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Computing the Folkman Number $F_v(2, 2, 3; 4)$

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Abstract

We discuss a branch of Ramsey theory concerning vertex Folkman numbers and how computer algorithms have been used to compute a new Folkman number. We write $G \rightarrow (a_1, \ldots, a_k)^v$ if for every vertex k-coloring of an undirected simple graph G, a monochromatic K_{a_i} is forced in color $i \in \{1, \ldots, k\}$. The vertex Folkman number is defined as $F_v(a_1, \ldots, a_k; p) = \min\{|V(G)| : G \rightarrow (a_1, \ldots, a_k)^v \wedge K_p \not\subseteq G\}$. Folkman showed in 1970 that this number exists for $p > \max\{a_1, \ldots, a_k\}$. Let $m = 1 + \sum_{i=1}^{k}(a_i - 1)$ and $a = \max\{a_1, \ldots, a_k\}$, then $F_v(a_1, \ldots, a_k; p) = m$ for $p > m$, and $F_v(a_1, \ldots, a_k; p) = a + m$ for $p = m$. For $p < m$ the situation is more difficult and much less is known. We show here that, for a case of $p = m-1$, $F_v(2, 2, 3; 4) = 14$.

1 Introduction

Let G be a simple, undirected graph with vertex set $V(G)$ and edge set $E(G)$. The chromatic number of G will be denoted by $\chi(G)$, and the independence number of G by $\alpha(G)$. For positive integers a_i, we write $G \rightarrow (a_1, \ldots, a_k)^v$ if for every vertex k-coloring of G, a monochromatic K_{a_i} is forced in some color $i \in \{1, \ldots, k\}$. Let

$$H_v(a_1, \ldots, a_k; p) = \{G : G \rightarrow (a_1, \ldots, a_k)^v \wedge K_p \not\subseteq G\} \ .$$

The graphs in the set $\mathcal{H} = H_v(a_1, \ldots, a_k; p)$ are called Folkman graphs. Folkman [2] (also [6]) showed that $\mathcal{H}$ is non-empty for $p > \max\{a_1, \ldots, a_k\}$.

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$H_v(a_1, \ldots, a_k; p; n)$ will denote the set of Folkman graphs with n vertices. Folkman graphs are maximal when the addition of any other edge will create the forbidden K_p. Similarly, a Folkman graph is minimal when the deletion of any edge causes the graph to lose the Folkman property. We define the vertex Folkman numbers by

$$F_v(a_1, \ldots, a_k; p) = \min\{|V(G)| : G \in H_v(a_1, \ldots, a_k; p)\} \ .$$

The Ramsey number $R(r, l)$ is the smallest number n such that all edge 2-colorings of K_n contain either a monochromatic K_r in the first color or a monochromatic K_l in the second color [3]. A graph G is an (r, l)-Ramsey graph if G has no K_r and $\alpha(G) < l$. The set $\mathscr{R}(r, l; n)$ is the set of all (r, l)-Ramsey graphs on n vertices.

Let $m = 1 + \sum_{i=1}^{k}(a_i - 1)$. If $G \to (a_1, \ldots, a_k; p)$ then $\chi(G) \geq m$ [13]. By the pigeon-hole principle it is easy to see that $F_v(a_1, \ldots, a_k; p) = m$ for $p > m$. Łuczak et al. [7] showed that $F_v(a_1, \ldots, a_k; p) = a_k + m$ for $p = m$. For $p < m$ much less is known. Only one nontrivial value is known for $p = m - 2$: $F_v(2, 2, 2, 2; 3) = 22$ computed by Jensen and Royle [4] in 1995. Most research has focused on the case $p = m - 1$. For a summary of other results see [6].

In 2000, Nenov [13] showed that $10 \leq F_v(2, 2, 3; 4) \leq 14$; he proved the upper bound using the 14-node graph Γ_3, depicted in Figure 1. No graph with fewer than 14 vertices is known to exist in $H_v(2, 2, 3; 4)$. It is shown here that there are *no* such graphs and thus, $F_v(2, 2, 3; 4) = 14$. Nenov and Nedialkov also studied several other parameter situations in [10, 11, 12, 13, 14, 15, 16, 17]. For the purposes of this paper, references to Folkman graphs will mean graphs in $H_v(2, 2, 3; 4)$ unless otherwise stated.

Proving exact values of Folkman numbers by hand is often very difficult since deriving the lower bound requires a non-existence proof. Computers can be of great help, but because showing non-existence often entails large searches, the algorithms must be carefully designed to work as efficiently as possible.

To find the exact value of $F_v(2, 2, 3; 4)$ with the aid of computers, there are two main approaches. One approach is to check all graphs of order < 14 for inclusion in $H_v(2, 2, 3; 4)$, starting with graphs of order 10. If a graph is found to be Folkman, then the current order is the Folkman number. If not, the next higher order must be checked. A second approach is to find all Folkman graphs on 14 vertices and drop a vertex from each one in all possible ways. If one of the resulting graphs is Folkman, then 13 is the new upper bound and the process can be repeated on the Folkman graphs with 13 vertices. Once no smaller Folkman graphs are obtained, the Folkman number has been found.

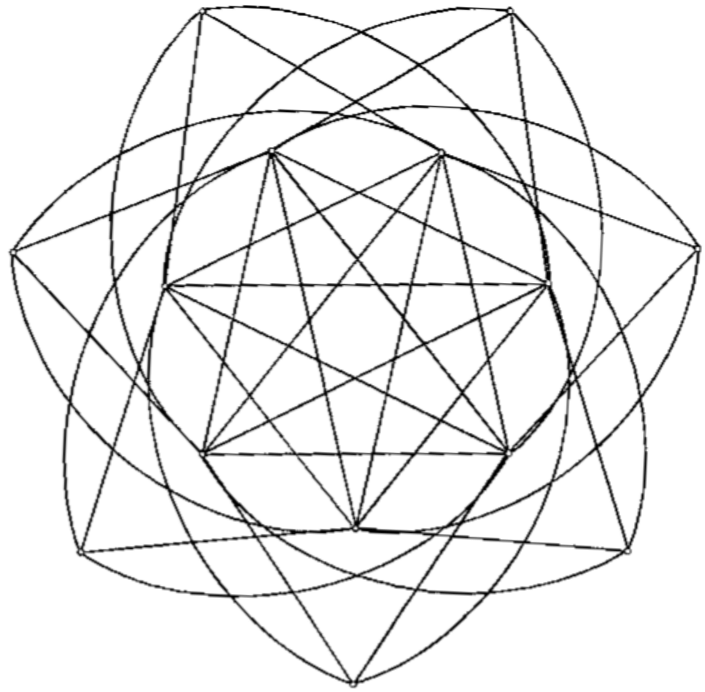

Figure 1: The Nenov graph Γ_3 from [13].

2 Testing for the Folkman Property

Regardless of the approach, we must have an algorithm for testing whether or not a graph is in $H_v(2,2,3;4)$. From the set parameters, any graph that contains K_4 can be discarded. For the remaining graphs, any colorings of a graph that don't have a K_2 in the first or second colors must force a K_3 in the third, otherwise the graph is not Folkman. Such colorings occur when two independent sets are colored with the first and second colors, forcing any remaining vertices not in the union of the two independent sets to contain a K_3 in the third color.

The algorithm for determining if $G \in H_v(2,2,3;4)$ is fairly straightforward: Discard G if it contains a K_4 or $\chi(G) \leq 4$. Otherwise, check that for each pair of independent sets $A, B \subset V(G)$, the induced subgraph on $V(G) \setminus (A \cup B)$ contains a K_3. A pseudo-code outline of the algorithm is presented as Algorithm 2.1, InH2234. For this to be efficient, we need Lemma 2.1.

Algorithm 2.1: INH2234(G)

if $K_4 \subseteq G$ **or** $\chi(G) \leq 4$
  then return (**false**)
$M = \text{AllMaxCliques}(\overline{G})$
$T = \{\{a, b, c\} : a, b, c \in V(G) \text{ and } abc \text{ is a triangle in } G\}$
for each $A, B \in M$
  do $\begin{cases} C \leftarrow V(G) \setminus (A \cup B) \\ \textbf{if } C \text{ doesn't contain a triangle in } T \\ \quad \textbf{then return} \text{ (} \textbf{false} \text{)} \end{cases}$
return (**true**)

Lemma 2.1 *In* INH2234 *it is sufficient to consider only maximal independent sets when determining whether* $G \in H_v(2, 2, 3; 4)$.

Proof. Let A, B be maximal independent sets in G and let $V(G) \setminus (A \cup B)$ induce a subgraph $\mathcal{S}$ in G. In order for $G \in H_v(2, 2, 3; 4)$, $\mathcal{S}$ must contain a K_3. Now let $A' \subset A$ and $B' \subset B$. Since K_3 is a subgraph of $\mathcal{S}$, it follows that the induced subgraph on $V(G) \setminus (A' \cup B')$ also contains a K_3. Thus, in Algorithm 2.1 it is sufficient to consider only the maximal independent sets A and B. □

For graphs of order 14, the number of maximal independent sets is usually very small (around 40), so the complexity of the algorithm is not a great obstacle. The algorithm ALLMAXCLIQUES [5] is used to find these sets. Although ALLMAXCLIQUES returns maximal cliques, if the input is $\overline{G}$, the result will be maximal independent sets in G.

For efficiency, it is also necessary to have a precomputed table of triangles in G. This table is used to check if an induced subgraph contains a triangle. The table is not very large, with an upper bound of 364 elements for K_{14}. The algorithm to build the table is a simple triple-nested search over all vertices. With at most 14 vertices, the $O(n^3)$ complexity is insignificant.

3 Computing $F_v(2, 2, 3; 4)$

For a single graph, INH2234 is virtually instantaneous, but the number of graphs that need to be examined explodes as the order increases. For example, there are 12,346 graphs on 8 vertices, which takes about 0.97 seconds to analyze. However, to analyze all 165,091,172,592 on 12 vertices would take about 96 days on a 1 GHz Pentium III CPU. Although distributing the computation over many computers is possible with graphs on 12 vertices, it would still not be a practical solution for graphs on 13 vertices.

There is an alternative, however, that avoids examining all graphs on less than 14 vertices. If it were possible to generate all the Folkman graphs on 14 vertices, the existence of Folkman graphs on 13 vertices could be decided by dropping a vertex in all possible ways from each graph and testing for the Folkman property.

We write $G - v$ to denote a graph with deleted vertex v and edges incident to v. Let $S = H_v(2,2,3;4;14)$ and $D = \{G-v : G \in S\}$, then there is a Folkman graph on 13 vertices if and only if $D \cap H_v(2,2,3;4;13) \neq \emptyset$.

The difficulty lies in generating S, but two observations help: (1) The Ramsey number $R(3,4) = R(4,3) = 9$ guarantees that all graphs of order $\geq$ 9 have either a K_4 or a $\overline{K_3}$, and (2) the Folkman graphs in $H_v(2,2,3;4;14)$, by definition, do not have a K_4, so they must have a $\overline{K_3}$ due to (1).

To find all maximal Folkman graphs we extend those graphs on 11 vertices without K_4 and $\chi \geq 4$ to graphs on 14 vertices. The restriction on χ is possible because all the Folkman graphs have $\chi \geq 5$. Each graph is extended by connecting three independent vertices to all triangle-free subsets in all possible ways. Subsets with triangles are avoided so that a K_4 is not formed. Furthermore, it is enough to consider extending only those graphs where the addition of any edge forms a triangle. The result will include all maximal Folkman graphs but $H_v(2,2,3;4)$ can easily be recovered from this subset. Figure 2 illustrates the extension. The algorithm is called Extend:

1. For each graph G of order 11 (a) that is K_4-free, (b) that has chromatic number at least 4, and (c) where the addition of any edge forms a triangle, do steps 2, 3, 4, and 5 below. The graphs are generated using *geng* from the *nauty* software package [8] and then filtered for (a), (b), and (c) using simple, custom algorithms. [easy]

2. Extend G to 14 vertices by efficiently adding $\overline{K_3}$ and incident edges. Each new vertex is incident with a maximal triangle-free subset to avoid creating a K_4. This is done in all possible ways with obvious isomorphs skipped (e.g., permutations of the new vertices). The output will contain all maximal Folkman graphs in addition to other Folkman and non-Folkman graphs. [hard]

3. Eliminate isomorphs using *nauty* software tools. [easy]

4. Collect graphs for which the addition of any edge forms a K_4. [easy]

5. Filter for Folkman property. This is a more expensive test so it should be performed after step 4. [easy]

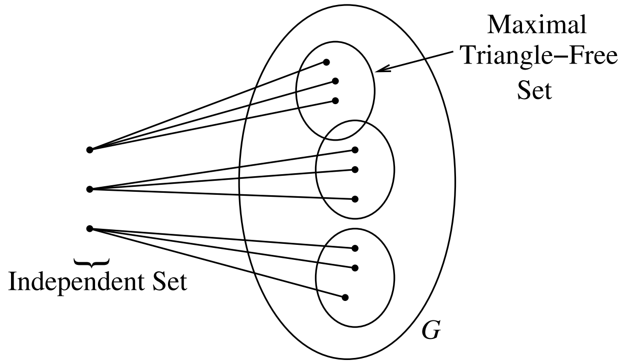

Figure 2: Illustration of the extension process.

In addition to the custom built algorithms InH2234 and Extend, the software packages *nauty* [8] and *Condor* [1] were crucial. *nauty* includes highly optimized and efficient tools for handling graphs. Developed by Brendan McKay from the Australian National University, *nauty* contains programs to quickly generate all non-isomorphic graphs of a given order as well as identify and eliminate isomorphic graphs. *nauty* has been used in numerous research projects for many years. *Condor* is a distributed processing package created at the Computer Science Department at the University of Wisconsin.

The advantage of the procedure Extend is that finding all graphs on 11 vertices without K_4 is feasible: There are 138,892,304 such graphs. The extension process is computationally challenging, yet easy to parallelize; the work was divided over the machines in the RIT Computer Science Department labs using *Condor*. Various types of machines were used: Sun Blade 150, Sun Blade 1500, and Sun Fire 880. With the distributed processing only moderate computational effort was required.

To find all Folkman graphs on 14 vertices, the maximal Folkman graphs were reduced using the algorithm ReduceSize. This algorithm inputs a graph and removes edges in all possible ways, outputting only those that are Folkman. ReduceSize was applied to all of the maximal Folkman graphs. The resulting set of non-isomorphic graphs combined with the maximal graphs is the complete set of Folkman graphs on 14 vertices. Isomorphs were eliminated using *nauty*. The set included the Nenov graph Γ_3. The processing time for this stage was small.

To verify the correctness of the results, the approach just described was slightly modified: All the Folkman graphs on 14 vertices were generated by extending graphs on 10 vertices instead of 11. The process is as follows:

Extend graphs on 10 vertices by adding an independent set of 4 vertices and connecting them to maximal triangle-free subsets of the graph in all possible ways. However, by using an independent set of order 4, the extensions avoid graphs which have no independent set of order 4. This is not a problem, as the missing graphs are those Folkman graphs which are in the Ramsey graph set $\mathscr{R}(4,4;14)$, computed in [9].

Since the extension from 10 to 14 vertices is guaranteed to generate all maximal Folkman graphs with $\alpha(G) \geq 4$, the other maximal Folkman graphs were extracted from $\mathscr{R}(4,4;14)$. This set was then reduced, using ReduceSize as before. The final set of non-isomorphic graphs was exactly the same as previously found after extending from 11 vertices and reducing.

To ensure that no graph on 13 vertices is in $H_v(2,2,3;4)$, the last algorithm ReduceOrder drops a vertex in all possible ways from each of the Folkman graphs on 14 vertices and the algorithm InH2234 then checks for the Folkman property. No Folkman graphs were found on 13 vertices, thus proving that $F_v(2,2,3;4) = 14$.

4 Results

$\lvert E(G)\rvert$	#	maxdeg(G)	#	mindeg(G)	#	$\alpha(G)$	#	$\lvert\mathrm{Aut}(G)\rvert$	#
42	1	7	527	4	451	3	1507	1	11367
43	6	8	11080	5	5759	4	10557	2	802
44	51	9	393	6	5996	5	160	4	44
45	453	10	227	7	21	6	2	7	1
46	2279					7	1	8	10
47	4555							14	2
48	3628							16	1
49	1138								
50	114								
51	2								

Table 1: Properties of graphs in $H_v(2,2,3;4;14)$.

G	$\lvert E(G)\rvert$	maxdeg(G)	mindeg(G)	$\alpha(G)$	$\lvert\mathrm{Aut}(G)\rvert$
Γ_3	42	8	4	7	14
F_{16}	45	7	5	4	16

Table 2: Properties of Γ_3 and F_{16}.

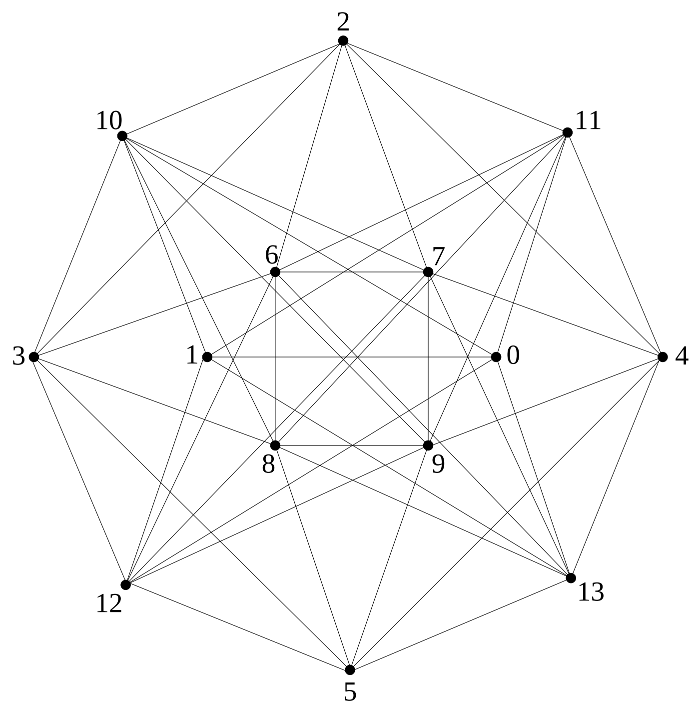

Figure 3: The Folkman graph F_{16} with $|\mathrm{Aut}(G)| = 16$.

Table 1 lists various properties of all Folkman graphs in $H_v(2, 2, 3; 4; 14)$. There are 12,227 such graphs in total; interestingly, all of them have chromatic number equal to 5. Among them, there are 591 maximal graphs, 1213 minimal graphs, and 8 bicritical graphs (those which are both maximal and minimal).

The order of the automorphism group of the Nenov graph Γ_3, pictured in Figure 1, is equal to 14. The graph F_{16}, presented in Figure 3, has the largest automorphism group among all 12,227 graphs in $H_v(2, 2, 3; 4)$, with $|\mathrm{Aut}(F_{16})| = 16$. Let $g_1 = (0\ 1)$, $g_2 = (3\ 4)(6\ 7)(8\ 9)(10\ 11)(12\ 13)$, and $g_3 = (2\ 3)(4\ 5)(7\ 8)(11\ 12)$ be the permutations of the set $\{0, \ldots, 13\}$. Then the full automorphism group of F_{16} is generated by g_1, g_2, and g_3.

Table 2 lists the specific properties of these two graphs in relation to the properties shown in Table 1. Notice that $H_v(3, 3; 4) \subset H_v(2, 2, 3; 4)$. It was shown in [12, 18] that $F_v(3, 3; 4) = 14$. The graph Γ_3 is in $H_v(3, 3; 4)$, but F_{16} is not.

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