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A NEW UPPER BOUND FOR THE RAMSEY NUMBER $R(5, 5)$

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Abstract.

We show that, in any colouring of the edges of K_{53} with two colours, there exists a monochromatic K_5 , and hence $R(5, 5) \leq 53$. This is accomplished in three stages: a full enumeration of (4,4)-good graphs, a derivation of some upper bounds for the maximum number of edges in (4,5)-good graphs, and a proof of the nonexistence of (5,5)-good graphs on 53 vertices. Only the first stage required extensive help from the computer.

1. Introduction.

The two-colour Ramsey number $R(k, l)$ is the smallest integer n such that, for any graph F on n vertices, either F contains K_k or \bar{F} contains K_l , where \bar{F} denotes the complement of F . A graph F is called (k, l) -good if F does not contain a K_k and \bar{F} does not contain a K_l . The best upper bound known previously, $R(5, 5) \leq 55$, is due to Walker (1971 [7]). The best lower bound, $R(5, 5) \geq 43$, was obtained by Exoo (1989 [1]), who constructed a (5,5)-good graph on 42 vertices.

Throughout this paper we will also use the following notation:

$N_F(x)$	— the neighbourhood of vertex x in graph F
$\deg_F(x)$	— the degree of vertex x in graph F
$n(F), e(F)$	— the number of vertices and edges in graph F
$t(F)$	— the number of triangles in F
$\bar{t}(F)$	— the number of independent 3-sets in graph F ; i.e. $t(\bar{F})$
$V(F)$	— the vertex set of graph F
(k, l, n) -good graph	— a (k, l) -good graph on n vertices
$e(k, l, n)$	— the minimum number of edges in any (k, l, n) -good graph
$E(k, l, n)$	— the maximum number of edges in any (k, l, n) -good graph
$t(k, l, n)$	— the minimum number of triangles in any (k, l, n) -good graph

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Let $n = |V(F)|$ and let n_i be the number of vertices of degree i in F . The well-known theorem of Goodman [2] says that

$$t(F) + \bar{t}(F) = \binom{n}{3} - \frac{1}{2} \sum_{i=0}^{n-1} i(n-i-1)n_i. \quad (1)$$

In his study of the Ramsey numbers $R(k, l)$, Walker [6] observed that if F is a (k, l, n) -good graph then

$$t(F) + \bar{t}(F) \leq \frac{1}{3} \sum_{i=0}^{n-1} \left(E(k-1, l, i) - e(k, l-1, n-i-1) + \binom{n-i-1}{2} \right) n_i.$$

Let $x \in V$ be a fixed vertex in a (k, l) -good graph F and consider the two induced subgraphs of F , G_x and H_x , where $V(G_x) = N_F(x)$ and $V(H_x) = V - (\{x\} \cup V(G_x))$. Note that G_x and H_x are $(k-1, l)$ -good and $(k, l-1)$ -good graphs, respectively. We define the *edge-deficiency* $\delta(x)$ of vertex x to be

$$\delta(x) = E(k-1, l, n(G_x)) - e(G_x) + e(H_x) - e(k, l-1, n(H_x)).$$

The edge deficiency $\delta(x)$ measures how close to extremal graphs the subgraphs G_x and H_x are. Clearly, $\delta(x) \geq 0$. One can also easily see that

$$\delta(x) = E(k-1, l, n(G_x)) - e(G_x) + E(l-1, k, n(H_x)) - e(\bar{H}_x). \quad (2)$$

It is convenient to define the *edge deficiency* $\Delta(F)$ of a (k, l) -good graph F by

$$\Delta(F) = \sum_{x \in V(F)} \delta(x). \quad (3)$$

The first lemma below, similar to (1) in [6], gives a strong condition which permits us to restrict the search space for (k, l) -good graphs.

Lemma 1. *If n_i is the number of vertices of degree i in a (k, l, n) -good graph F then*

$$0 \leq 2\Delta(F) = \sum_{i=0}^{n-1} (2E(k-1, l, i) + 2E(l-1, k, n-i-1) + 3i(n-i-1) - (n-1)(n-2))n_i. \quad (4)$$

Proof. Observe that for all $x \in V(F)$ the number of triangles containing x is equal to $e(G_x)$ and the number of independent 3-sets containing x is equal to $e(\bar{H}_x)$. Hence by (2),

$$\begin{aligned} 3(t(F) + \bar{t}(F)) &= \sum_{x \in V(F)} (e(G_x) + e(\bar{H}_x)) \\ &= \sum_{x \in V(F)} (E(k-1, l, n(G_x)) + E(l-1, k, n(H_x)) - \delta(x)), \end{aligned}$$

and so by (3) we have

$$0 \leq \Delta(F) = \sum_{i=0}^{n-1} (E(k-1, l, i) + E(l-1, k, n-i-1))n_i - 3(t(F) + \bar{t}(F)).$$

Now using (1) and $\sum_{i=0}^{n-1} n_i = n$, we obtain (4). \blacksquare

2. Generation of all (4, 4)-good graphs.

This section describes how we generated the set of all (4,4)-good graphs. Let us denote by $R(4, 4, n)$ the set of all (4,4, n)-good graphs and let $R'(4, 4, n)$ be the subset of those $F \in R(4, 4, n)$ with maximum degree D at most $(n - 1)/2$. The result of applying the permutation α to the labels of any labelled object X will be denoted by X^α , and also $\text{Aut}(F)$ is the automorphism group of the graph F , as a group of permutations of $V(F)$.

Suppose that θ is a function defined on $\bigcup_{n \geq 2} R'(4, 4, n)$ which satisfies these properties:

- (i) $\theta(F)$ is an orbit of $\text{Aut}(F)$,
- (ii) the vertices in $\theta(F)$ have maximum degree in F , and
- (iii) for any F , and any permutation α of $V(F)$, $\theta(F^\alpha) = \theta(F)^\alpha$.

It is easy to implement a function satisfying the requirements for θ by using the program *nauty* [3]. Given θ , and $F \in R'(4, 4, n)$ for some $n \geq 2$, the *parent* of F is the graph $\text{par}(F)$ formed from F by removing the first vertex in $\theta(F)$ and its incident edges. The properties of θ imply that isomorphic graphs have isomorphic parents. It is also easily seen that $\text{par}(F) \in R'(4, 4, n-1)$. Since $R'(4, 4, 1) = \{K_1\}$, we find that the relationship “par” defines a rooted directed tree T whose vertices are the isomorphism classes of $\bigcup_{n \geq 1} R'(4, 4, n)$, with the graph K_1 at the root. If ν is a node of T , then the *children* of ν are those nodes ν' of T such that for some $F \in \nu'$ we have $\text{par}(F) \in \nu$. The set of children of ν can be found by the following algorithm, whose correctness follows easily from the definitions:

- (a) Let F be any representative of the isomorphism class ν .
Suppose that F has n vertices and maximum degree D .
- (b) Let $L = L(F)$ be a list of all subsets X of $V(F)$ such that
 - (b.1) either $|X| > D$, or $|X| = D$ and X does not include any vertex of degree D ,
 - (b.2) X intersects every independent set of size 3 in F ,
 - (b.3) X does not include any triangle of F , and
 - (b.4) if $F(X)$ is the graph of order $n + 1$ formed by joining a new vertex x to X , then $x \in \theta(F(X))$.
- (c) Remove isomorphs from amongst the set $\{F(X) \mid X \in L\}$.

The remaining graphs form a set of distinct representatives for the children of ν .

The primary advantage of this method is that isomorph rejection need only be performed within very restricted sets of graphs. For example, even though $|R'(4, 4, 12)| = 909767$, no isomorphism class of $R'(4, 4, 11)$ has more than 58 children.

The full set $\bigcup_{n \geq 1} R'(4, 4, n)$ was found by this method. Altogether, 5623547 sets X passed conditions (b.1)-(b.3), and 2165034 passed condition (b.4) as well. The total size of $R'(4, 4, n)$ for all n is 2065740, which is only slightly less because most (4,4)-good graphs have no nontrivial automorphisms. There are altogether 3432184 nonisomorphic (4,4)-good

graphs. The total execution time on a 12-mip computer was 9.4 hours, or 6 milliseconds per invocation of the program *nauty*. In particular, we obtained the information gathered in Table I.

n	4	5	6	7	8	9	10
$ R(4, 4, n) $	9	24	84	362	2079	14701	103706
$E(4, 4, n)$	5	8	12	16	21	27	31
$t(4, 4, n)$	0	0	0	0	0	1	4
n	11	12	13	14	15	16	17
$ R(4, 4, n) $	546356	1449166	1184231	130816	640	2	1
$E(4, 4, n)$	36	40	45	50	55	60	68
$t(4, 4, n)$	7	10	17	25	38	56	68

Table I. Some data on (4,4)-good graphs

3. Upper bounds for $E(4, 5, n)$.

Walker [7] established the best upper bound so far of 28 for $R(4, 5)$, so we know that any (4,5)-good graph has at most 27 vertices. No (4, 5, n)-good graph is known for $n \geq 25$. The goal of this section is to derive some upper bounds for $E(4, 5, n)$ for $24 \leq n \leq 27$, provided such graphs exist.

Let F be a (4, 5, n)-good graph and let a_i denote the number of edges in F contained in i triangles. Note that $a_i = 0$ for $i \geq 5$ since F is (4,5)-good. For each $x \in V(F)$ consider induced subgraphs G_x and H_x as in Section 1, which in this case are (3,5)-good and (4,4)-good graphs, respectively.

Lemma 2.

$$\sum_{x \in V(F)} t(H_x) = 4a_4 - 2a_2 - 2a_1 + \sum_{x \in V(F)} (n/3 + 3 - \deg_F(x))e(G_x). \quad (5)$$

Proof. For an arbitrary triangle $T = ABC$ in F let $b_i(T)$ denote the number of vertices in $V(F) - T$ adjacent to exactly i vertices in T , and let $\deg_F(T) = \deg_F(A) + \deg_F(B) + \deg_F(C)$. Note that $b_i(T) = 0$ for $i \geq 3$, since F has no K_4 . By counting the 4-sets of vertices formed by any triangle T and any vertex x not adjacent to T in two different ways we have

$$\sum_{x \in V(F)} t(H_x) = \sum_{T\text{-triangle}} b_0(T), \quad (6)$$

and one also easily notes that for each triangle T

$$b_0(T) = n - 3 - b_1(T) - b_2(T) \quad (7)$$

and

$$b_1(T) + 2b_2(T) + 6 = \deg_F(T). \quad (8)$$

Now (7) and (8) give

$$b_0(T) = n + 3 + b_2(T) - \deg_F(T). \quad (9)$$

Using (9) in (6) we obtain

$$\sum_{x \in V(F)} t(H_x) = (n + 3)t(F) + \sum_{T\text{-triangle}} (b_2(T) - \deg_F(T)). \quad (10)$$

Counting edges adjacent to points in triangles by two methods gives

$$\sum_{T\text{-triangle}} \deg_F(T) = \sum_{x \in V(F)} \deg_F(x)e(G_x), \quad (11)$$

and one can also easily see that

$$3t(F) = \sum_{x \in V(F)} e(G_x) = \sum_{i=1}^4 ia_i. \quad (12)$$

By recalling the definitions of $b_2(T)$ and a_i we conclude that

$$\sum_{T\text{-triangle}} b_2(T) = \sum_{i=2}^4 i(i-1)a_i = 4a_4 - 2a_2 - 2a_1 + 2 \sum_{i=1}^4 ia_i. \quad (13)$$

Now applying (11), (12) and (13) in (10) we obtain

$$\sum_{x \in V(F)} t(H_x) = \frac{1}{3}(n+3) \sum_{x \in V(F)} e(G_x) + 4a_4 - 2a_2 - 2a_1 + 2 \sum_{x \in V(F)} e(G_x) - \sum_{x \in V(F)} \deg_F(x)e(G_x),$$

which can be easily converted to (5). ■

We know that for each vertex x the number of triangles in H_x is at least $t(4, 4, n(H_x))$, where $n(H_x) = n - 1 - \deg_F(x)$. Define the *triangle deficiencies* $\gamma(x)$ of a vertex x and $\Gamma(F)$ of a graph F as

$$\gamma(x) = t(H_x) - t(4, 4, n(H_x)), \quad \Gamma(F) = \sum_{x \in V(F)} \gamma(x). \quad (14)$$

For any vertex x we obviously have $\gamma(x) \geq 0$.

Lemma 3. *If F is any $(4, 5, n)$ -good graph on at least 24 vertices and F has n_i vertices of degree i for each i , then*

$$0 \leq 3\Gamma(F) \leq \sum_{i=6}^{13} ((n+9-3i)E(3, 5, i) + 6i - 3t(4, 4, n-i-1))n_i. \quad (15)$$

Proof. Since $R(3, 5) = 14$ and $R(4, 4) = 18$, by (5) we have

$$3 \sum_{x \in V(F)} t(H_x) = 12a_4 - 6a_2 - 6a_1 + \sum_{i=6}^{13} \sum_{\deg_F(x)=i} (n+9-3i)e(G_x).$$

Note that for $n \geq 24$ the coefficient $n+9-3i$ is negative only for $i=13$ or for $i=12$ and $n=24, 25, 26$, hence we can use $E(3, 5, i)$ in place of $e(G_x)$ in the following inequality except in those cases.

$$\begin{aligned} 3 \sum_{x \in V(F)} t(H_x) &\leq 12a_4 + \sum_{i=6}^{13} (n+9-3i)E(3, 5, i)n_i \\ &\quad + \sum_{\deg_F(x) \geq 12} (E(3, 5, \deg_F(x)) - e(G_x))(3 \deg_F(x) - n - 9). \end{aligned} \quad (16)$$

All $(3, 5)$ -good graphs are known ([5] and independently [4]). In particular, there exists a unique $(3, 5, 13)$ -good graph, which implies that the terms in the last summation for $\deg_F(x) \geq 13$ are equal to zero. It is also known that $E(3, 5, 12) = 24$ is achieved only by 4-regular graphs, and furthermore any $(3, 5, 12)$ -good graph has only vertices of degree 3 and/or 4. Thus if for some vertex x of degree 12 in F the graph G_x is not maximal, i.e. $e(G_x) < 24$, then for each vertex y of degree 3 in G_x the edge $\{x, y\}$ contributes to a_3 , and each edge appearing in three triangles can be accounted at most twice this way. Thus the second summation in the right hand side of (16) is at most $3a_3$ for $n \geq 24$. Hence by $e(F) \geq a_4 + a_3$ and (16) we find

$$3 \sum_{x \in V(F)} t(H_x) \leq 12e(F) + \sum_{i=6}^{13} (n+9-3i)E(3, 5, i)n_i. \quad (17)$$

Finally, we can easily obtain (15) by using (14), (17) and $12e(F) = \sum_{i=6}^{13} 6in_i$. \blacksquare

Theorem 1. *If we interpret $e(k, l, n)$ as ∞ and $E(k, l, n)$ as 0 for $n \geq R(k, l)$ then $153 \leq e(4, 5, 27)$ and $E(4, 5, 27) \leq 160$, $130 \leq e(4, 5, 26)$ and $E(4, 5, 26) \leq 154$, $116 \leq e(4, 5, 25)$ and $E(4, 5, 25) \leq 148$, $101 \leq e(4, 5, 24)$ and $E(4, 5, 24) \leq 139$.*

Proof. Let F be any $(4, 5, n)$ -good graph for some $24 \leq n \leq 27$ with e edges and n_i vertices of degree i . Consider the set of constraints formed by $\sum_{i=6}^{13} n_i = n$ and the conditions for $\Delta(F)$ and $\Gamma(F)$ given by Lemmas 1 and 3, respectively. This gives a simple instance

(for a computer) of a non-negative integer linear programming optimization problem with variables n_i and objective function $2e = \sum_{i=6}^{13} in_i$. For $n = 27$ we have to minimize or maximize

$$9n_9 + 10n_{10} + 11n_{11} + 12n_{12} + 13n_{13}$$

subject to

$$\begin{aligned} 27 &= n_9 + n_{10} + n_{11} + n_{12} + n_{13}, \\ 0 &\leq -21n_9 - 10n_{10} - n_{11} + 2n_{12} - n_{13}, \end{aligned} \tag{18}$$

and

$$0 \leq n_9 + 4n_{10} + 6n_{11} - n_{12} - 17n_{13}, \tag{19}$$

where constraint (18) is obtained from (4) and constraint (19) is obtained from (15), using the numerical data from Table I for $t(4, 4, j)$, $E(4, 4, i)$, and some of the results listed in [5], namely $E(3, 5, i) = 2i$ for $10 \leq i \leq 13$ and $E(3, 5, 9) = 17$. Also in [5] we find the values $E(3, 5, 8) = 16$, $E(3, 5, 7) = 12$ and $E(3, 5, 6) = 9$, which are needed for the calculations in the cases of $24 \leq n \leq 26$. For $n = 27$ the maximal number of edges e is 160 with the unique possible degree sequence $n_{12} = 23$ and $n_{11} = 4$. The other bounds are obtained similarly. We used a simple computer program to perform these calculations, and another to check them. ■

The numbers of edges in the known (4,5,24)-good graphs range from 118 to 132 (personal communication from G. Exoo). The lower bounds for $e(4, 5, n)$ are not needed for the proof of $R(5, 5) \leq 53$; they are included in Theorem 1 for completeness.

4. An upper bound for $R(5, 5)$.

We are now in a position to prove our major result.

Theorem 2. $R(5, 5) \leq 53$.

Proof. Assume that F is a (5,5)-good graph on 53 vertices and let n_i be the number of vertices of degree i in F . Since $R(4, 5) \leq 28$ we have in this case $n_{25} + n_{26} + n_{27} = 53$. The calculation of bounds for $2\Delta(F)$ from Lemma 1, using Theorem 1, gives

$$\begin{aligned} 0 &\leq (2 \cdot 308 + 3 \cdot 25 \cdot 27 - 52 \cdot 51)(n_{25} + n_{27}) + (2 \cdot 308 + 3 \cdot 26 \cdot 26 - 52 \cdot 51)n_{26} \\ &= -11(n_{25} + n_{27}) - 8n_{26}, \end{aligned}$$

which is a contradiction. ■

The same method does not disprove the existence of a (5,5,52)-good graph, but such a result would be possible if we could sufficiently improve the bounds of Theorem 1.

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