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Solids Whose Parallel Cross-Sections are Regular Polygons

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Abstract

Given a solid in \mathbb{R}^3 whose base is contained in the xy -plane and parallel cross-sections perpendicular to the xy -plane are regular polygons, the volume and surface area are determined. Additionally, the asymptotic behavior of the solid as the number of sides of the cross-sectional polygon increases is analyzed.

1 Introduction

In many introductory calculus textbooks (such as [1] and [3]) there is the traditional section on calculating volumes and surface areas of regions in the xy -plane that are rotated about one of the coordinate axes or some other line in the plane.

One conventional way of introducing volumes by slicing is by defining a hard-to-imagine, let alone draw freehand on a blackboard, solid whose base is a region R in the xy -plane such that the z -coordinate of any point in the solid is never negative and parallel cross-sections of the solid perpendicular to the

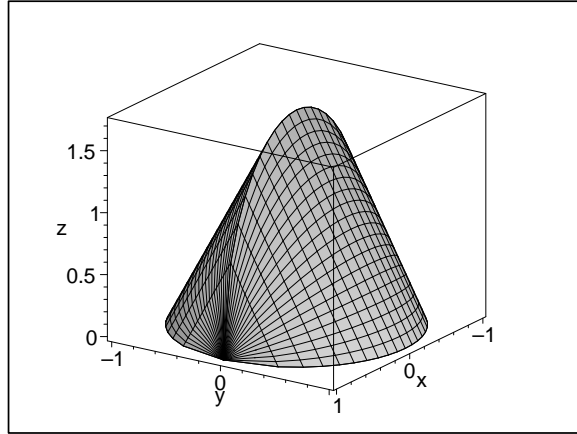


Figure 1: Base is the unit disk and cross-sections are equilateral triangles.

x -axis (or y -axis) are semi-circles, squares, or triangles. The most common solid is to take the base to be a unit disk with parallel cross-sections perpendicular to the x -axis as either equilateral triangles (Figure 1) or squares (Figure 2).

A natural question would be to generalize the problem so that cross-sections are regular polygons with n sides for $n \geq 3$. See Figures 3 and 4 for two additional examples. Determining the height and volume of these solids are simple, but the surface area is not as straightforward.

To generalize further, given differentiable functions f and g defined on the interval $[a, b]$ such that $f(x) \geq g(x)$ for all $x \in [a, b]$, let R be the region defined as

$$R := \{(x, y, 0) : a \leq x \leq b, g(x) \leq y \leq f(x)\} \quad (1.1)$$

and let \mathcal{S}_R^n be the solid whose base is R and cross-sections parallel to the yz -plane are regular n -gons.

2 Height, Volume and Surface Area

Let R be a region given by (1.1) and let $n \geq 3$. Represent the height, volume and *lateral* surface area (that is, the surface area not including any regular polygons formed at $x = a$ and $x = b$) of \mathcal{S}_R^n as, respectively, $H(\mathcal{S}_R^n)$, $V(\mathcal{S}_R^n)$

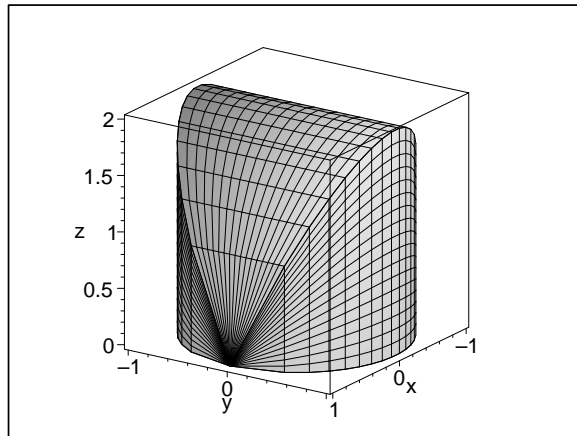


Figure 2: Base is the unit disk and cross-sections are squares.

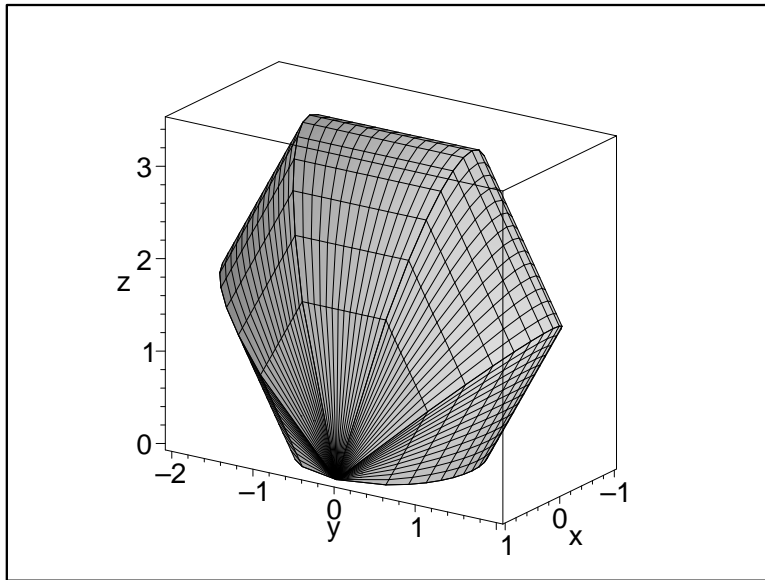


Figure 3: Base is the unit disk and cross-sections are regular hexagons.

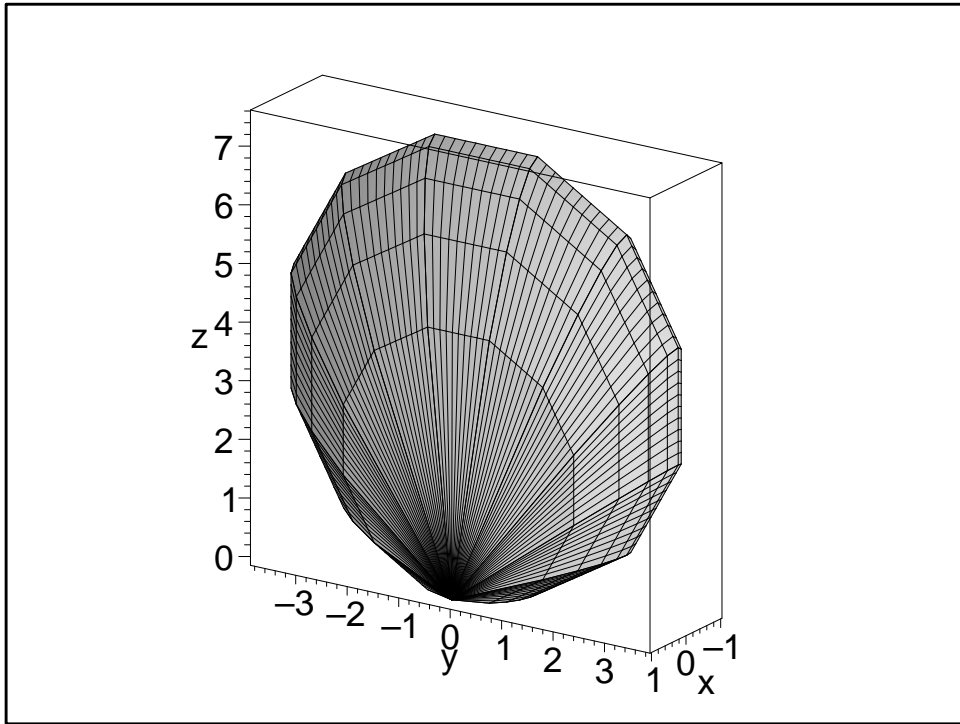


Figure 4: Base is the unit disk and cross-sections are regular dodecagons.

and $A(\mathcal{S}_R^n)$.

2.1 Height

The height is determined initially to provide some of the common notation needed to calculate the volume and surface area, the formula is also necessary for the asymptotic analysis in section 3.

The highest point on \mathcal{S}_R^n will take place wherever the longest vertical slice in R occurs. The length of this slice is

$$M := \sup_{x \in [a, b]} \{(f - g)(x)\}. \quad (2.1)$$

Let $C \equiv C(x)$ be the circle that circumscribes the cross-sectional polygon formed at $x \in [a, b]$. The circle has radius $r \equiv r(x) = \frac{(f-g)(x)}{2} \csc \frac{\pi}{n}$ with the center at (x, y_c, z_c) , where $y_c \equiv y_c(x) = \frac{(f+g)(x)}{2}$ and $z_c \equiv z_c(x) = \frac{(f-g)(x)}{2} \cot \frac{\pi}{n}$. Each polygonal slice can be subdivided into n isosceles triangles with a vertex at the center of C . The altitude of each triangle is $h \equiv h(x) = \frac{(f-g)(x)}{2} \cot \frac{\pi}{n}$.

The overall height of the solid depends on the parity of n . The two cases are illustrated in Figures 5 and 6. Consequently, the height of \mathcal{S}_R^n is

$$H(\mathcal{S}_R^n) = \frac{c_n}{\sin \frac{\pi}{n}} M, \quad (2.2)$$

where

$$c_n = \begin{cases} \cos \frac{\pi}{n}, & \text{if } n \text{ is even;} \\ \frac{1}{2}(1 + \cos \frac{\pi}{n}), & \text{if } n \text{ is odd.} \end{cases} \quad (2.3)$$

Note that $c_n \rightarrow 1$ as $n \rightarrow \infty$. As a result, for fixed R , $H(\mathcal{S}_R^n)$ is $\mathcal{O}(n)$.

2.2 Volume

To determine $V(\mathcal{S}_R^n)$, the cross-sectional area of a polygonal slice at $x \in [a, b]$, $A(x)$, is needed. The area of each of the n sub-triangles at x is $\frac{1}{2}r^2 \sin \frac{2\pi}{n}$, where r is the radius of the circle that circumscribes the polygon at x given

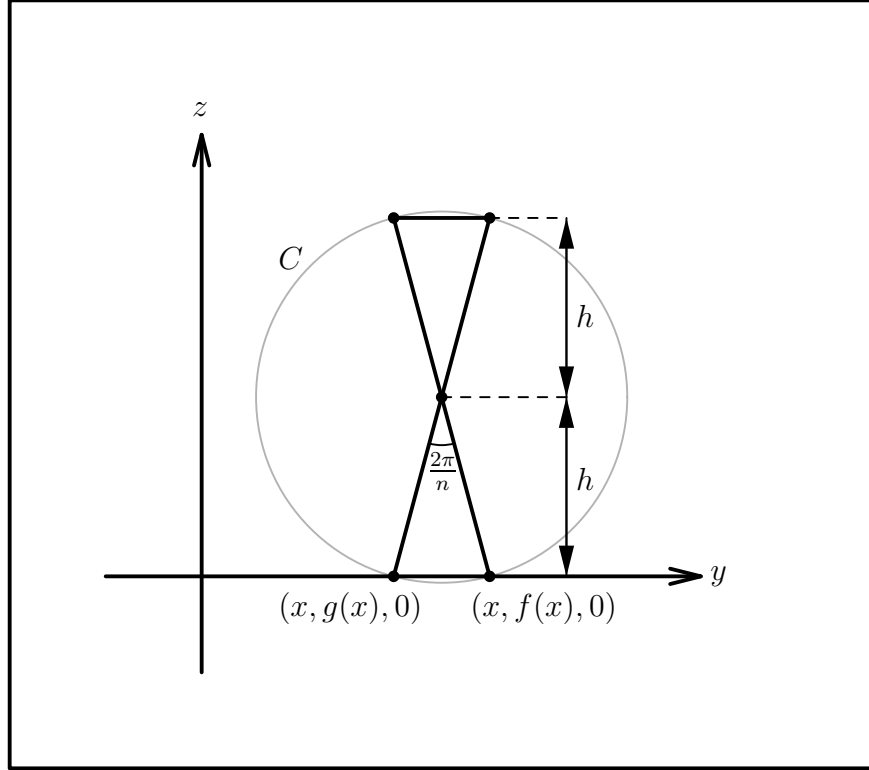


Figure 5: Cross-section of \mathcal{S}_R^n at $x \in [a, b]$ to determine height for n even.

in the previous section. And so the cross-sectional area of \mathcal{S}_R^n at x is $A(x) := \frac{n}{2}r^2 \sin \frac{2\pi}{n} = \frac{n}{4}(\cot \frac{\pi}{n})[(f - g)(x)]^2$.

Following standard techniques in defining integrals from beginning calculus textbooks, let $\Delta = \{x_0, x_1, \dots, x_m\}$ be a partition of $[a, b]$ such that $a = x_0 < x_1 < \dots < x_m = b$ where Δx_i is the length of the i -th subinterval. The *norm* of Δ , denoted $\|\Delta\|$, is the length of the largest subinterval formed by the partition. For each i , choose a sample point $x_i^* \in [x_{i-1}, x_i]$. An approximation of the volume is given by the Riemann sum

$$V(\mathcal{S}_R^n) \approx \sum_{i=1}^m A(x_i^*) \Delta x_i.$$

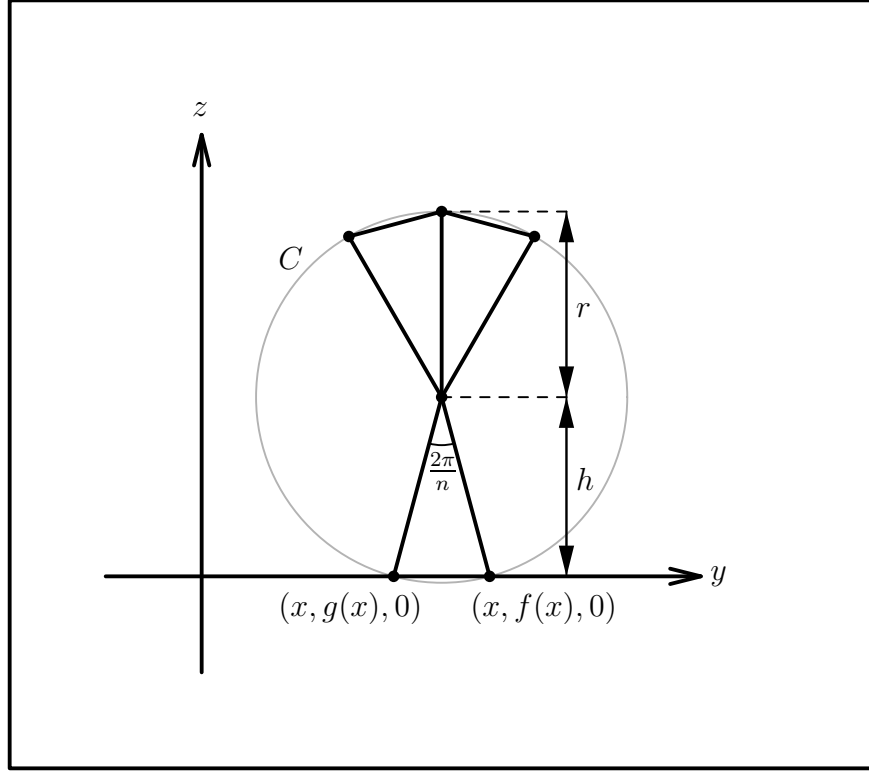


Figure 6: Cross-section of \mathcal{S}_R^n at $x \in [a, b]$ to determine height for n odd.

The exact volume,

$$V(\mathcal{S}_R^n) = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^m A(x_i^*) \Delta x_i = \int_a^b A(x) dx = \frac{n}{4} (\cot \frac{\pi}{n}) \int_a^b [(f - g)(x)]^2 dx, \quad (2.4)$$

is the integral over $[a, b]$ of the cross-sectional areas. For fixed R , $V(\mathcal{S}_R^n)$ is $\mathcal{O}(n^2)$.

2.3 Surface Area

The surface of \mathcal{S}_R^n can be approximated as the union of quadrilaterals. The goal is therefore to determine the area of each quadrilateral. To do so, we need to find the coordinates of the vertices of the cross-sectional polygons. Likewise, it is necessary to parameterize the circle that circumscribes the

polygon.

Given $x \in [a, b]$, recall from section 2.1 that the center of C is given by (x, y_c, z_c) . Let θ be the angle measured counterclockwise (as viewed from the positive x -axis) from the half-line beginning at the center of C and directed downward (in the negative z direction). The angle that corresponds to the vertices on the polygon are $\theta_k = \frac{\pi}{n}(2k + 1)$ for $k \in \{0, \dots, n\}$. Define the points $P^k \equiv P^k(x)$ to have coordinates $(x, y^k(x), z^k(x))$, where

$$\begin{aligned} y^k(x) &:= y_c + r \cos(\theta_k - \frac{\pi}{2}) \\ &= y_c + r \sin \theta_k \\ &= \frac{(f+g)(x)}{2} + \frac{(f-g)(x)}{2} (\csc \frac{\pi}{n}) \sin \frac{(2k+1)\pi}{n} \end{aligned} \quad (2.5a)$$

and

$$\begin{aligned} z^k(x) &:= z_c + r \sin(\theta_k - \frac{\pi}{2}) \\ &= z_c - r \cos \theta_k \\ &= \frac{(f-g)(x)}{2} (\csc \frac{\pi}{n}) \left(\cos \frac{\pi}{n} - \cos \frac{(2k+1)\pi}{n} \right). \end{aligned} \quad (2.5b)$$

The collection of points $\{P^k : k = 0, \dots, n\}$ are the vertices of the cross-sectional polygon at x (Figure 7). It is readily verified that the base vertices P^{n-1} and P^0 are, respectively, located at $(x, g(x), 0)$ and $(x, f(x), 0)$, as expected. Also, note that $P^n \equiv P^0$.

Given $k \in \{1, \dots, n\}$, the change in the y -coordinate and z -coordinate from P^{k-1} to P^k is, respectively,

$$y^{\Delta k}(x) := y^k(x) - y^{k-1}(x) = (f - g)(x) \cos \frac{2k\pi}{n} \quad (2.6a)$$

and

$$z^{\Delta k}(x) := z^k(x) - z^{k-1}(x) = (f - g)(x) \sin \frac{2k\pi}{n}. \quad (2.6b)$$

As in the volume problem, let $\Delta = \{x_0, \dots, x_m\}$ be a partition of $[a, b]$ where Δx_i is the length of the i -th subinterval.

To simplify notation that appears later, define each of the following for all $i \in \{0, \dots, m\}$ and $k \in \{0, \dots, n\}$.

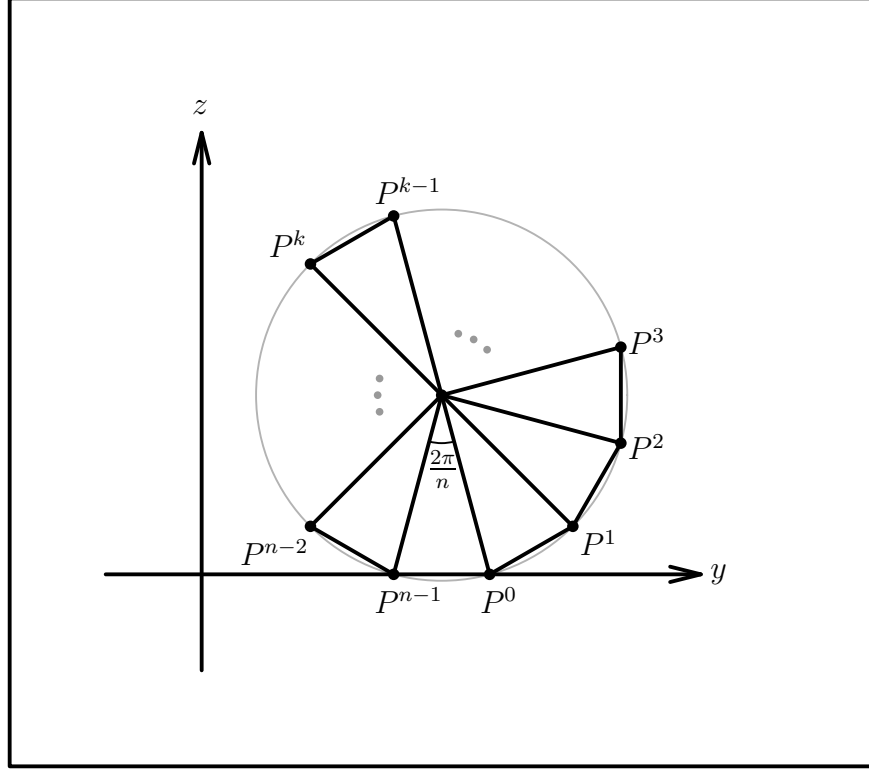


Figure 7: Cross-section of \mathcal{S}_R^n at $x \in [a, b]$.

- $f_i := f(x_i)$ and $g_i := g(x_i)$.
- $\Delta f_i := f_i - f_{i-1}$ and $\Delta g_i := g_i - g_{i-1}$ for $i \neq 0$.
- $y_i^k := y^k(x_i)$ and $z_i^k := z^k(x_i)$.
- P_i^k is the point (x_i, y_i^k, z_i^k) .
- $y_i^{\Delta k} := y^{\Delta k}(x_i)$ and $z_i^{\Delta k} := z^{\Delta k}(x_i)$ for $k \neq 0$.
- $y_{\Delta i}^k := y_i^k - y_{i-1}^k$ and $z_{\Delta i}^k := z_i^k - z_{i-1}^k$ for $i \neq 0$.

Given $k \in \{1, \dots, n\}$, the non-zero vectors in the collection $\{\overrightarrow{P_i^{k-1} P_i^k} = \langle 0, y_i^{\Delta k}, z_i^{\Delta k} \rangle : i = 0, \dots, m\}$ are mutually parallel because every nondegenerate vector is in the same direction as the unit vector $\mathbf{u}^k := \langle 0, \cos \frac{2k\pi}{n}, \sin \frac{2k\pi}{n} \rangle$. So, for any $i \in \{1, \dots, m\}$, the quadrilateral $P_{i-1}^{k-1} P_i^{k-1} P_i^k P_{i-1}^k$ forms one of

three shapes: a trapezoid (as in Figure 8), a triangle (in the case that either $f_i = g_i$ or $f_{i-1} = g_{i-1}$), or is degenerate (in the case that both are true). To determine the area of this quadrilateral, let L_i^k be the line passing through P_i^k and parallel to \mathbf{u}^k (and hence L_i^k also passes through P_i^{k-1}). The area of this quadrilateral, A_i^k , is

$$A_i^k = \frac{1}{2} \left(|P_{i-1}^{k-1} P_{i-1}^k| + |P_i^{k-1} P_i^k| \right) d_i^k = \left(\frac{f_i + f_{i-1}}{2} - \frac{g_i + g_{i-1}}{2} \right) d_i^k,$$

where d_i^k is the distance between L_{i-1}^k and L_i^k . This distance is the magnitude of the orthogonal projection of $\mathbf{v}_i^k := \overrightarrow{P_{i-1}^k P_i^k} = \langle \Delta x_i, y_{\Delta_i}^k, z_{\Delta_i}^k \rangle$ in the direction of \mathbf{u}^k . That is,

$$d_i^k = |\mathbf{v}_i^k - (\mathbf{u}^k \cdot \mathbf{v}_i^k) \mathbf{u}^k| = \left| \langle \Delta x_i, Q_i^k \sin \frac{2k\pi}{n}, -Q_i^k \cos \frac{2k\pi}{n} \rangle \right| = \sqrt{(\Delta x_i)^2 + (Q_i^k)^2},$$

where

$$Q_i^k = y_{\Delta_i}^k \left(\sin \frac{2k\pi}{n} \right) - z_{\Delta_i}^k \left(\cos \frac{2k\pi}{n} \right) = \frac{1}{2} \sin \left(\frac{2k\pi}{n} \right) \left(\Delta f_i + \Delta g_i + \frac{\tan(k\pi/n)}{\tan(\pi/n)} (\Delta f_i - \Delta g_i) \right).$$

If A_i is the sum of the area of all the quadrilaterals in $[x_{i-1}, x_i]$, then

$$\begin{aligned} A_i &= \sum_{k=1}^n A_i^k \\ &= \left(\frac{f_i + f_{i-1}}{2} - \frac{g_i + g_{i-1}}{2} \right) \sum_{k=1}^n \sqrt{1 + \left[\frac{1}{2} \sin \left(\frac{2k\pi}{n} \right) \left(\frac{\Delta f_i + \Delta g_i}{\Delta x_i} + \frac{\tan(k\pi/n)}{\tan(\pi/n)} \frac{\Delta f_i - \Delta g_i}{\Delta x_i} \right) \right]^2} \Delta x_i. \end{aligned}$$

Since f and g are differentiable on $[a, b]$, apply the Mean Value Theorem and the Intermediate Value Theorem to each interval $[x_{i-1}, x_i]$, where $i = 1, \dots, m$. There exists $u_i, v_i, \mu_i, \nu_i \in (x_{i-1}, x_i)$ such that $\Delta f_i = f'(u_i) \Delta x_i$, $\Delta g_i = g'(v_i) \Delta x_i$, $f(\mu_i) = \frac{1}{2}(f_i + f_{i-1})$ and $g(\nu_i) = \frac{1}{2}(g_i + g_{i-1})$.

An approximation of the lateral surface area is given by the Riemann sum

$$A(\mathcal{S}_R^n) \approx \sum_{i=1}^m A_i,$$

where

$$A_i = (f(\mu_i) - g(\nu_i)) \sum_{k=1}^n \sqrt{1 + \left[\frac{f'(u_i) + g'(v_i)}{2} \sin \left(\frac{2k\pi}{n} \right) + \frac{f'(u_i) - g'(v_i)}{\tan(\pi/n)} \sin^2 \left(\frac{k\pi}{n} \right) \right]^2} \Delta x_i.$$

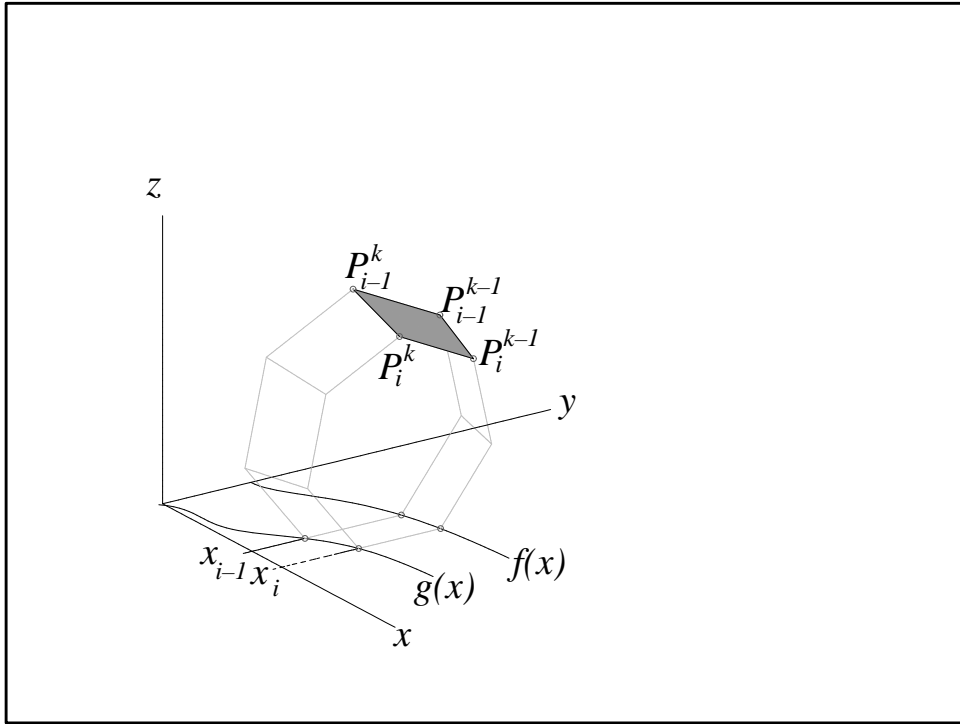


Figure 8: Approximating the surface of \mathcal{S}_R^n area using trapezoids.

Duhamel's principle [4, pages 515–8] is the generalized technique of representing a limit of certain sums as an integral. In essence, it states that anything which looks like a good approximation to an integral really is. The lateral surface area is therefore

$$\begin{aligned} A(\mathcal{S}_R^n) &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^m A_i \\ &= \int_a^b (f - g)(x) \sum_{k=1}^n \sqrt{1 + \left[\frac{(f+g)'(x)}{2} \sin\left(\frac{2\pi k}{n}\right) + \frac{(f-g)'(x)}{\tan(\pi/n)} \sin^2\left(\frac{k\pi}{n}\right) \right]^2} dx. \end{aligned} \tag{2.7}$$

We will see in the next section that $A(\mathcal{S}_R^n)$ is $\mathcal{O}(n^2)$ for fixed R .

The equation for the lateral surface area is also valid if f and g are non-differentiable for a finite number of points in $[a, b]$ by choosing appropriate partitions to include the non-differentiable points.

The general form of the formula for the surface area (2.7) can be somewhat confirmed by appealing to the the well-known formula for the area of a surface of revolution. Given a differentiable function F defined on $[a, b]$ rotated about the x -axis, the surface area is

$$A = 2\pi \int_a^b F(x) \sqrt{1 + [F'(x)]^2} dx,$$

which is remarkably similar to (2.7) in appearance. An alternative approach to derive the formula for the lateral surface area of \mathcal{S}_R^n is outlined in the appendix.

3 Asymptotic Behavior

Here we would like to determine the asymptotic behavior of the volume and lateral surface area of \mathcal{S}_R^n as n increases with R fixed. Both volume and lateral surface area grow without bound, so the solid must be normalized to make the analysis meaningful by *rescaling* the base region in such a way that the height of the solid is always 1. This rescaling will be justified later in the section after looking at two examples.

Example 3.1. Let R be the unit disk centered at the origin. Some of the solids generated by this base have already been given in Figures 1 through 4. R is defined by taking $f(x) = \sqrt{1 - x^2}$ and $g(x) = -\sqrt{1 - x^2}$ on $[-1, 1]$. For each n , rescale the dimensions of the base so that the height of the solid is 1. As a consequence, the base will be smaller and the overall appearance of the solid will be very thin for n large. The volume will be very close to zero, but the surface should approach that of two disks with unit diameter (since regular polygons for large number of sides approximate a circle). ■

Example 3.2. Let R be the polygon with vertices at the points $(-2, 0)$, $(-1, 2)$, $(0, 1)$, $(1, 2)$ and $(2, 0)$. This region is shown in Figure 9. R is defined by taking f as a piecewise linear function or, more explicitly, as $f(x) = |x| - \frac{3}{2}(|x - 1| + |x + 1|) + |x - 2| + |x + 2|$ and $g(x) = 0$ on $[-2, 2]$. Examples of this solid with various values of n is given in Figures 10 through 12. For each n , rescale the dimensions of the base so that the height of the solid is 1. As in Example 3.1, the volume will be close to zero as n increases, but the surface area is not as simple. Since the height increases at a linear rate, the height of the rescaled solid at $x = 0$ will always be $1/2$. But as n increases, the base will get thinner and the surface between the two maximums will be “sandwiched” between the two outside “disks”. Can you determine the limiting surface area? The answer will be given at the end of this section. ■

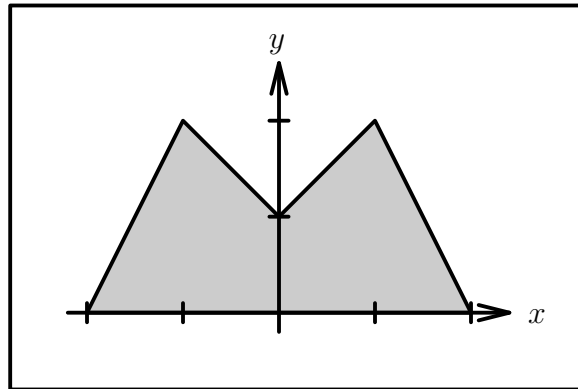


Figure 9: Polygon base for Example 3.2.

The general problem of determining the asymptotic behavior of the volume and surface area of any region defined by (1.1) are given in Theorems

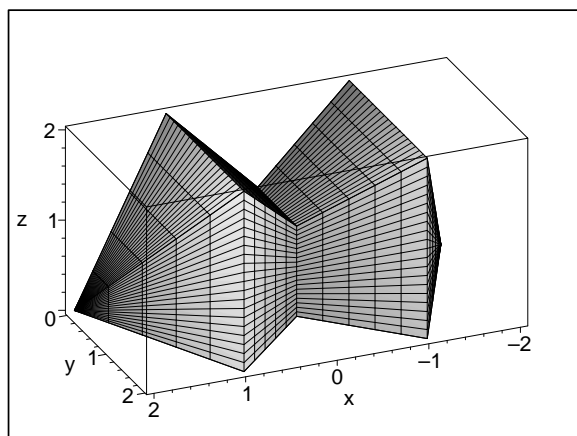


Figure 10: The non-normalized solid \mathcal{S}_R^4 for Example 3.2.

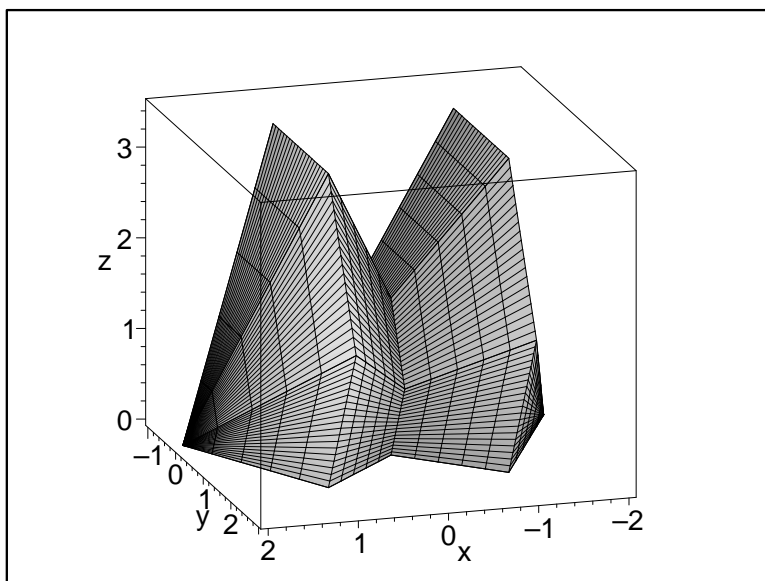


Figure 11: The non-normalized solid \mathcal{S}_R^6 for Example 3.2.

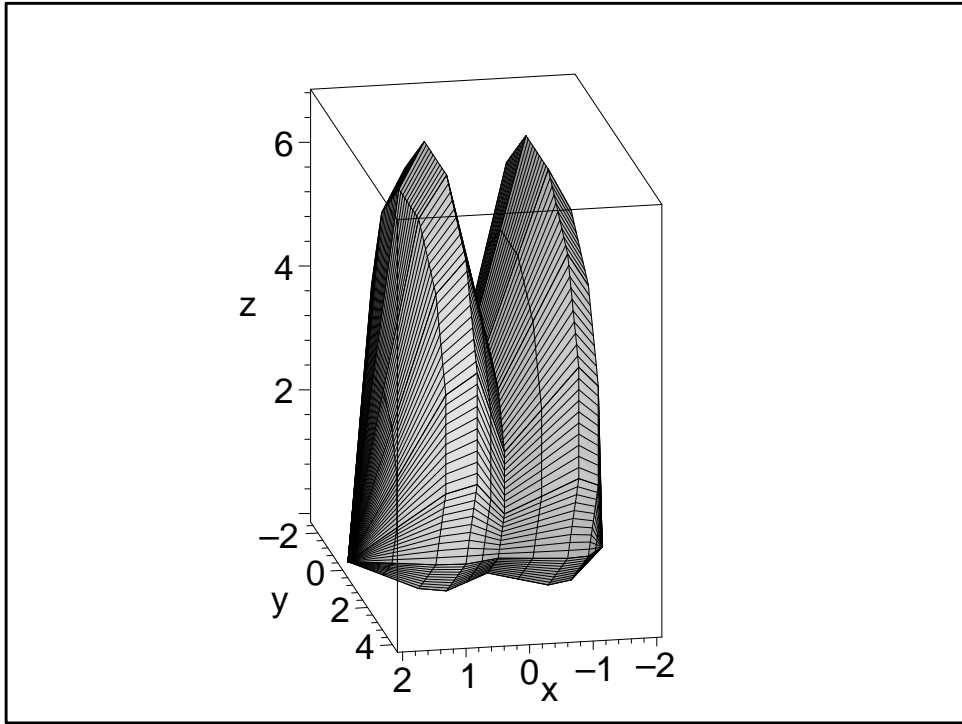


Figure 12: The non-normalized solid \mathcal{S}_R^{12} for Example 3.2.

3.4 and 3.5. But first we will need a lemma to describe how the solids are to be normalized.

Lemma 3.3. *Given a solid \mathcal{S}_R^n , let $\overline{\mathcal{S}}_R^n$ be the solid normalized to a height of 1. Then*

$$V(\overline{\mathcal{S}}_R^n) = \frac{V(\mathcal{S}_R^n)}{H(\mathcal{S}_R^n)^3} \text{ and } A(\overline{\mathcal{S}}_R^n) = \frac{A(\mathcal{S}_R^n)}{H(\mathcal{S}_R^n)^2}.$$

Proof. First scale the x and y coordinates of R by a factor of $t \in (0, \infty)$. Replace $f(x)$ and $g(x)$ by, respectively, $\frac{1}{t}f(tx)$ and $\frac{1}{t}g(tx)$. These new functions are defined on the interval $[\frac{a}{t}, \frac{b}{t}]$. Define the new base, $R(t)$, scaled by t as

$$R(t) := \{(x, y, 0) : \frac{a}{t} \leq x \leq \frac{b}{t}, \frac{1}{t}g(tx) \leq y \leq \frac{1}{t}f(tx)\}.$$

The longest vertical strip in $R(t)$ is $M(t) := \sup_{x \in [a/t, b/t]} \{\frac{1}{t}(f - g)(tx)\} = M/t$, where $M \equiv M(1)$ is the longest vertical strip in $R \equiv R(1)$ given by (2.1). The height of $\mathcal{S}_{R(t)}^n$ is given by (2.2) and is

$$H(\mathcal{S}_{R(t)}^n) = \frac{c_n M(t)}{\sin \frac{\pi}{n}} = \frac{c_n M}{t \sin \frac{\pi}{n}}.$$

We need to find a positive number t_0 such that $H(\mathcal{S}_{R(t_0)}^n) = 1$. Taking $t_0 = \frac{c_n}{\sin(\pi/n)} M = H(\mathcal{S}_R^n)$ will give the desired height.

From (2.4), the volume is

$$\begin{aligned} V(\overline{\mathcal{S}}_R^n) &= V(\mathcal{S}_{R(t_0)}^n) = \frac{n}{4} (\cot \frac{\pi}{n}) \int_{a/t_0}^{b/t_0} [\frac{1}{t_0}(f - g)(t_0 x)]^2 dx \\ &= \frac{n}{4t_0^3} (\cot \frac{\pi}{n}) \int_a^b [(f - g)(u)]^2 du = \frac{V(\mathcal{S}_R^n)}{H(\mathcal{S}_R^n)^3}. \end{aligned}$$

To find the lateral surface area, use (2.7) along with the identity $\frac{d}{dx} [\frac{1}{t_0}(f \pm g)(t_0 x)] = (f \pm g)'(t_0 x)$ followed by the substitution $u = t_0 x$ to obtain

$$A(\overline{\mathcal{S}}_R^n) = A(\mathcal{S}_{R(t_0)}^n) = \frac{1}{t_0^2} A(\mathcal{S}_R^n) = \frac{A(\mathcal{S}_R^n)}{H(\mathcal{S}_R^n)^2}.$$

□

Theorem 3.4. *Given R , the volume of \mathcal{S}_R^n , normalized so that the height is 1, converges to 0 as $n \rightarrow \infty$.*

Proof. By Lemma 3.3,

$$\lim_{n \rightarrow \infty} V(\overline{\mathcal{S}_R^n}) = \lim_{n \rightarrow \infty} \frac{V(\mathcal{S}_R^n)}{H(\mathcal{S}_R^n)^3} = \frac{\int_a^b [(f-g)(x)]^2 dx}{4M^3} \lim_{n \rightarrow \infty} \frac{n \sin^3(\frac{\pi}{n})}{c_n^3 \tan(\frac{\pi}{n})} = 0.$$

□

This result should not be surprising since $V(\mathcal{S}_R^n)$ is $\mathcal{O}(n^2)$ and $H(\mathcal{S}_R^n)$ is $\mathcal{O}(n)$.

Theorem 3.5. *Given R , the lateral surface area of \mathcal{S}_R^n , normalized so that the height is 1, converges provided $(f-g)|(f-g)'$ is integrable on $[a, b]$ and converges to $\frac{\pi K}{4M^2}$ as $n \rightarrow \infty$, where K is total variation of $(f-g)^2$ on $[a, b]$ and M is given by (2.1).*

Proof. Since

$$|\xi_k| \leq \sqrt{1 + \xi_k^2} \leq 1 + |\xi_k|$$

for any $\xi_k \in \mathbb{R}$, then summing all terms over ξ_1 through ξ_n yields

$$\sum_{k=1}^n |\xi_k| \leq \sum_{k=1}^n \sqrt{1 + \xi_k^2} \leq n + \sum_{k=1}^n |\xi_k|.$$

By the triangle inequality,

$$\sum_{k=1}^n |\xi_k| \geq \left| \sum_{k=1}^n \xi_k \right|,$$

so

$$\left| \sum_{k=1}^n \xi_k \right| \leq \sum_{k=1}^n \sqrt{1 + \xi_k^2} \leq n + \sum_{k=1}^n |\xi_k|. \quad (3.1)$$

Let $\xi_k \equiv \xi_k(x) := \alpha \sin(\frac{2k\pi}{n}) + \beta_n \sin^2(\frac{k\pi}{n})$, where $\alpha \equiv \alpha(x) := \frac{1}{2}(f+g)'(x)$ and $\beta_n \equiv \beta_n(x) := (f-g)'(x) \cot(\frac{\pi}{n})$.

Since $\sum_{k=1}^n \sin(\frac{2k\pi}{n}) = \sum_{k=1}^n \cos(\frac{2k\pi}{n}) = 0$, then $\sum_{k=1}^n \sin^2(\frac{k\pi}{n}) = \frac{1}{2}n$ for all $n \geq 3$ and so

$$\sum_{k=1}^n \xi_k = \alpha \sum_{k=1}^n \sin(\frac{2k\pi}{n}) + \beta_n \sum_{k=1}^n \sin^2(\frac{k\pi}{n}) = \frac{1}{2}n\beta_n.$$

Again, by the triangle inequality,

$$\sum_{k=1}^n |\xi_k| \leq |\alpha| \sum_{k=1}^n \left| \sin\left(\frac{2k\pi}{n}\right) \right| + |\beta_n| \sum_{k=1}^n \sin^2\left(\frac{k\pi}{n}\right) = |\alpha| \sum_{k=1}^n \left| \sin\left(\frac{2k\pi}{n}\right) \right| + \frac{1}{2}n|\beta_n|.$$

Thus (3.1) can be written as

$$\frac{1}{2}n|\beta_n| \leq \sum_{k=1}^n \sqrt{1 + \xi_k^2} \leq n\left(1 + \frac{1}{2}|\beta_n|\right) + |\alpha| \sum_{k=1}^n \left| \sin\left(\frac{2k\pi}{n}\right) \right|. \quad (3.2)$$

Multiply all terms in the previous inequality by $\frac{\sin^2(\pi/n)}{c_n^2}$, where c_n is given by (2.3), and take the limit as n goes to infinity. The term on the left of (3.2) becomes

$$\lim_{n \rightarrow \infty} \frac{\sin^2(\frac{\pi}{n})}{c_n^2} \frac{n|\beta_n|}{2} = \frac{|(f-g)'(x)|}{4} \lim_{n \rightarrow \infty} \frac{n \sin(\frac{2\pi}{n})}{c_n^2} = \frac{1}{2}\pi |(f-g)'(x)|.$$

The first term on the right of (3.2) becomes

$$\lim_{n \rightarrow \infty} \frac{\sin^2(\frac{\pi}{n})}{c_n^2} n\left(1 + \frac{1}{2}|\beta_n|\right) = \frac{1}{2}\pi |(f-g)'(x)|.$$

The second term on the right of (3.2) becomes

$$\lim_{n \rightarrow \infty} \frac{\sin^2(\frac{\pi}{n})}{c_n^2} |\alpha| \sum_{k=1}^n \left| \sin\left(\frac{2k\pi}{n}\right) \right| = 0$$

since $\sin^2(\frac{\pi}{n})$ is $\mathcal{O}(\frac{1}{n^2})$, c_n is $\mathcal{O}(1)$ and $\sum_{k=1}^n \left| \sin\left(\frac{2k\pi}{n}\right) \right|$ is at most $\mathcal{O}(n)$.

Now, to determine the asymptotic behavior of $A(\overline{\mathcal{S}_R^n})$, utilize Lemma 3.3 followed by the Squeeze Theorem applied to the inequality (3.2) and the previous three limits. We have

$$\begin{aligned} \lim_{n \rightarrow \infty} A(\overline{\mathcal{S}_R^n}) &= \lim_{n \rightarrow \infty} \frac{A(\mathcal{S}_R^n)}{H(\mathcal{S}_R^n)^2} \\ &= \lim_{n \rightarrow \infty} \frac{\sin^2(\frac{\pi}{n})}{M^2 c_n^2} \int_a^b (f-g)(x) \sum_{k=1}^n \sqrt{1 + \xi_k^2} dx \\ &= \frac{1}{M^2} \int_a^b (f-g)(x) \left(\lim_{n \rightarrow \infty} \frac{\sin^2(\frac{\pi}{n})}{c_n^2} \sum_{k=1}^n \sqrt{1 + \xi_k^2} \right) dx \\ &= \frac{\pi}{2M^2} \int_a^b (f-g)(x) |(f-g)'(x)| dx, \end{aligned}$$

where it was necessary to interchange the limit and integral.

In [2, page 176], it is given that the total variation of a differentiable function F on $[a, b]$ is $\int_a^b |F'(x)| dx$ provided that $|F'|$ is integrable. Since $f - g \geq 0$ on $[a, b]$, then, taking $F = (f - g)^2$, we have

$$\lim_{n \rightarrow \infty} A(\overline{\mathcal{S}_R^n}) = \frac{\pi K}{4M^2}.$$

□

Since $A(\mathcal{S}_R^n)$ and $H(\mathcal{S}_R^n)^2$ behave in a similar manner asymptotically and $H(\mathcal{S}_R^n)$ is $\mathcal{O}(n)$, then $A(\mathcal{S}_R^n)$ is $\mathcal{O}(n^2)$. It is an interesting coincidence that, for fixed R , the volume and lateral surface area of \mathcal{S}_R^n are both $\mathcal{O}(n^2)$.

Answer. For Example 3.2, $M = 2$ and $K = 14$ implies $A(\overline{\mathcal{S}_R^n}) \sim \frac{7\pi}{8}$ by Theorem 3.5. This is the same as the area of four disks of unit diameter minus the area of two disks of half-unit diameter.

4 Open Questions

A few problems are left open to the gentle and curious readers. Here, we will assume we are provided a region R given by (1.1).

- Let $\theta \in (0, \pi]$ and consider the solid \mathcal{C}_R^θ generated by the region R whose cross-sections parallel to the yz -plane at $x \in [a, b]$ are disks such that θ is the central angle of the sector of the circle formed by the intersection of the boundary of the cross-sectional disk with f and g in the xy -plane (see Figure 13 for a typical cross-section). Determine the height, volume and surface area of \mathcal{C}_R^θ . (A specific example of this type of solid is given in Figure 14 where the solid is generated by the unit disk with $\theta = \frac{\pi}{3}$.)
- Calculate the centroid of \mathcal{S}_R^n for $n \geq 3$ or \mathcal{C}_R^θ for $\theta \in (0, \pi]$.
- Describe a geodesic on the surface of \mathcal{S}_R^n for $n \geq 3$ or \mathcal{C}_R^θ for $\theta \in (0, \pi]$.
- Consider the solid *dynamically* generated by a region R . That is, move R along a simple path in \mathbb{R}^3 and let \mathcal{D}_R be the solid swept out by

R . Care must be taken to define the orientation of R as it is moving. Determine the volume and surface area of \mathcal{D}_R . (This is the generalized case seen in calculus of generating a solid by rotating a region about a line.)

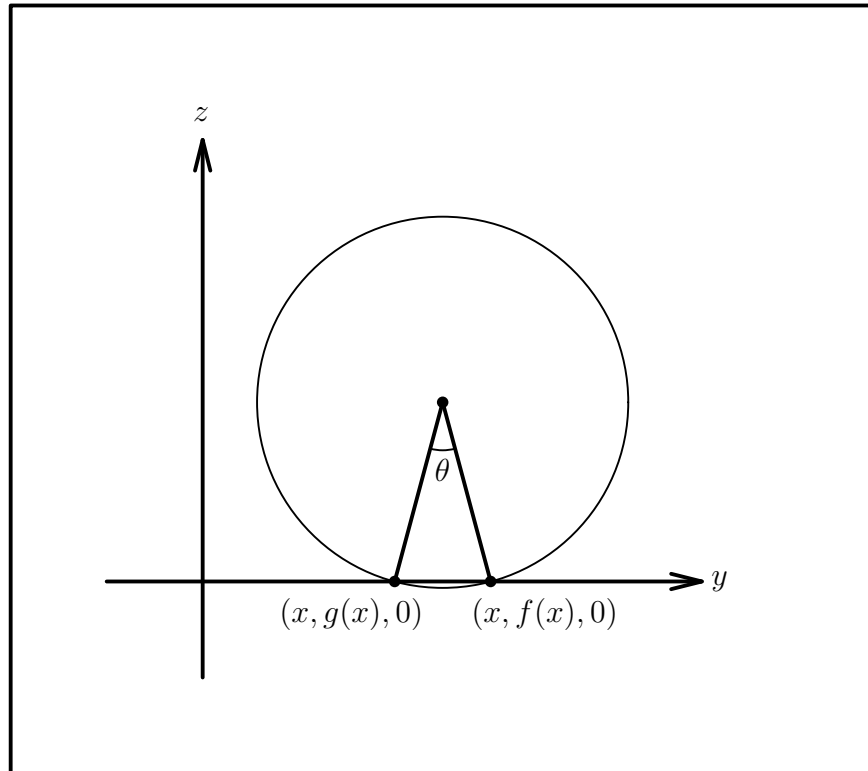


Figure 13: Cross-section of \mathcal{C}_R^θ at $x \in [a, b]$.

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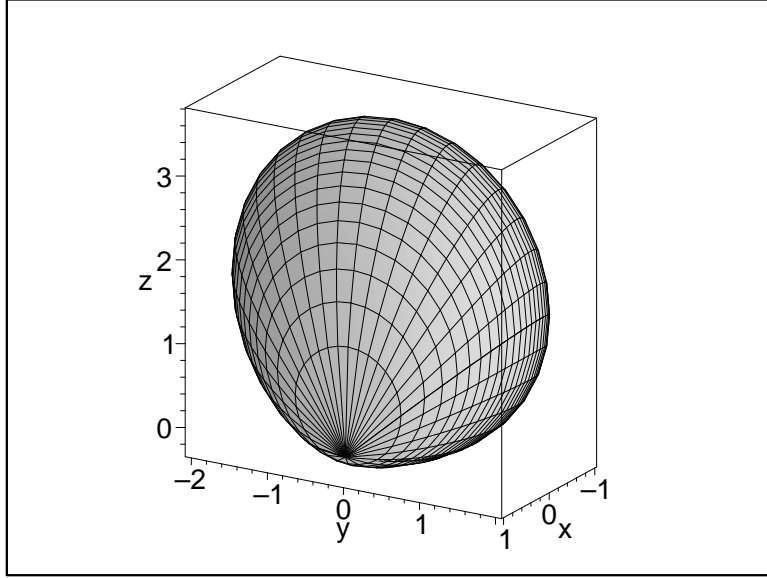


Figure 14: $\mathcal{C}_R^{\pi/3}$ where R is the unit disk.

Appendix - Parametric Equations Representing the Lateral Surface of \mathcal{S}_R^n

Let R be the base given by (1.1) and let $n \geq 3$. Referring to (2.5) and (2.6), for $k \in \{1, \dots, n\}$, the parametric equations of the k -th lateral surface of \mathcal{S}_R^n are

$$\begin{aligned}
 x(u, v) &= u \\
 y(u, v) &= y^k(u) - vy^{\Delta k}(u) \\
 &= \frac{(f+g)(u)}{2} + \frac{(f-g)(u)}{2} \left[(\cot \frac{\pi}{n}) (\sin \frac{2k\pi}{n}) + (1-2v) (\cos \frac{2k\pi}{n}) \right] \\
 z(u, v) &= z^k(u) - vz^{\Delta k}(u) \\
 &= \frac{(f-g)(u)}{2} (\sin \frac{2k\pi}{n}) \left[(\cot \frac{\pi}{n}) (\tan \frac{k\pi}{n}) + 1 - 2v \right],
 \end{aligned}$$

where $u \in [a, b]$ and $v \in [0, 1]$.

The area of a smooth parametric surface, S , is given in [3, page 1104] by

$$A(S) = \iint_D \left| \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \times \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle \right| dA, \quad (4.1)$$

where D is the domain and S is covered just once as (u, v) ranges throughout D .

Substituting the parametric equations into (4.1), integrating over v , and summing over the n surfaces will result in an equivalent version of (2.7) with the integral and summation reversed.

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