2007

Rook polynomials for chessboards of two and three dimensions

Benjamin Zindle

Rook Polynomials for Chessboards of Two and Three Dimensions

Benjamin Zindle

School of Mathematical Sciences

Rochester Institute of Technology

Rochester, NY 14623

e-mail: btz0342@rit.edu

Graphics by Benjamin Zindle and Sarah Schnepf

Master's Thesis under Supervision of Dr. Hossein Shahmohamad

1 Introduction

Let A be a set in some universe U. Then we say $N(A)$ is the number of elements in A, and $N = N(U)$. Furthermore, we will represent $N(A_1 \bigcap A_2 \bigcap A_3 \bigcap ... \bigcap A_n)$ as $N(A_1A_2A_3...A_n)$, for the sake of simplicity. Therefore, the number of elements contained in none of the sets $A_1, A_2, A_3, ..., A_n$ can be expressed as $N(\overline{A}_1\overline{A}_2\overline{A}_3...\overline{A}_n)$. To determine this quantity with regard to future counting problems, we will require the **Inclusion-Exclusion Formula**, which is as follows:

Theorem 1.1: Let $A_1, A_2, A_3, ..., A_n$ be sets in some universe U, and let S_k represent the sum of the sizes of all k-tuple intersections of the A_i's. Then

$$N(\overline{A}_1\overline{A}_2\overline{A}_3...\overline{A}_n) = N - S_1 + S_2 - S_3 + ... + (-1)^k S_k + ... + (-1)^n S_n. \tag{1.1}$$

Proof: We begin with an explanation of what the S_k's represent. $S_1 = \sum_i N(A_i)$, the sum of the sizes of all intersections of a single A_i. $S_2 = \sum_{ij} N(A_iA_j)$, the sum of the sizes of all intersections of any two A_i's. S_k is likewise the sum of the sizes of all intersections of any k A_i's, and S_n is simply $N(A_1A_2A_3...A_n)$. Now, in order to prove Eq.(1.1)'s validity, we must demonstrate that the right-hand side of the formula counts every element in none of the A_i's once while counting every element in at least one of the A_i's 0 times, which is to say, for every time an element contained in at least one A_i is counted, it must at some point be subtracted, so that in the final reckoning it is not counted at all.

If an element is not in any of the A_i's, and is thus contained in $\overline{A}_1\overline{A}_2\overline{A}_3...\overline{A}_n$, then it is counted once in the right-hand side of Eq.(1.1) by the term N, and is not counted again in any of the S_k's. Therefore the count for each element in $\overline{A}_1\overline{A}_2\overline{A}_3...\overline{A}_n$ is 1, as desired. Also, if an element is in exactly one A_i, then it is counted once by N, but subtracted once by S_1. As it is in only one of the A_i's, the element would not be counted in any S_k for $k \geq 2$. Thus its net count would be 0, again as desired.

To proceed from here we will require a more general approach. Suppose that an element x is contained in exactly m A_i's. Then x is counted once by N, m [or $\binom{m}{1}$] times by S_1, $\binom{m}{2}$ times by S_2 [as x is contained in each of the $\binom{m}{2}$ intersections of two of the m A_i's containing x], and so on. We see that for any $k \leq m$, S_k counts x $\binom{m}{k}$ times, and that for any $k > m$, S_k counts x 0 times. Thus the net count of x in Eq.(1.1) is

$$1 - \binom{m}{1} + \binom{m}{2} - \binom{m}{3} + ... + (-1)^k \binom{m}{k} + ... + (-1)^m \binom{m}{m}. \tag{1.2}$$

This sum of binomial coefficients can be determined using the binomial expansion

$$(1 + x)^m = 1 + \binom{m}{1}x + \binom{m}{2}x^2 + \binom{m}{3}x^3 + ... + \binom{m}{k}x^k + ... + \binom{m}{m}x^m. \tag{1.3}$$

Plugging x = -1 into the left-hand side of Eq.(1.3) yields

$$[1 + (-1)]^m = 0^m = 0.$$

Plugging x = -1 into the right-hand side of Eq.(1.3) yields

$$1 + \binom{m}{1}(-1) + \binom{m}{2}(-1)^2 + \binom{m}{3}(-1)^3 + ... + \binom{m}{k}(-1)^k + ... + \binom{m}{m}(-1)^m =$$
$$1 - \binom{m}{1} + \binom{m}{2} - \binom{m}{3} + ... + (-1)^k\binom{m}{k} + ... + (-1)^m\binom{m}{m}.$$

Thus the expression in Eq.(1.2) is equal to 0, as desired, and since m was arbitrarily chosen, any element that appears in at least one of the A_i's will ultimately have a net count of 0 with regard to the right-hand side of Eq.(1.1).■

Example 1 - Fun with Inclusion-Exclusion and Cards

In how many ways can one select a 5-card hand from a regular 52-card deck such that the hand contains at least one card from each suit?

In this case, our universe U is the set of all 5-card hands, and we define our A_i's so that any hand with at least one card of every suit [our target demographic] is not contained in any of the A_i's. This we will do as follows: let A_1 be the set of all 5-card hands that contain no hearts; A_2 no clubs; A_3 no spades; and A_4 no diamonds. Thus to answer the problem, we must now find the number of hands that cannot be found in any of the A_i's, i.e. $N(\overline{A}_1\overline{A}_2\overline{A}_3...\overline{A}_n)$.

Note that the total number of possible 5-card hands is $\binom{52}{5}$, therefore, $N = \binom{52}{5}$. The number of elements in A_1 is, quite simply, the number of 5-card hands one may choose from among the 39 cards of a suit other than hearts, and so $N(A_1) = \binom{39}{5}$. Since there are 13 cards of every suit, we will see that $N(A_i) = \binom{39}{5}$ for all i. Therefore $S_1 = \sum_i N(A_i) = 4 \times \binom{39}{5}$. Now we consider $S_2 = \sum_{ij} N(A_iA_j)$. Note that for any i, j, the 5-card hands contained in A_iA_j are chosen from the 26 cards not of the suits forbidden in A_i and A_j, yielding $\binom{26}{5}$ possibilities. Since there are $\binom{4}{2} = 6$ intersections of two sets from the four A_i's, $S_2 = 6 \times \binom{26}{5}$. Similarly, there are $\binom{4}{3}$ intersections of three sets, each of which will contain $\binom{13}{5}$ hands, and thus $S_3 = 4 \times \binom{13}{5}$. It is impossible to have a 5-card hand with no cards of any suit, and so $S_4 = 0$.

Therefore, by Eq.(1.1),

$$N(\overline{A}_1\overline{A}_2\overline{A}_3\overline{A}_4) = \binom{52}{5} - 4\binom{39}{5} + 6\binom{26}{5} - 4\binom{13}{5} + 0 = 685,464.■$$

The Inclusion-Exclusion Formula can be extremely useful when dealing with restrictions that bear some kind of simple symmetry, for example, in dealing with a deck of cards. However, this is not always the case, and for solving problems regarding arrangements of n objects in which particular objects may only appear in certain positions, something a little more powerful is required.

Consider the following problem: in how many ways can the letters a, b, c, d, and e be arranged if a cannot be placed first, b cannot be placed third or fourth, d cannot be placed first or fifth, and e cannot be placed second or third? In essence, we are searching for the number of possible **complete matchings**, or bijective mappings between the sets $\{a, b, c, d, e\}$ and $\{1, 2, 3, 4, 5\}$ that do not violate our original restrictions. This problem can be modeled by the following two-dimensional array, with restricted positions indicated by darkened squares:

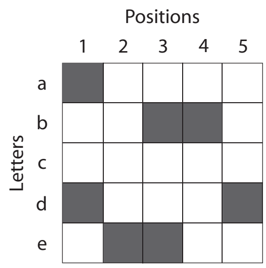

Figure 1.1: Array Modeling Restricted Positions For Matchings

Attempting to solve this problem by use of the Inclusion-Exclusion Formula requires that we define five sets A_i [in this case, we'll choose A_i to be the set of all arrangements of the five letters such that there exists a forbidden letter in position i] and then proceed to calculate the number of elements in S_1, S_2, S_3, S_4 and S_5. The number of arrangements such that no restriction is violated will then simply be $N(\overline{A}_1\overline{A}_2\overline{A}_3\overline{A}_4\overline{A}_5)$.

$N(A_i)$ will be the number of ways to violate the restrictions on position i, which we find by multiplying the number of letters restricted in that position by the 4! ways to order the remaining four letters. [We do not concern ourselves with whether the remaining letters are placed in restricted positions, as our goal here is only to count all arrangements that contain one restricted placement. Our over- or under-counting of identical arrangements will be resolved by the Inclusion-Exclusion Formula.] Therefore, $N(A_1) = 2 \times 4!$, $N(A_2) = 1 \times 4!$, $N(A_3) = 2 \times 4!$, $N(A_4) = 1 \times 4!$ and $N(A_5) = 1 \times 4!$. Summing our results, we get

$$S_1 = \sum_{i=1}^{5} N(A_i) = 2 \times 4! + 1 \times 4! + 2 \times 4! + 1 \times 4! + 1 \times 4! =$$

$$(2+1+2+1+1) \times 4! = 7 \times 4!$$

Note that the number multiplied by 4! is equivalent to the number of darkened squares on the board in Fig.(1.1). This is because every choice of a restricted position yields 4! possibilities for arranging the other four letters. Therefore, S_1 =(number of restricted positions)$\times$4!.

Now we must calculate all the $N(A_iA_j)$'s, each of which equals the number of ways to place two different forbidden letters in the i^{th} and j^{th} positions multiplied by the 3! ways to arrange the remaining letters. Similarly, this is equivalent to finding the number of ways to choose two darkened squares in Fig.(1.1) in columns i and j, and in different rows, then multiplying that result by 3!. Thus we find that $N(A_1A_2) = 2\times 3!$, $N(A_1A_3) = 4\times 3!$, $N(A_1A_4) = 2\times 3!$, $N(A_1A_5) = 1\times 3!$, $N(A_2A_3) = 1\times 3!$, $N(A_2A_4) = 1\times 3!$, $N(A_2A_5) = 1\times 3!$, $N(A_3A_4) = 1\times 3!$, $N(A_3A_5) = 2 \times 3!$ and $N(A_4A_5) = 1 \times 3!$. Therefore

$$S_2 = \sum_{ij} N(A_iA_j) = (2+4+2+1+1+1+1+1+2+1) \times 3! = 16 \times 3!.$$

In other words, $S_2 =$ (# of ways to pick 2 darkened squares in distinct rows and columns)$\times$3!. In a more generalized form, for an $n\times n$ array, what we have is the following result:

$$S_k = (\#\text{of ways to pick } k \text{ darkened squares in distinct rows and columns})(n-k)! \quad (1.4)$$

In our current problem, S_5 is simply 0, as since we have no restrictions on the letter c it is impossible to have a forbidden letter in every position of an arrangement. However, in determining S_3 and S_4 we face a cumbersome undertaking, and for problems dealing with matchings between even larger sets, using this method borders on insanity. Fortunately, there is a simpler way to go about finding a solution.

Note that to obtain a complete matching requires that each letter be paired with a distinct number. Therefore, in terms of the array in Fig.(1.1), if we were to place a marker on the row-column intersection designating such a pairing, [e.g. $(a, 3)$] it would have to be the only marker in that row and column. We can thus recast this problem using our observation in Eq.(1.4).

In chess, a **rook** is a piece that can capture any opponent's piece in the same row or column, provided there are no other pieces positioned between them. A normal chessboard is an 8×8 array with no restrictions, but in this problem we will essentially play chess using the restricted positions of our 5×5 array. Counting the number of ways to pick k darkened squares from Fig.(1.1) in distinct rows and columns is equivalent to counting the number of ways one may place k rooks on the restricted positions of the board such that each rook occupies its own row and column, i.e. rooks that cannot capture one another. For brevity, we will refer to these as **mutually noncapturing rooks**.

Interchanging rows or columns in Fig.(1.1) does not alter the number of ways to place k mutually noncapturing rooks, and in some cases can be used to break up a board into **disjoint subboards**, meaning groupings of darkened squares that share no rows or columns with any darkened squares not of their own grouping. We will consider the arrangement of darkened squares in Fig.(1.1) our original board B, and observe the result of interchanging rows b and d, and columns 2 and 5, and 3 and 4:

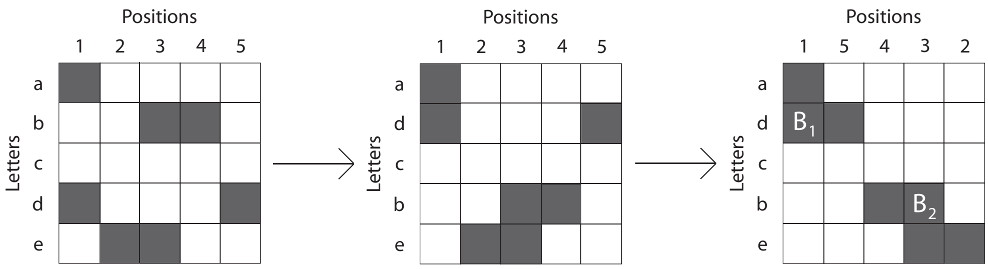

Figure 1.2: Rearranging B

Note that the resultant board in Fig.(1.2) [which we will still call B, as it is in essence identical to the original] is revealed to be decomposable into the disjoint subboards B_1 and B_2. The value of this property is that choosing rook placements in B_1 does not affect rook placements in B_2, except in terms of the total number of rooks being placed. [For example, if we are trying to find out how many ways to place two rooks in B, placing two rooks in B_1 means we can place no rooks in B_2.]

We will now define the following terms: let $r_k(B)$ represent the number of ways to place k mutually noncapturing rooks on board B. Likewise, let $r_k(B_1)$ and $r_k(B_2)$ represent the number of ways to place k mutually noncapturing rooks on subboards B_1 and B_2, respectively. Consulting our last board B from Fig.(1.2), it is evident that $r_1(B_1) = 3$ and $r_1(B_2) = 4$. It is not much more difficult to see that there is only one way to place two rooks in B_1, and three ways to place two rooks in B_2, yielding that $r_2(B_1) = 1$ and $r_2(B_2) = 3$. For $k \geq 3$, $r_k(B_1) = r_k(B_2) = 0$, since both B_1 and B_2 have only two rows, which serves as an upper bound on the possible number of rooks that may be placed upon them. $r_0(B) = 1$ for any board B, since there is obviously only one way to place no rooks on a board.

Looking again at the final board of Fig.(1.2), we see that in order to determine all the ways to place 2 mutually noncapturing rooks on B, we must break our counting into three cases: placing both rooks on B_1, placing one rook on each subboard, and placing both rooks on

B_2. From this, we see that

$$r_2(B) = r_2(B_1) + r_1(B_1)r_1(B_2) + r_2(B_2).$$

Incorporating that $r_0(B_1) = r_0(B_2) = 1$ yields:

$$r_2(B) = r_2(B_1)r_0(B_2) + r_1(B_1)r_1(B_2) + r_0(B_1)r_2(B_2) =$$
$$(1 \times 1) + (3 \times 4) + (1 \times 3) = 1 + 12 + 3 = 16.$$

This agrees with our earlier result using the method of counting darkened squares, as it should. Yet our equation for $r_2(B)$ can be generalized for far greater applicability. For our purposes here, we will simply state the generalized form.

Lemma 1.2: Let B be a board of darkened squares that decomposes into disjoint subboards B_1 and B_2. Then

$$r_k(B) = r_k(B_1)r_0(B_2) + r_{k-1}(B_1)r_1(B_2) + ... + r_0(B_1)r_k(B_2).$$

Now we define the **rook polynomial** $R(x, B)$ of the board B as follows:

$$R(x, B) = r_0(B) + r_1(B)x + r_2(B)x^2 + r_3(B)x^3 + ... + r_n(B)x^n + ...$$

Theorem 1.3: Let B be a board of darkened squares that decomposes into disjoint subboards B_1 and B_2. Then

$$R(x, B) = R(x, B_1)R(x, B_2).$$

Proof:

$$R(x, B) = r_0(B) + r_1(B)x + r_2(B)x^2 + r_3(B)x^3 + ... =$$
$$1 + [r_1(B_1)r_0(B_2) + r_0(B_1)r_1(B_2)]x + [r_2(B_1)r_0(B_2) + r_1(B_1)r_1(B_2) + r_0(B_1)r_2(B_2)]x^2 + ... =$$
$$[r_0(B_1) + r_1(B_1)x + r_2(B_1)x^2 + ...] \times [r_0(B_2) + r_1(B_2)x + r_2(B_2)x^2 + ...] =$$
$$R(x, B_1)R(x, B_2). \blacksquare$$

It is interesting to note that the rook polynomial of a board depends only on the darkened squares, and not on the size of the array containing them. Thus the same groupings of darkened squares placed in a 5×5 array and an 8×8 array would yield the same rook polynomials. Only the final counts attained using the coefficients of the rook polynomials would differ, which we will now proceed to show.

Returning to our earlier problem with regard to the B, B_1 and B_2 defined in the last board of Fig.(1.2), if we apply the $r_k(B_i)$ values we obtained, we get that $R(x, B_1) = 1 + 3x + x^2$ and $R(x, B_2) = 1 + 4x + 3x^2$. Thus, using Theorem 1.3 and Lemma 1.2 we see that:

$$R(x, B) = R(x, B_1)R(x, B_2) = (1 + 3x + x^2)(1 + 4x + 3x^2) =$$

$$1 + [(3 \times 1) + (1 \times 4)]x + [(1 \times 1) + (3 \times 4) + (1 \times 3)]x^2 + [(1 \times 4) + (3 \times 3)]x^3 + (1 \times 3)x^4 =$$

$$1 + 7x + 16x^2 + 13x^3 + 3x^4.$$

Armed with our rook polynomial, we can now use the Inclusion-Exclusion Formula to determine how many arrangements of a, b, c, d and e there are that satisfy the restrictions represented in Fig.(1.1). Using the notation of rook polynomials, we can now rewrite Eq.(1.4) as follows:

$$S_k = r_k(B)(n - k)! \tag{1.5}$$

And for this problem, where $n = 5$, we have, by Eq.(1.5) and the Inclusion-Exclusion Formula, that

$$N(\overline{A}_1\overline{A}_2\overline{A}_3\overline{A}_4\overline{A}_5) = N - S_1 + S_2 - S_3 + S_4 - S_5 =$$

$$5! - r_1(B) \times 4! + r_2(B) \times 3! - r_3(B) \times 2! + r_4(B) \times 1! - r_5(B) \times 0! =$$

$$5! - 7 \times 4! + 16 \times 3! - 13 \times 2! + 3 \times 1! - 0 \times 0! =$$

$$120 - 168 + 96 - 26 + 3 - 0 = 25.$$

Thus we have that there are 25 arrangements of the five letters that will satisfy the restrictions displayed in Fig.(1.1). Combining Theorem 1.1 and Eq.(1.5) yields the following generalized result:

Theorem 1.4: Let B be a board with restricted positions and rook polynomial $R(x, B)$. Then the number of ways to arrange n distinct objects given the restrictions on B is equivalent to

$$n! - r_1(B)(n - 1)! + r_2(B)(n - 2)! + ... + (-1)^k r_k(B)(n - k)! + ... + (-1)^n r_n(B)0!. \tag{1.6}$$

Before proceeding, let's review the method for using rook polynomials to solve counting problems involving matchings or arrangements with restricted positions. First, we represent the problem with an array, darkening those positions that are restricted, then we attempt to rearrange the board through row and column interchanges to determine whether or not it can be decomposed into disjoint subboards. If it cannot be decomposed, we determine the $r_k(B)$'s and use those to generate the rook polynomial of the board, $R(x, B)$. If the board can be decomposed, we examine our decomposed board to determine the $r_k(B_i)$'s, which we can then use to generate the rook polynomials for the B_i's. Then we multiply the rook polynomials of the subboards to obtain the rook polynomial of the entire board, $R(x, B)$. Finally, to determine the number of matchings or arrangements we substitute the coefficients of $R(x, B)$ into Eq.(1.6) and solve. Before we move on to more complicated applications involving rook polynomials, we will explore without interruption a simple example from start to finish.

Example 2 - Mentoring Program

At a university, seven freshmen, $F_1, F_2, F_3, F_4, F_5, F_6$ and F_7, enter the same academic program. Their department head, eager to retain these new students, wants to assign each incoming freshman a mentor from among the upperclassmen of the program. Seven mentors are chosen, $M_1, M_2, M_3, M_4, M_5, M_6$ and M_7, but there are some scheduling conflicts. M_1 cannot work with F_1 or F_3, M_2 cannot work with F_1 or F_5, M_4 cannot work with F_3 or F_6, M_5 cannot work with F_2 or F_7, and M_7 cannot work with F_4. In how many ways can the department head assign the mentors so that each incoming freshman has a different mentor?

First, let's model the problem with the following chessboard, which we will denote as B:

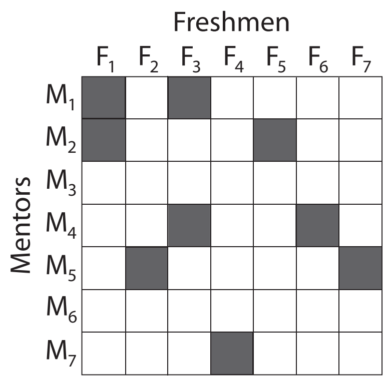

Figure 1.3: B

Now let's try interchanging some rows and columns to see if we can decompose this board into disjoint subboards.

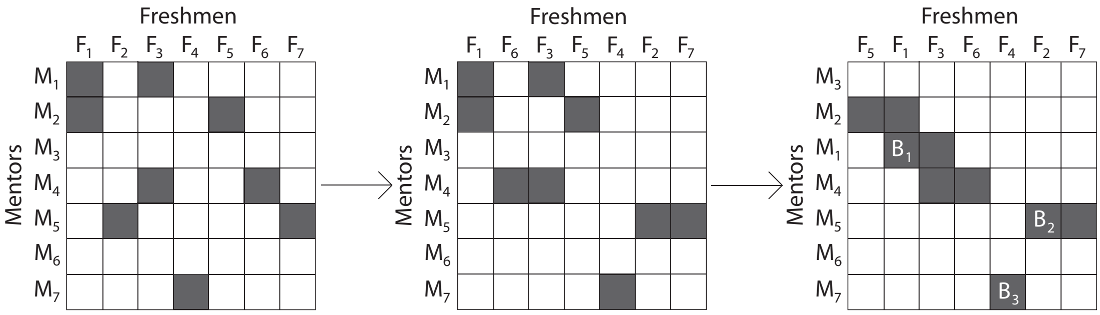

Figure 1.4: The Evolution of B

Interchanging columns F_2 and F_6, F_4 and F_5, F_1 and F_5, F_1 and F_6, and rows M_1 and M_3 yields a decomposition of the original board into the three subboards displayed in the final board of Fig.(1.4). Now we set about calculating the $r_k(B_i)$'s for these subboards, and arrive at the following: $r_1(B_1) = 6$, $r_2(B_1) = 10$, $r_3(B_1) = 4$; $r_1(B_2) = 2$; $r_1(B_3) = 1$. Thus we arrive with the following rook polynomials for B_1, B_2 and B_3:

$$R(x, B_1) = 1 + 6x + 10x^2 + 4x^3$$
$$R(x, B_2) = 1 + 2x$$
$$R(x, B_3) = 1 + x$$

Multiplying these rook polynomials yields:

$$R(x, B) = R(x, B_1)R(x, B_2)R(x, B_3) =$$
$$(1 + 6x + 10x^2 + 4x^3)(1 + 2x)(1 + x) =$$
$$(1 + 6x + 10x^2 + 4x^3)(1 + 3x + 2x^2) =$$
$$1 + 3x + 2x^2 + 6x + 18x^2 + 12x^3 + 10x^2 + 30x^3 + 20x^4 + 4x^3 + 12x^4 + 8x^5 =$$
$$1 + 9x + 30x^2 + 46x^3 + 32x^4 + 8x^5.$$

Now we plug the coefficients of $R(x, B)$ into Eq.(1.6), which gives us the following:

$$7! - 9 \times 6! + 30 \times 5! - 46 \times 4! + 32 \times 3! - 8 \times 2! + 0 \times 1! - 0 \times 0! =$$
$$5,040 - 6,480 + 3,600 - 1,104 + 192 - 16 = 1,232.$$

Thus there are 1,232 ways to assign each freshman his or her own mentor, in accordance with the given restrictions.■

Now we're ready to use our material on rook polynomials to attack some more complicated counting problems.

2 Rook Polynomials for Two-Dimensional Chessboards

In this section we will develop polynomial expressions for two-dimensional chessboards suited to two cumbersome counting problems with restricted positions. To aid in that task, we first consider a chessboard *without* restrictions.

Theorem 2.1: Let $\mathbf{B}_{m,n}$ denote an $m \times n$ chessboard with no restricted positions, and let $s = min\{m,n\}$. Then

$$R(x, \mathbf{B}_{m,n}) = \sum_{k=0}^{s} \binom{m}{k} P(n,k) x^k.$$

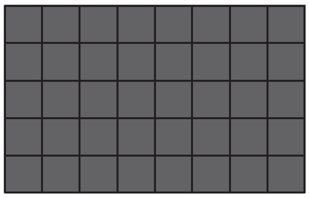

Figure 2.1: $\mathbf{B}_{5,8}$

Proof: Suppose we want to place k rooks on $\mathbf{B}_{m,n}$. There are $\binom{m}{k}$ ways to select the k rows in which the rooks will be placed. For each choice of k rows, we must select k columns and permute them to obtain every possible row-column matching, as the intersection of each distinct row-column pair indicates a distinct rook placement on $\mathbf{B}_{m,n}$. Thus, the number of ways to place k rooks on $\mathbf{B}_{m,n}$ is $\binom{m}{k} P(n,k)$. Therefore, since the total number of rooks placed cannot exceed s,

$$R(x, \mathbf{B}_{m,n}) = \sum_{k=0}^{s} \binom{m}{k} P(n,k) x^k. \blacksquare$$

We will be using rook polynomials to determine the number of matchings between sets of equal size. Thus, more useful to us will be the following corollary of Theorem 2.1:

Corollary 2.2:

$$R(x, \mathbf{B}_{n,n}) = \sum_{k=0}^{n} \binom{n}{k} P(n,k) x^k. \tag{2.7}$$

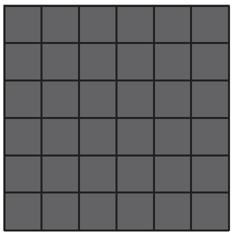

Figure 2.2: $\mathbf{B}_{6,6}$

Example 3 - Game Time in the Bates Center

Four RIT math students/alumni, Amir Barghi, Jeremy Nieman, Nate Reff, and Ben Zindle walk into the Bates Study Center to find four different card games about to start, each at its own table and with one open seat yet to be filled. The four games are, respectively, poker, spades, euchre, and Oh Shwat. If Amir doesn't want to play poker, Nate and Jeremy don't want to play spades or euchre, and Ben doesn't want to play Oh Shwat, in how many ways can each student/alumnus choose a different table such that all end up satisfied? (It should be noted that not one of the students in question is scheduled for tutor duty at this time.)

We start by modeling the problem with the following chessboard:

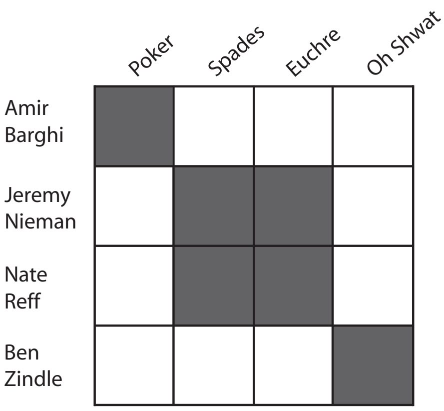

Figure 2.3: Bates Center Brouhaha

Note that the board is already decomposed into three disjoint subboards: two copies of $\mathbf{B}_{1,1}$ and one copy of $\mathbf{B}_{2,2}$. Therefore we see that the rook polynomial for this board B is:

$$R(x, B) = R(x, \mathbf{B}_{2,2})[R(x, \mathbf{B}_{1,1})]^2 =$$

$$[\sum_{k=0}^{2}\binom{2}{k}P(2,k)x^k][\sum_{k=0}^{1}\binom{1}{k}P(1,k)x^k]^2 =$$
$$(1+4x+2x^2)(1+x)^2 = 1+6x+11x^2+8x^3+2x^4.$$

Plugging the coefficients of $R(x,B)$ into Eq.(1.6) gives us the following:

$$4! - 6\times 3! + 11\times 2! - 8\times 1! + 2\times 0! = 24 - 36 + 22 - 8 + 2 = 4.$$

Thus there are 4 ways to assign each student/alumnus a card game so that everyone is content, that is, apart from the students sitting at other tables attempting to get work done.■

We now commence with a more complicated counting problem involving a simple game.

Problem 1 - Odds and Evens

Two friends, Joe and Dave, are on their way to spend spring break in St. George, Florida, but must first spend a five-hour layover in St. Paul, Minnesota. To pass the time, they decide to play an extended game of Odds and Evens, where each person writes down a number from 1 to n on a piece of paper, n being even. If the sum of the two written numbers is odd, Joe wins the round; if the sum of the numbers is even, Dave wins. If the game consists of n rounds, over which each player must use every number once and only once, in how many ways can Joe win every round for the following values of n?

a.) n = 6 **b.)** n = 10

Handling these two subproblems case by case could prove extremely tedious. Therefore, we will develop an expression that gives us the rook polynomial for this problem at every value of n, and then plug in the values of n the problem specifies. Toward that end, we present the following theorem:

Theorem 2.3: Let n be an even number, and let $\mathbf{C}_{n,n}$ denote an $n\times n$ chessboard with restrictions on any position whose row and column values sum to an even number. Then

$$R(x,\mathbf{C}_{n,n}) = [R(x,\mathbf{B}_{\frac{n}{2},\frac{n}{2}})]^2 = [\sum_{k=0}^{\frac{n}{2}}\binom{\frac{n}{2}}{k}P(\frac{n}{2},k)x^k]^2. \qquad (2.8)$$

Proof: To begin, let us consider the chessboard that models the problem, of which we offer a concrete example below, for the case where $n = 10$:

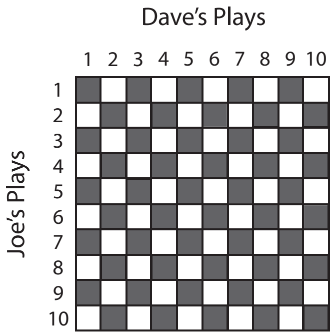

Figure 2.4a: $C_{10,10}$

Rearranging the columns so that all even-numbered columns are gathered on the right half of the board yields the following chessboard:

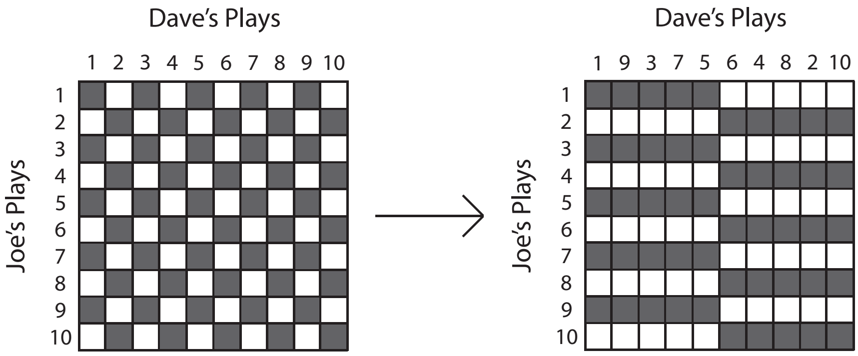

Figure 2.4b: $C_{10,10}$ In Transition

Now, rearranging the rows so that all odd-numbered rows are on the upper half of the board yields the final result:

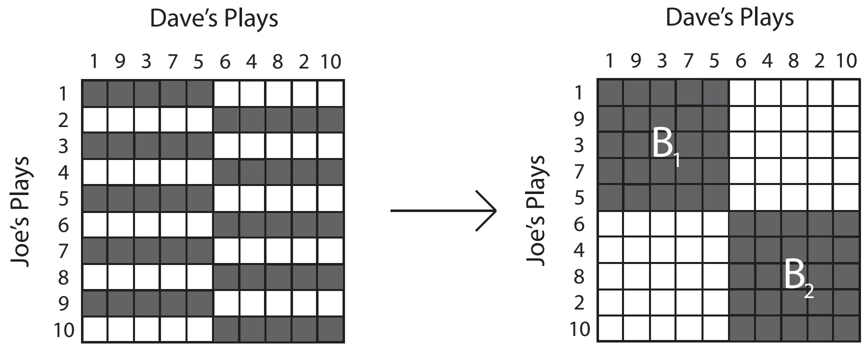

Figure 2.4c: $\mathbf{C}_{10,10}$ Decomposed

Note that in general, $\mathbf{C}_{n,n}$ can be rearranged into a chessboard consisting of two disjoint subboards: twin copies of $\mathbf{B}_{\frac{n}{2},\frac{n}{2}}$. Thus,

$$R(x,\mathbf{C}_{n,n}) = [R(x,\mathbf{B}_{\frac{n}{2},\frac{n}{2}})]^2 = [\sum_{k=0}^{\frac{n}{2}} \binom{\frac{n}{2}}{k} P(\frac{n}{2},k)x^k]^2. \blacksquare$$

Now we're ready to tackle the subproblems. Let's begin with **#1a**, where $n = 6$:

$$R(x,\mathbf{C}_{6,6}) = [\sum_{k=0}^{3} \binom{3}{k} P(3,k)x^k]^2 =$$

$$[1 + (3 \times 3)x + (3 \times 6)x^2 + (1 \times 6)x^3]^2 = [1 + 9x + 18x^2 + 6x^3]^2 =$$

$$1 + 18x + 117x^2 + 336x^3 + 432x^4 + 216x^5 + 36x^6.$$

Plugging the coefficients of $R(x,\mathbf{C}_{6,6})$ into Eq.(1.6) gives us:

$$6! - 18 \times 5! + 117 \times 4! - 336 \times 3! + 432 \times 2! - 216 \times 1! + 36 \times 0! =$$

$$720 - 2,160 + 2,808 - 2,016 + 864 - 216 + 36 = 36. \blacksquare$$

Therefore, if Joe and Dave play 6 rounds, there are 36 ways that Joe can win every round. But what if they play 10? We move on now to **#1b**, where $n = 10$:

$$R(x,\mathbf{C}_{10,10}) = [\sum_{k=0}^{5} \binom{5}{k} P(5,k)x^k]^2 =$$

$$[1 + (5 \times 5)x + (10 \times 20)x^2 + (10 \times 60)x^3 + (5 \times 120)x^4 + (1 \times 120)x^5]^2 =$$

$$[1 + 25x + 200x^2 + 600x^3 + 600x^4 + 120x^5]^2 =$$
$$1 + 50x + 1,025x^2 + 11,200x^3 + 71,200x^4 + 270,240x^5 + 606,000x^6 + 768,000x^7$$
$$+504,000x^8 + 144,000x^9 + 14,400x^{10}.$$

Plugging the coefficients of $R(x, \mathbf{C}_{10,10})$ into Eq.(1.6) yields:

$$10! - 50 \times 9! + 1,025 \times 8! - 11,200 \times 7! + 71,200 \times 6! - 270,240 \times 5! + 606,000 \times 4!$$
$$-768,000 \times 3! + 504,000 \times 2! - 144,000 \times 1! + 14,400 \times 0! =$$
$$3,628,800 - 18,144,000 + 41,328,000 - 56,448,000 + 51,264,000 - 32,428,800 + 14,544,000$$
$$-4,608,000 + 1,008,000 - 144,000 + 14,400 = 14,400. \blacksquare$$

Thus if Joe and Dave play 10 rounds of Odds and Evens under the rules specified above, there are 14,400 ways for Joe to win every round. Note the peculiarity that for both **#1a** and **#1b** the result was equivalent to the number of ways to place the maximum number of rooks on our board of darkened squares. We proceed now to the next problem.

Problem 2 - The Lottery

A Niagara Falls, New York lodge, the Order of the Rhinos, holds a weekly lottery for its n members, n an even number. Upon joining, each member is assigned a number, based on the order in which they have joined [e.g. the i^{th} member to join is assigned the number i]. In the weekly lottery drawing, each member is assigned randomly a number from 1 to n, such that each number is used once and only once, and winners are determined as follows: if Member i draws either i or $n + 1 - i$ for the week, he or she wins. In the case of multiple winners, the pot would be split among them. The problem we will consider is the following: in how many ways can there be no winners the first week of the lottery for the following values of n?

a.) n = 10 **b.)** n = 20

As with Problem 1, it will again be useful to develop an expression that gives us the rook polynomial for this problem at every value of n, so that we may then plug in the values of n the problem specifies. Thus we present the following theorem:

Theorem 2.4: Let n be an even number, and let $\mathbf{S}_{n,n}$ denote an $n \times n$ chessboard with the following specifications: for every row i, the i^{th} and $(n + 1 - i)^{th}$ positions are restricted. Then

$$R(x, \mathbf{S}_{n,n}) = (1 + 4x + 2x^2)^{\frac{n}{2}}. \tag{2.9}$$

Proof: To begin, we will consider $\mathbf{S}_{n,n}$, the chessboard that models the lottery problem. We introduce the following concrete example for the sake of simplicity:

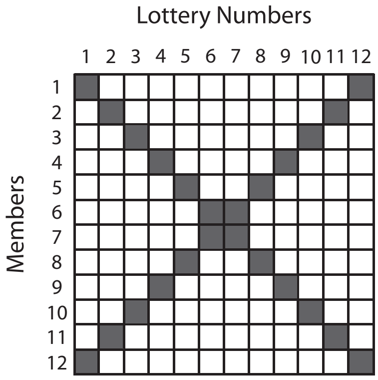

Figure 2.5a: $\mathbf{S}_{12,12}$

Note that each pairing of the i^{th} and $(n+1-i)^{th}$ rows constitutes a disjoint subboard, which can be rearranged to yield $\mathbf{B}_{2,2}$. As in general there are $\frac{n}{2}$ such pairings, we have that $\mathbf{S}_{n,n}$ can be rearranged into $\frac{n}{2}$ disjoint copies of $\mathbf{B}_{2,2}$.

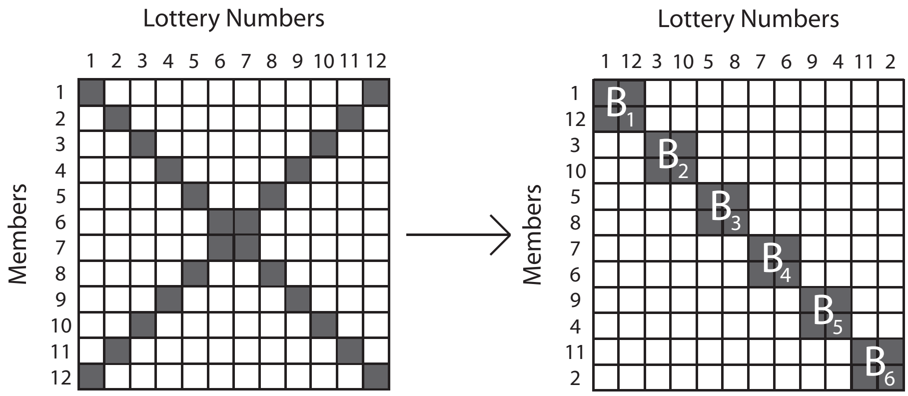

Figure 2.5b: $\mathbf{S}_{12,12}$ Decomposed

Thus,

$$R(x, \mathbf{S}_{n,n}) = [R(x, \mathbf{B}_{2,2})]^{\frac{n}{2}} = [\sum_{k=0}^{2} \binom{2}{k} P(2,k) x^k]^{\frac{n}{2}} = (1 + 4x + 2x^2)^{\frac{n}{2}}. \blacksquare$$

Now, we begin with **#2a**, where $n = 10$:

$$R(x, \mathbf{S}_{10,10}) = (1 + 4x + 2x^2)^5 =$$
$$1 + 20x + 170x^2 + 800x^3 + 2,280x^4 + 4,064x^5 + 4,560x^6 + 3,200x^7 + 1,360x^8 + 320x^9 + 32x^{10}.$$

Plugging the coefficients of $R(x, \mathbf{S}_{10,10})$ into Eq.(1.6), we get:

$$10! - 20 \times 9! + 170 \times 8! - 800 \times 7! + 2,280 \times 6! - 4,064 \times 5! + 4,560 \times 4!$$
$$-3,200 \times 3! + 1,360 \times 2! - 320 \times 1! + 32 \times 0! =$$
$$3,628,800 - 7,257,600 + 6,854,400 - 4,032,000 + 1,641,600 - 487,680 + 109,440$$
$$-19,200 + 2,720 - 320 + 32 = 440,192. \blacksquare$$

Thus if the lodge has 10 members, there are 440, 192 ways for no one to win the first week's lottery. Let's proceed to **#1b**, with $n = 20$:

$$R(x, \mathbf{S}_{20,20}) = (1 + 4x + 2x^2)^{10} =$$
$$1 + 40x + 740x^2 + 8,400x^3 + 65,460x^4 + 371,328x^5 + 1,586,880x^6 + 5,218,560x^7$$
$$+13,381,920x^8 + 26,970,880x^9 + 42,904,960x^{10} + 53,941,760x^{11} + 53,527,680x^{12}$$
$$+41,748,480x^{13} + 25,390,080x^{14} + 11,882,496x^{15} + 4,189,440x^{16}$$
$$+1,075,200x^{17} + 189,440x^{18} + 20,480x^{19} + 1,024x^{20}.$$

Plugging the many coefficients of $R(x, \mathbf{S}_{20,20})$ into Eq.(1.6) gives us

$$20! - 40 \times 19! + 740 \times 18! - 8,400 \times 17! + 65,460 \times 16! - 371,328 \times 15! + 1,586,880 \times 14!$$
$$-5,218,560 \times 13! + 13,381,920 \times 12! - 26,970,880 \times 11! + 42,904,960 \times 10!$$
$$-53,941,760 \times 9! + 53,527,680 \times 8! - 41,748,480 \times 7! + 25,390,080 \times 6!$$
$$-11,882,496 \times 5! + 4,189,440 \times 4! - 1,075,200 \times 3! + 189,440 \times 2!$$
$$-20,480 \times 1! + 1,024 \times 0! =$$
$$2,432,902,008,176,640,000 - 4,865,804,016,353,280,000 + 4,737,756,542,238,720,000$$
$$-2,987,774,396,006,400,000 + 1,369,605,826,068,480,000 - 485,576,107,720,704,000$$
$$+138,341,486,739,456,000 - 32,496,081,666,048,000 + 6,409,961,091,072,000$$
$$-1,076,591,222,784,000 + 155,693,518,848,000 - 19,574,385,868,800 + 2,158,236,057,600$$

$$-210,412,339,200 + 18,280,857,600 - 1,425,899,520 + 100,546,560 - 6,451,200$$
$$+378,880 - 20,480 + 1,024 = 312,426,715,251,262,464. \blacksquare$$

Thus if the lodge has 20 members, there are 312, 426, 715, 251, 262, 464 ways for no one to win the first week's lottery.

In this section, rook polynomials effectively transformed a few of the ugliest, most tedious counting problems into fairly straightforward calculations. But what happens when trying to determine the number of complete matchings between not two sets, but three? In the next section, we will solve some problems of this nature using rook polynomials of chessboards in three dimensions.

3 Rook Polynomials for Three-Dimensional Chessboards

In this section we will develop polynomial expressions for three-dimensional chessboards suited to two cumbersome counting problems with restricted positions, but first we require a little background.

A two-dimensional chessboard is defined by its rows and columns, and we may consider a three-dimensional chessboard to be simply a pile of two-dimensional chessboards stacked one upon another. Considering every two-dimensional level of a three-dimensional chessboard, we still have rows and columns in the traditional sense, each of which is a one-dimensional array used to describe position within the board. We will likewise define a **tower** as a one-dimensional array used to describe position along the added third dimension.

In dealing with rook polynomials of two-dimensional chessboards, we treated every rook in position {i,j} as the only object to occupy row i and column j. Translating from two to three dimensions, however, complicates things a little bit. On a two-dimensional board, each rook placement prohibits any further rook placements in the union of that rook's row and column, i.e. two intersecting lines. On a three-dimensional board, however, each rook placement prohibits any further rook placements in the union of three intersecting planes formed by the $\binom{3}{2} = 3$ intersections of the dimensions, i.e. the plane formed using as a basis that rook's row and column, the plane formed using the rook's row and tower, and the plane formed using the rook's column and tower.

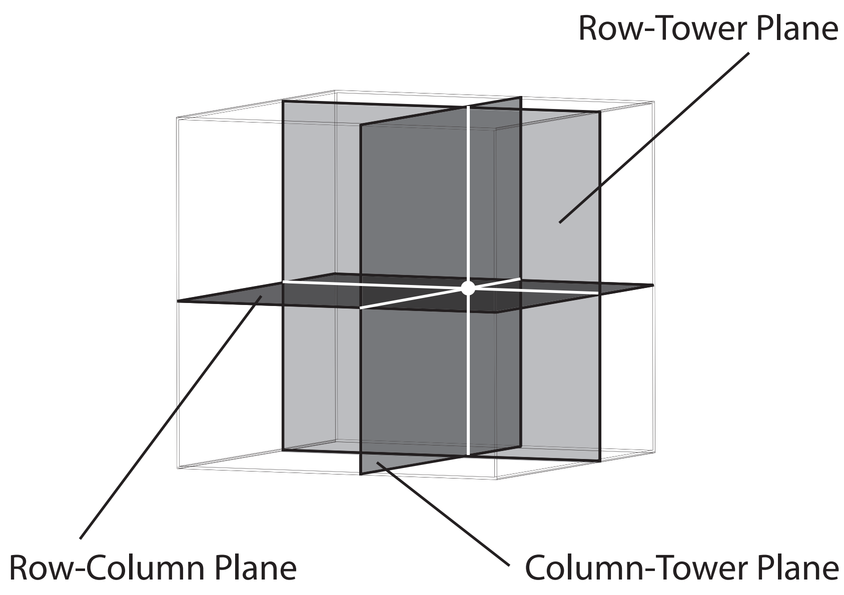

Figure 3.1: The Range of Rooks in Three Dimensions

With that in mind, we begin with the following theorem, concerning a three-dimensional chessboard without restrictions.

Theorem 3.1: Let $\mathbf{B}_{m,n,r}$ denote an $m \times n \times r$ chessboard with no restricted positions, and let $s = min\{m, n, r\}$. Then

$$R(x, \mathbf{B}_{m,n,r}) = \sum_{k=0}^{s} \binom{m}{k} P(n,k) P(r,k) x^k.$$

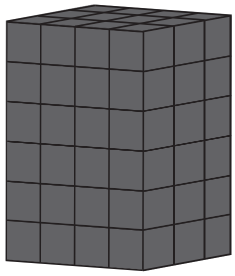

Figure 3.2: $\mathbf{B}_{3,4,6}$

Proof: Suppose we want to place k rooks in $\mathbf{B}_{m,n,r}$. To illustrate this, we will consider the following analogy: suppose that the three-dimensional chessboard is transparent, the rooks placed within it are opaque, and a lamp has been positioned directly above the center of the board so that its light is shining down upon the entire construct. The k rooks that we have placed within the board will thus cast shadows onto the floor. These shadows may be regarded as a projection of the three-dimensional board onto a two-dimensional subboard. Note that since the rooks cannot share a row or column with one another, the number of arrangements for the k shadows on the floor is equivalent to $r_k(x, \mathbf{B}_{m,n})$, and therefore, the rook polynomial whose coefficients enumerate the number of ways to arrange the k shadows on the floor [for any k] is none other than $R(x, \mathbf{B}_{m,n})$.

Therefore, to place k rooks in $\mathbf{B}_{m,n,r}$, we know we have $r_k(x, \mathbf{B}_{m,n}) = \binom{m}{k} P(n,k)$ ways to choose the row and column positions for each rook. To determine the total number of ways to place those k rooks within the three-dimensional board, we need only select k tower positions from the r available and permute them to obtain every possible row-column-tower matching, as the intersection of each distinct row-column-tower triple indicates a distinct rook placement within $\mathbf{B}_{m,n,r}$. Therefore, the number of ways to place k rooks within $\mathbf{B}_{m,n,r}$ is $r_k(x, \mathbf{B}_{m,n}) \times P(r,k)$, or $\binom{m}{k} P(n,k) P(r,k)$. And finally, since the total number of rooks placed cannot exceed s,

$$R(x, \mathbf{B}_{m,n,r}) = \sum_{k=0}^{s} \binom{m}{k} P(n,k) P(r,k) x^k. \blacksquare$$

As was in the case in Section 2, we will be using rook polynomials to determine the number of matchings between sets of equal size. Thus, more useful to us will be the following corollary of Theorem 3.1:

Corollary 3.2:

$$R(x, \mathbf{B}_{n,n,n}) = \sum_{k=0}^{n} \binom{n}{k} [P(n,k)]^2 x^k. \tag{3.10}$$

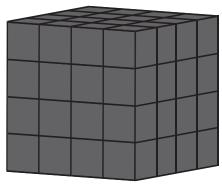

Figure 3.3: $\mathbf{B}_{4,4,4}$

Before we look at some problems, we require the following extension of Eq.(1.6), the Inclusion-Exclusion Formula for rook polynomials, so that we have the appropriate tool for counting with regard to three-dimensional chessboards.

Theorem 3.3: Let B be a three-dimensional chessboard with restricted positions and rook polynomial $R(x, B)$. Then the number of ways to arrange n distinct objects given the restrictions on B is equivalent to

$$(n!)^2 - r_1(B)(n-1)!^2 + ... + (-1)^k r_k(B)(n-k)!^2 + ... + (-1)^n r_n(B)(0!)^2. \tag{3.11}$$

Proof: When counting using inclusion-exclusion for two-dimensional chessboards, to get the terms of our final sum we multiplied [for every $k \leq n$] the number of ways to place k rooks on the darkened squares of the board by the $(n-k)!$ ways to place the remaining $(n-k)$ rooks, disregarding restricted positions for the latter. For counting on three-dimensional chessboards, once we have determined all the ways to place k rooks within our network of restricted positions there are $(n-k)!$ ways to fix the row-column positions of the remaining $(n-k)$ rooks, but then we must also consider the $(n-k)!$ ways to arrange their tower positions. Therefore, the number of ways to arrange n distinct objects given the restrictions on B is

$$(n!)^2 - r_1(B)(n-1)!^2 + ... + (-1)^k r_k(B)(n-k)!^2 + ... + (-1)^n r_n(B)(0!)^2. \blacksquare$$

Example 4 - A Game of Clue in Building 8

Following their rollicking card games in the Bates Study Center, and a string of complaints from students unable to get their work done with all the noise, Amir Barghi, Jeremy Nieman, Nate Reff and Ben Zindle each file reports with Campus Safety claiming to have been beaten simultaneously in different locations by a masked faculty member of the School of Mathematical Sciences. Four scuffed and bloodied objects were found in garbage cans around the campus: a compass, a TI-82 graphing calculator, a protractor, and a James Stewart calculus textbook. The only four faculty members unaccounted for during the hour in question are David Hart, Sophia Maggelakis, Hossein Shahmohamad and Tamas Wiandt, all of whom are known for their violent tempers.

Assuming that Campus Safety has the authority to conduct a criminal investigation of any kind, the following facts are established: David Hart could not have attacked Amir Barghi with the compass, Sophia Maggelakis and Hossein Shahmohamad could not have attacked Jeremy Nieman or Nate Reff with either the TI-82 calculator or the protractor, and Tamas Wiandt could not have attacked Ben Zindle with the calculus textbook.

Given that Campus Safety personnel lack access to equipment for testing DNA or fingerprints, and are unable to tell the difference between trauma caused by a compass and trauma caused by a textbook, in how many ways can they randomly guess who was beaten by whom and with what so that they don't contradict the evidence and embarrass themselves?

We start by modeling the problem with the following chessboard:

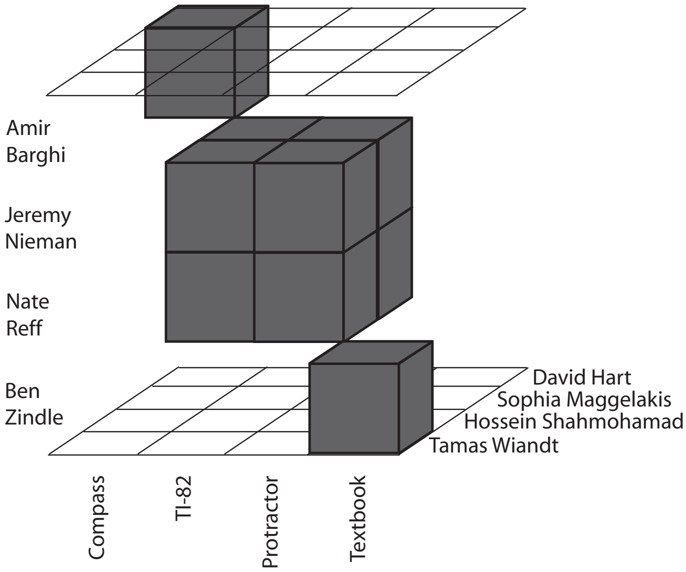

Figure 3.4: Getting a Clue

Note that the board is already decomposed into three disjoint subboards: two copies of $\mathbf{B}_{1,1,1}$ and one copy of $\mathbf{B}_{2,2,2}$. Therefore we see that the rook polynomial for this board B is:

$$R(x,B) = R(x,\mathbf{B}_{2,2,2})[R(x,\mathbf{B}_{1,1,1})]^2 =$$
$$[\sum_{k=0}^{2}\tbinom{2}{k}[P(2,k)]^2x^k][\sum_{k=0}^{1}\tbinom{1}{k}[P(1,k)]^2x^k]^2 =$$
$$(1+8x+4x^2)(1+x)^2 = 1+10x+21x^2+16x^3+4x^4.$$

Plugging the coefficients of $R(x,B)$ into Eq.(3.11) gives us the following:

$$(4!)^2 - 10\times(3!)^2 + 21\times(2!)^2 - 16\times(1!)^2 + 4\times(0!)^2 = 576 - 360 + 84 - 16 + 4 = 288.$$

Thus there are 288 ways to match each assault victim to a faculty member and mathematical implement of destruction that don't contradict the evidence at hand.■

We will now return to the counting problems from Section 2, but with the introduction of an added level of complexity.

Problem 3 - The Lottery Revisited

The Order of the Rhinos has just inducted its newest member, an affluent mathematician who promises to donate a million dollars toward the Lodge's next charity fundraiser if one of his Lodge brothers or sisters can correctly answer the following question: if the Lodge has n members, n an even number, and Member i wins a share of the weekly lottery only if he or she draws either i or $n+1-i$, then in how many ways can an n-week cycle of lottery drawings pass in which no Member i wins a share of the pot during the i^{th} or $(n+1-i)^{th}$ weeks of the cycle for the following values of n?

a.) n = 6 **b.)** n = 12

As with the earlier problems, we will develop an expression for any n, then plug in the values the problem specifies.

Theorem 3.4: Let n be an even number, and let $\mathbf{S}_{n,n,n}$ denote an $n\times n\times n$ chessboard where for every tower level i, the following positions are restricted: **1.)** row i, column i; **2.)** row i, column $n+1-i$; **3.)** row $n+1-i$, column i; and **4.)** row $n+1-i$, column $n+1-i$. Then

$$R(x, \mathbf{S}_{n,n,n}) = (1+8x+4x^2)^{\frac{n}{2}}. \tag{3.12}$$

Proof: We consider the board $\mathbf{S}_{n,n,n}$, and offer the case for $n=6$ as an example:

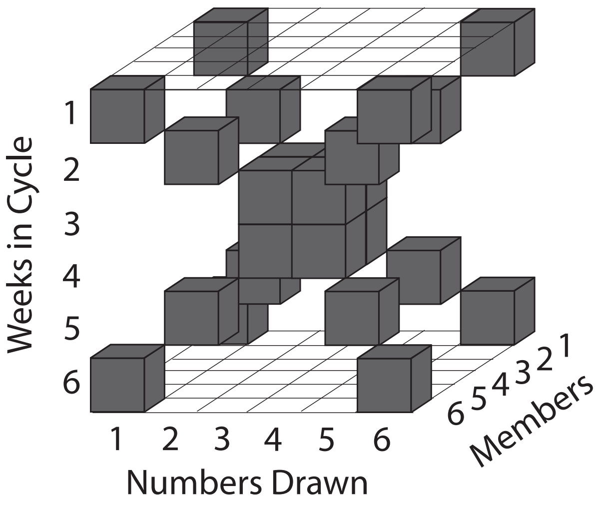

Figure 3.5a: $\mathbf{S}_{6,6,6}$

Note that for every pairing of tower levels i and $n+1-i$ the darkened squares on those levels combined constitute a disjoint subboard of $\mathbf{S}_{n,n,n}$, as none of those squares, when considered as a grouping, share any row-column plane, row-tower plane or column-tower plane with any other darkened squares within the board. Thus through a brief series of row, column and tower exchanges, we arrive at the following decomposed representation of $\mathbf{S}_{n,n,n}$:

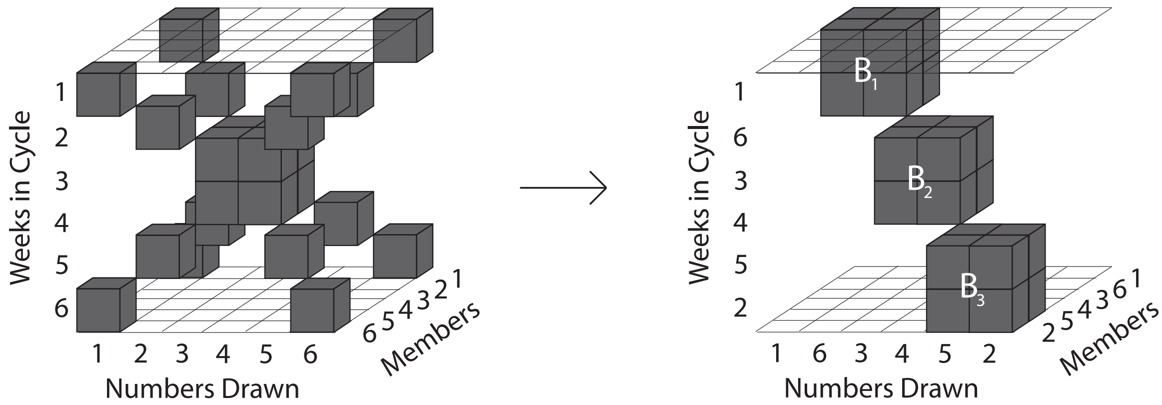

Figure 3.5b: $\mathbf{S}_{6,6,6}$ Decomposed

It is evident that in general, $\mathbf{S}_{n,n,n}$ can be decomposed into $\frac{n}{2}$ copies of $\mathbf{B}_{2,2,2}$. Therefore:

$$R(x, \mathbf{S}_{n,n,n}) = [R(x, \mathbf{B}_{2,2,2})]^{\frac{n}{2}} = [\sum_{k=0}^{2} \binom{2}{k} [P(2,k)]^2 x^k]^{\frac{n}{2}} = (1 + 8x + 4x^2)^{\frac{n}{2}}.\blacksquare$$

We begin by approaching **#3a**, where $n = 6$:

$$R(x, \mathbf{S}_{6,6,6}) = (1 + 8x + 4x^2)^3 =$$
$$1 + 24x + 204x^2 + 704x^3 + 816x^4 + 384x^5 + 64x^6.$$

Entering the coefficients of $R(x, \mathbf{S}_{6,6,6})$ into Eq.(3.11) gives:

$$(6!)^2 - 24 \times (5!)^2 + 204 \times (4!)^2 - 704 \times (3!)^2 + 816 \times (2!)^2 - 384 \times (1!)^2 + 64 \times (0!)^2 =$$

$$518,400 - 345,600 + 117,504 - 25,344 + 3,264 - 384 + 64 = 267,904.\blacksquare$$

Thus if the Lodge has 6 members, there are 267,904 ways for no member to win during a week whose number would give them a win if drawn in the lottery. Now we consider **#3b**, for $n = 12$:

$$R(x, \mathbf{S}_{12,12,12}) = (1 + 8x + 4x^2)^6 =$$
$$1 + 48x + 984x^2 + 11,200x^3 + 77,040x^4 + 327,168x^5 + 847,104x^6 + 1,308,672x^7$$
$$+1,232,640x^8 + 716,800x^9 + 251,904x^{10} + 49,152x^{11} + 4,096x^{12}.$$

Plugging the coefficients of $R(x, \mathbf{S}_{12,12,12})$ into Eq.(3.11) gives:

$$(12!)^2 - 48 \times (11!)^2 + 984 \times (10!)^2 - 11,200 \times (9!)^2 + 77,040 \times (8!)^2 - 327,168 \times (7!)^2$$
$$+847,104 \times (6!)^2 - 1,308,672 \times (5!)^2 + 1,232,640 \times (4!)^2 - 716,800 \times (3!)^2$$
$$+251,904 \times (2!)^2 - 49,152 \times (1!)^2 + 4,096 \times (0!)^2 =$$
$$229,442,532,802,560,000 - 76,480,844,267,520,000 + 12,957,498,408,960,000$$
$$-1,474,837,217,280,000 + 125,244,112,896,000 - 8,310,590,668,800 + 439,138,713,600$$
$$-18,844,876,800 + 710,000,640 - 25,804,800 + 1,007,616$$
$$-49,152 + 4,096 = 164,561,704,227,942,400. \blacksquare$$

Thus if there are 12 members in the Lodge, there are 164, 561, 704, 227, 942, 400 ways for no member to win during a week whose number would give them a win if drawn in the lottery. We now move on to the next problem.

Problem 4 - Odds and Evens Tournament

Joe and Dave, excited by their impromptu St. Paul, Minnesota Odds and Evens game, decide to enter an Odds and Evens tournament in New York City after returning from their vacation in St. George, Florida. The tournament plays host to $2n$ players [$n = 2^m$, $m \in \mathbb{N}$] who are seated randomly at one of n numbered tables in the hall where the game is to be conducted. One player at each table will be designated Odd, meaning that that player wins a round if the sum of the numbers played at that table during the round is odd. The other player will be designated Even. Two bins each containing n folded slips of paper numbered 1 to n are placed in the center of the room, one marked Odds and the other marked Evens. At the beginning of each round, all Odd players draw a number from the Odds bin and all Even players draw a number from the Evens bin. The number each player draws will be the number he or she plays for that round. At the conclusion of the round the winner at each table proceeds, while the loser is eliminated. For the next round, play moves to the first $\frac{n}{2}$ tables, with the procedure of designating players Odd or Even repeated, and the Odds and Evens bins emptied of all but the folded slips of paper numbered from 1 to $\frac{n}{2}$. Play will continue in this fashion round to round until only two players remain, who will then participate in a final round at a single table, where each player has as an option the numbers 1 and 2. Whichever player wins the final round is dubbed the winner of the tournament. We consider the following problems:

a.) If 16 players play in the tournament, in how many ways can the first round end with every Odd player a winner?

b.) In how many ways can the same tournament end with every round at every table won by an Odd player?

Given that with the conclusion of every round the size of the tournament is halved, it would be beneficial to develop an expression modeling a single round for any value of n, where $n = 2^m$, $m \in \mathbb{N}$. Thus we introduce the following theorem:

Theorem 3.5: Let n be an even number, and let $\mathbf{C}_{n,n,n}$ denote an $n \times n \times n$ chessboard with restrictions on any position whose row and column values sum to an even number. Then

$$R(x, \mathbf{C}_{n,n,n}) = \sum_{k=0}^{n} r_k(x, \mathbf{C}_{n,n}) P(n,k) x^k. \tag{3.13}$$

Proof: We will begin by considering a concrete example of the chessboard that models the problem, for $n = 4$.

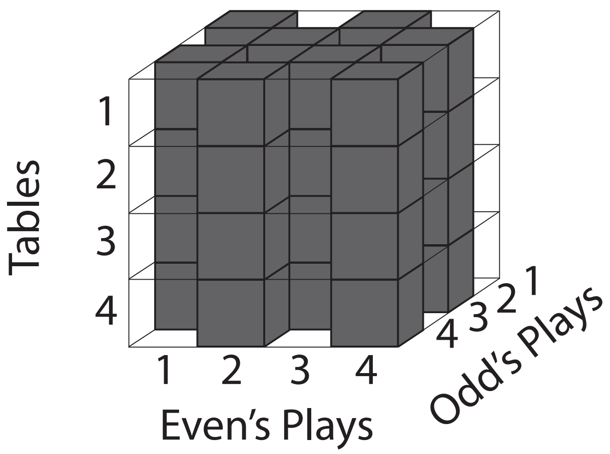

Figure 3.6a: $\mathbf{C}_{4,4,4}$

Rearranging the columns so that all even-numbered columns are gathered on the right half of the board yields the following chessboard:

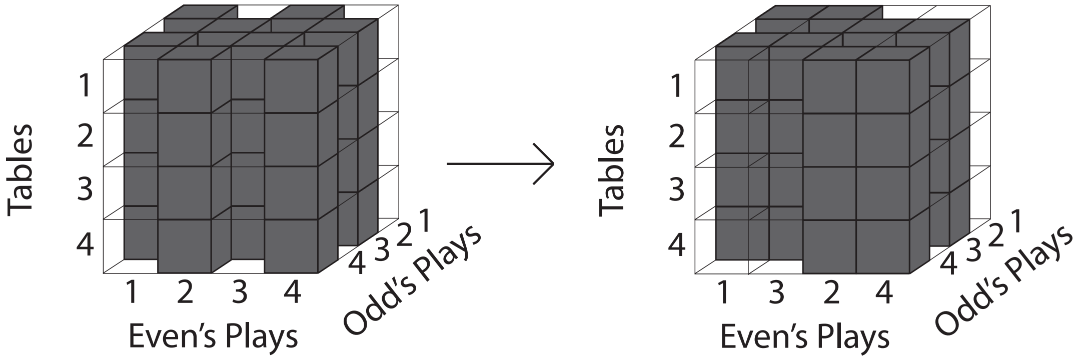

Figure 3.6b: $C_{4,4,4}$ In Transition

Now, rearranging the rows so that all odd-numbered rows are on the upper half of the board yields the final result:

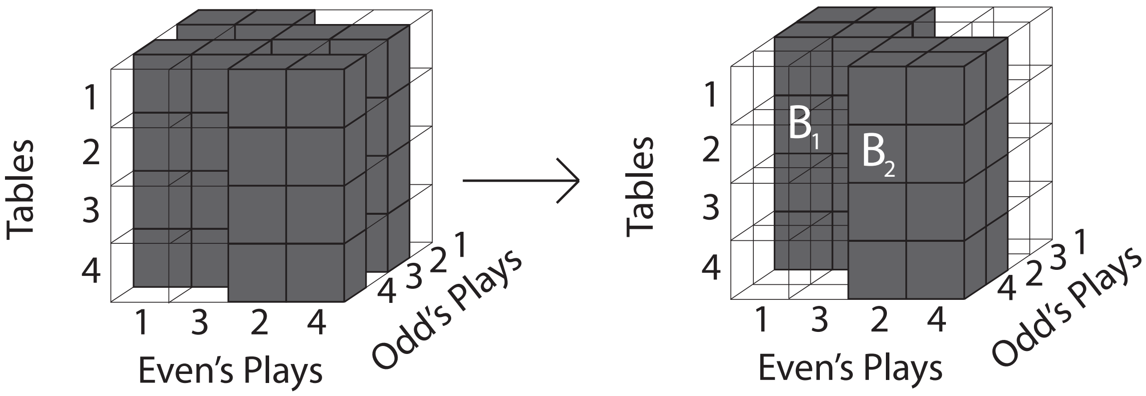

Figure 3.6c: $C_{4,4,4}$ Rearranged

Note that $\mathbf{C}_{n,n,n}$ can be rearranged into two pillars, which, although they cannot simultaneously be touched by any row-tower or column-tower planes, are not disjoint because at every level they can both be touched by a row-column plane. However, this will not prove much of a setback. We will again use the analogy of the overhead lamp and the rook shadows on the floor, from which the result will emerge simply.

If we were to place k rooks in the resultant three-dimensional board from Fig.(3.6c), the shadows of those rooks would be projected onto a two-dimensional copy of the decomposed form of $\mathbf{C}_{n,n}$, and so the number of ways to arrange the k shadows on the floor would be $r_k(x, \mathbf{C}_{n,n})$. Thus we obtain the number of ways to place k rooks in $\mathbf{C}_{n,n,n}$ by multiplying the $r_k(x, \mathbf{C}_{n,n})$ ways to determine the row-column positions of the rooks by the $P(n, k)$ ways to choose and permute the rooks' k tower positions, yielding every possible collection of rook placements within the board. Therefore, the rook polynomial for $\mathbf{C}_{n,n,n}$ over all k is

$$R(x, \mathbf{C}_{n,n,n}) = \sum_{k=0}^{n} r_k(x, \mathbf{C}_{n,n}) P(n,k) x^k. \blacksquare$$

To start, let's consider **#4a**, which will prove part of the calculation for **#4b**. Since there are $2n = 16$ players, we have that $n = 8$ for the first round.

$$R(x, \mathbf{C}_{8,8}) = [R(x, \mathbf{B}_{4,4})]^2 = [\sum_{k=0}^{4} \tbinom{4}{k} P(4,k) x^k]^2 =$$

$$[1 + (4 \times 4)x + (6 \times 12)x^2 + (4 \times 24)x^3 + (1 \times 24)x^4]^2 =$$

$$[1 + 16x + 72x^2 + 96x^3 + 24x^4]^2 =$$

$$1 + 32x + 400x^2 + 2,496x^3 + 8,304x^4 + 14,592x^5 + 12,672x^6 + 4,608x^7 + 576x^8.$$

Now that we have $R(x, \mathbf{C}_{8,8})$, we can proceed to determine $R(x, \mathbf{C}_{8,8,8})$:

$$R(x, \mathbf{C}_{8,8,8}) = \sum_{k=0}^{8} r_k(x, \mathbf{C}_{8,8}) P(8,k) x^k =$$

$$1 + (32 \times 8)x + (400 \times 56)x^2 + (2,496 \times 336)x^3 + (8,304 \times 1,680)x^4$$

$$+(14,592 \times 6,720)x^5 + (12,672 \times 20,160)x^6 + (4,608 \times 40,320)x^7 + (576 \times 40,320)x^8 =$$

$$1 + 256x + 22,400x^2 + 838,656x^3 + 13,950,720x^4 + 98,058,240x^5$$

$$+255,467,520x^6 + 185,794,560x^7 + 23,224,320x^8.$$

Plugging the coefficients of $R(x, \mathbf{C}_{8,8,8})$ into Eq.(3.11) yields:

$$(8!)^2 - 256 \times (7!)^2 + 22,400 \times (6!)^2 - 838,656 \times (5!)^2 + 13,950,720 \times (4!)^2$$

$$-98,058,240 \times (3!)^2 + 255,467,520 \times (2!)^2 - 185,794,560 \times (1!)^2 + 23,224,320 \times (0!)^2 =$$

$$1,625,702,400 - 6,502,809,600 + 11,612,160,000 - 12,076,646,400 + 8,035,614,720$$

$$-3,530,096,640 + 1,021,870,080 - 185,794,560 + 23,224,320 = 23,224,320. \blacksquare$$

Therefore there are 23,224,320 ways for every Odd player to win the first round of the tournament. To determine the answer to **#4b** [the number of ways for every Odd player to win every round of an entire 16-player tournament] requires we first determine $R(x, \mathbf{C}_{4,4,4})$ and $R(x, \mathbf{C}_{2,2,2})$, which will represent Rounds 2 and 3 of the tournament, respectively.

$$R(x, \mathbf{C}_{4,4}) = [R(x, \mathbf{B}_{2,2})]^2 = [\sum_{k=0}^{2} \binom{2}{k} P(2,k) x^k]^2 =$$
$$(1 + 4x + 2x^2)^2 = 1 + 8x + 20x^2 + 16x^3 + 4x^4.$$

$$R(x, \mathbf{C}_{2,2}) = [R(x, \mathbf{B}_{1,1})]^2 = [\sum_{k=0}^{1} \binom{1}{k} P(1,k) x^k]^2 =$$
$$(1 + x)^2 = 1 + 2x + x^2.$$

Now that we have $R(x, \mathbf{C}_{4,4})$ and $R(x, \mathbf{C}_{2,2})$, we can determine $R(x, \mathbf{C}_{4,4,4})$ and $R(x, \mathbf{C}_{2,2,2})$:

$$R(x, \mathbf{C}_{4,4,4}) = 1 + (8 \times 4)x + (20 \times 12)x^2 + (16 \times 24)x^3 + (4 \times 24)x^4 =$$
$$1 + 32x + 240x^2 + 384x^3 + 96x^4.$$

$$R(x, \mathbf{C}_{2,2,2}) = 1 + (2 \times 2)x + (1 \times 2)x^2 = 1 + 4x + 2x^2.$$

Plugging the coefficients of $R(x, \mathbf{C}_{4,4,4})$ into Eq.(3.11) yields:

$$(4!)^2 - 32 \times (3!)^2 + 240 \times (2!)^2 - 384 \times (1!)^2 + 96 \times (0!)^2 =$$
$$576 - 1,152 + 960 - 384 + 96 = 96.$$

Thus there are 96 ways for an Odd player to win at every table in Round 2. Plugging the coefficients of $R(x, \mathbf{C}_{2,2,2})$ into Eq.(3.11) yields:

$$(2!)^2 - 4 \times (1!)^2 + 2 \times (0!)^2 = 2.$$

And so there are 2 ways for the Odd players in Round 3 to win. Round 4, the final round of the tournament, is played at a single table between the two remaining players, in which there are four possible combinations of plays: (1,1), (1,2), (2,1), and (2,2). Thus there are 2 ways for the Odd player to win the final round.

To determine the number of ways that the tournament ends with every round at every table won by an Odd player, we simply multiply our results for each round, arriving at the following:

$$23,224,320 \times 96 \times 2 \times 2 = 8,918,138,880. \blacksquare$$

Thus there are $8,918,138,880$ ways for every round at every table to be won by an Odd player.

In our translation from Problem **#2** to Problem **#3** [from $\mathbf{S}_{n,n}$ to $\mathbf{S}_{n,n,n}$] we see an interesting correlation. $\mathbf{S}_{n,n}$ decomposed into $\frac{n}{2}$ copies of $\mathbf{B}_{2,2}$, while $\mathbf{S}_{n,n,n}$ decomposed into $\frac{n}{2}$ copies of $\mathbf{B}_{2,2,2}$. Likewise, for our translation from Problem **#1** to Problem **#4** we see that obtaining $R(x, \mathbf{C}_{n,n,n})$ is as simple as multiplying the coefficients r_k of $R(x, \mathbf{C}_{n,n})$ by the simple permutation $P(n,k)$. It might be interesting to determine whether such similarities persist when exploring rook polynomials for similar problems on n-dimensional chessboards.

Bibliography

Tucker, Alan. Applied Combinatorics. New York: Wiley, 2001.