td>
<td>4.6332</td>
<td></td>
</tr>
<tr>
<td>MULD3</td>
<td>(\omega_1)</td>
<td>0.9967</td>
<td>0.9942</td>
<td>1.0134</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(\omega_2)</td>
<td>2.9197</td>
<td>3.1316</td>
<td>3.3069</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(\omega_3)</td>
<td>4.3633</td>
<td>4.8782</td>
<td>5.0946</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.9 Approximate solutions of an undamped three-degree-of-freedom system.
Figure 4.13  Time series of $y_1(t)$, $y_2(t)$ and $y_3(t)$ with respect to each mode for an undamped three-degree-of-freedom system. (a) the first mode (b) the second mode (c) the third mode.
Table 4.9 shows that favorable results are obtained for various values of $e$. As is apparent, the comparison also shows that less accuracy of approximation is incurred from increasing the value of $e$. Since we construct the system with two nonlinear springs and three degrees of freedom, it is expected that the system will exhibit a stronger nonlinearity, therefore an expected inaccuracy of approximation occurs while increasing the value of $e$. However, the accuracy of approximation can be judged through the error of optimization which is shown in the output solution of program OPTMDS (f). As to a nonlinear system which possesses weaker nonlinearity or quasi-linear properties, it is demonstrated that the proposed method provides an easy and effective tool for approximating the solutions.
V Conclusions

In the preceding chapters, it is shown that the geometric linearization of nonlinear oscillators can be successfully performed through optimization techniques and the associated computer implementation. For an autonomous system, favorable approximations of the solution can be acquired by the proposed methods in a very straight-forward and efficient manner. In contrast to the conventional methods, the proposed methods suggest very generalized equations for obtaining the solution with greater ease of application. Moreover, we leave the lengthy and tedious computation to computers.

With respect to a single-degree-of-freedom system, the geometric nature of phase curves is utilized for the method of optimal ellipse. Therefore, the development of the approximate solution becomes so simple that only the data points (x and y coordinates) are required. There is no need to obtain the governing equation of the system beforehand for application of this method. If an experiment is conducted without having the exact governing equation of the system, the proposed method becomes very helpful in approximating the solution of the system. On the other hand, when the nonlinearity of a system is small, the proposed method is able to generate a better approximation than the conventional methods. This judgement can be discerned by the results in Table 4.2, obtained for the pendulum oscillator. Moreover, from Example 4.4, the method of optimal ellipse is very effective in the approximation of solutions for a damped system possessing a limit cycle. The result of this approximation is that we have generated an undamped linear system which possesses the same or very similar frequency and amplitude associated with the limit
cycle of the damped nonlinear system. Thus both undamped and certain damped systems are appropriate for using the proposed method. The major setback of the proposed method is that the accuracy of approximation greatly depends on the topology of the phase plane. However, the deviation of accuracy can be justified by the results of optimization from computers. For instance, in each previous example, the total distance between the phase curves of the nonlinear and the generated linear systems (from optimal semi-axes) is calculated by program OPTELP. This total distance represents the difference between the two systems and also reveals the accuracy of approximation.

We have shown that the vector field of a phase plane not only illustrates the dynamic characteristics of the system, but also assists in the approximation of the solution. The previous examples clearly show that favorable results are obtained for both undamped and certain damped systems by the method of optimal vector field. Compared to the conventional methods, this proposed method gives much the same results for systems with weak nonlinearities, as shown in previous examples. Nevertheless, the proposed method is more accommodating to systems with higher nonlinearities, which is demonstrated in Table 4.5 pertaining to the pendulum oscillator. For undamped systems, we have developed a very simple and generalized equation (Eq. (3.30)) which readily facilitates the approximation of the frequencies. On the other hand, since a relatively large amount of derivation and computation is required for approximating the limit cycle of the damped system, it becomes the unfavorable feature of this method. The previous examples also show that the accuracy of this method is not as strongly dependent on the topology of the phase plane as is the method of optimal ellipse.
Based on the proposed geometric methods, we have extended our results to undamped multiple-degree-of-freedom systems. Two generalized equations, Eqs. (3.58) and (3.59) are developed for implementation by numerical optimization. Assisted by these numerical methods, the solution can be approximated without much difficulty. For undamped systems with weak nonlinearities, very favorable results are obtained as shown in Tables 4.8 and 4.9. The accuracy of approximation for the proposed method is also dependent on the degree of nonlinearity. Further, the error of approximate solutions can be inspected by the result of computer optimization. As shown by the illustrated examples, higher degree of freedom systems are also conveniently analyzed using this proposed method.

In summary, the proposed geometric linearization methods and computational examples clearly demonstrate that it is possible to achieve favorable approximate solutions for nonlinear autonomous systems. For each of the proposed methods, the generalized equations for approximating the solutions are developed, which incorporate computer implementation to greatly simplify and facilitate the derivation of the approximate solutions. Moreover, from the results of the examples, the proposed methods are particularly effective for systems having weak nonlinearities. However, for systems with higher order nonlinearities, these proposed methods still can be applicable with the errors of approximation given by the particular discretization. In conclusion, the proposed methods effectively provide several attractive approaches for the analysis of nonlinear autonomous systems.
VI References


Appendix A

Computer Programs and Outputs
For Method of Optimal Ellipse
A.1 Program PHASE

******************************************************************************
*  Program: PHASE.FOR
*  *
*  A program to solve a system of two autonomous
*  differential equations by using the Runge-Kutta
*  algorithm which computes values of the state
*  variables and stores data for "time" plots and
*  "phase plane" plots.
*  *
*  Input to the program:
*  n   - step size
*  tfinal - time of termination
*  numcond - number of sets of initial conditions
*  xic, yic - initial values of x and y
*  *
*  Output of the program:
*  t   - time
*  w1  - amplitude of the system
*  w2  - velocity of the system
*  *
*  Output is written to three files:
*  for010.dat - x(t) vs. t
*  for015.dat - y(t) vs. t
*  for020.dat - y(t) vs. x(t) [phase plane]
*  *
*  Program in double precision arithmetic.
* *
******************************************************************************

Implicit real*8(a-h, o-z)
read(5,*), h, tfinal, numcond, xic, yic

  t = 0.0
  n = cint(tfinal/h)
do 10 i = 1, numcond
    w1 = xic
    w2 = yic

  Iteration of R-k algorithm

  do 20 i = 1, n
    t = t + h
    rk11 = h*f(w1, w2)
    rk12 = h*g(w1, w2)
    rk21 = h*f(w1 + 0.5*rk11, w2 + 0.5*rk12)
    rk22 = h*g(w1 + 0.5*rk11, w2 + 0.5*rk12)
    rk31 = h*f(w1 + 0.5*rk21, w2 + 0.5*rk22)
    rk32 = h*g(w1 + 0.5*rk21, w2 + 0.5*rk22)
    rk41 = h*f(w1 + rk31, w2 + rk32)

 10 continue
rk42 = h*g(w1 + rk31, w2 + rk32)
w1 = w1 + (rk11 + 2.0*rk21 + 2.0*rk31 +
+ rk41)/6.0
w2 = w2 + (rk12 + 2.0*rk22 + 2.0*rk32 +
+ rk42)/6.0
write(10, 100) t, w1
write(15, 100) t, w2
write(20, 100) w1, w2
100 format(5x,d15.7,5x,d15.7)
20 continue
10 continue
stop
end

real*8 function f(x, y)
* Calculate the time derivative of x.
* For Examples 4.1, 4.2, 4.3 and 4.4: f = y
implicit real*8(a-h, o-z)
*
Calculate Example 4.1.
f = y
return
end

real*8 function g(x, y)
* Calculate the time derivative of y.
* For Example 4.1: g = -0.5x
* For Example 4.2: g = -sin(x)
* For Example 4.3: g = -sgn(x)
* For Example 4.4: g = -0.2*(x**2 - 1.0)*y - x
implicit real*8(a-h, o-z)
*
Calculate Example 4.1.
g = -0.5x
return
end
A.2 Program OPTELP

* Program: OPTELP.FOR

* A program to estimate the frequency of a nonlinear system by using the Method of Optimal Ellipse.

* Input to the program:
  * n - number of points on the phase plane
  * x, y - coordinates of each point on the phase plane

* Output of the program:
  * a, b - optimal semi-axes
  * freq - approximate frequency

* Program in double precision arithmetic.

Implicit real*8(a-h, o-z)
dimension x(1000), y(1000), th(1000)
parameter(pi = 3.141592654)
logical error

* Input and convert the coordinates of each point into the first quadrant of the phase plane.

read(5,*) n
do 10 i = 1, n
   read(5,*) x(i), y(i)
   x(i) = dabs(x(i))
   y(i) = dabs(y(i))
10 continue

* Obtain the initial approximation of semi-axes.

sumr = 0.0
do 20 i = 1, n
   sumr = sumr + dsqrt(x(i)**2 + y(i)**2)
20 continue
r = sumr/dfloat(n)
a = r
b = r

* Set and print limits of search iteration.

itmax = 100
eps = 1.0d-7
diff = 2.0*eps
write(6,100) n, itmax, eps
Start iteration.

dowhile (diff .gt. eps .and. .not. error)
   if (k .gt. itmax) then
      write(6,200) k, a, b
   else
      do 30 i = 1, n
   endif

Obtain the optimal angles by using Newton's Method.

call newton1(a, b, x(i), y(i), th(i), error, pi)
   if (error) then
      write(6,300)
   endif
   goto 1000
endif

continue
sx = 0.0
sy = 0.0
sc = 0.0
ss = 0.0
do 40 i = 1, n
   sx = sx + x(i)*dcos(th(i))
   sy = sy + y(i)*dsin(th(i))
   sc = sc + dcos(th(i))**2
   ss = ss + dsin(th(i))**2
continue

tempa = a
tempb = b
a = sx/sc
b = sy/ss
diffa = dabs(a - tempa)
diffb = dabs(b - tempb)
diff = dmax1(diffa, diffb)
write(6,400) k, a, b
format(/7x,i3,3x,d12.5,2x,d12.5)
k = k + 1
endif

enddo
* Calculate the total distance between the optimal ellipse and the given points on the phase plane.

\[
d_{\text{dist}} = 0.0 \\
\text{do 50 } i = 1, n \\
\quad d_1 = (x(i) - a*\cos(\theta(i)))^2 \\
\quad d_2 = (y(i) - b*\sin(\theta(i)))^2 \\
\quad \text{dist} = \text{dist} + d\sqrt{d_1 + d_2} \\
\text{50 continue} \\
\text{freq} = b/a \\
* Output the solution.
\]

```
write(6,500) a, b, dist, freq
500 format(//6x,'optimal semi-axis(a)='t50,d12.5//
+ 6x,'optimal semi-axis(b)='t50,d12.5//
+ 6x,'optimal total distance(dist)='
+ ,t50,d12.5// 6x,'approximate
+ frequency(w)=',t50,d12.5)
stop
end
```

Subroutine newtonl(a, b, x, y, root, error, pi)

* A subroutine uses Newton's Method to find a zero for a given function.

```
imPLICIT REAL*8(A-H, O-Z) 
LOGICAL ERROR 
* Set limits of search and obtain the initial approximation. 
ITMAX = 100 
EPSL = 1.0d-6 
CORR = 2.0*EPSL 
I = 1 
IF (X .EQ. 0.0) THEN 
\quad Z = PI/2.0 
ELSE 
\quad Z = DATAN(Y/X) 
* Start iteration. 
DO WHILE (ABS(CORR) .GT. EPSL .AND. .NOT. ERROR) 
\quad IF (I .GT. ITMAX) THEN 
\quad \quad ERROR = .TRUE. 
\quad ELSE 
\quad \quad CORR = FP(A, B, X, Y, Z)/DFP(A, B, X, Z) 
\quad \quad Z = Z - CORR 
\quad \quad I = I + 1 
```
endif
endif
root = z
return
end

real*8 function fp(a, b, x, y, z)
implicit real*8(a-h, o-z)

* Calculate the distance function.
fp = (b*y + (a**2 - b**2)*dsin(z))/(a*x) - dtan(z)
return
end

real*8 function dfp(a, b, x, z)
implicit real*8(a-h, o-z)

* Calculate the derivative of distance function.
dfp = ((a**2 - b**2)*dcos(z))/(a*x) - 1.0/dcos(z)**2
return
end

A.3 Output from program OPTELP for Example 4.2

<table>
<thead>
<tr>
<th>iter</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.98444D+00</td>
<td>0.96390D+00</td>
</tr>
<tr>
<td>2</td>
<td>0.98952D+00</td>
<td>0.95870D+00</td>
</tr>
<tr>
<td>3</td>
<td>0.99208D+00</td>
<td>0.95611D+00</td>
</tr>
<tr>
<td>4</td>
<td>0.99337D+00</td>
<td>0.95482D+00</td>
</tr>
<tr>
<td>5</td>
<td>0.99402D+00</td>
<td>0.95418D+00</td>
</tr>
<tr>
<td>6</td>
<td>0.99434D+00</td>
<td>0.95386D+00</td>
</tr>
<tr>
<td>7</td>
<td>0.99451D+00</td>
<td>0.95369D+00</td>
</tr>
<tr>
<td>8</td>
<td>0.99459D+00</td>
<td>0.95361D+00</td>
</tr>
<tr>
<td>9</td>
<td>0.99463D+00</td>
<td>0.95357D+00</td>
</tr>
<tr>
<td>10</td>
<td>0.99465D+00</td>
<td>0.95355D+00</td>
</tr>
<tr>
<td>11</td>
<td>0.99466D+00</td>
<td>0.95354D+00</td>
</tr>
<tr>
<td>12</td>
<td>0.99467D+00</td>
<td>0.95354D+00</td>
</tr>
<tr>
<td></td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>---</td>
<td>-------------------</td>
<td>-------------------</td>
</tr>
<tr>
<td>1</td>
<td>0.10388D+01</td>
<td>0.11718D+01</td>
</tr>
<tr>
<td>2</td>
<td>0.10107D+01</td>
<td>0.12149D+01</td>
</tr>
<tr>
<td>3</td>
<td>0.99655D+00</td>
<td>0.12390D+01</td>
</tr>
<tr>
<td>4</td>
<td>0.98908D+00</td>
<td>0.12523D+01</td>
</tr>
<tr>
<td>5</td>
<td>0.98504D+00</td>
<td>0.12597D+01</td>
</tr>
<tr>
<td>6</td>
<td>0.98283D+00</td>
<td>0.12638D+01</td>
</tr>
<tr>
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<td>0.12661D+01</td>
</tr>
<tr>
<td>8</td>
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<td>0.12674D+01</td>
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<td>0.12681D+01</td>
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<td>0.12685D+01</td>
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</tr>
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<td>15</td>
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<td>0.12690D+01</td>
</tr>
<tr>
<td>16</td>
<td>0.98009D+00</td>
<td>0.12690D+01</td>
</tr>
<tr>
<td>17</td>
<td>0.98008D+00</td>
<td>0.12690D+01</td>
</tr>
<tr>
<td>18</td>
<td>0.98008D+00</td>
<td>0.12690D+01</td>
</tr>
<tr>
<td>19</td>
<td>0.98008D+00</td>
<td>0.12690D+01</td>
</tr>
<tr>
<td>20</td>
<td>0.98008D+00</td>
<td>0.12690D+01</td>
</tr>
<tr>
<td>21</td>
<td>0.98008D+00</td>
<td>0.12690D+01</td>
</tr>
<tr>
<td>22</td>
<td>0.98008D+00</td>
<td>0.12690D+01</td>
</tr>
<tr>
<td>23</td>
<td>0.98008D+00</td>
<td>0.12690D+01</td>
</tr>
<tr>
<td>24</td>
<td>0.98008D+00</td>
<td>0.12690D+01</td>
</tr>
<tr>
<td>25</td>
<td>0.98008D+00</td>
<td>0.12690D+01</td>
</tr>
</tbody>
</table>
optimal semi-axis(a) = 0.98008D+00
optimal semi-axis(b) = 0.12690D+01
optimal total distance(dist) = 0.76205D+01
approximate frequency(w) = 0.12948D+01

A.5 Output from program OPTEbP for Example 4.4

number of points(n) = 200
maximum number of iteration(itmax) = 100
tolerance of search(eps) = 0.10000D-06

<table>
<thead>
<tr>
<th>iter</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.19925D+01</td>
<td>0.20009D+01</td>
</tr>
<tr>
<td>2</td>
<td>0.19904D+01</td>
<td>0.20031D+01</td>
</tr>
<tr>
<td>3</td>
<td>0.19893D+01</td>
<td>0.20042D+01</td>
</tr>
<tr>
<td>4</td>
<td>0.19888D+01</td>
<td>0.20048D+01</td>
</tr>
<tr>
<td>5</td>
<td>0.19885D+01</td>
<td>0.20051D+01</td>
</tr>
<tr>
<td>6</td>
<td>0.19883D+01</td>
<td>0.20053D+01</td>
</tr>
<tr>
<td>7</td>
<td>0.19883D+01</td>
<td>0.20054D+01</td>
</tr>
<tr>
<td>8</td>
<td>0.19882D+01</td>
<td>0.20054D+01</td>
</tr>
<tr>
<td>9</td>
<td>0.19882D+01</td>
<td>0.20054D+01</td>
</tr>
<tr>
<td>10</td>
<td>0.19882D+01</td>
<td>0.20054D+01</td>
</tr>
<tr>
<td>11</td>
<td>0.19882D+01</td>
<td>0.20055D+01</td>
</tr>
<tr>
<td>12</td>
<td>0.19882D+01</td>
<td>0.20055D+01</td>
</tr>
<tr>
<td>13</td>
<td>0.19882D+01</td>
<td>0.20055D+01</td>
</tr>
<tr>
<td>14</td>
<td>0.19882D+01</td>
<td>0.20055D+01</td>
</tr>
<tr>
<td>15</td>
<td>0.19882D+01</td>
<td>0.20055D+01</td>
</tr>
<tr>
<td>16</td>
<td>0.19882D+01</td>
<td>0.20055D+01</td>
</tr>
<tr>
<td>17</td>
<td>0.19882D+01</td>
<td>0.20055D+01</td>
</tr>
<tr>
<td>18</td>
<td>0.19882D+01</td>
<td>0.20055D+01</td>
</tr>
</tbody>
</table>

optimal semi-axis(a) = 0.19882D+01
optimal semi-axis(b) = 0.20055D+01
optimal total distance(dist) = 0.12804D+02
approximate frequency(w) = 0.10087D+01
Appendix B

Computer Programs and Outputs for the Method of Optimal Vector Field
**B.1 Program OPTVF**

```
* Program: OPTVF.FOR
* A program to estimate the frequency of a nonlinear system without damping by using the Method of Optimal Vector Field.
* Input to the program:
  * n - number of points on the phase plane
  * a - amplitude of the system
* Output of the program:
  * freq - approximate frequency
* Program in double precision arithmetic.
*
Implicit real*8(a-h, o-z)
dimension th(1000)
parameter(pi = 3.141592654)

* Obtain and print values of input.

real(5,*) n, a
write(6,100) n, a
100 format(///6x,'number of points(n)='t50,i3//
       + 6x,'amplitude of the system(a)='t50,
       + d12.5)
sm1 = 0.0
sm2 = 0.0
do 10 i = 1, n
   th(i) = (dfloat(i - 1)*2.0*pi)/dfloat(n)
   cth = dcos(th(i))
   sm1 = sm1 + fv(a, cth)
   sm2 = sm2 + ctn**2
10 continue
* Compute and output the approximate frequency.

freq = dsqrt(sm1/(a*sm2))
write(6,200) freq
200 format(///6x,'approximate frequency(freq)='t50,d12.5)
stop
end
```
real*8 function fv(a, cth)
*
Calculate the value of specified equation.
* For Example 4.5: fv = cth*dsin(a*cth)
* For Example 4.6: fv = dabs(cth)
*
implicit real*8(a-h, o-z)
*
Calculate Example 4.5.

fv = cth*dsin(a*cth)
return
end

B.2 Output from program OPTVF for Example 4.5

number of points(n)= 200
amplitude of the system(a)= 0.10000D+01
approximate frequency(freq)= 0.93814D+00

B.3 Output from program OPTVF for Example 4.6

number of points(n)= 200
amplitude of the system(a)= 0.10000D+01
approximate frequency(freq)= 0.11283D+01
B.4 Program OPTVFD

**************************************************************************************************************************************

* 
* Program: OPTVFD.FOR
*
* A program to estimate the frequency of a nonlinear system with damping by using the Method of Optimal Vector Field.
*
* Input to the program:
* n - number of points on the phase plane
* a - amplitude of the system
* w - initial estimate of frequency
*
* Output of the program:
* freq - approximate frequency
*
* Program in double precision arithmetic.
*
**************************************************************************************************************************************

Implicit real*8(a-h, o-z)
dimension th(1000), c(4)
parameter(pi=3.141592654)
logical error

* Obtain and print values of input.

real(5,*) n, a, w
write(6,100) n, a, w
100 format(/'number of points(n)=',i3,'/ 
+ 'amplitude of the system(a)=',f5.1,'/ 
+ 'initial estimate of frequency(w)=',f5.1)

error = .false.

* Calculate value of each angle

do 10 i = 1, n
   th(i) = (float(i-1)*2.0*pi)/float(n)
10 continue

* Use Subroutine Vandp to acquire constants of frequency polynomial for Example 4.7.

call vandp(n, a, th, c)

* Use Subroutine Newton2 to solve the frequency polynomial.
call newton2(w. freq, c, error)

* Print the results.

if (error) then
    write(6,200)
    format('///10x, 'Program did not converge in
+   Newton's Method')
else
    write(6,300) (c(i), i = 1, 4)
    format(/6x,'constants of frequency polynomial'///
+   6x,'1st constant(c1)='t50,d12.5/
+   6x,'2nd constant(c2)='t50,d12.5/
+   6x,'3rd constant(c3)='t50,d12.5/
+   6x,'4th constant(c4)='t50,d12.5)
write(6,400) freq
    format(/6x,'approximate frequency(freq)=',
+   t50,d12.5)
endif
stop
end

Subroutine vandp(n, a, z, ct)

* A subroutine to obtain the frequency polynomial of
  the Van der Pol oscillator.

implicit real*8(a-h, o-z)
dimension z(1000), ct(4)

read(5,*) eps
write(6,100) eps
100 format(/6x,'value of nonlinearity constant(eps)=',
+   t50,d12.5/)
    sml = 0.0
    sm2 = 0.0
    sm3 = 0.0
    sm4 = 0.0
    do 10 i=1, n
        csq = dcos(z(i))**2
        ssq = dsin(z(i))**2
        csm = dcos(z(i))*dsin(z(i))
        sml = sml + csq
    sm2 = sm2 + csm*(a**2*csq - 1.0)
    sm3 = sm3 + ssq*(a**4*csq**2 - 2.0*a**2*csq + 1.0)
    sm4 = sm4 + csm*(a**2*csq - 1.0)
10  continue
    ct(1) = 2.0*sml
    ct(2) = 3.0*eps*sm2
    ct(3) = eps**2*sm3 - 2.0*sml
    ct(4) = -1.0*eps*sm4
return
end

Subroutine newton2(x, root, c, error)
* A subroutine uses Newton's Method to find a zero
  for a given function.
implicit real*8(a-h, o-z)
dimension c(4)
logical error
* Set limits of search.
  itmax = 100
  epsl = 1.0d-6
  corr = 2.0*epsl
* Start iteration.
dowhile (abs(corr) .gt. epsl .and. .not. error)
  if (i .gt. itmax) then
    error = .true.
  else
    ft = c(1)*x**3 + c(2)*x**2 + c(3)*x + c(4)
    dft = 3.0*c(1)*x**2 + 2.0*c(2)*x + c(3)
    corr = ft/dft
    x = x - corr
    i = i + 1
  endif
enddo
root = x
return
end

B.5 Output from program OPTVFD for Example 4.7

number of points(n)= 200
amplitude of the system(a)= 0.20000D+01
initial estimate of frequency(w)= 1.00000D+00
value of nonlinearity constant(eps)= 0.20000D+00
constants of frequency polynomial
1st constant(c1)= 0.20000D+03
2nd constant(c2)= -0.73837D-09
<table>
<thead>
<tr>
<th>Term</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>3rd constant ($c_3$)</td>
<td>-0.19600D+03</td>
</tr>
<tr>
<td>4th constant ($c_4$)</td>
<td>0.24612D-09</td>
</tr>
<tr>
<td>Approximate frequency ($freq$)</td>
<td>-0.98995D+00</td>
</tr>
</tbody>
</table>
Appendix C

Computer programs and outputs for the Method of Optimal Equivalent Linearization for MDOF Systems
C.1 Program OPTMDS

Program: OPTMDS.FOR

This program incorporates the subroutines of the ADS PROGRAM the IMSL MATH/LIBRARY to optimize a constrained nonlinear equation.

Input to the program:
- n - number of modes
- st - value of stiffness
- stm - stiffness matrix for initial guess
- eps - nonlinearity coefficient
- ap - amplitude of each mode

Output of the program:
- x - optimized values of variables
- f - value of optimized equation
- g - value of constrained equation
- frq - value of optimized frequency for each mode
- stln - value of linearized stiffness

Subroutine of the ADS PROGRAM:
- ads (info, istrat, iopt, ioned, iprint, igrad, ndv, ncon, x, vlb, vub, obj, g, idg, ngt. ic, df, a, nra, ncola, wk, nrwk, iwk, nriwk)

Subroutines of the IMSL MATH/LIBRARY:
- evcrg (n, stm, n, eval, evec, n)
- linrg (n, u, n, uinv, n)
- mrrrr (n, n, u, n, n, n, w, n, n, n, c, n)
- mrrrr (n, n, c, n, n, n, uinv, n, n, n, stln, n)

For the two-degree-of-freedom system in Example 4.8.

parameter(n = 2, st = 5.0)
dimension ap(n), x(n+1), xg(n+1), stm(n, n), u(n, n), +
    uinv(n, n), w(n, n), c(n, n), stln(n, n)
complex eval(n), evec(n, n)
data stm/10.0, -5.0, -5.0, 5.0/

Note: For the three-degree-of-freedom system in Example 4.9:

    n = 3
data = stm/ 10.0, -5.0,  0.0,  
    -5.0,  10.0, -5.0,  
    0.0,  -5.0,  5.0/
* Obtain the input values.

read(5,*) (ap(i), i = 1, n), eps

* Print the input values.

write(6,100) n, st, eps
100 format(/6x,'number of modes(n)='t50,i3/
  + 6x,'value of stiffness constant(st)=',t50,e12.5
  + /6x,'value of nonlinearity constant(eps)=',t50,
  + e12.5/)

* Compute all the eigenvalues and eigenvectors of
  the given matrix(stm) by subroutine of IMSL.

call evcrg (n, stm, n. eval, evec, n)
do 10 i = 1, n

* Normalize the eigenvectors and eigenvalues of each
  mode for initial approximation.

do 20 j = 1, n
  x(j) = real(evec(j,i))/real(evec(i,i))
  xg(j) = x(j)
20 continue

x(n+1) = sqrt(real(eval(eval(i))/st)
xg(n+1) = x(n+1)

* Print input values of each mode.

write(6,200) i, ap(i)
200 format(/6x,'****** mode number: ',i3,' ******'/
  + 6x,'amplitude(ap)=',t50,e12.5/
  + 6x,'i ',8x,'initial guess x(i)',
  + 8x,'optimal value x(i)/

* Optimize the constrained nonlinear equation.

call optimz(n+1, x, eps/st, ap(i), f, g)

* Obtain the optimal frequency.

frq = sqrt(st)*x(n+1)

* Output the results of optimization.

do 30 m = 1, n+1
  write(6,300) m, xg(m), x(m)
300 format(6x,i3,11x,e12.5,14x,e12.5)
30 continue

write(6,400) f, g, frq
400 format(/6x,'value of optimized equation(f)='t50,
+ e12.5/6x,'value of constrained equation(g)=',
+ t50,e12.5/6x,'optimal frequency(frq)=',
+ t50.e12.5)

* Compute the modal(u) and the frequency(w) matrices.

    do 40 k = 1, n
    u(k,i) = x(k)/ap(i)
    if(i .ne. k) then
        w(k,i) = 0.0
    endif
    40 continue
    w(i,i) = frq**2
    continue

* Compute the inverse of modal matrix(uinv) by
* subroutine of IMSL.

    call linrg(n, u, n, uinv, n)

* Multiply two matrices(u*w = c) by subroutine of IMSL.

    call mrrrr(n, n, u, n, n, w, n, n, n, c, n)

* Multiply two matrices(c*uinv = stln) by subroutine of
* IMSL.

    call mrrrr (n, n, c, n, n, n, uinv, n, n, n, stln, n)

* Output the equivalent linearized stiffness matrix.

    write(6,*)
    do 50 i = 1, n
        do 60 j = 1, n
            write(6,500) i, j, stln(i,j)
        500 format(6x,'linearized stiffness k(\',i2,\',1x,
            i2,\')='t50,e12.5)
    60 continue
    50 continue
    stop

end

subroutine optimz(ndv, x, var, ap, obj, eqct)

* This subroutine uses ADS PROGRAM(ADS) to optimize a
* constrained nonlinear equation.

* Required arrays.

dimension x(11), vlb(11), vub(11), g(1), idg(20),

Array dimensions.
nra = 11
ncola = 20
nrwk = 1000
nriwk = 500

Parameters.
igrad = 0
ncon = 1
istrat = 0
iopt = 5
ioned = 7
iprint = 000

Bounds
vlb(1)=-1.0e+20
vlb(2)=-1.0e+20
vlb(3)=-1.0e+20
vub(1)=1.0e+20
vub(2)=1.0e+20
vub(3)=1.0e+20

Note: For the three-degree-of-freedom system in Example 4.9:
add vlb(4) = -1.0e+20
vub(4) = 1.0e+20

Identify constraint as nonlinear, equality.
idg(1) = -1

Optimize.
info = 0

Evaluate objective and constraint.
obj = (2.0*x(1) - x(2) - x(3)**2*x(1) +
+ 0.75*var*x(1)**3)**2 + (-x(1) + x(2) -
\[ x(3)\times 2\times x(2)\times 2 \]
\[ g(1) = x(1)\times 2 + x(2)\times 2 - ap\times 2 \]

* Note: For the three-degree-of-freedom system in Example 4.9:

\[ \text{obj} = (2.0\times x(1) - x(2) - x(4)\times 2\times x(1) +\]
\[ + 0.75\times var\times x(1)\times 3\times 2 + (-x(1) + 2.0\times x(2) -\]
\[ + x(3) - x(4)\times 2\times x(2) - 0.75\times var\times x(3) -\]
\[ + x(2)\times 3\times 2 + (-x(2) + x(3) -\]
\[ + x(4)\times 2\times x(3) + 0.75\times var\times x(3) -\]
\[ + x(2)\times 3\times 2 \]

\[ g(1) = x(1)\times 2 + x(2)\times 2 + x(3)\times 2 - ap\times 2 \]

* Go continue with optimization.

```
go to 10
continue
eqct = g(1)
return
end
```

C.2 Output from program OPTMDS for Example 4.8

number of modes(n)= 2
value of stiffness constant(st)= 0.50000E+01
value of nonlinearity constant(eps)= 0.10000E+01

***** mode number: 1 *****

amplitude(ap)= 0.40000E+01

<table>
<thead>
<tr>
<th>i</th>
<th>initial guess x(i)</th>
<th>optimal value x(i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.10000E+01</td>
<td>0.17947E+01</td>
</tr>
<tr>
<td>2</td>
<td>0.16180E+01</td>
<td>0.35748E+01</td>
</tr>
<tr>
<td>3</td>
<td>0.61803E+00</td>
<td>0.70368E+00</td>
</tr>
</tbody>
</table>

value of optimized equation(f)= 0.14857E-03
value of constrained equation(g)= 0.26321E-03

optimal frequency(frq)= 0.15735E+01

***** mode number: 2 *****

amplitude(ap)= 0.40000E+01
i  initial guess x(i)  optimal value x(i)

1  -0.16180E+01  -0.37940E+01  
2   0.10000E+01   0.12673E+01  
3   0.16180E+01   0.21054E+01  

value of optimized equation(f)=  0.36192E+00  
value of constrained equation(g)=  0.68665E-04  

optimal frequency(frq)=  0.47078E+01  
linearized stiffness k(1, 1)=  0.19336E+02  
linearized stiffness k(1, 2)= -0.84646E+01  
linearized stiffness k(2, 1)= -0.56318E+01  
linearized stiffness k(2, 2)=  0.53032E+01  

C.3  Output from program OPTMDS for Example 4.9  
number of modes(n)=  3  
value of stiffness constant(st)=  0.50000E+01  
value of nonlinearity constant(eps)=  0.50000E+00  

*****  mode number:  1  *****  
amplitude(ap)=  0.20000E+01  

i  initial guess x(i)  optimal value x(i)

1   0.10000E+01   0.64810E+00  
2   0.18019E+01   0.11811E+01  
3   0.22470E+01   0.14789E+01  
4   0.44504E+00   0.45080E+00  

value of optimized equation(f)=  0.59546E-04  
value of constrained equation(g)=  0.21539E-02  

optimal frequency(frq)=  0.10080E+01  

*****  mode number:  2  *****  
amplitude(ap)=  0.20000E+01  

i  initial guess x(i)  optimal value x(i)

1   0.22470E+01   0.15710E+01  
2   0.10000E+01   0.72210E+00  
3  -0.18019E+01  -0.10050E+01  
4   0.12470E+01   0.13386E+01  

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value of optimized equation(f) = 0.10949E+00
value of constrained equation(g) = -0.39697E-03

optimal frequency(frq) = 0.29932E+01

****** mode number:  3  ******

amplitude(ap) = 0.20000E+01

<table>
<thead>
<tr>
<th>i</th>
<th>initial guess x(i)</th>
<th>optimal value x(i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.18019E+01</td>
<td>0.10617E+01</td>
</tr>
<tr>
<td>2</td>
<td>-0.22470E+01</td>
<td>-0.14893E+01</td>
</tr>
<tr>
<td>3</td>
<td>0.10000E+01</td>
<td>0.80953E+00</td>
</tr>
<tr>
<td>4</td>
<td>0.18019E+01</td>
<td>0.19542E+01</td>
</tr>
</tbody>
</table>

value of optimized equation(f) = 0.14333E+00
value of constrained equation(g) = 0.43488E+00

optimal frequency(frq) = 0.43697E+01

linearized stiffness k(1, 1) = 0.11421E+02
linearized stiffness k(1, 2) = -0.55424E+01
linearized stiffness k(1, 3) = -0.13339E+00
linearized stiffness k(2, 1) = -0.53597E+01
linearized stiffness k(2, 2) = 0.11848E+02
linearized stiffness k(2, 3) = -0.63025E+01
linearized stiffness k(3, 1) = 0.97971E+00
linearized stiffness k(3, 2) = -0.65278E+01
linearized stiffness k(3, 3) = 0.58005E+01

C.4 Program MULD1

PROGRAM MULD1

DERIVATIVE

" THIS PROGRAM USES ACSL LANGUAGE TO MODEL A"  
" NONLINEAR TWO-DEGREE-OF-FREEDOM SYSTEM. "  
" DEFINE ALL PRESET VARIABLES"  
" K ......... STIFFNESS"  
" EPS ....... NONLINEARITY COEFFICIENT"  
" Y1IC .... INITIAL DISPLACEMENT OF Y1"  
" Y2IC .... INITIAL DISPLACEMENT OF Y2"  
" Y1DIC ... INITIAL VELOCITY OF Y1"  
" Y2DIC ... INITIAL VELOCITY OF Y2"  
" TSTP .... TIME OF TERMINATING CALCULATION"  
" CINT .... TIME STEP"
CONSTANT  \( K = 5.0 \), \( \varepsilon = 1.0 \)
CONSTANT  \( Y_{1IC} = 1.7947 \), \( Y_{2IC} = 3.5748 \), ...
\( Y_{1DIC} = 0.0 \), \( Y_{2DIC} = 0.0 \), \( TSTP = 20.0 \)
CINTERVAL  CINT = 0.01

" EQUATION OF MOTION"

\[
Y_{1DD} = K \times (Y_2 - 2.0 \times Y_1) - \varepsilon \times Y_1^{\times 3}
\]
\[
Y_{2DD} = K \times (Y_1 - Y_2)
\]

" INTEGRATION"

\[
Y_1D = \text{INTEG}(Y_{1DD}, Y_{1DIC})
\]
\[
Y_2D = \text{INTEG}(Y_{2DD}, Y_{2DIC})
\]
\[
Y_1 = \text{INTEG}(Y_1D, Y_{1IC})
\]
\[
Y_2 = \text{INTEG}(Y_2D, Y_{2IC})
\]

" TERMINATE CALCULATION"

TERMT (T .GE. TSTP)

END

END

C.5  Program MULD2

PROGRAM MULD2

DERIVATIVE

" THIS PROGRAM USES ACSL LANGUAGE TO MODEL A"
" LINEAR TWO-DEGREE-OF-FREEDOM SYSTEM. "

" DEFINE ALL PRESET VARIABLES"
" K ......... STIFFNESS"
" Y_{1IC} .... INITIAL DISPLACEMENT OF Y_1"
" Y_{2IC} .... INITIAL DISPLACEMENT OF Y_2"
" Y_{1DIC} .... INITIAL VELOCITY OF Y_1"
" Y_{2DIC} .... INITIAL VELOCITY OF Y_2"
" TSTP .... TIME OF TERMINATING CALCULATION"
" CINT .... TIME STEP"

CONSTANT  \( K = 5.0 \), \( Y_{1IC} = 1.7947 \), \( Y_{2IC} = 3.5748 \),
\( Y_{1DIC} = 0.0 \), \( Y_{2DIC} = 0.0 \), \( TSTP = 20.0 \)
CINTERVAL  CINT = 0.01

" EQUATION OF MOTION"

\[
Y_{1DD} = -19.3360 \times Y_1 + 8.4642 \times Y_2
\]
\[
Y_{2DD} = 5.6318 \times Y_1 - 5.3032 \times Y_2
\]
" INTEGRATION"

Y1D = INTEG(Y1DD, Y1DIC)
Y2D = INTEG(Y2DD, Y2DIC)
Y1 = INTEG(Y1D, Y1IC)
Y2 = INTEG(Y2D, Y2IC)

" TERMINATE CALCULATION"

TERMT (T .GE. TSTP)

END

C.6 Program MULD3

PROGRAM MULD3

DERIVATIVE

" THIS PROGRAM USES ACSL LANGUAGE TO MODEL A" 
" NONLINEAR THREE-DEGREE-OF-FREEDOM SYSTEM."

" DEFINE ALL PRESET VARIABLES"
" K ......... STIFFNESS"
" EPS ......... NONLINEARITY COEFFICIENT"
" Y1IC .... INITIAL DISPLACEMENT OF Y1"
" Y2IC .... INITIAL DISPLACEMENT OF Y2"
" Y3IC .... INITIAL DISPLACEMENT OF Y3"
" Y1DIC ... INITIAL VELOCITY OF Y1"
" Y2DIC ... INITIAL VELOCITY OF Y2"
" Y3DIC ... INITIAL VELOCITY OF Y3"
" TSTP .... TIME OF TERMINATING CALCULATION"
" CINT .... TIME STEP"

CONSTANT  K = 5.0, EPS = 0.5
CONSTANT  Y1IC = 0.6481, Y2IC = 1.1811, ...
          Y3IC = 1.4789, Y1DIC = 0.0, ...
          Y2DIC = 0.0, Y3DIC = 0.0. TSTP = 20.0
CINTERVAL CINT = 0.01

" EQUATION OF MOTION"

Y1DD = K*(-2.0*Y1 + Y2) - EPS*Y1**3
Y2DD = K*(Y1 - 2.0*Y2 + Y3) + EPS*(Y3 - Y2)**3
Y3DD = K*(Y2 - Y3) - EPS*(Y3 - Y2)**3

" INTEGRATION"

Y1D = INTEG(Y1DD, Y1DIC)
\[ Y2D = \text{INTEG}(Y2DD, Y2DIC) \]
\[ Y3D = \text{INTEG}(Y3DD, Y3DIC) \]
\[ Y1 = \text{INTEG}(Y1D, Y1IC) \]
\[ Y2 = \text{INTEG}(Y2D, Y2IC) \]
\[ Y3 = \text{INTEG}(Y3D, Y3IC) \]

"TERMINATE CALCULATION"

TERMT (T GE. TSTP)

END

END