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The application of the conjugate gradient method to the solution of transient electromagnetic scattering from thin wires

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Previous approaches to the problem of computing scattering by conducting bodies have utilized the well-known marching-on-in-time solution procedures. However, these procedures are very dependent on discretization techniques and sometimes lead to instabilities as the time progresses. Moreover, the accuracy of the solution cannot be verified easily, and usually there is no error estimation. In this paper we describe the conjugate gradient method for solving transient problems. For this method, the time and space discretizations are independent of one another. The method has the advantage of a direct method as the solution is obtained in a finite number of steps and also of an iterative method since the roundoff and truncation errors are limited only to the last stage of iteration. The conjugate gradient method converges for any initial guess; however, a good initial guess may significantly reduce the computation time. Also, explicit error formulas are given for the rate of convergence of this method. Hence any problem may be solved to a prespecified degree of accuracy. The procedure is stable with respect to roundoff and truncation errors and simple to apply. As an example, we apply the method of conjugate gradient to the problem of scattering from a thin conducting wire illuminated by a Gaussian pulse. The results compare well with the marching-on-in-time procedure.

1. INTRODUCTION

The primary objective of this paper is to obtain the current distribution on a thin wire as a function of time when the wire is irradiated by a narrow electromagnetic pulse. It is well known that when an incident electric field E^{inc} impinges on a thin wire of length L and radius a , the current distribution on the structure $I(z, t)$ satisfies the following integrodifferential equation:

$$\frac{1}{4\pi\epsilon_0} \left[\frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \left\{ \int_0^L \frac{I[z', t - (R/c)]}{[(z - z')^2 + a^2]^{1/2}} dz' \right\} = - \frac{\partial E^{\text{inc}}(z, t)}{\partial t} \quad (1)$$

where

$$c = \text{velocity of light} = (\mu_0 \epsilon_0)^{-1/2} \quad (2)$$

$$R = [(z - z')^2 + a^2]^{1/2} \quad (3)$$

In operator notation, (1) can be rewritten as

$$AI = Y \quad (4)$$

where A is the integrodifferential operator acting on the current I which produces the excitation Y for $0 \leq z \leq L$ and $0 \leq t \leq \infty$.

Classically, equation (1) has been solved by the marching-on-in-time procedure. The procedure is very efficient and simple to program. However, for proper implementation of the procedure the discretization in space and time has to be done with great care. Moreover, sometimes as time progresses, the solution becomes unstable. Hence in this paper we apply the method of conjugate gradient to solve equation (1).

Equation (1) has been solved in various ways. We first discuss the marching-on-in-time technique.

2. MARCHING-ON-IN-TIME SOLUTION

Marching-on-in-time has been applied previously to several types of transient scattering problems. For example, in acoustics it has been applied by Shaw [1967], Mitzner [1967], Neilson *et al.* [1978], Cole *et al.* [1978], Herman [1980, 1981], and Bennett and Mieres [1981].

For the transient analysis from wire antennas, the marching-on-in-time solution was first applied by Sayre and Harrington [1972]. However, the first working computer program for transient analysis from wire antennas is contained in the works by Ben-

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nett et al. [1970], and Auckenthaler and Bennett [1971]. Other researchers, such as Bennett [1968], Bennett and Weeks [1970], Bennett and Mieres [1981], Miller and Landt [1980], and Herman [1980, 1981], have also applied the marching-on-in-time solution with much success. A short description for the marching-on-in-time procedure is presented here for completeness.

In the forthcoming analysis we assume that the electric field is incident on the wire antenna from the broadside direction. Moreover, we assume that the current distribution on the wire can be written as

$$I(z, t) = \sum_{m,n} I(z_m, t_n) P_m(z) Q_n(t) \quad (5)$$

where

$$P_m(z) = 1 \quad m\Delta z - \frac{\Delta z}{2} \leq z < m\Delta z + \frac{\Delta z}{2} \quad (6)$$

$$P_m(z) = 0 \quad \text{otherwise}$$

and

$$Q_n(t) = 1 \quad n\Delta t - \frac{\Delta t}{2} \leq t < n\Delta t + \frac{\Delta t}{2} \quad (7)$$

$$Q_n(t) = 0 \quad \text{otherwise}$$

We observe in (1) that at $z = z'$, $R = a$. By utilizing (5)–(7), we can write

$$\begin{aligned} & \frac{1}{4\pi\epsilon_0} \left[\frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \left\{ \int_{z'=z-(\Delta z/2)}^{z+\Delta z/2} \frac{I(z', t) dz'}{[(z-z')^2 + a^2]^{1/2}} \right\} \\ & + \frac{1}{4\pi\epsilon_0} \left[\frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \left\{ \int_{z'=0}^L \frac{I[z', t - (R/c)] dz'}{[(z-z')^2 + a^2]^{1/2}} \right\} \\ & = - \frac{\partial E^{\text{inc}}}{\partial t} \end{aligned} \quad (8)$$

where the second integral indicates that the effect of the self term is not included. By approximating the current to be constant in the interval Δz at a given time we can rewrite (8) as

$$\begin{aligned} I(z_m, t_n) &= 2I(z_m, t_{n-1}) - I(z_m, t_{n-2}) - \left(\frac{c\Delta t}{\Delta z} \right)^2 G(z_m, t_{n-1}) \\ &+ \left(\frac{c\Delta t}{\Delta z} \right)^2 [I(z_{m+1}, t_{n-1}) - 2I(z_m, t_{n-1}) + I(z_{m-1}, t_{n-1})] \end{aligned} \quad (9)$$

where

$$\begin{aligned} G(z_m, t_n) &= - \frac{\epsilon_0(\Delta z)^2}{\alpha} \frac{\partial E^{\text{inc}}(t_n)}{\partial t} - \frac{1}{4\pi\alpha} \\ &\cdot [F(z_{m+1}, \tau_n) - 2F(z_m, \tau_n) + F(z_{m-1}, \tau_n)] \\ &+ \frac{1}{4\pi\alpha} \left(\frac{\Delta z}{c\Delta t} \right)^2 [F(z_m, \tau_{n+1}) - 2F(z_m, \tau_n) + F(z_m, \tau_{n-1})] \end{aligned} \quad (10)$$

$$\begin{aligned} \alpha &= \frac{1}{4\pi} \int_{z'=z-\Delta z/2}^{z+\Delta z/2} \frac{dz'}{[(z-z')^2 + a^2]^{1/2}} \\ &= \frac{1}{2\pi} \log \left[\frac{(\Delta z/2) + [(\Delta z/2)^2 + a^2]^{1/2}}{a} \right] \end{aligned} \quad (11)$$

$$F(z_m, \tau_n) = \int_0^L \frac{I[z', t_n - (R/c)] dz'}{[(z_m - z')^2 + a^2]^{1/2}} \quad (12)$$

From equations (9)–(12) it is clear that this procedure is based upon calculation, at successive instants, of the unknown field values that have been calculated at earlier instants. The method is quite efficient for computing transient responses of objects which are of the order of the pulse width of the incident field. An important advantage of this method, which has been frequently stressed [Bennett, 1968; Bennett and Weeks, 1970; Mitzner, 1967; Herman, 1980, 1981] is the fact that no matrix inversion is required if the time step Δt is smaller than a certain upper value which is, among others, determined by spatial discretization of the object.

An important disadvantage with this procedure is the possible occurrence of rapidly growing spurious oscillations at later instants which is apparently due to the accumulation of errors during the calculations. These instabilities have been touched upon by several authors [Miller and Landt, 1980; Neilson et al., 1978; Herman, 1980, 1981; Bennett and Mieres, 1981]. Although it may be possible to eliminate these instabilities in many cases by a smoothing procedure, the accumulation of errors during calculation puts limitation on the applicability of this technique [Herman, 1980, 1981]. Moreover, with the marching-on-in-time procedure the accuracy of the solution cannot easily be verified and usually there is no error estimation. That is why we discuss iterative methods in which the accumulation of errors in time does not take place. Also, for the iterative methods the time and space discretizations are independent of one another. There are two iterative methods which we will

discuss, namely, the method of steepest descent and the method of conjugate gradient.

3. METHOD OF STEEPEST DESCENT

A serious limitation on the applicability of the marching-on-in-time method is the accumulation of errors during the calculation, which can lead to a spurious oscillatory behavior of rapid growth at later instants as illustrated by *Herman* [1980, 1981]. The iterative methods, however, do not suffer from instabilities of this kind [Herman, 1980, 1981]. This is because the iterative methods are based upon the successive minimization of a suitably defined integrated squared error. The method is described and presented by *Huizer et al.* [1981]. The method they utilized is a modified version of the method of steepest descent. In their papers, the following functional E has been minimized by an iterative method.

$$E = \int_0^T dt \int_0^L dz [AI - Y]^2 \quad (13)$$

In (13), T is large enough so that the transients have died down. *Herman* [1980, 1981] and P. Vandenberg (personal communication, 1981) applied the method of steepest descent to solve the operator equation of (4) and minimize the error of (13). Their method is identical to the steepest descent method described by *Sarkar and Rao* [1982]. The method of steepest descent is described by the iteration

$$I_{k+1} = I_k - \frac{\|A^*(AI_k - Y)\|^2}{\|AA^*(AI_k - Y)\|^2} A^*(AI_k - Y) \quad (14)$$

where A^* is the adjoint operator for (1) and has been derived in the appendix. In addition,

$$\|a\|^2 = \langle a, a \rangle = \int_0^T dt \int_0^L dz a^2(z, t) \quad (15)$$

The method of steepest descent converges with a geometric progression and the rate of convergence is given by

$$\frac{\|I_k - I_{\text{exact}}\|}{\|I_0 - I_{\text{exact}}\|} \leq C \left[\left(\frac{B-b}{B+b} \right)^k + \left(\frac{B+b}{B-b} \right)^k \right]^{-1} \quad (16)$$

where B and b are the maximum and the minimum eigenvalues of A^*A in the finite dimensional space in which the problem is solved, where C is a constant.

It has been shown by *Sarkar et al.* [1981] and *Sarkar and Rao* [1982, 1984] that the method of steepest descent is quite slow compared to the method of

conjugate gradient. In an earlier paper the method of conjugate gradient has been applied to solve for electromagnetic scattering from wire antennas [Sarkar, this issue]. In this paper we apply the method of conjugate gradient for the solution of transient electromagnetic problems.

4. METHOD OF CONJUGATE GRADIENT

For the conjugate gradient method, we try to minimize the error defined by (13) at each iteration. The conjugate gradient method starts with an initial guess $I_0(z, t)$ and generates

$$P_0 = -b_0 A^*R_0 = -b_0 A^*(AI_0 - Y) \quad (17)$$

where A^* represents the adjoint operator for A . The expression for the adjoint operator has been developed in the appendix. The conjugate gradient method then develops

$$I_{k+1} = I_k + \alpha_k P_k \quad (18)$$

$$R_{k+1} = R_k + \alpha_k AP_k \quad (19)$$

$$\alpha_k = \frac{1}{\|AP_k\|^2} \quad (20)$$

$$P_{k+1} = P_k - b_{k+1} A^*R_{k+1} \quad (21)$$

and

$$b_k = \frac{1}{\|A^*R_k\|^2} \quad (22)$$

The conjugate gradient method described by (17)–(22) converges for any initial guess in at most M steps, where M is the number of distinct eigenvalues of the operator A in the finite dimensional space in which the problem is solved.

The conjugate gradient method converges as fast as a geometric series [Sarkar et al., 1981; Sarkar and Rao, 1984], and the rate of convergence is given by

$$\frac{\|I_k - I_{\text{exact}}\|}{\|I_0 - I_{\text{exact}}\|} \leq D \left[\left(\frac{B^{1/2} + b^{1/2}}{B^{1/2} - b^{1/2}} \right)^k + \left(\frac{B^{1/2} - b^{1/2}}{B^{1/2} + b^{1/2}} \right)^k \right]^{-1} \quad (23)$$

where B and b are the maximum and the minimum eigenvalues of the operator A^*A , in the finite dimensional space in which the problem is being solved, where D is a constant.

Even though the conjugate gradient requires considerably more work than the method of steepest descent (equations (17)–(22) as compared to (14)), the

Fig. 1a. Transient waveform at location 1 of Figure 1c.

reward of doing more work lies in the fact that the conjugate gradient method has a faster rate of convergence than the method of steepest descent ((23) as compared to (16)).

Because of rounding errors in a numerical calculation and of iterative calculations of R_k in (19) and P_k of (21), instability may result, just as in the case of the marching-in-on-time solution procedure. One way to detect numerical instability in the computation process, automatically, is to look at the ratio α_k/α_{k+1} . All the scalars α_k , $k = 1, 2, \dots$, lie in the range

$$\frac{1}{B} < \alpha_k < \frac{1}{b} \quad (24)$$

where B and b are the maximum and the minimum eigenvalues of A^*A in the finite dimensional space, in which the problem is solved. An upper bound for α_k/α_{k-1} is B/b and hence stability may be low if B/b is large.

However, *Hestenes and Steifel* [1952] have shown that when A^*A has distinct eigenvalues and if one starts with an initial guess I_0 such that the residual R_0 is near the eigenvector corresponding to the minimum eigenvalue b of A^*A , then α_k/α_{k-1} is always less than unity and the conjugate gradient method is always stable with respect to roundoff errors. This is an assurance that for certain initial guesses, propagation of roundoff errors has very little effect on the computational accuracy.

Fig. 1b. Transient waveform at location 2 of Figure 1c.

Fig. 1c. Antenna configuration.

5. NUMERICAL RESULTS

As an example, consider a 1-m-dipole of radius 0.005 m irradiated by an electromagnetic pulse of the form

$$E^{inc} = \frac{\eta}{\pi^{1/2} c \sigma} \exp \left[-\frac{(t - t_0)^2}{\sigma^2} \right] \quad (25)$$

from the broadside direction. In (25), η is the characteristic impedance of free space and is given by

$$\begin{aligned} \eta &= 377 \text{ ohms} \\ c &= 3 \times 10^8 \text{ m/s} \\ \sigma &= 0.5c \\ t_0 &= 6\sigma \end{aligned} \quad (26)$$

The 1-m-long antenna is divided into four segments so that $\Delta z = 0.25$ m. The marching-on-in-time solution is next applied to solve for $I(z, t)$ with $\Delta t = 0.25$ light-meter. The solution obtained by the marching-on-in-time procedure at different points on the antenna is shown in Figures 1a and 1b.

We then applied the conjugate gradient method for the solution of the same transient problem. In this case we develop a two-dimensional grid in the (z, t) plane. There are four equispaced divisions on the z axis separated by the following points at $(z = 0, \Delta z, 2\Delta z, 3\Delta z, 4\Delta z = 1)$ and 49 divisions on the t axis separated by $\Delta t = 0.25$. On the grid of 5×49 points we minimize the square of the residual defined by (13) in an iterative fashion by the conjugate gradient method. We start with an initial guess $I_0(z, t)$ on the grid, which is identically zero for our case. We then perform the following integration operation and

obtain

$$Q(z, t) = \int_0^L \frac{I_0[z', t - (R/c)]}{[(z - z')^2 + a^2]^{1/2}} dz' \quad (27)$$

with the assumption that

$$I(z = 0, t) = 0 \quad I(z = L, t) = 0$$

$$I(z, t = 0) = 0 \quad I(z, t = 12) = 0$$

We next perform the double derivative operation by the finite difference operation on $Q(z, t)$, so that

$$AI_0 = \frac{1}{4\pi\epsilon_0} \left[\frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] Q(z, t) \quad (28)$$

Then we obtain Y analytically by performing the derivative operation on E^{inc} to obtain $\partial E^{\text{inc}}(z, t)/\partial t$. This completes the evaluation of $R_0 = AI_0 - Y$. The next step is the computation of A^*R_0 . In this case we require the adjoint operator. The adjoint operator has been evaluated in the appendix and it can be shown to be the advance convolution operator. Therefore

$$A^*R_0 = \frac{1}{4\pi\epsilon_0} \left[\frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \int_0^L \frac{R_0[z', t + (R/c)]}{[(z - z')^2 + a^2]^{1/2}} dz' \quad (29)$$

Observe that the difference between (27) and (29) is that (29) contains the advance convolution in the integrand as opposed to the classical retarded convolution in (27). The steps (17)–(22) are successively applied until a desired convergence is achieved. In our case we continued the iteration until the following error criterion was satisfied:

$$\frac{\|AI_k - Y\|}{\|Y\|} \leq 10^{-4} \quad (30)$$

The results for the conjugate gradient method are visually indistinguishable from the results obtained by the marching-on-in-time solution, presented in Figure 1. In this case, the conjugate gradient method took approximately 5 times more CPU time than the marching-on-in-time solution. The method of steepest descent was considerably slower than the conjugate gradient method.

6. EPILOGUE

In summary, the method of conjugate gradient is not generally going to replace the marching-on-in-time solution procedure for obtaining transient response of objects. However, for cases when the

marching-on-in-time solution procedure fails, particularly for late times in low- Q circuits, and for fast rise time incident pulses we advocate the utilization of the conjugate gradient method. The other advantage of the conjugate gradient method over the marching-on-in-time solution procedure is that the time and space discretizations are independent of one another and not related as in the marching-on-in-time solution.

It is important to point out that *Herman* [1981] demonstrated in his thesis that for low- Q transient acoustic scattering the method of steepest descent converges much faster than the marching-on-in-time solution for the same degree of accuracy in the solution. This was achieved by choosing a time step which was approximately 3 times larger than the time step required for the marching-on-in-time solution. In our case, we could not make the time step too large, because we had to obtain $\partial^2/\partial t^2$ numerically. However, it is conjectured that if one utilizes the conjugate gradient method for solving transient problems utilizing H field integral equations (as was done by Herman), it is quite possible that the conjugate gradient may take less time than the marching-on-in-time solution procedure. This is because for the H field integral equation there is no derivative operator and one could choose a larger step size for the variable for which the finite differencing is not required. Work is now in progress to verify this conjecture.

APPENDIX: DERIVATION OF THE ADJOINT OPERATOR

In order to obtain the adjoint operator we have to utilize the following identity:

$$\langle AI, J \rangle = \langle I, A^*J \rangle + B(I, J) \quad (A1)$$

where A^* is the adjoint operator, J is the adjoint function, and $B(I, J)$ is the bilinear concomitant and yields the adjoint boundary conditions when equated to zero. We have

$$\langle AI, J \rangle = \int_0^\infty dt \int_0^L dz \frac{J(z, t)}{4\pi\epsilon_0} \left[\frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \int_0^L dz' \frac{I[z', t - (R/c)]}{R} \quad (A2)$$

where

$$R = [(z - z')^2 + a^2]^{1/2} \quad (A3)$$

What makes the derivation of the adjoint operator a

little complicated is the presence of the term $(t - R/c)$. We also know that the spatial and temporal variables are separable, i.e.,

$$I(z', t) = g(z')h(t) \quad (\text{A4})$$

where $g(z')$ is a function of space only and $h(t)$ varies with time and not with space. By utilizing (A4) into (A2) we obtain

$$\langle AI, J \rangle = \int_0^\infty dt \int_0^L dz \frac{J(z, t)}{4\pi\epsilon_0} \left[\frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \cdot \int_0^L dz' \frac{g(z')h[t - (R/c)]}{R} \quad (\text{A5})$$

$$\langle AI, J \rangle \triangleq \frac{1}{4\pi\epsilon_0} \left[\int_0^\infty dt P - \frac{1}{c^2} \int_0^L dz \int_0^L dz' Q \right] \quad (\text{A6})$$

where

$$P \triangleq \int_0^L dz J(z, t) \frac{\partial^2}{\partial z^2} \int_0^L dz' \frac{g(z')h[t - (R/c)]}{R} \quad (\text{A7})$$

$$Q \triangleq \int_0^\infty dt J(z, t) \frac{\partial^2}{\partial t^2} \left[\frac{g(z')h[t - (R/c)]}{R} \right] \quad (\text{A8})$$

We rewrite (A7) as

$$P = \int_0^L dz J(z, t) \frac{\partial}{\partial z} \int_0^L dz' g(z') \frac{\partial}{\partial z} \left[\frac{h[t - (R/c)]}{R} \right] \quad (\text{A9})$$

Since

$$\frac{\partial}{\partial z} \left[\frac{h[t - (R/c)]}{R} \right] = - \frac{\partial}{\partial z'} \left[\frac{h[t - (R/c)]}{R} \right] \quad (\text{A10})$$

(A9) can be rewritten as

$$P = - \int_0^L dz J(z, t) \frac{\partial}{\partial z} \int_0^L dz' g(z') \frac{\partial}{\partial z'} \left[\frac{h[t - (R/c)]}{R} \right] \quad (\text{A11})$$

We now perform an integration by parts on the second integral to obtain

$$P = \int_0^L dz J(z, t) \frac{\partial}{\partial z} \int_0^L dz' \frac{dg(z')}{dz'} \left[\frac{h[t - (R/c)]}{R} \right] \quad (\text{A12})$$

as $g(z') = 0$ at $z' = 0$ and L .

We now perform another integration by parts on the first integral. This results in

$$P = \left[J(z, t) \int_0^L dz' \frac{dg(z')}{dz'} \left\{ \frac{h[t - (R/c)]}{R} \right\} \right]_{z=0}^L - \int_0^L dz \frac{\partial J(z, t)}{\partial z} \int_0^L dz' \frac{dg(z')}{dz'} \frac{h[t - (R/c)]}{R} \quad (\text{A13})$$

We now interchange the order of integration to obtain

$$P = \left[J(z, t) \int_0^L dz' \frac{dg(z')}{dz'} \frac{h[t - (R/c)]}{R} \right]_{z=0}^L - \int_0^L dz' \frac{dg(z')}{dz'} \int_0^L dz \frac{\partial J(z, t)}{\partial z} \frac{h[t - (R/c)]}{R} \quad (\text{A14})$$

We now perform an integration by parts on the second integral and obtain

$$P = \left[J(z, t) \int_0^L dz' \frac{dg(z')}{dz'} \frac{h[t - (R/c)]}{R} \right]_{z=0}^L - \left[\int_0^L dz' \frac{dg(z')}{dz'} J(z, t) \frac{h[t - (R/c)]}{R} \right]_{z=0}^L + \int_0^L dz' \frac{dg(z')}{dz'} \int_0^L dz J(z, t) \frac{\partial}{\partial z} \left[\frac{h[t - (R/c)]}{R} \right] \quad (\text{A15})$$

Observe that the first two terms cancel each other and the expression can be finally rewritten as

$$P = - \int_0^L dz' \frac{dg(z')}{dz'} \frac{\partial}{\partial z'} \int_0^L dz J(z, t) \frac{h[t - (R/c)]}{R} \quad (\text{A16})$$

We now integrate the first integral by parts and by utilizing $g(z') = 0$ at $z' = 0$ and L , we get

$$P = \int_0^L dz' g(z') \frac{\partial^2}{(\partial z')^2} \int_0^L dz J(z, t) \frac{h[t - (R/c)]}{R} \quad (\text{A17})$$

We now evaluate Q by integrating (A8) by parts. This yields

$$Q = \frac{\partial}{\partial t} \left[g(z') \frac{h[t - (R/c)]}{R} J(z, t) \right]_{t=0}^\infty - \int_0^\infty dt g(z') \frac{\partial J(z, t)}{\partial t} \frac{\partial}{\partial t} \left[\frac{h[t - (R/c)]}{R} \right] \quad (\text{A18})$$

Since $I(z', t)$ is causal, i.e.,

$$I(z', t) = 0 \quad t \leq 0 \quad (\text{A19})$$

and also of finite energy, the first term of (A18) must be zero. Therefore

$$Q = - \int_0^\infty dt g(z') \frac{\partial J(z, t)}{\partial t} \frac{\partial}{\partial t} \left[\frac{h[t - (R/c)]}{R} \right] \quad (\text{A20})$$

We further integrate Q by parts to obtain

$$Q = \int_0^\infty dt g(z') \frac{h[t - (R/c)]}{R} \frac{\partial^2 J(z, t)}{\partial t^2} \quad (\text{A21})$$

By utilizing (A17) and (A21) in (A6) we obtain

$$\begin{aligned} \langle AI, J \rangle = & \int_0^\infty dt \int_0^L dz' \frac{g(z')}{4\pi\epsilon_0} \frac{\partial^2}{(\partial z')^2} \\ & \cdot \int_0^L dz \frac{J(z, t)h[t - (R/c)]}{R} \\ & - \frac{1}{4\pi\epsilon_0 c^2} \int_0^L dz' \int_0^L dz \int_0^\infty dt g(z') \\ & \cdot \frac{h[t - (R/c)]}{R} \frac{\partial^2 J(z, t)}{\partial t^2} \end{aligned} \quad (A22)$$

We now introduce a new variable τ , such that

$$\tau = t - (R/c) \quad (A23)$$

This results in

$$\begin{aligned} \langle AI, J \rangle = & \int_0^L dz' \frac{g(z')}{4\pi\epsilon_0} \\ & \cdot \frac{\partial^2}{(\partial z')^2} \int_0^L dz \int_{-R/c}^\infty \frac{J[z, \tau + (R/c)]}{R} h(\tau) d\tau \\ & - \frac{1}{4\pi\epsilon_0 c^2} \int_0^L dz' g(z') \int_0^L dz \int_{-R/c}^\infty \frac{h(\tau)}{R} \\ & \cdot \frac{\partial^2 J[z, \tau + (R/c)]}{\partial \tau^2} d\tau \end{aligned} \quad (A24)$$

The lower limit for τ should now be set equal to zero as $I(z', \tau)$ is causal, i.e.,

$$I(z', \tau) = g(z')h(\tau) = 0 \quad \tau \leq 0 \quad (A25)$$

This results in

$$\begin{aligned} \langle AI, J \rangle = & \int_0^L dz' \frac{g(z')}{4\pi\epsilon_0} \\ & \cdot \frac{\partial^2}{(\partial z')^2} \int_0^L dz \int_0^\infty d\tau \frac{J[z, \tau + (R/c)]h(\tau)}{R} \\ & - \frac{1}{4\pi\epsilon_0 c^2} \int_0^L dz' g(z') \int_0^L dz \int_0^\infty \frac{h(\tau)}{R} \\ & \cdot \frac{\partial^2 J[z, \tau + (R/c)]}{\partial \tau^2} d\tau \end{aligned} \quad (A26)$$

Now, replacing τ by the dummy variable t we obtain

$$\begin{aligned} \langle AI, J \rangle = & \int_0^L dz' \int_0^\infty dt \frac{I(z', t)}{4\pi\epsilon_0} \\ & \cdot \frac{\partial^2}{(\partial z')^2} \int_0^L dz \frac{J[z, t + (R/c)]}{R} \end{aligned}$$

$$\begin{aligned} & - \frac{1}{4\pi\epsilon_0 c^2} \int_0^L dz' \int_0^\infty dt I(z', t) \\ & \cdot \frac{\partial^2}{\partial t^2} \int_0^L dz \frac{J[z, t + (R/c)]}{R} \\ & = \int_0^\infty dt \int_0^L dz' \frac{I(z', t)}{4\pi\epsilon_0} \\ & \cdot \left[\frac{\partial^2}{(\partial z')^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \int_0^L dz \frac{J[z, t + (R/c)]}{R} \end{aligned}$$

$$= \langle I, A^*J \rangle + B(I, J) \quad (A27)$$

Now we obtain the adjoint operator as

$$A^*J = \frac{1}{4\pi\epsilon_0} \left[\frac{\partial^2}{(\partial z')^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \int_0^L dz \frac{J[z, t + (R/c)]}{R} \quad (A28)$$

and there are no adjoint boundary conditions for the adjoint operator since $B(I, J) \equiv 0$. It is important, therefore, to point out that even though the operator A has four boundary conditions, namely,

$$I(0, t) = 0 \quad I(L, t) = 0$$

$$I(z', 0) = 0 \quad I(z', \infty) = 0$$

the adjoint operator has no boundary conditions. By comparing (A2) and (A28) it is seen that the adjoint operator is merely the advance convolution operator except that it has no boundary conditions. Another important point we would like to point out is that in deriving the adjoint operator we assumed that the upper limit of $t = \infty$. This can never be achieved in actual computations as we can only compute up to a finite time, $t = T$. Therefore the adjoint operator given by (A28) is not strictly valid for numerical computations. However, if we take our time T long enough such that

$$I(z', T) \simeq 0$$

then (A28) becomes the adjoint operator and the derivation is pragmatically valid with T replacing ∞ in the deviations.

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REFERENCES

- Auckenthaler, A. M., and C. L. Bennett, Computer solution of transient time domain thin-wire antenna problems, *IEEE Trans. Microwave Theory Tech.*, 19, 892-893, 1971.
- Bennett, C. L., A technique for computing approximate electromagnetic pulse response of conducting bodies, Ph.D. thesis, Purdue Univ., Lafayette, Indiana, 1968.

- Bennett, C. L., and A. M. Auckenthaler, Transient and time-domain solutions for antennas and scatterers, paper presented at IEEE International Convention, New York, 1971.
- Bennett, C. L., and J. Martine, A space-time integral equation for currents on wire structures with arbitrary excitations, paper presented at International Conference, URSI, Columbus, Ohio, 1970.
- Bennett, C. L., and H. Mieres, Time-domain integral equation solution of acoustic scattering from fluid targets, *J. Acoust. Soc. Am.*, **69**, 1261-1265, 1981.
- Bennett, C. L., and G. F. Ross, Time-domain electromagnetics and its applications, *Proc. IEEE*, **66**, 299-318, 1978.
- Bennett, C. L., and W. L. Weeks, Transient scattering from conducting cylinders, *IEEE Trans. Antennas Propag.*, **AP-18**, 627-633, 1970.
- Bennett, C. L., J. D. DeLorenzo, and A. M. Auckenthaler, Integral equation approach to wideband inverse scattering, final report on contract F30602-69-C-0332; TR SCRC-CR-70-16; vol. I, II, AD876849, AD876627, Defense Documentation Center, Alexandria, Va., June 1970.
- Cole, D. M., D. D. Kosloff, and J. B. Minister, A numerical boundary-integral equation method for elastodynamics I, *Bull. Seismol. Soc. Am.*, **68**, 1331-1357, 1978.
- Herman, G. C., Scattering of transient acoustic waves by an inhomogeneous obstacle, paper presented at Symposium, URSI, Munich, 1980.
- Herman, G. C., Scattering of transient acoustic waves in fluids and solids, Ph.D. thesis, Delft Inst. of Technol., Delft, The Netherlands, 1981.
- Hestenes, M., and E. Steifel, Method of conjugate gradients for solving linear systems, *J. Res. Natl. Bur. Stand.*, **49**, 409-436, 1952.
- Huizer, A. J. M., A. Quattrapani, and H. P. Baltes, Numerical solution of electromagnetic scattering problems. Case of perfectly conducting cylinders, *Opt. Commun.*, **37**, 307-310, 1981.
- Miller, E. K., and J. A. Landt, Direct time domain techniques for transient radiation and scattering from wires, *Proc. IEEE*, **68**, 1396-1423, 1980.
- Mitzner, K. M., Numerical solution for transient scattering from a hard surface by arbitrary shape-retarded potential technique, *J. Acoust. Soc. Am.*, **42**, 391-397, 1967.
- Neilson, H. C., Y. P. Lu, and Y. F. Wang, Transient scattering by arbitrary isymmetric surfaces, *J. Acoust. Soc. Am.*, **63**, 1719-1726, 1978.
- Sarkar, T. K., The application of the conjugate gradient method for the solution of operator equations arising in electromagnetic scattering from wire antennas, *Radio Sci.*, this issue.
- Sarkar, T. K., and S. M. Rao, An iterative method for solving electrostatic problems, *IEEE Trans. Antennas Propag.*, **30**, 611-616, 1982.
- Sarkar, T. K., and S. M. Rao, The application of the conjugate gradient method for the solution of electromagnetic scattering from wire antennas, *IEEE Trans. Antennas Propag.*, **32**, 398-403, 1984.
- Sarkar, T. K., K. Siarkiewicz, and R. Stratton, Survey of numerical methods for solution of large systems of linear equations for electromagnetic field problems, *IEEE Trans. Antennas Propag.*, **29**, 847-856, 1981.
- Sayre, E., and R. F. Harrington, Time domain radiation and scattering by thin wires, *Appl. Sci. Res.*, **26**, 413-444, 1972.
- Shaw, R. P., Diffraction of acoustic pulses by obstacles of arbitrary shape with a Robin boundary condition, *J. Acoust. Soc. Am.*, **41**, 855-859, 1967.

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