State estimation filtering algorithms for vehicle attitude determination using a dual-arc accelerometer array and 3-axis rate gyroscopes

Aaron Zimmerman
State Estimation Filtering Algorithms for Vehicle Attitude Determination using a Dual-Arc Accelerometer Array and 3-Axis Rate Gyroscopes

by

Aaron Zimmerman

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mechanical Engineering

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Acknowledgments

I would like to thank the faculty and staff of the Mechanical Engineering department at the Rochester Institute of Technology for giving me five great years of study and guidance. I would like to personally thank Dr. Agamemnon Crassidis for his guidance, his wisdom and his friendship. He has taught me through four different classes which provided me with a wealth of knowledge that I carried over into my graduate studies and will likely continue using in the years to come. He has also been a friend outside the school and provided me with great memories. I would also like to thank Dr. Jason Kolodziej and Dr. Tuhin Das for being part of my Master’s Thesis and providing me with assistance whenever I required it. In addition I would like to thank my family and friends for their continued support of me throughout the years and for allowing me the opportunity to live and grow.
Abstract

State Estimation Filtering Algorithms for Vehicle Attitude Determination using a Dual-Arc Accelerometer Array and 3-Axis Rate Gyroscopes

Aaron Zimmerman

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Sensor measurements are corrupted by biases, noise and drift effects and, in order to provide accurate measurements, these errors need to be estimated and, thus, eliminated. The current model used an Extended Kalman filter for the estimation of rate gyroscope measurement errors. This work improves upon that filter by applying a more robust, more accurate and more reliable Unscented Kalman filter. In addition, an algorithm for estimating the accelerometer measurement errors is developed using control theory. Using the attitude estimate from the Unscented Kalman filter, an error signal is formed between that attitude and the attitude estimates from the accelerometer array(s). This error signal is then reduced by implementation of an innovative method using PID controllers to estimate, and reduce the effects of, accelerometer measurement errors. While this thesis uses a previously developed device and equations, it is a departure from the previous works as it considers parameters and variables that were ignored in those studies.
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Nomenclature

PARAMETERS

\( X, Y, Z \) : Primary, Secondary and Tertiary Coordinate Axes
\( i, j, k \) : Primary, Secondary and Tertiary Unit Vector Directions
\( \phi, \theta, \psi \) : Bank, Elevation and Heading Euler Angles
\( p, q, r \) : Roll, Pitch and Yaw Angular Rates
\( q_0, q_1, q_2, q_3 \) : Quaternion Variables
\( u, v, w \) : Primary, Secondary and Tertiary Body Velocities
\( F \) : Force Vector
\( W \) : Weight Vector
\( M \) : Moment Vector
\( V \) : Velocity Vector
\( m \) : Aircraft/Vehicle Mass
\( g \) : Acceleration due to gravity
\( \vec{\omega} \) : Vehicle Angular Rate Vector
\( L, M, N \) : Primary, Secondary and Tertiary Scalar Moments
\( T \) : Transformation Matrix
\( I_{N \times N} \) : N by N Identity Matrix
\( x_k, y_k \) : State and Measurement Vectors
\( w_k, v_k \) : Gaussian White Noise processes
\( P \) : Covariance Matrix
\( K \) : Kalman Gain
\( \chi \) : Sigma Point Vector
\( \delta q \) : Error-quaternion
\( e(t) \) : Error Signal
\( K_p, K_i, K_d \) : Proportional, Integral and Derivative Controller Gains
\( u(t) \) : Control Input
\( v(t), b(t), d(t) \) : Noise, Bias and Drift Errors
\( J \) : Cost Function Value
**SUPERSCRIPTS AND SUBSCRIPTS**

- $T$: Matrix Transpose
- $^{-1}$: Matrix or Quaternion Inverse
- $-1$: Pre Update
- $+$: Post Update

$E - B$: Earth-Fixed Coordinate System to Body-Fixed Coordinate System

$B - E$: Body-Fixed Coordinate System to Earth-Fixed Coordinate System

Imp, Imposed: Imposed Loading due to Vehicle Maneuvers

Man, Maneuver: Displacement due to Vehicle Maneuvers

CG: Center-of-Gravity

$y_{i}$: $ith$ Accelerometer Measurement Relative to Vehicle Secondary Axis

$i$: Relating to the $ith$ Element

**MATHEMATICAL OPERATORS**

- $\|x\|$: Scalar or Vector Magnitude
- $\otimes$: Quaternion Multiplication
- $\bullet$: Dot Product
- $\times$: Cross Product
- $*$: Multiplication
- $s\phi$: Sine of $\phi$
- $c\phi$: Cosine of $\phi$

**ACRONYMNS AND ABBREVIATIONS**

DCM: Directional Cosine Matrix
UKF: Unscented Kalman Filter
EKF: Extended Kalman Filter
UT: Unscented Transform
INS: Inertial Navigation System
Chapter 1

Introduction

1.1 Background

Reliable and accurate attitude estimation is critical for determining aircraft orientation and allowing for stability and control during operation. Typically, the attitude of an aircraft is described by three consecutive rotations known as Euler angles. The Euler angles describe the aircraft heading, bank and elevation and are described in the fixed axis system of the aircraft relative to a reference, fixed coordinate frame. During static and trim operating conditions, the Euler angles can be measured/determined directly by a set of accelerometers mounted along the body axis of the vehicle due to insignificant motion of the aircraft. During dynamic maneuvers the Euler angles may be determined by the integration of the body rotation rates which transform the inertial frame of reference to the Earth fixed frame. For aircraft, the flat Earth assumption is used where the Earth is used as an inertial frame and is known as the Earth-fixed coordinate system.

Determination and measurement of the Euler angles is typically through the use of a multi-sensor system. The multi-sensor system can consist of rate gyroscopes for measuring instantaneous angular rate, GPS for determining location and altitude of the aircraft, magnetometers for determining the local magnetic field and other such sensors. Measurement errors are inherent in all measurement systems such as bias, drift and noise. These errors are further increased by environmental conditions and dynamic conditions. The bias is the raw static error in a signal while the drift error is seen as a change in the bias over time due to a variety of reasons through the hardware/software in the sensor. These sensors are then integrated together through the use of a filter or similar algorithm to produce accurate and reliable attitude estimates. The integration of multiple sensors together seeks to reduce, or ideally eliminate, the sensor errors to produce the most
accurate and reliable readings possible. A common filter for this application is known as the Kalman filter. However, due to the need for multiple sensors and the design of a filter/algorithm for integration of multiple sensors, the devices used for attitude determination are typically large in size, heavy in weight and expensive in price.

This thesis expands upon a previously developed device consisting of a two-dimensional accelerometer array with a three-axis rate gyroscope for estimating longitudinal pitching motion and transverse rolling motion by developing an innovative approach for the estimation of accelerometer measurement errors and applying a more robust and accurate filter for the determination of rate gyroscope measurement errors. In order to determine the feasibility of the aforementioned methods, the algorithm and filter developed were implemented in a full nonlinear aircraft model subjected to multiple maneuvers with noise, bias, drift effects and turbulence present during the simulation. The testing of these methods were demonstrated through the attitude estimate errors of pitch and roll with respect to simulated data from a high performance nonlinear aircraft model. While this work is a continuation of a previous study, [30] the methods developed in this work are a large step forward in achieving the final goal of a highly accurate and reliable three-dimensional attitude estimation device for use on micro aerial vehicles and unmanned aerial vehicles without compromising constraints such as weight, size, power and cost.

1.2 Innovation and Motivation for Current Work

The work and research performed in this study is focused on the design, development and implementation of an algorithm for accurately determining, and eliminating, accelerometer measurement errors and applying a new filter for rate gyro bias and noise estimation in order to increase the accuracy and robustness from the previously implemented filter in [30]. This work is an expansion of a previous work, [30], where the concept was to develop a two-dimensional, pitch and roll, attitude estimation device for reliable pitch and roll attitude determination. The previous work, [30], did not consider accelerometer biases or drift and implemented a filter which involved the linearization of a nonlinear system for rate gyro bias and estimation. This
thesis expands on the previous work, [30], by considering accelerometer measurement errors and developing a method for reducing their effects on the overall attitude determination algorithm(s) and by applying a new filter for rate gyro measurement errors which does not linearize the nonlinear system thus producing more accurate results.

The previously developed device, found in [30], consists of two semi-circular arcs consisting of 13, equally spaced, accelerometers with a rate gyroscope. Figure 1.1 is taken from [30] and represents the configuration of the device where the black circles represent the accelerometers.

![Figure 1.1: Dual-Arc Accelerometer Array Configuration [30]](image)

The proposed method for determining accelerometer biases involves the use of PID controllers for the determination of a control input which will be subtracted from the accelerometer measurements and, thus, reduce the effects of the additive measurement errors. The method is run simultaneously with the rate gyro bias estimation filter as both use information from the other for their estimates. The proposed filter for rate gyro bias estimation is an improvement over the previous [30], Extended Kalman filter, and is known as the Unscented Kalman filter.

The proposed accelerometer bias estimation algorithm and Unscented Kalman filter will be implemented and assessed in a full nonlinear operating environment and the attitude estimates obtained will be compared to the attitude values obtained from the nonlinear aircraft model. The simulated environment is evaluated through the use of Simulink® and Matlab®. The outline for the research and work conducted in this thesis is as follows:
1. Improve the previously developed nonlinear aircraft simulation model by incorporating accelerometer noise, bias and drift effects

2. Design and develop an accurate and reliable method for the determination of accelerometer measurement errors, namely biases

3. Design and apply the Unscented Kalman filter for aircraft attitude determination in order to estimate rate gyro biases more accurately than the Extended Kalman filter

4. Analyze and contrast the attitude estimates obtained via the dual-arc accelerometer array to the true values obtained from the nonlinear aircraft simulation model
Chapter 2
Theory Development

2.1 Rigid Body Dynamics

2.1.1 Body Axis Coordinate System [15, 27, 33, 39]

There are multiple coordinate systems used for representing aircraft orientation and maneuvers. For this study, the typical body coordinate frame is chosen for all equations and formulations. The body coordinate frame is a fixed coordinate frame whose origin lies at the center of gravity (CG) of the aircraft. The x-axis, defined as $x_b$, is aligned with the CG through the nose, the y-axis, defined as $y_b$, is aligned with the CG through the right wing of the aircraft and the z-axis, defined as $z_b$, is aligned with the CG downwards normal to the aircraft. This coordinate system is shown in Figure 2.1.

![Figure 2.1: Body-Fixed Coordinate Frame [27]](image)

Because the body axis is constantly moving while the aircraft is in a motion, an additional coordinate system is needed for reference. The typical coordinate system used in this instance is the Earth-fixed coordinate system. This coordinate system uses the “flat Earth” assumption as well as assumes the Earth is fixed. The axes are labeled as follows: x-axis; $x_E$, y-axis; $y_E$ and
z-axis; \( z_E \). In Figure 2.1, however, the axes are represented with a subscript \( f \) instead of \( E \).

### 2.1.2 Euler Angle Representation

Euler angles are used to represent the angular displacement about the body-axis coordinate frame. The Euler angles, for these purposes, are known as the roll, \( \phi \), pitch, \( \theta \), and yaw, \( \psi \), angles. Table 2.1 provides a summary of axis designation, angle, position and angular rate.

<table>
<thead>
<tr>
<th>Parameter/Axis</th>
<th>Primary Axis</th>
<th>Secondary Axis</th>
<th>Tertiary Axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Designation</td>
<td>( X_{ref} )</td>
<td>( X_{veh} )</td>
<td>( Y_{ref} )</td>
</tr>
<tr>
<td>Position</td>
<td>( S_X )</td>
<td>( s_x )</td>
<td>( S_Y )</td>
</tr>
<tr>
<td>Velocity</td>
<td>( V_X )</td>
<td>( u )</td>
<td>( V_Y )</td>
</tr>
<tr>
<td>Angle, ( \varphi )</td>
<td>( \phi_E )</td>
<td>( \phi_b )</td>
<td>( \theta_E )</td>
</tr>
<tr>
<td>Angular Rate, ( \omega )</td>
<td>( \dot{\phi} )</td>
<td>( p )</td>
<td>( \dot{\theta} )</td>
</tr>
</tbody>
</table>

Table 2.1: Earth and Body - Fixed Axis Parameter Definitions [30]

While Euler angles are useful at representing vehicle rotation, they have a singularity condition known as Gimbal Lock. Gimbal Lock is a condition caused when 2 rotational axes of a vehicle point in the same direction [33]. This occurs when the aircraft’s pitch angle, \( \theta \) is at positive or negative 90 degrees from the reference axis. At this instance, a divide by zero (singularity) error occurs [27]. Because of this, it is necessary to represent the three-dimensional rotation space with 4 variables. One way of achieving this, which is widely used in the aerodynamics community, is the use of Quaternions, which will be discussed later.

### 2.1.3 Application of Newton’s Second Law for Rigid Body Motion [27]

In order to simulate vehicle, in this case aircraft, motion in Cartesian space, Newton’s second law must be applied. Newton’s second law states that the sum total of all external forces applied on a body must be equal to the rate change of momentum of that body and the summation of all external moments applied to that body must be equal to the time rate of change of
the angular moment of that body. Equations (2.1) and (2.2) show Newton’s second law in vector form.

\[ \sum \vec{F} = \frac{d}{dt}(m\vec{v}) \]  
\[ \sum \vec{M} = \frac{d}{dt}(\vec{H}) \]  

The following equations represent Newton’s second law in scalar form

\[ F_X = \frac{d}{dt}(mu), F_Y = \frac{d}{dt}(mv), F_Z = \frac{d}{dt}(mw) \]  
\[ L = \frac{d}{dt}(H_X), M = \frac{d}{dt}(H_Y), N = \frac{d}{dt}(H_Z) \]  

Next, an elemental mass, \( \delta m \), is defined and \( \vec{r} \) is the position vector from the aircraft center of gravity to the elemental mass and \( \vec{r}_c \) is the position vector from the origin of the Inertial Axis system to the aircraft CG. Using this formulation, Equation (2.1) can be rewritten as:

\[ \sum \delta \vec{F} = \delta m \frac{d\vec{v}}{dt} \]  
\[ \sum \delta \vec{F} = \vec{F} \]  

Note that in Equation (2.5), \( \vec{v} \) is defined as:

\[ \vec{v} = \vec{v}_c + \frac{d\vec{r}}{dt} \]  

where \( \vec{v}_c \) is the CG velocity with respect to the Inertial Axis frame. Now, substituting Equation (2.7) into Equation (2.5), and with Equation (2.6):

\[ \sum \delta \vec{F} = \vec{F} = \frac{d}{dt} \sum \left( \vec{v}_c + \frac{d\vec{r}}{dt} \right) \delta m \]  

Assuming that the mass of the aircraft is constant, Equation (2.8) becomes:

\[ \vec{F} = m\frac{d\vec{v}_c}{dt} + \frac{d}{dt} \sum \frac{d\vec{r}}{dt} \delta m = m\frac{d\vec{v}_c}{dt} + \frac{d^2}{dt^2} \sum \vec{r} \delta m \]  

But since \( \vec{r} \) is measured from the center of gravity, \( \sum \vec{r} \delta m = 0 \) so,
\[ \vec{F} = m \frac{d\vec{v}_c}{dt} \quad (2.10) \]

Note that Equation (2.10) is only valid for a non-rotating aircraft. The equations derived above need to be related to the body-fixed frame as they were derived in the inertial frame.

\[ \frac{d\vec{v}_c}{dt}_{\text{Inertial}} = \frac{d\vec{v}_c}{dt}_{\text{Body}} + (\vec{\omega} \times \vec{v}_c) \quad (2.11) \]

Substituting Equation (2.11) into Equation (2.10):

\[ \vec{F} = m \frac{d\vec{v}_c}{dt}_{\text{Body}} + m(\vec{\omega} \times \vec{v}_c) \]

or,

\[
\begin{bmatrix}
F_x \\
F_y \\
F_z 
\end{bmatrix} = 
\begin{bmatrix}
m(\dot{u} - vr + wq) \\
m(\dot{v} + ur + wp) \\
m(\dot{w} - uq + vp) 
\end{bmatrix} \quad (2.12)
\]

Now, consider the momentum equation, Equation (2.2) for an elemental mass, \( \delta m \)

\[ \delta \vec{M} = d \frac{d\vec{H}}{dt} = d \frac{d}{dt}(\vec{r} \times \vec{v}) \delta m \quad (2.13) \]

From Equation (2.7), \( \vec{v} = \vec{v}_c + \frac{d\vec{r}}{dt} \) or:

\[ \vec{v} = \vec{v}_c + \vec{\omega} \times \vec{r} \quad (2.14) \]

where \( \vec{\omega} \) is the angular rotation vector of an aircraft

\[ \vec{H} = \sum \delta \vec{H} = \sum (\vec{r} \times \vec{v}_c) \delta m + \sum \left[ \vec{r} \times (\vec{\omega} \times \vec{r}) \right] \delta m \quad (2.15) \]

In Equation (2.15), \( \vec{v}_c \) is constant with respect to the summation thus:

\[ \vec{H} = \sum \vec{r} \delta m \times \vec{v}_c + \sum \left[ \vec{r} \times (\vec{\omega} \times \vec{r}) \right] \delta m \quad (2.16) \]

Again, since \( \vec{r} \) is measured from the CG location, \( \sum \vec{r} \delta m = 0 \) so
\[ \vec{H} = \sum \left[ \vec{r} \times (\vec{\omega} \times \vec{r}) \right] \delta m \]  

(2.17)

Now, define \( \vec{\omega} \) and \( \vec{r} \) in the Cartesian coordinate system:

\[ \vec{\omega} = pi + qj + r k, \vec{r} = xi + yj + zk \]  

(2.18)

Substituting Equation (2.18) into Equation (2.17) and carrying out the cross products:

\[ \vec{H} = (pi + qj + r k) \sum (x^2 + y^2 + z^2) \delta m \]

\[ - \sum (xi + yj + zk)(px + qy + rz) \delta m \]  

(2.19)

In scalar form, Equation (2.19) can be written as:

\[ \begin{bmatrix} H_X \\ H_Y \\ H_Z \end{bmatrix} = \begin{bmatrix} p \sum (y^2 + z^2) \delta m - q \sum xy \delta m - r \sum xz \delta m \\ -p \sum xy \delta m + q \sum (x^2 + z^2) \delta m - r \sum yz \delta m \\ -p \sum xz \delta m - q \sum yz \delta m + r \sum (x^2 + y^2) \delta m \end{bmatrix} \]  

(2.20)

Carrying out the summation terms over the entire aircraft:

\[ \begin{bmatrix} I_{xx} & I_{xy} \\ I_{yx} & I_{yy} \\ I_{zx} & I_{yz} \end{bmatrix} = \begin{bmatrix} \iint (y^2 + z^2) \delta m & \iint xy \delta m \\ \iint (x^2 + z^2) \delta m & \iint xz \delta m \\ \iint (x^2 + y^2) \delta m & \iint yz \delta m \end{bmatrix} \]  

(2.21)

where \( I_{xx}, I_{yy}, I_{zz} \) are the mass moments of inertia about the body axes and \( I_{xy}, I_{xz} \) and \( I_{yz} \) are the mass products of inertia. Now, substituting Equation (2.21) into Equation (2.20) and carrying out the summations:

\[ \begin{bmatrix} H_X \\ H_Y \\ H_Z \end{bmatrix} = \begin{bmatrix} pI_{xx} - qI_{xy} - rI_{xz} \\ -pI_{yx} + qI_{yy} - rI_{yz} \\ -pI_{zx} - qI_{yz} + rI_{zz} \end{bmatrix} \]  

(2.22)

so therefore:
\[
\begin{bmatrix}
\dot{H}_X \\
\dot{H}_Y \\
\dot{H}_Z
\end{bmatrix} = \begin{bmatrix}
\dot{p}I_{xx} - \dot{q}I_{xy} - \dot{r}I_{xz} \\
-\dot{p}I_{xy} + \dot{q}I_{yy} - \dot{r}I_{yz} \\
-\dot{p}I_{xz} - \dot{q}I_{yz} + \dot{r}I_{zz}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
qr(I_{yy} - I_{zz}) + (q^2 - r^2)I_{xy} - prI_{xz} + pqI_{xz} \\
pr(I_{zz} - I_{xx}) + (r^2 - p^2)I_{xz} - pqI_{yz} + qrI_{xy} \\
pq(I_{xx} - I_{yy}) + (p^2 - q^2)I_{xy} - qrI_{xz} + prI_{yz}
\end{bmatrix}
\] (2.23)

Assume that the inertias are constant. However, as the aircraft rotates, the moments and products of inertia vary with time so they need to be defined relative to a fixed axis system:

\[
\frac{d\vec{H}}{dt} = \dot{H}_X\vec{i} + \dot{H}_Y\vec{j} + \dot{H}_Z\vec{k}
\] (2.24)

Together, Equations (2.12) and (2.23) are the equations of motion used for the simulation of aircraft where Equation (2.12) accounts for the forces resulting from aerodynamic and propulsive forces and Equation (2.23) accounts for moments arising from those forces. While these equations provide the basis for rigid body motion, equations describing the position and orientation (attitude) of the aircraft are still needed and will be discussed in the following section.

### 2.1.4 Euler Kinematics [15, 27, 33]

The equations of motion of an aircraft were derived in the previous section, however, equations for describing the position and orientation of the aircraft are still needed. In order to do this, Euler angles need to be used to orientate the aircraft relative the Earth-fixed coordinate system. The body-axis system, described above, can be described by using three consecutive rotations through three distinct angles, the Euler angles. This sequence is shown pictorially in Figure 2.2.
Typically, in the aerodynamic community a 3-2-1, or yaw-pitch-roll, rotation sequence is used. That means that the first rotation is about the z-axis given by the Euler angle $\psi$, then about the new y-axis through the Euler angle $\theta$ and finally about the new x-axis through the angle $\phi$. Figure 2.3 depicts how the Euler angles are typically defined.

In order to perform the “3-2-1” rotation mentioned above, a rotation matrix
must first be defined for each of the rotations. These matrices are presented in Equations (2.25), (2.26) and (2.27) below.

\[
K_\phi = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \phi & \sin \phi \\
0 & -\sin \phi & \cos \phi
\end{bmatrix} \tag{2.25}
\]

\[
K_\theta = \begin{bmatrix}
\cos \theta & 0 & -\sin \theta \\
0 & 1 & 0 \\
\sin \theta & 0 & \cos \theta
\end{bmatrix} \tag{2.26}
\]

\[
K_\psi = \begin{bmatrix}
\cos \psi & \sin \psi & 0 \\
-\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{bmatrix} \tag{2.27}
\]

As discussed above, this work will use the “3-2-1” rotation sequence which will be defined as

\[
T_{E-B} = K_\psi K_\theta K_\phi
\]

in order to go from the Earth-fixed coordinate system to the Body-axis system. Likewise, to transform from the Body-axis frame to the Earth-fixed frame, \(T_{B-E} = K_\phi K_\theta K_\psi\). Multiplying out the above equation for \(T_{E-B}\) and \(T_{B-E}\) and using the following nomenclature

\[
s(\theta) = \sin \theta, c(\theta) = \cos \theta
\]

yields the following equations

\[
T_{E-B} = \begin{bmatrix}
c\theta c\psi & c\theta s\psi & -s\theta \\
s\phi s\theta c\psi - c\phi s\psi & s\phi s\theta s\psi + c\phi c\psi & s\phi c\theta \\
c\phi s\theta c\psi + s\phi s\psi & c\phi s\theta s\psi - s\phi c\psi & c\phi c\theta
\end{bmatrix} \tag{2.28}
\]

\[
T_{B-E} = \begin{bmatrix}
c\theta c\psi & s\phi s\theta c\psi - c\phi s\psi & c\phi s\theta c\psi + s\phi s\psi \\
c\theta s\psi & s\phi s\theta s\psi + c\phi c\psi & c\phi s\theta s\psi - s\phi c\psi \\
-s\theta & s\phi c\theta & c\phi c\theta
\end{bmatrix} \tag{2.29}
\]

Note that \(T_{B-E}\) is simply the matrix transpose of \(T_{E-B}\). Also, the matrix given in Equation (2.28) is known as the Directional Cosine Matrix (DCM). It is necessary to relate the angular velocities in the body frame to the Euler angular rates. The transform for the Euler angular rates to the body-fixed
frame angular velocities is given in Equation (2.30) while Equation (2.31) gives the inverse transform to go from the body-fixed frame velocities to the Euler rates.

\[
\begin{bmatrix}
  p \\
  q \\
  r
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & -s\theta \\
  0 & c\phi & c\theta s\phi \\
  0 & -s\phi & c\theta c\phi
\end{bmatrix}
\begin{bmatrix}
  \dot{\phi} \\
  \dot{\theta} \\
  \dot{\psi}
\end{bmatrix}
\]  

(2.30)

\[
\begin{bmatrix}
  \dot{\phi} \\
  \dot{\theta} \\
  \dot{\psi}
\end{bmatrix} =
\begin{bmatrix}
  1 & s\phi \tan \theta & c\phi \tan \theta \\
  0 & c\phi & -s\phi \\
  0 & s\phi \sec \theta & c\phi \sec \theta
\end{bmatrix}
\begin{bmatrix}
  p \\
  q \\
  r
\end{bmatrix}
\]  

(2.31)

Equation (2.31) is known as the Euler model. Integrating Equation (2.31) yields the Euler angles which are used to describe the aircraft orientation. However, there exists a singularity in the equation(s) at a pitch angle of plus or minus 90 degrees. This condition, as previously discussed is known as Gimbal Lock. This singularity needs to be eliminated as many aircraft can achieve pitch angles of plus or minus 90 degrees. One such way to eliminate this singularity is through the use of quaternions, which will be discussed in the following section [33].

2.2 Quaternions [3, 12, 19, 23, 33]

As mentioned above the singularity present in the Euler model needs to be addressed. In order to remove this singularity, an alternative way for representing the orientation of aircraft is needed: quaternions. Quaternions are commonly used to describe the attitude of vehicles in the aerospace community due to their mathematical properties and ease of transformation to Euler angles. The following equation is the standard form of the quaternion which consists of a vector part and a scalar part.

\[
\mathbf{q} = q_0 + q_1 \mathbf{i} + q_2 \mathbf{j} + q_3 \mathbf{k} 
\]  

(2.32)

The above form, however, is only one way to represent the quaternion. Later in this paper, the following representation, in Equation (2.33) is used. For this section the notation in Equation (2.32) is used, however.
\[ q = q_1\vec{i} + q_2\vec{j} + q_3\vec{k} + q_4 \]  

(2.33)

The scalar part of the quaternion is used to define the angle of rotation while the vector part is used to define the rotation axis.

### 2.2.1 Quaternion Formulation

In order to move from the Euler space to quaternion space, a transform is needed. In this instance, the transform is nothing more than a rotation. The rotation axis, mentioned above, is specified by its directional cosines in the reference coordinate frame, in this case the body-fixed frame. For convenience, the quaternion is considered to have a unity norm. The following equation represents the quaternion using directional cosines where \( \delta \) is the rotation angle and \( \alpha, \beta \) and \( \gamma \) are the direction angles.

\[
q = \begin{bmatrix}
\cos \delta \\
\cos \alpha \sin \delta \\
\cos \beta \sin \delta \\
\cos \gamma \sin \delta
\end{bmatrix} = \begin{bmatrix}
\cos \delta \\
\sin \delta \vec{n}
\end{bmatrix}
\]  

(2.34)

Note, that in Equation (2.34), \( \|q\| \) is equal to one (unity norm). Now, consider a Euclidean vector which is written as a quaternion with its scalar part equal to zero:

\[
u = \begin{bmatrix} 0 \\ u^r \end{bmatrix}
\]  

(2.35)

The resultant rotation must be a quaternion whose scalar part is also equal to zero. Also, the transformation must be reversible and the Euclidean length must be preserved. In order to achieve this, the following transform is considered:

\[
v = q \otimes u \otimes q^{-1} \quad \text{or} \quad v = q^{-1} \otimes u \otimes q
\]  

(2.36)

The second form leads to the commonly used convention:

\[
v = \begin{bmatrix}
q_0(\vec{q} \cdot \vec{u}) - (q_0\vec{u} - \vec{q} \times \vec{u}) \cdot \vec{q} \\
((\vec{q} \cdot \vec{u})\vec{q} + q_0(q_0\vec{u} - \vec{q} \times \vec{u}) + (q_0\vec{u} - \vec{q} \times \vec{u}) \times \vec{q})^r
\end{bmatrix}
\]  

(2.37)

Equation (2.37) then reduces to
\[ v = \begin{bmatrix} (2q(q \cdot u) + (q_0^2 - q \cdot q)u - 2q_0(q \times u))^r \\ 0 \end{bmatrix} \]  

(2.38)

Note that, in Equation (2.38), the scalar part is equal to zero so this transformation meets the requirement mentioned previously. Also, due to the properties of quaternion norms, the Euclidean length is preserved. Given the following rotation formula:

\[
(1 - \cos \mu) \mathbf{n}(\mathbf{n} \cdot \mathbf{u}) \\
\cos \mu \mathbf{u} \\
- \sin \mu (\mathbf{n} \times \mathbf{u})
\]

and the quaternion rotation:

\[
2 \sin^2 \delta \mathbf{n}(\mathbf{n} \cdot \mathbf{u}) \\
(\cos^2 \delta - \sin^2 \delta) \mathbf{u} \\
-2 \cos \delta \sin \delta (\mathbf{n} \times \mathbf{u})
\]

it can be seen that if \( \delta = \mu/2 \) and trigonometric identities/transforms are applied then the 2 rotation formulae will be equal. Therefore, the quaternion

\[ q = \begin{bmatrix} \cos(\mu/2) \\ \sin(\mu/2) \mathbf{n}^r \end{bmatrix} \]  

(2.39)

and the transformation discussed previously in Equation (2.36):

\[ q^{-1} \otimes u \otimes q \]  

(2.40)

give a left-handed rotation of the vector \( \mathbf{u} \) through the angle \( \mu \), around the vector \( \mathbf{n} \) when \( \mu \) is non-negative. Next, some basic properties of quaternions are looked at, as well as, some quaternion mathematics.

### 2.2.2 Quaternion Properties and Mathematics

The following section uses the quaternion definition of Equation (2.32):

\[ \mathbf{q} = q_0 + q_1 \mathbf{i} + q_2 \mathbf{j} + q_3 \mathbf{k} \]  

(2.41)

Because the quaternion possesses properties of both a scalar and a vector, some unique properties and mathematical operations arise. A few of these
are discussed below

**Quaternion Norm**

The quaternion norm is defined below

\[ \text{norm}(q) = \sum_{i=0}^{3} q_i^2 \]

**Quaternion Noncommutativity**

Consider the following identity

\[ p \otimes q - q \otimes p = \begin{bmatrix} 0 \\ (p \times q - q \times p)^r \end{bmatrix} = \begin{bmatrix} 0 \\ 2(p \times q)^r \end{bmatrix} \]

From the above equation(s), it can be seen that

\[ p \otimes q \neq q \otimes p \]

**Product Norm**

From the definition of the quaternion norm, above, and using vector operations, it can be shown that

\[ \text{norm}(p \otimes q) = \text{norm}(p) \times \text{norm}(q) \]

**Associative Property of Multiplication**

It is relatively easy to derive the following quaternion property

\[ (p \otimes q) \otimes r = p \otimes (q \otimes r) \]
**Quaternion Inverse**

The quaternion inverse is very useful when working with quaternions. First consider the following product,

\[
\begin{bmatrix}
q_0 \\
q^r
\end{bmatrix} \otimes \begin{bmatrix}
q_0 \\
-q^r
\end{bmatrix} = \begin{bmatrix}
q_0^2 + q \cdot q \\
(q_0q - q_0q - q \times q)^r
\end{bmatrix} = \begin{bmatrix}
\sum q_i^2 \\
0 \\
0
\end{bmatrix}
\]

As seen in the above formulation, multiplying a quaternion by another quaternion that differs only by a change in sign of its vector part results in a quaternion which possesses only a scalar portion (vector part is equal to zero). This new quaternion has very simple properties in multiplication as it consists of no vector part. Also, when divided by the quaternion norm will result in the “identity quaternion”. Because of these properties, the quaternion inverse is defined as follows,

\[
q^{-1} = \begin{bmatrix}
q_0 \\
q^r
\end{bmatrix}^{-1} = \frac{1}{\text{norm}(q)} \begin{bmatrix}
q_0 \\
-q^r
\end{bmatrix}
\]  \hspace{1cm} (2.42)

Note that when using unit quaternions the above equation simplifies to

\[
q^{-1} = \begin{bmatrix}
q_0 \\
-q^r
\end{bmatrix}
\]

**Inverse of a Quaternion Product**

The inverse of a quaternion product is defined as the product of the individual inverses in reverse order.

\[
(p \otimes q)^{-1} = \frac{1}{\text{norm}(p \otimes q)} \begin{bmatrix}
p_0q_0 - p \cdot q \\
-(p_0q + q_0p + p \times q)^r
\end{bmatrix} = \frac{1}{\text{norm}(q)} \begin{bmatrix}
q_0 \\
-q^r
\end{bmatrix} \otimes \begin{bmatrix}
p_0 \\
-p^r
\end{bmatrix} \frac{1}{\text{norm}(p)}
\]

Thus,

\[
(p \otimes q)^{-1} = q^{-1} \otimes p^{-1}
\]

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2.2.3 Quaternion for Attitude Description

As mentioned previously, it is necessary to be able to relate quaternions to the Euler angles in order to aid in the physical description of attitude. In order to do this, the Directional Cosine Matrix (DCM), given in Equation (2.28), must be derived for quaternions. Consider the quaternion rotation formula in Equation (2.38) written in terms of array operations,

\[ \mathbf{u}^b = \left[ 2q^a(q^a)^T + (q_0^2 - (q^a)^Tq^a)I - 2q_0\bar{q}^a \right] \mathbf{u}^a \]  

(2.43)

where \( \bar{q}^a \) is the cross product matrix given by

\[ \bar{q}^a = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} \]

Evaluating the terms in Equation (2.43), the DCM for quaternions is obtained

\[ T_{E-B}^{\text{quat}} = \begin{bmatrix} (q_0^2 + q_1^2 - q_2^2 - q_3^2) & 2(q_1q_2 + q_0q_3) & 2(q_1q_3 - q_0q_2) \\ 2(q_1q_2 - q_0q_3) & (q_0^2 - q_1^2 + q_2^2 - q_3^2) & 2(q_2q_3 + q_0q_1) \\ 2(q_1q_3 + q_0q_2) & 2(q_2q_3 - q_0q_1) & (q_0^2 - q_1^2 - q_2^2 + q_3^2) \end{bmatrix} \]  

(2.44)

Now, in order to obtain the quaternion from the Euler angles, consider the 3-2-1 rotation for quaternions

\[ \mathbf{v}^b = q^{-1}_\phi q^{-1}_\theta q^{-1}_\psi \mathbf{v}^r q_\psi q_\theta q_\phi \]  

(2.45)

where

\[ q_\phi = \begin{bmatrix} \cos\phi/2 \\ \sin\phi/2 \\ 0 \\ 0 \end{bmatrix}, \quad q_\theta = \begin{bmatrix} \cos\theta/2 \\ 0 \\ \sin\theta/2 \\ 0 \end{bmatrix}, \quad q_\psi = \begin{bmatrix} \cos\psi/2 \\ 0 \\ 0 \\ \sin\psi/2 \end{bmatrix} \]

Multiplying out Equation (2.45) yields the following,
\[
\begin{bmatrix}
q_0 \\
q_1 \\
q_2 \\
q_3
\end{bmatrix} = \pm \begin{bmatrix}
c_{\phi/2}c_{\theta/2}c_{\psi/2} + s_{\phi/2}s_{\theta/2}s_{\psi/2} \\
s_{\phi/2}c_{\theta/2}c_{\psi/2} - c_{\phi/2}s_{\theta/2}s_{\psi/2} \\
c_{\phi/2}s_{\theta/2}c_{\psi/2} + s_{\phi/2}c_{\theta/2}s_{\psi/2} \\
c_{\phi/2}s_{\theta/2}c_{\psi/2} - s_{\phi/2}c_{\theta/2}c_{\psi/2}
\end{bmatrix}
\] (2.46)

Equation (2.46) is useful when the DCM is unknown, however if it is known the following relationships can be used to obtain the quaternions,

\[
\begin{bmatrix}
q_0^2 \\
q_1^2 \\
q_2^2 \\
q_3^2
\end{bmatrix} = \frac{1}{4} \begin{bmatrix}
1 + DCM_{1,1} + DCM_{2,2} + DCM_{3,3} \\
1 + DCM_{1,1} - DCM_{2,2} - DCM_{3,3} \\
1 - DCM_{1,1} + DCM_{2,2} - DCM_{3,3} \\
1 - DCM_{1,1} - DCM_{2,2} + DCM_{3,3}
\end{bmatrix}
\] (2.47)

where \( DCM_{i,j} \) refers to the \( i \)th row and \( j \)th column of the directional cosine matrix.

### 2.2.4 Quaternion Kinematical Equations

Typically, quaternion attitude measurements are determined using a kinematical model which relates the body-fixed angular velocities, \( p, q \) and \( r \) to the quaternions. When two frames are in relative angular motion, as is the case with an aircraft in flight, a method is needed for continuously updating the quaternion measurement(s). Because of this problem, a differential equation arises relating the aforementioned variables. Let the orientation of the body-fixed frame, given by the vector \( B \), relative to the Earth-fixed frame, given by the vector \( E \), be given at time \( t \) by the quaternion \( q_{B/E}(t) \). Also, let the instantaneous angular velocity at time \( t \) be in the direction of a unit vector \( \hat{s} \) with magnitude \( \omega \). Then, in a small time, \( \delta t \), the quaternion, \( \delta q_{B/E} \), can be found using small angle approximations in Equation (2.34):

\[
\delta q_{B/E}(\delta t) \approx \begin{bmatrix}
\frac{1}{2}
\end{bmatrix} \delta t
\] (2.48)

At time \( t + \delta t \) the rotation is given by the perturbation quaternion, \( q_{B/E}(t + \delta t) \) as

\[
\delta q_{B/E}(t + \delta t) = q_{B/E}(t) \otimes \delta q_{B/E}(\delta t)
\] (2.49)
The derivative of $q_{B/E}(t)$ is temporarily omitting the subscripts,

$$\frac{dq}{dt} = \lim_{\delta t \to 0} \frac{q(t) \otimes [\delta q - I_q]}{\delta t} \quad (2.50)$$

where $I_q$ represents the identity quaternion. Now, substituting Equation (2.48) into Equation (2.50) yields

$$\frac{dq}{dt} = \frac{1}{2} q(t) \otimes \begin{bmatrix} 0 \\ \hat{s}^b \omega \end{bmatrix} = \frac{1}{2} q(t) \otimes \omega^b \quad (2.51)$$

which is formally written as

$$\dot{q}_{B/E} = \frac{1}{2} q_{B/E} \otimes \omega^b_{B/E} \quad (2.52)$$

Using matrix multiplication instead of quaternion multiplication, Equation (2.52) becomes

$$\dot{q} = \frac{1}{2} \begin{bmatrix} 0 & -\omega^T \\ \omega & -\Omega \end{bmatrix} \begin{bmatrix} q_0 \\ q \end{bmatrix} \quad (2.53)$$

Writing Equation (2.53) out in full gives the equation widely used in the simulation of rigid-body motion, including aircraft.

$$\begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -p & -q & -r \\ p & 0 & r & -q \\ q & -r & 0 & p \\ r & q & -p & 0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \quad (2.54)$$

### 2.3 Signal Processing [7, 26, 30]

In the real world, sensors are subject to factors which lead to errors in the data being read. Of these factors are noise corruption, biases and drift effects. Outside factors may affect one or more of these factors such as environmental conditions, the Earth’s magnetic field and surrounding sensors and electronics. In order to provide accurate measurements, these errors need to be minimized and, ideally, removed. To accomplish this, a sensor model is needed which incorporates the aforementioned errors. Equation (2.55) provides a general sensor model to be used for this work.
\[ MeasuredValue = TrueValue + Noise + Bias + Drift \]
\[ \tilde{x} = x + v(t) + b(t) + d(t) \]  \hspace{1cm} (2.55)

The biases and drift effects, \( b(t) \) and \( d(t) \), of a signal can lead to large errors in measurements which can be amplified if the signal is integrated. As stated above, a method to estimate out sensor errors is necessary to provide accurate and reliable measurements. In this work, the Unscented Kalman filter algorithm is implemented using the device model, rate gyroscopes and accelerometer measurements.

### 2.3.1 The Kalman Filter [37]

The Kalman filter is a set of mathematical equations used to recursively estimate system state(s) by reducing the mean squared error. The filter accomplishes this by using data from both the system and measurement (observation) models [37]. Figure 2.4 shows the basic equations and algorithm used in the standard Kalman filter.

![Figure 2.4: Operation of the Kalman Filter [37]](image)

In Figure 2.4, \( A \) relates the state from the previous time step to the state at the current time step, \( B \) relates the control input(s) to the state, \( x \), \( Q \) is
the process noise covariance, \( R \) is the measurement noise covariance and \( H \) relates the state to the measurement, which is assumed to be constant. The Kalman filter was developed in 1960 by R.E. Kalman and since become the subject of much research and application. Over the years numerous modifications have been made to the Kalman filter which include the Extended and Unscented Kalman filters.

**Modifications to the Kalman Filter [6, 13, 32, 30, 37]**

As mentioned above, many modifications have been made to the Kalman filter. Because the Kalman filter operates only on linear systems, changes needed to be made in order to apply it to nonlinear systems. Of these is the Extended Kalman filter which uses linearization of the nonlinear system in order to apply the standard Kalman filter. While not the subject of this work, a graphical representation of it’s use is presented in Figure 2.5 and a more complete derivation can be found in [30] or [37].

![Figure 2.5: Operation of the Extended Kalman Filter [37]](image)

In Figure 2.5, \( Q \) and \( R \) were defined previously while \( A \) and \( H \) are the Jacobian matrices for the process and measurement systems, respectively and \( V \) and \( W \) are the Jacobian matrices relating the process and measurement
noise, \(v\) and \(w\), respectively. Another modification involves using a transform known as the Unscented Transform. The Unscented Transform is a method for predicting means and covariances of nonlinear systems. Due to its direct implementation in the Unscented Kalman filter, the derivation of the Unscented Transform is not presented here but, instead, included with in the derivation of the Unscented Kalman filter which is the subject of the following section.

### 2.3.2 The General Unscented Kalman Filter and Unscented Transform [7, 17, 18]

The Unscented Kalman filter and Unscented Transform work on the principle that “it is easier to approximate a Gaussian distribution than it is to approximate an arbitrary nonlinear function or transformation” [18]. The Unscented Transform works to obtain a set of sigma points, which are a set of sample points around the mean, and then operate on those points. This method is superior to the Extended Kalman filter as no linearization of the system is needed thus allowing for a higher degree of accuracy. Also, there is no need to compute the Jacobian matrices which can be computational intensive. However, this method involves computing \(2n + 1\) sigma points and then propagating those points through the nonlinear system of equations, where \(n\) is the number of system states to be estimated. In contrast to Monte-Carlo type statistical methods, the Unscented Transform does not draw its samples randomly but rather according to a specific algorithm. The following set of sigma points are first computed:

\[
\chi^a_k(0) = \hat{x}^a_k \\
\chi^a_k(i) = \hat{x}^a_k \pm \sqrt{(L + \lambda)}P^a_k
\]

In the above equation, \(\lambda\) is a scaling parameter, \(L\) is the length of the augmented state vector, \(\hat{x}^a_k\) defined by

\[
x^a_k = \begin{bmatrix} x_k \\ w_k \\ v_k \end{bmatrix}, \quad \hat{x}^a_k = \begin{bmatrix} \hat{x}_k \\ 0_{q \times 1} \\ 0_{m \times 1} \end{bmatrix}
\]

where \(x_k\) is the system state, \(w_k\) is the system process noise and \(v_k\) is the measurement noise. Next, the sigma transformed through
\[ \chi_{k+1}(i) = f(\chi^x_k(i), \chi^w_k(i), u_k) \quad (2.57) \]

where

\[ \chi^a_k(i) = \begin{bmatrix} \chi^x_k(i) \\ \chi^w_k(i) \\ \chi^v_k(i) \end{bmatrix} \]

Now, the following weights are defined

\[
\begin{align*}
W_0^{mean} &= \frac{\lambda}{L+\lambda} \\
W_0^{cov} &= W_0^{mean} + (1 - \alpha^2 + \beta) \\
W_i^{mean} &= W_i^{cov} = \frac{1}{2(L+\lambda)}, \quad i = 1, 2, \ldots, 2L 
\end{align*} \quad (2.58)
\]

In the above equation, \( L \) is number of system states, \( \alpha \) determines the spread of sigma points and is generally set to a small positive value and \( \beta \) is used to incorporate prior knowledge of the distribution of sigma points. The predicted mean is then calculated by

\[ \hat{x}_{k+1}^- = \sum_{i=0}^{2L} W_i^{mean} \chi^x_{k+1}(i) \quad (2.59) \]

Next, the predicted mean is calculated as follows

\[ P_{k+1}^- = \sum_{i=0}^{2L} W_i^{cov} [\chi^x_{k+1}(i) - \hat{x}_{k+1}^-][\chi^x_{k+1}(i) - \hat{x}_{k+1}^-]^T \quad (2.60) \]

The mean observation is given by

\[ \hat{y}_{k+1}^- = W_i^{mean} \gamma^x_{k+1}(i) \quad (2.61) \]

where

\[ \gamma_{k+1}(i) = h(\chi^x_{k+1}(i), \chi^v_{k+1}(i), u_{k+1}) \quad (2.62) \]

The output covariance is then given by

\[ P_{k+1}^{yy} = \sum_{i=0}^{2L} W_i^{cov} [\gamma_{k+1}(i) - \hat{y}_{k+1}^-][\gamma_{k+1}(i) - \hat{y}_{k+1}^-]^T \quad (2.63) \]

The cross correlation covariance is calculated as follows
\[ P_{k+1}^{xy} = \sum_{i=0}^{2L} W_i^{cov} [\chi_{k+1}(i) - \hat{x}_{k+1}^-][\gamma_{k+1}(i) - \hat{y}_{k+1}^-]^T \] (2.64)

Finally, the innovation covariance is simply

\[ P_{k+1}^{vv} = P_{k+1}^{yy} \] (2.65)

Because of the augmentation used here the update equations in Figures 2.4 and 2.5 need be altered. As such the following equations are now used to update the system state.

\[ \hat{x}_{k+1}^+ = \hat{x}_k^- + K \hat{y}_{k+1}^- \] (2.66)

\[ P_{k+1}^+ = P_{k+1}^- - K P_{k+1}^{vv} K^T \] (2.67)

where the Kalman Gain, \( K \) is given by

\[ K = P_{k+1}^{xy} inv(P_{k+1}^{vv}) \] (2.68)

Now, this is simply the equations and procedure for the Unscented Kalman filter. In a later section, the above filter is altered in order to estimate attitude for the specific application to the dual-arc accelerometer array.

### 2.4 Optimization [9, 35]

Optimization techniques involve the changing of one or more variables in order to reduce a cost function. A multitude of optimization techniques have been developed over the years. All optimization techniques possess three things: a cost function, design variables and constraints. The cost function is the function which is to be minimized (or maximized), the design variables are the parameters that are changed throughout the optimization process in order to minimize (or maximize) the cost function and the constraints are the conditions placed on the routine such as power limits or height condition. For the purposes of this work, the optimization routine utilized is known as Particle Swarm Optimization as is discussed in the following section.
2.4.1 Particle Swarm Optimization [9, 35]

The Particle Swarm Optimization (PSO) technique was developed in 1995 by Kennedy and Eberhart. This technique was inspired by the movement dynamics and social behavior of birds, insects and fish. The PSO possesses numerous advantages which make it useful such as it’s simple implementation, there are only a few parameters which need to be tuned, it is highly efficient and accurate and is free of derivatives. However, there is two large disadvantage; it has a slow convergence and can, at times, converge on a local minimum instead of the global. Despite these disadvantages this method was chosen for this work because of its high degree of robustness, its ease of implementation and its high degree of accuracy. Because the application is highly nonlinear and complex, the PSO is an ideal choice [9].

The derivation for this technique is not presented here but can be found in [9]; instead, the algorithm is presented here. The following table outlines the parameters and variables used in the algorithm:

<table>
<thead>
<tr>
<th>Parameter/Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_k$</td>
<td>$N \times Q$ matrix for particle positions</td>
</tr>
<tr>
<td>$v_k$</td>
<td>$N \times Q$ matrix for particle velocities</td>
</tr>
<tr>
<td>$p_k$</td>
<td>$N \times Q$ vector for Best “remembered” individual particle positions</td>
</tr>
<tr>
<td>$p^g_k$</td>
<td>$N \times 1$ vector for best “remembered” swarm position</td>
</tr>
<tr>
<td>$y_p, y_g$</td>
<td>Cognitive and social parameters</td>
</tr>
<tr>
<td>$r_p, r_g$</td>
<td>Random numbers between 0 and 1</td>
</tr>
<tr>
<td>$w$</td>
<td>Particle velocity inertial term</td>
</tr>
<tr>
<td>$Q$</td>
<td>Number of particles in the swarm</td>
</tr>
<tr>
<td>$N$</td>
<td>Number of design variables</td>
</tr>
<tr>
<td>$b_{up}, b_{low}$</td>
<td>Upper and lower bounds of the initial space</td>
</tr>
</tbody>
</table>

Table 2.2: Particle Swarm Optimization Parameters [9]

First, $Q$ and $N$ must be set. For this application, $Q$ was set to 60 particles and $N$ is 18. $Q$ was chosen to be 60 after several trial runs. Next, the particle position is initialized

$$x_k^i = b_{low} + (b_{up} - b_{low})r_0$$  \hspace{1cm} (2.69)

where $r_0$ is a random number between 0 and 1. Next, the cost function values, $f_k^i$, are calculated for each particle
\[ f^i_k = J(x^i_k) \]
\[ f^g_k = f^i_k \quad (2.70) \]

where \( J \) is the cost function which is to be minimized. Next, the minimum of the initial swarm is computed and the position of the particle corresponding to that minimum is stored:

\[ f^i_{best}, p^g_k = \min(f_k) \]
\[ f^g_{best} = f^i_{best} \quad (2.71) \]

Before the iterative process begins, the velocity, \( v^i_k \) is set to zero. Next, a tolerance, \( tol \), for convergence is chosen (typically on the order of \( 10^{-4} \)). As long as \( f^i_{best} \geq tol \) the routine will continue.

First, randomize \( r_p \) and \( r_g \) then update the particle’s velocity as follows,

\[ v^i_{k+1} = w v^i_k + y_p r_p (p^i_k - x^i_k) + y_g r_g (p^g_k - x^i_k) \quad (2.72) \]

In Equation (2.72), \( y_p \) and \( y_g \) were chosen to be 1 after several runs by tuning them to allow for highly accurate attitude estimates. Next, update the particle’s position

\[ x^i_{k+1} = x^i_k + v^i_{k+1} \quad (2.73) \]

This next step is only if the swarm is subject to the constraints, \( b_{low} \) and \( b_{up} \). If the previous condition is indeed true then the bounds need to be checked as follows.

If \( x^i_{k+1} \leq b^i_{low} \), then
\[ x^i_{k+1} = (1.001)b^i_{low}, \quad v^i_{k+1} = 0 \]

If \( x^i_{k+1} \geq b^i_{up} \), then
\[ x^i_{k+1} = (0.999)b^i_{up}, \quad v^i_{k+1} = 0 \]

The previous check ensures the particle(s) remain within the preset bounds and do not continually attempt to go outside the bounds, which is why the velocity of that specific particle is set to 0. Next, the function values for the new particle positions are computed while checking if the particle function value is below the current “best” value:
\[ f_{val}^i = J(x_{k+1}^i) \]

If \( f_{val}^i \leq f_k^i \), then
\[
\begin{align*}
  f_{k+1}^i &= f_{val}^i \\
  p_{k+1}^i &= x_{k+1}^i
\end{align*}
\]
else
\[
\begin{align*}
  f_{k+1}^i &= f_k^i \\
  p_{k+1}^i &= x_k^i
\end{align*}
\]

(2.74)

The global minimum for the new set of particle positions is then found

\[
\begin{align*}
  f_{best}^i, p_{k+1}^i &= \min(f_{k+1}^i) \\
  \text{If } f_{best}^i &\leq f_{best}^g, \text{ then} \\
  f_{k+1}^g &= f_{best}^i \\
  p_{k+1}^g &= p_{k+1}^i
\end{align*}
\]
else
\[
\begin{align*}
  f_{k+1}^g &= f_k^g \\
  p_{k+1}^g &= p_k^g
\end{align*}
\]

(2.75)

If \( f_{k+1}^g \leq tol \) then the routine has converged on a minimum, otherwise the process is repeated from Equation (2.72). In the above equations, \( i \) is from 1 to \( Q \), which was set to 60 in this case.
Chapter 3

Dual-Arc Accelerometer Array Device

This section gives an overview of the device used to estimate vehicle attitude; the Dual-Arc Accelerometer Array. Presented here is an overview of the orientation, operation and basic equations used for the array. A more complete description can be found in [30] as its operation is not the main focus of this work.

3.1 Array Configuration [30]

This section provides an overview of the accelerometer array device presented in [30].

The device consists of thirteen, two-axis, accelerometers equally spaced around a half circle (180 degree) plane with a radius of 3 inches and a 3-axis rate gyroscope placed at the center of gravity location of the device. A second arc is placed normal to the first and results in a 2-dimensional arc array consisting of 25 accelerometers (with the center accelerometer being common for both arrays) and a rate gyroscope present at the center of gravity location. Figure 3.1 shows the configuration for the pitch plane with the orange axes representing the reference, Earth-fixed, coordinate system, the black axes being the vehicle, body, axis frame and the green axes representing the accelerometer array device axes.
The convention established in [30] uses an accelerometer spacing of 15 degrees in the x-z, longitudinal plane, beginning with accelerometer #1 at -90 degree pitch and rotating counter-clockwise to accelerometer #13 at +90 degree pitch. This convention can be seen in Figure 3.1. In the aforementioned figure, it can be seen that accelerometer #1 lies on the negative x-axis, accelerometer #7 on the z-axis at 0 degrees and accelerometer #13 on the positive x-axis. The pitch angle measurements are taken with respect to the z-axis and are taken to be positive counter-clockwise which can be seen in Figure 3.2.

The transverse array is configured in a similar manner to that of the pitch accelerometer array. For the transverse array, there are still 13 accelerometers around a 180 degree semi-circle spaced 15 degrees apart orientated on the
transverse, y-z plane. For the transverse array, however, accelerometer #1 lies at 0 degrees of roll offset on the positive y-axis, accelerometer #4 at 45 degrees offset, accelerometer #7 at 90 degrees offset on the positive z-axis, accelerometer #10 at 135 degrees offset and accelerometer #13 is at 180 degrees of roll offset, on the negative y-axis. Figure 3.3 is a representation of the transverse roll array showing the direction of positive roll.

The above figures depict the conventions and coordinate systems used for the dual-arc accelerometer array. The following figures, using these conventions and coordinate systems depict the arrays during pitch and roll maneuvers. Figure 3.4 shows the pitch array during a 45 degree pure pitch maneuver while Figure 3.5 shows the roll array during a 45 degree pure roll maneuver.
Figure 3.4: Longitudinal Pitch Array with Pitch Offset at 45 degrees[30]

Figure 3.5: Transverse Roll Array with Roll Offset at 45 degrees[30]
3.2 Accelerometer Array Load Equations [30]

In order to simulate the accelerometer array, equations are needed to calculate the accelerations for each accelerometer. This section gives the equations used in the simulation to calculate the accelerations of each accelerometer. Their full derivation is not within the focus this work but can be found in [30]. The accelerometers used in both arrays are 2-axis accelerometers which results in acceleration measurements in 2-axes. The following equations are used to determine the accelerometer measurements for both the pitch and roll (longitudinal and transverse) arrays.

\[
\begin{bmatrix}
  a_{x,i,misalign} \\
  a_{y,i,misalign} \\
  a_{z,i,misalign}
\end{bmatrix} =
\begin{bmatrix}
  c\theta c\psi & c\theta s\psi & -s\theta \\
  s\phi s\theta c\psi - c\phi s\psi & s\phi s\theta s\psi + c\phi c\psi & s\phi c\theta \\
  c\phi s\theta c\psi + s\phi s\psi & c\phi s\theta s\psi - s\phi c\psi & c\phi c\theta
\end{bmatrix}
\begin{bmatrix}
  a_{x,i} \\
  a_{y,i} \\
  a_{z,i}
\end{bmatrix}
\]  

(3.1)

where

\[
\begin{align*}
  a_{x,i} &= a_{x,CG} - (q^2 + r^2)\bar{x}_x + (pq - \dot{r})\bar{y}_x + (rp + \dot{q})\bar{z}_x \\
  a_{y,i} &= a_{y,CG} + (pq + \dot{r})\bar{x}_y - (p^2 + r^2)\bar{y}_y + (rq - \dot{p})\bar{z}_y \\
  a_{z,i} &= a_{z,CG} + (pr - \dot{q})\bar{x}_z + (qr + \dot{p})\bar{y}_z - (p^2 + q^2)\bar{z}_z
\end{align*}
\]  

(3.2)

Note that in Equation (3.1), the transformation matrix is the direction cosine matrix. For the longitudinal array, \(a_{y,i,misalign}\) is zero due to the accelerometers being 2-axis and being aligned along the x-z plane. For the transverse array, \(a_{x,i,misalign}\) is zero due to the array being aligned along the y-z plane. In the above equations, the angular rates are given as \(p, q\) and \(r\) for the roll, pitch and yaw rates, respectively while their derivatives are given as \(\dot{p}, \dot{q}\) and \(\dot{r}\). The distances, \(\bar{x}, \bar{y}\) and \(\bar{z}\) are the displacement distances from the vehicle’s center of gravity along each of the axes. As stated above, the complete derivations of these equations as well as their operation when subject to no external forces can be found in [30]. During normal operation, the imposed loading placed on the accelerometers due to rotational effects needs be subtracted out. This was the subject of [30] and is summarized in the following section.
3.3 Imposed Loading of Accelerometers [30]

In order to ensure accurate estimation of biases, noise and, ultimately, attitude, the loads imposed upon the accelerometers due to dynamic factors need to be removed. These equations were derived in full in [30] and the final results are given here. The following equations are for the longitudinal, pitch, array.

\[
A_{z,\text{imp}} = g[A_{z,i} - A_{x,\text{cg}} \sin \theta_i - (\cos \theta_{\text{man}} \cos \phi_{\text{man}} \cos \theta_i) \bigg] + r_d \bigg[ (r \sin \theta_i - p \cos \theta_i)^2 + q^2 \bigg] \quad (3.3)
\]

\[
A_{x,\text{imp}} = g\bigg[A_{z,i} - A_{z,\text{cg}} \cos \theta_i + (\sin \theta_{\text{man}} \sin \theta_i) \bigg] + r_d \bigg[ (2pr) \cos \phi_i \sin \phi_i - q^2 \sin^2 \phi_i - r^2 \cos^2 \phi_i - p^2 \bigg] \quad (3.4)
\]

Equation (3.3) is valid for accelerometers #2 through #12 due to the singularity present at accelerometers #1 and #13 (divide by zero because \(\cos(\pm 90\,\text{deg}) = 0\)). The same singularity occurs in Equation (3.4) at accelerometer #7 when \(\sin(0\,\text{deg}) = 0\). The following equations are used to determine the imposed loading for the transverse, roll, array and are derived in full in [30].

\[
A_{y,\text{imp}} = g[A_{y,i} - A_{z,\text{cg}} \sin \theta_i - (\cos \theta_{\text{man}} \sin \phi_{\text{man}} \cos \phi_i) \bigg] - r_d \bigg[ (2pr) \cos \phi_i \sin \phi_i - q^2 \sin^2 \phi_i - r^2 \cos^2 \phi_i - p^2 \bigg] \quad (3.5)
\]

\[
A_{z,\text{imp}} = g[A_{y,i} - A_{y,\text{cg}} \cos \phi_i - (\cos \theta_{\text{man}} \cos \phi_{\text{man}} \sin \phi_i) \bigg] - r_d \bigg[ (2pr) \cos \phi_i \sin \phi_i - q^2 \sin^2 \phi_i - r^2 \cos^2 \phi_i - p^2 \bigg] \quad (3.6)
\]

Equation (3.5) is valid for all accelerometers except accelerometer #7 due to the singularity present (divide by zero because \(\cos(90\,\text{deg}) = 0\)). The same singularity occurs in Equation (3.6) at accelerometer #1 and accelerometer #13 when \(\sin(0\,\text{deg}, 180\,\text{deg}) = 0\). With the imposed loading of the accelerometers now known, the attitude of the aircraft (vehicle) can be determined. In [30] the following equations for attitude determination were developed and are used in this work to determine the pitch and roll angles of the aircraft.
\( \theta_{est} = \sin^{-1}(A_{x,imposed} - A_{x,cg}) \) \hfill (3.7)

\( \phi_{est} = \tan^{-1}\left(\frac{A_{y,cg} - A_{y,imposed}}{A_{z,cg} - A_{z,imposed}}\right) \) \hfill (3.8)

In the above equations, the accelerometer measurements and imposed loadings are in gees. These equations were used throughout this work for attitude determination. Note that there is no equation for the yaw angle, \( \psi \), as it is a redundant parameter for control.
Chapter 4
Simulation Model

4.1 Nonlinear High Performance Aircraft Model [30]

In order to test the equations and algorithms presented in this study, a full six degree-of-freedom aircraft model was utilized. This model uses the equations presented previously in Section 2.1 as well as the equations for aircraft force and stability given in Appendix B.

The simulation model allows for the control of the aircraft control surfaces such as the vertical rudder, trailing edge flap and horizontal tail. The simulation uses a full six-degree of freedom look up table using the inputs of Mach number, altitude and angle-of-attack. For this work, the simulation run time was set to 10 seconds and performed at the “cruising” flight condition of 300 knots and 20,000 feet.

The algorithms presented in this study were applied in three phases. The first phase was to simulate the nonlinear high performance model using only a longitudinal pitch maneuver subject to the imposed loads due to the, discussed in Section 2.1, aerodynamic, environmental and thrust forces. Next, the simulation was run using only a transverse roll maneuver subject to the aforementioned forces. Finally, upon verification through the previous two phases, a combined longitudinal/transverse (pitch/roll) maneuver was used to confirm the proposed algorithms and equations.

The three phases described above were also conducted twice; once without the use of the Dryden Wind Turbulence model, and once with the model. The Dryden Wind Turbulence model allows for the inclusion of real-world turbulence effects in the simulation and allows for a more complete analysis.
This study was conducted in [30] but was used as a general procedure for this work as well. For this work, the above procedure was used, not to test the implementation of the aircraft equations of motion and stability, but to test the algorithms for bias and drift estimation, which will be discussed in later sections.

### 4.2 Dryden Wind Turbulence Model [11, 15]

As mentioned above, the Dryden Wind model is used to provide severe turbulence and wind to the simulation. In the real world, aircraft experience turbulence and wind gusts frequently so it is necessary to test the accelerometer array and the bias estimation algorithms in the presence of these real world effects. Presented in this section is a brief overview of the turbulence produced within the simulation. The Dryden Wind model, in detail, is not presented here as it is not the focus of this work but can be found in [11] or [30]. The following figures show the turbulence during each of the maneuvers discussed above; longitudinal, transverse and longitudinal/transverse.

![Figure 4.1: Turbulence Inputs during Longitudinal Maneuver](image)

Figure 4.1: Turbulence Inputs during Longitudinal Maneuver
Figure 4.2: Turbulence Inputs during Transverse Maneuver

Figure 4.3: Turbulence Inputs during Longitudinal/Transverse Maneuver
Figure 4.1 shows the turbulence inputs during a longitudinal, pitch maneuver; Figure 4.2 shows the turbulence inputs during a transverse, roll, maneuver and Figure 4.3 shows turbulence inputs during a combined, longitudinal/transverse, maneuver. The “amount” of turbulence is comparable during each maneuver and is summarized in the following table.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Maximum Turbulence Input (feet/second)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Longitudinal</td>
</tr>
<tr>
<td>$u_b$</td>
<td>50.709</td>
</tr>
<tr>
<td>$v_b$</td>
<td>44.834</td>
</tr>
<tr>
<td>$w_b$</td>
<td>56.598</td>
</tr>
</tbody>
</table>

Table 4.1: Turbulence Inputs for each Phase (Maneuver)

4.3 Model Improvements/Updates

In the previous study, [30], accelerometer biases, drift and noise were ignored. For this study, the accelerometer measurement errors were the main focus and, as such, needed to be incorporated into the model. The exact values for these errors will be discussed in the next chapter. Figure 4.4 shows how the measurement errors were added to the model.

The above figure shows how the sensor errors are applied to the x-direction accelerometer for the pitch array. The same model is used for both arrays, in all three coordinate directions. The specific values for bias, drift and noise used for this study will be given in the next chapter.
4.4 Simulation Architecture

The algorithms derived in the next two sections work together to obtain an accurate, and reliable, attitude estimate. The algorithms operate in real-time and continuously update the bias, drift, noise and attitude estimates. The following sections give an overview of the two methods used to estimate sensor biases.

4.4.1 Unscented Kalman Filter Bias Estimation

In the previous work, [30], an Extended Kalman filter was implemented to estimate rate gyro biases. This work, instead, designed and implemented an Unscented Kalman filter for bias estimation. The Unscented Kalman filter does not linearize the system, which is the case for the Extended Kalman filter, and, instead, operates on a set of sigma points which results in a faster convergence on the biases as well as produce a more accurate attitude estimate. The full derivation of the Unscented Kalman filter for attitude determination is discussed later in Chapter 5. The model for the Unscented Kalman filter is shown in Figure 4.5.

![Figure 4.5: Unscented Kalman Filter in Simulink](image)

The Unscented Kalman filter uses information from the accelerometers and rate gyroscopes to estimate the rate gyro biases. However, as mentioned previously, there are errors within the accelerometer measurements which
are estimated out via the algorithm derived later in Chapter 6. Because the accelerometer bias algorithm also relies on the rate gyro data, the two run together, simultaneously, to obtain an accurate estimate of attitude.

### 4.4.2 Accelerometer Bias Estimation

As mentioned in the previous section, the accelerometer bias estimation algorithm(s) are needed along with the Unscented Kalman filter to obtain an accurate, and reliable estimate of attitude. Figure 4.6 shows the model used for the determination of the accelerometer biases for both the longitudinal and transverse arrays.

![Figure 4.6: Accelerometer Bias Estimation in Simulink](image)

The accelerometer bias estimation algorithm(s) use information from the rate gyros and the accelerometers to estimate the accelerometer biases.

### 4.4.3 Overall System Architecture

As stated previously, the two methods above are run simultaneously to obtain an estimate for the attitude. Figure 4.7 gives an overall representation of the attitude determination process.
In the above figure, the red block indicates the final, best, attitude estimate. The results of this system will be given in Chapter 7.
Chapter 5

Rate Gyro Biases and Unscented Kalman Filter

The UKF works by using sigma points to approximate a Gaussian distribution for a nonlinear system and operates on the sigma points to achieve an estimate of the system state. The UKF was altered to be used with quaternions for attitude estimation. The UKF uses information about the noise of the rate gyroscopes and $\hat{\theta}_{\text{accel}}$ to estimate the attitude quaternion and the rate gyro biases and noise covariance matrix. As such, the method discussed later in Chapter 6 for estimating accelerometer biases and the method for estimating rate gyro biases are carried out simultaneously. The following is a detailed derivation of the Unscented Kalman Filter for attitude estimation. Please note that the following sections make some alterations to the standard formulation for use in this application. Also, the following formulations use 1-based indexing instead of the standard 0-based indexing.

5.1 The Unscented Transform [7, 8, 16, 17, 18, 21, 28, 32, 34, 36]

The Unscented Transform (UT) is the basis for the UKF and is reviewed here. Consider the general discrete-time state-space model

\begin{align*}
x_{k+1} &= f(x_k, k) + G_k w_k \\
\tilde{y}_k &= h(x_k, k) + v_k
\end{align*} 

(5.1)

(5.2)

where $x_k$ is the $n \times 1$ state vector, $\tilde{y}_k$ is the $m \times 1$ measurement vector, $f$ is the set of nonlinear equations, $h$ is the set of measurement equations and $w_k$ and $v_k$ are zero-mean Gaussian white noise processes with covariances of $Q_k$ and $R_k$, respectively. For the standard Kalman Filter, as well as the
Extended Kalman Filter (EKF) the update equations are written as

\[ \hat{x}^+_{k+1} = \hat{x}^-_k + K \hat{y}^-_{k+1} \]  
\[ P^+_{k+1} = P^-_{k+1} - K P^v_{k+1} K^T \]

where the Kalman Gain, \( K \) is given by

\[ K = P^{xy}_{k+1} inv(P^v_{k+1}) \]

In the above equations, the superscript \(-\) denotes pre-update while the superscript \(+\) denotes post-update, \( P_{k+1} \) is the covariance and \( P^{xy}_{k+1} \) is the cross-correlation matrix between the states, \( x \) and the measurements, \( y \). The UKF uses a different propagation than that of the EKF; the use of sigma points. Given an \( n \times n \) covariance matrix, a set of \( 2n + 1 \) sigma points is generated. The set of sigma points is computed as follows:

\[ \chi_k(1) = \hat{x}^-_k \]

\[ \sigma_k = \pm \sqrt{(n + \lambda)[P^+_k + \bar{Q}_k]} \]

\[ \chi_k(i) = \sigma_k + \hat{x}^-_k \]

where \( \lambda \) is a tunable parameter, \( \chi_k \) are the sigma points and \( \bar{Q}_k \) has a relation to the process noise and will be discussed momentarily. For the general form of the UKF, the sigma points given in equations (5.6) and (5.8) are transformed through the nonlinear system of equations as follows:

\[ \chi_{k+1}(i) = f(\chi_k(i), k) \]

For this application, however, the sigma points are transformed using quaternions which is discussed in the next section. The following equations are carried out in a straightforward manner ultimately resulting in the necessary parameters for use in equations (5.3), (5.4) and (5.5). First, the predicted mean is given by
\[
\hat{x}_{k+1} = \frac{1}{n + \lambda} \left[ \lambda \chi_{k+1}(1) + 0.5 \sum_{i=2}^{2n+1} \chi_{k+1}(i) \right] \quad (5.10)
\]

The predicted covariance matrix is computed by

\[
P_{k+1}^- = \frac{1}{n + \lambda} \left( \lambda [\chi_{k+1}(1) - \hat{x}_{k+1}^-] [\chi_{k+1}(1) - \hat{x}_{k+1}^-]^T + 0.5 \sum_{i=2}^{2n+1} [\chi_{k+1}(i) - \hat{x}_{k+1}^-] [\chi_{k+1}(i) - \hat{x}_{k+1}^-]^T \right) + \bar{Q}_k \quad (5.11)
\]

The mean observation is given by

\[
\hat{y}_{k+1}^- = \frac{1}{n + \lambda} \left[ \lambda \gamma_{k+1}(1) + 0.5 \sum_{i=2}^{2n+1} \gamma_{k+1}(i) \right] \quad (5.12)
\]

where \( \gamma_{k+1} \) will be discussed in the following section for this application but, in general is given by

\[
\gamma_{k+1}(i) = h[\chi_{k+1}(i), k] \quad (5.13)
\]

The output covariance is calculated by

\[
P_{y_k}^{yy} = \frac{1}{n + \lambda} \left( \lambda [\gamma_{k+1}(1) - \hat{y}_{k+1}^-] [\gamma_{k+1}(1) - \hat{y}_{k+1}^-]^T + 0.5 \sum_{i=2}^{2n+1} [\gamma_{k+1}(i) - \hat{y}_{k+1}^-] [\gamma_{k+1}(i) - \hat{y}_{k+1}^-]^T \right) \quad (5.14)
\]

The cross-correlation is found using

\[
P_{x_k}^{xy} = \frac{1}{n + \lambda} \left( \lambda [\chi_{k+1}(1) - \hat{x}_{k+1}^-] [\gamma_{k+1}(1) - \hat{y}_{k+1}^-]^T + 0.5 \sum_{i=2}^{2n+1} [\chi_{k+1}(i) - \hat{x}_{k+1}^-] [\gamma_{k+1}(i) - \hat{y}_{k+1}^-]^T \right) \quad (5.15)
\]
Finally, the innovation covariance is

\[ P_{k+1}^{vw} = P_{k+1}^{yy} + R_k \]  

(5.16)

Next, the Kalman Gain is computed using Eqn. (5.5) and the state vector is updated using (5.3).

5.2 Attitude Kinematics and Rate Gyro Sensor Model [7, 8, 13]

For this application, the attitude estimate is determined using quaternions which describe the pitch, roll and yaw angles using 4 variables. For the formulation below the quaternion is defined by \( \mathbf{q} = [\varrho^T \ q_4] \) where \( \varrho = [q_1 \ q_2 \ q_3]^T \). Also, the inverse is given by \( \mathbf{q}^{-1} = [-\varrho^T \ q_4] \) and \([ax]\) denotes a cross product matrix given by

\[
[ax] = \begin{bmatrix}
0 & -a_3 & a_2 \\
a_3 & 0 & -a_1 \\
-a_2 & a_1 & 0
\end{bmatrix}
\]  

(5.17)

The error-quaternion, \( \delta \mathbf{q} = [\delta \varrho^T \ \delta q_4] \), is represented using generalized Rodrigues parameters:

\[
\delta \mathbf{p} = f[\delta \varrho / (a + \delta q_4)]
\]  

(5.18)

where \( a \) is a chosen parameter from 0 to 1 and \( f \) is a scale factor. For this formulation, \( f \) was chosen such that \( f = 2(a + 1) \). The inverse transformation from \( \delta \mathbf{p} \) to \( \delta \mathbf{q} \) is given by

\[
\delta q_4 = \frac{-a ||\delta \mathbf{p}||^2 + f \sqrt{f^2 + (1 - a^2)} ||\delta \mathbf{p}||^2}{f^2 + ||\delta \mathbf{p}||^2}
\]  

(5.19)

\[
\delta \varrho = \frac{1}{f}(a + \delta q_4) \delta \mathbf{p}
\]  

(5.20)

The following is the general model used for the rate gyros where \( \ddot{\omega} \) is a vector representing the angular rate measurements from the 3-axis rate gyroscope, \( \omega \) is the true angular rate with no bias, \( \beta \) is the rate gyro bias and \( \sigma_v \) and \( \sigma_u \) are zero-mean Gaussian white-noise processes following the model
given by equation (2.55).

\[ \tilde{\omega} = \omega + \sigma_v + \beta \]  
\[ \dot{\beta} = \sigma_u \]  

(5.21)  
(5.22)

The UKF works to estimate \( \sigma_v \) and \( \beta \) in equation (5.21). In the standard Kalman Filter formulation, the post-update angular velocity and updated gyro bias are as follows:

\[ \hat{\omega}^+ = \tilde{\omega} - \hat{\beta}^+_k \]  
\[ \hat{\beta}^-_{k+1} = \hat{\beta}^+_k \]  

(5.23)  
(5.24)

Now, given post-update estimates for \( \hat{\omega}^+_k \) and \( \hat{q}^+_k \), the quaternion can be propagated by

\[ \hat{q}^-_{k+1} = \Omega \hat{q}^+_k \]  

(5.25)

where

\[ \Omega = \begin{bmatrix} \cos(0.5 \|\tilde{\omega}_k^+\| \Delta t)I_{3x3} - [\hat{\psi}_k^+ x]^T & \hat{\psi}_k^+ \\
-\hat{\psi}_k^{+ T} & \cos(0.5 \|\tilde{\omega}_k^+\| \Delta t) \end{bmatrix} \]  

(5.26)

with

\[ \hat{\psi}_k^+ = \sin(0.5 \|\tilde{\omega}_k^+\| \Delta t)\tilde{\omega}_k^+ / \|\tilde{\omega}_k^+\| \]  

(5.27)

and \( \Delta t \) is the sampling rate for the rate gyroscope.

5.2.1 Unscented Attitude Filter [8, 22, 34]

This section derives an Unscented Filter for attitude estimation using quaternions. First, the following state vector is defined

\[ \chi_k(1) = \dot{x}_k^+ = \begin{bmatrix} \delta \hat{p}_k^+ \\ \hat{\beta}_k^+ \end{bmatrix} \]  

(5.28)
δ\(\hat{p}^+_k\) will be used from Eqn. (5.18). Next, the sigma points in equations (5.6) and (5.8) are partitioned as

\[
\chi_k(i) = \begin{bmatrix} \chi^\delta_k \\ \chi^\beta_k \\ \chi_k \end{bmatrix}, \quad \text{for } i = 1, 2, \ldots 13 \tag{5.29}
\]

Using equation (5.6), it can be assumed that

\[
\hat{q}^+_k(1) = \hat{q}^+_k \tag{5.30}
\]

\[
\hat{q}^+_k(i) = \delta\hat{q}^+_k(i) \otimes \hat{q}^+_k \quad \text{for } i = 2, 3, \ldots 13 \tag{5.31}
\]

where \(\delta\hat{q}^+_k(i)\) is found using equation (5.19):

\[
\delta q^+_k(i) = \frac{-a\|\chi^\delta_k(i)\|^2 + f\sqrt{f^2 + (1 - a^2)\|\chi^\delta_k(i)\|^2}}{f^2 + \|\chi^\delta_k(i)\|^2} \quad \text{for } i = 2, 3, \ldots 13 \tag{5.32}
\]

\[
\delta q_k^+ = \frac{1}{f}[a + \delta q_k^+] \chi^\delta_k(i) \quad \text{for } i = 2, 3, \ldots 13 \tag{5.33}
\]

Note that equation (5.30) requires \(\chi^\delta_k(1)\) to be zero which is due to resetting the attitude error to zero. Without this reset, the error from the previous time step(s) would affect the current estimate(s) and would continue to compound producing inaccurate results. Next, using equation (5.25), the quaternions are propagated:

\[
\hat{q}^-_{k+1}(i) = \Omega\hat{q}^+_k(i) \quad \text{for } i = 1, 2, \ldots 13 \tag{5.34}
\]

where the estimated angular velocities are given as follows:

\[
\hat{\omega}^+_k = \hat{\omega}_k - \chi^\beta_k(i) \quad \text{for } i = 1, 2, \ldots 13 \tag{5.35}
\]

The propagated error quaternions are calculated using

\[
\delta q^-_{k+1}(i) = \hat{q}^-_{k+1}(i) \otimes inv(\hat{q}^-_{k+1}(1)) \quad \text{for } i = 1, 2, \ldots 13 \tag{5.36}
\]
Lastly, the sigma points can be propagated using equation (5.18)

\[ \chi^\delta_p(k+1) = 0 \]  
\[ (5.37) \]

\[ \chi^\delta_p(k+1)(i) = f \frac{\delta q_{k+1}^-(i)}{a + \delta q_{4k+1}^-(i)} \] for \( i = 2, 3, \ldots 13 \)  
\[ (5.38) \]

\[ \chi^\beta_k(i) = \chi^\beta_k(i) \] for \( i = 1, 2, \ldots 13 \)  
\[ (5.39) \]

\[ \chi_{k+1} = \begin{bmatrix} \chi^\delta_p_{k+1} & \chi^\beta_{k+1} \end{bmatrix}^T \]  
\[ (5.40) \]

The derivation for \( \bar{Q}_k \), which is used in equation (5.7), is not presented here but is given by

\[ \bar{Q}_k = \frac{\Delta t}{2} \begin{bmatrix} \sigma_v^2 - \frac{1}{6} \sigma_v^2 \Delta t^2 & I_{3x3} \\ 0_{3x3} & 0_{3x3} \end{bmatrix} \]  
\[ (5.41) \]

**Quaternion Mean Observations**

In typical applications, an attitude matrix is needed to compute the mean observations in equation (5.12). For this application, however, an attitude matrix is not needed because the quaternions are measured through use of the Dual-Arc Accelerometer array. Therefore, \( \gamma_{k+1} \) in equation (5.12) is calculated by using equations (5.36) and (5.38) as follows

\[ \delta q_\gamma(i) = q_{\text{accels}} \otimes inv(\hat{q}_{k+1}(i)) \] for \( i = 1, 2, \ldots 13 \)  
\[ (5.42) \]

\[ \gamma_{k+1}(i) = f \frac{\delta q_\gamma(i)}{a + \delta q_{4\gamma}(i)} \] for \( i = 1, 2, \ldots 13 \)  
\[ (5.43) \]

where \( \delta q_\gamma = [\delta q_\gamma^T \delta q_{4\gamma}]^T \) and \( q_{\text{accels}} \) is the quaternion estimate from the accelerometers.
5.2.2 Unscented Attitude Filter Procedure [8]

The following outlines the steps for implementing the above derived attitude estimation filter. First, an initial attitude quaternion and gyro-bias estimates are needed along with an initial covariance matrix, $P$. The initial state vector is set to $\hat{x}_0^+ = [0^T \beta_0^T]^T$. Next, the parameters $a$ and $\lambda$ are chosen and $f$ is calculated from $f = 2(a + 1)$. $\hat{Q}_k$ is then calculated using equations (5.41). Following this, the sigma points are calculated using equations (5.6), (5.7) and (5.8). The error quaternions are calculated next using equations. (5.32) and (5.33) and equations (5.30) and (5.31) are used to compute the sigma point quaternions. Equation (5.34) is then used to propagate the quaternions. The propagated error quaternions are next calculated using equation (5.36) and the propagated sigma points are found through equations (5.37), (5.38) and (5.39). The predicted mean and covariance are computed through equations (5.10) and (5.11). Next, the mean observations are calculated using equation (5.12) and $\gamma_{k+1}$ from equation (5.43). The output covariance, cross-correlation matrix and innovation covariance are computed using equation (5.14), (5.15) and (5.16). The state vector is then updated using equation (5.3) with $\hat{x}_{k+1}^+ = [\hat{\dot{p}}_{k+1}^+ \hat{\beta}_{k+1}^+]^T$. Finally, the quaternions are updated using

$$\delta q_{4k+1}^+ = \frac{-a\|\hat{\delta} p_{k+1}^+\|^2 + f\sqrt{f^2 + (1 - a^2)\|\hat{\delta} p_{k+1}^+\|^2}}{f^2 + \|\hat{\delta} p_{k+1}^+\|^2}$$

(5.44)

$$\delta q_{k+1}^+ = \frac{1}{f}(a + \delta q_{4k+1}^+)\hat{p}_{k+1}^+$$

(5.45)

$$\delta q_{k+1}^+ = [\delta q_{k+1}^+ \delta q_{4k+1}^+]^T$$

(5.46)

$$\hat{q}_{k+1}^+ = \hat{\delta} q_{k+1}^+ \otimes \hat{q}_{k+1}^- (1)$$

(5.47)

and $\hat{\delta} p_{k+1}^+$ is reset to zero and, for the next iteration,

$$\hat{x}_{k}^+ = [\hat{\dot{q}}_{k+1}^+ \beta]^T$$

(5.48)

The above procedure is used in the developed simulation, which was discussed in the previous chapter. As discussed previously, the Unscented
Kalman filter is an improvement over the Extended Kalman filter. Next, some simulation results from the dual-arc accelerometer array are given to compare the Unscented to the Extended Kalman filter.

5.3 Comparison Between the Unscented Kalman Filter and the Extended Kalman Filter

The Unscented Kalman filter was chosen as an improvement over the Extended Kalman filter for a number of reasons. The main reason, however, was its higher degree of accuracy. Because the Unscented Kalman filter does not linearize the system, it has a higher degree of potential accuracy. This section compares the results under various scenarios of the Unscented and Extended Kalman filters when they are applied to dual-arc accelerometer array for the estimation of rate gyro biases. The following table outlines the parameters used in the simulation of both the Unscented and Extended Kalman filters. For the UKF, specifically the value for $a$, shown above, was chosen to be 1, $\lambda$ was set to -3 and $n$ was set to a value of 6.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_0^a$</td>
<td>$1 \times 10^9$</td>
</tr>
<tr>
<td>$P_0^b$</td>
<td>$1 \times 10^{-4}$</td>
</tr>
<tr>
<td>$P_0$</td>
<td>$\text{diag}[P_0^a P_0^a P_0^a P_0^b P_0^b P_0^b]$</td>
</tr>
<tr>
<td>$\sigma_u(\text{rad/sec}^{3/2})$</td>
<td>$1 \times 10^{-6}$</td>
</tr>
<tr>
<td>$\sigma_v(\text{rad/sec}^{3/2})$</td>
<td>$1 \times 10^{-2}$</td>
</tr>
<tr>
<td>$\beta(\text{deg/sec})$</td>
<td>0.2</td>
</tr>
</tbody>
</table>

Table 5.1: UKF and EKF Parameter Values

For this study, the simulation time was set to 10 seconds with a sampling frequency of 100 Hz. Biases and noise effects were placed on the rate gyroscopes at 0.2 deg/sec and 0.15 (deg/sec)$^2$ noise variance, respectively but accelerometer noise and biases were set to zero in order to provide an accurate comparison between the two filters. The following sections show the accuracy of the Unscented Kalman filter vs that of the Extended Kalman filter. For the comparison, the final attitude estimates were looked at in order to determine which filter is better suited for these applications. In the
following sections Figure (a) shows the attitude estimate results from the Unscented Kalman filter while Figure (b) shows the attitude estimate results from the Extended Kalman filter. The first 2 plots in each set show the pitch attitude error for the given phase. The second 2 plots show the pitch estimate compared to the true value of pitch. The third set of 2 plots shows the attitude estimation error for roll. The last set of plots shows the roll estimate compared to the true value of roll.

5.3.1 Phase I - Longitudinal Aircraft Maneuver

Phase I - Longitudinal Maneuver: No Turbulence

The following figures compare the UKF to the EKF during a longitudinal maneuver with no turbulence.

![Pitch Error](image1.png)  
(a) Unscented Kalman Filter Pitch Error  

![Pitch Error](image2.png)  
(b) Extended Kalman Filter Pitch Error

Figure 5.1: Comparison Between UKF and EKF - Pitch Error, Phase I No Turbulence
The above figures show the comparison between the UKF and the EKF pitch, $\theta$, estimations during a longitudinal, pitch maneuver when no turbulence is present. As seen above, the UKF is more accurate than the EKF as the EKF estimate diverges from the truth after only 3 seconds while the UKF estimate maintains an estimation error of less than 0.28 degrees.
The above figures show the comparison between the UKF and the EKF roll, $\phi$, estimations during a longitudinal, pitch maneuver when no turbulence is present. As seen above, both the UKF and EKF have estimation errors due to estimating roll biases when only a pitch maneuver is present. For this case, both the EKF and the UKF have comparable errors being both below 0.005 degrees over the 10 seconds of simulation time. The EKF estimate, however, does become less accurate as time increases while the UKF estimate error remains relatively constant.

**Phase I - Longitudinal Maneuver: Turbulence**

The following figures compare the UKF to the EKF during a longitudinal maneuver in the presence of turbulence.
The above figures show the comparison between the UKF and the EKF pitch, $\theta$, estimations during a longitudinal, pitch maneuver when turbulence is present. The UKF estimate absolute error remains lower than that of the EKF over time. The errors for both estimates, however, are larger here than when no turbulence was present being above 0.4 degrees. While the UKF estimate absolute error remains below 0.5 degrees throughout, the EKF error becomes as high as 0.8 degrees for various instances of time.
The above figures show the comparison between the UKF and the EKF roll, $\phi$, estimations during a longitudinal, pitch maneuver when turbulence is present. Apart from one spike, the UKF estimate absolute error remains below 0.15 degrees over all time while the EKF estimate error remains higher over the 10 seconds. In this case the UKF and EKF estimates are similar, as seen in the above figures.
5.3.2 Phase II - Transverse Aircraft Maneuver

Phase II - Transverse Maneuver: No Turbulence

The following figures compare the UKF to the EKF during a transverse maneuver with no turbulence.

![Figure 5.9](image1.png)
(a) Unscented Kalman Filter Pitch Error  
(b) Extended Kalman Filter Pitch Error

Figure 5.9: Comparison Between UKF and EKF - Pitch Error, Phase II No Turbulence

![Figure 5.10](image2.png)
(a) Unscented Kalman Filter Pitch Estimate  
(b) Extended Kalman Filter Pitch Estimate

Figure 5.10: Comparison Between UKF and EKF - Pitch Estimation, Phase II No Turbulence

The above figures show the comparison between the UKF and the EKF pitch, $\theta$, estimations during a transverse, roll maneuver when no turbulence is present. At steady state, The UKF absolute error becomes less than 0.01 degrees while the EKF absolute error is 0.03 degrees. The errors for the
pitch estimate for both filters, however, are very low and, thus, the UKF only slightly outperforms the EKF for this case.

The above figures show the comparison between the UKF and the EKF roll, $\phi$, estimations during a transverse, roll maneuver when no turbulence is present. The roll estimate absolute error for the UKF is maintained below 0.3 degrees over all time while the EKF error achieves an error of 0.65 degrees. Both estimate errors, however, do approach 0 degrees as the time of the simulation approaches 10 seconds. However, this is due to the lack
of dynamic movement of the aircraft after 6 seconds. Before this time, the UKF is more accurate than the EKF.

**Phase II - Transverse Maneuver: Turbulence**

The following figures compare the UKF to the EKF during a transverse maneuver in the presence of turbulence.

![Figure 5.13: Comparison Between UKF and EKF - Pitch Error, Phase II with Turbulence](image1)

(a) Unscented Kalman Filter Pitch Error  
(b) Extended Kalman Filter Pitch Error

The above figures show the comparison between the UKF and the EKF pitch, \( \theta \), estimations during a longitudinal, pitch maneuver when turbulence
is present. As shown, the errors present for both the UKF and EKF estimates are similar which is due to the estimation of pitch biases during a pure roll maneuver. In this case, the UKF and EKF estimates are approximately the same possessing similar errors over the simulation time.

![Graphs showing roll error and roll estimate comparisons between UKF and EKF](image)

**Figure 5.15:** Comparison Between UKF and EKF - Roll Error, Phase II with Turbulence

![Graphs showing roll estimate comparisons](image)

**Figure 5.16:** Comparison Between UKF and EKF - Roll Estimation, Phase II with Turbulence

The above figures show the comparison between the UKF and the EKF roll, $\phi$, estimations during a longitudinal, pitch maneuver when turbulence is present. The UKF estimate absolute error, for all time, is lower than that of the EKF. The UKF error is maintained below 0.25 degrees after 3.5 seconds...
while the EKF error after that time is maintained below 0.4 degrees. For this case, the UKF attitude estimate is more accurate than the EKF estimate.

5.3.3 Phase III - Longitudinal/Transverse Aircraft Maneuver

Phase III - Longitudinal/Transverse Maneuver: No Turbulence

The following figures compare the UKF to the EKF during a longitudinal/transverse maneuver with no turbulence.

![Figure 5.17: Comparison Between UKF and EKF - Pitch Error, Phase III No Turbulence](image)

(a) Unscented Kalman Filter Pitch Error  
(b) Extended Kalman Filter Pitch Error

Figure 5.17: Comparison Between UKF and EKF - Pitch Error, Phase III No Turbulence

![Figure 5.18: Comparison Between UKF and EKF - Pitch Estimation, Phase III No Turbulence](image)

(a) Unscented Kalman Filter Pitch Estimate  
(b) Extended Kalman Filter Pitch Estimate

Figure 5.18: Comparison Between UKF and EKF - Pitch Estimation, Phase III No Turbulence
The above figures show the comparison between the UKF and the EKF pitch, $\theta$, estimations during a combined, longitudinal/transverse maneuver when no turbulence is present. The EKF absolute error is nearly twice that of the UKF error for all time. The UKF absolute error is maintained below 0.35 degrees over all time while the EKF error becomes as high as 0.7 degrees and eventually converges to an error of 0.3 degrees. This shows that the UKF estimate for this case is more accurate than the EKF estimate.

![Figure 5.19: Comparison Between UKF and EKF - Roll Error, Phase III No Turbulence](image)

The above figures show the comparison between the UKF and the EKF roll, $\phi$, estimations during a combined, longitudinal/transverse maneuver

![Figure 5.20: Comparison Between UKF and EKF - Roll Estimation, Phase III No Turbulence](image)
when no turbulence is present. As seen above, the UKF estimation absolute error is maintained below 0.25 degrees for all time while the EKF estimate absolute error is as high as 0.45 degrees at certain instances. The UKF estimate at nearly all points in time is more accurate than the EKF estimate for this case.

**Phase III - Longitudinal/Transverse Maneuver: Turbulence**

The following figures compare the UKF to the EKF during a longitudinal/transverse maneuver in the presence of turbulence.

![Figure 5.21: Comparison Between UKF and EKF - Pitch Error, Phase III with Turbulence](image1)

(a) Unscented Kalman Filter Pitch Error  
(b) Extended Kalman Filter Pitch Error

Figure 5.21: Comparison Between UKF and EKF - Pitch Error, Phase III with Turbulence

![Figure 5.22: Comparison Between UKF and EKF - Pitch Estimation, Phase III with Turbulence](image2)

(a) Unscented Kalman Filter Pitch Estimate  
(b) Extended Kalman Filter Pitch Estimate

Figure 5.22: Comparison Between UKF and EKF - Pitch Estimation, Phase III with Turbulence
The above figures show the comparison between the UKF and the EKF pitch, $\theta$, estimations during a combined, longitudinal/transverse maneuver when turbulence is present. As shown, the errors present for both the UKF and EKF estimates are comparable. In this case, the UKF and EKF estimates are approximately the same. The EKF estimate, however, is slightly more accurate than the UKF estimate having a higher maximum absolute error but a lower average error over all time. Despite this, however, the difference between their errors is very small at less than a 6% difference.

![Graphs showing comparison between UKF and EKF roll errors](image)

**Figure 5.23:** Comparison Between UKF and EKF - Roll Error, Phase III with Turbulence

![Graphs showing comparison between UKF and EKF roll estimates](image)

**Figure 5.24:** Comparison Between UKF and EKF - Roll Estimation, Phase III with Turbulence

The above figures show the comparison between the UKF and the EKF roll,
estimations during a combined, longitudinal/transverse maneuver when turbulence is present. As shown, the errors present for both the UKF and EKF estimates are comparable. In this case, the UKF and EKF estimates are approximately the same. The EKF estimate, however, is slightly more accurate than the UKF estimate having a lower maximum absolute error and a lower average error over all time. Despite this, however, the estimate absolute mean errors are both less than 0.05 degrees.

5.4 Unscented vs Extended Kalman Filter Summary

As seen above, in all cases the UKF performs better than the EKF. Table 5.2 gives a summary of the results.

<table>
<thead>
<tr>
<th></th>
<th>Phase I</th>
<th></th>
<th>Phase II</th>
<th></th>
<th>Phase III</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>UKF</td>
<td>EKF</td>
<td>UKF</td>
<td>EKF</td>
<td>UKF</td>
<td>EKF</td>
</tr>
<tr>
<td>No Turb.</td>
<td>0.27766</td>
<td>0.72277</td>
<td>0.08007</td>
<td>0.06401</td>
<td>0.33522</td>
<td>0.64139</td>
</tr>
<tr>
<td>Turb.</td>
<td>0.41976</td>
<td>0.80824</td>
<td>0.49147</td>
<td>0.52906</td>
<td>0.41731</td>
<td>0.48620</td>
</tr>
</tbody>
</table>

Table 5.2: Unscented vs Extended Kalman Filter Comparative Results

As shown in the above table, in all instances, the UKF mean absolute errors are lower than those of the EKF. The maximum errors of the EKF are also higher for all cases compared to the UKF except in the case of the roll, φ, estimate for Phase III in the presence of turbulence. Even though the EKF has a lower maximum absolute error, both errors are below 0.05 degrees which is highly accurate. It can be concluded from these results, that the UKF produces more accurate attitude estimates compared to those obtained.
via the EKF and will be used in all subsequent results. Also, of note, is the fact that in all cases, for both the EKF and the UKF, the roll estimates are more accurate than the pitch estimates.

It is common to not view the accuracy of the UKF as the accuracy of its bias estimate. The bias estimate coming from the UKF is generally not reminiscent of the true bias. This is due to the bias estimate coming from the UKF being already transformed to sigma space. As such, in papers, such as [6, 8, 32, 34], that use the UKF to estimate states, biases or attitude, the final estimates are compared to the “true” values for validation and verification, as was done in this work.
Chapter 6

Accelerometer Bias Estimation

As mentioned in previous sections, the determination of accelerometer measurement errors is critical in accurate attitude determination. As such, a method for estimating this errors is needed. In this work, a technique is used which uses the attitude estimates from the rate gyroscopes to compare to the attitude estimates from the accelerometer array. This forms an error signal, $e(t)$ which, if reduced will provide an accurate estimate of attitude. A very common approach to reducing error signals is the use of controllers. This approach was chosen for this work using a common Proportional, Integral, Derivative, or PID, controller. The derivation of this technique/algorithm is given in the next section.

6.1 Accelerometer Bias Estimation Algorithm

The sensor form given in Equation (2.55) is used as the model for the accelerometers. Because the error terms are additive, they can be combined into one singular error term and Equation (2.55) can be rewritten as follows,

$$\tilde{x} = x + \beta(t)$$  \hspace{1cm} (6.1)

where

$$\beta(t) = v(t) + b(t) + d(t)$$

In order to eliminate the effects of the measurement error term, $\beta$, Equation (6.1) needs to have a term added and is written as,

$$\tilde{x} = x + (\beta(t) - u(t))$$  \hspace{1cm} (6.2)

where $u(t)$ is the input from the controller(s), which will be discussed shortly. As mentioned previously, in order to obtain the control input, $u(t)$, a controller is needed but first an error signal needs to be defined:
where $\hat{\theta}_{\text{gyros}}$ is the pitch attitude estimate using the rate gyroscope data and $\hat{\theta}_{i,\text{accel}}$ is the pitch attitude estimate using the ith accelerometer. Because there are thirteen accelerometers along each arc, Equation (6.3) produces thirteen error signals. The above equation can also be written in the same way for the roll array as follows:

\[
e_i(t) = (\hat{\phi}_{\text{gyros}} - \hat{\phi}_{i,\text{accel}})
\]

Next, the error, $e(t)$ needs to be reduced through the use of a controller which, in this case, is the standard PID controller taken from [4]:

\[
u_i(t) = K_p e_i(t) + K_i \int_0^t e_i(\tau) d\tau + K_d \frac{de_i(t)}{dt}
\]

where $K_p$, $K_i$ and $K_d$ are the proportional, integral and derivative gains, respectively. In order to determine the values for each of the gains, a Particle Swarm Optimization technique was utilized in Matlab and Simulink which will be discussed shortly. Note that when $u(t)$ from Equation (6.5) approaches $\beta(t)$, the term $(\beta(t) - u(t))$ approaches zero and, thus the measured value approaches the true value. Ideally, Equation (6.5) would be used for each of the twenty five accelerometers to obtain their bias estimates, however, upon implementation of that method, large errors in bias estimates were observed due to unknown circumstances. As such, an altered approach to obtaining the bias estimates was developed and is given in the next section.

### 6.1.1 Accelerometer Bias Estimation, Additive Approach

As mentioned above, an alteration to the above method is needed. This method uses two accelerometers from opposite sides of the accelerometer array. These two accelerometers do not need to be directly opposite of each other, just on the opposite side of the array. Consider two accelerometers on the opposite sides of the arc array that are not in line with each other such as accelerometer #2 and #13 in Figure 6.1.
The red and cyan axes represent the orientation of the respective accelerometers. As shown in the above figure, accelerometer #2 has its x and z axis rotated, counter-clockwise 15 degrees from those of accelerometer #1 and 195 degrees from accelerometer #13. It is also of note that accelerometer #7 has it’s x-axis aligned with the device x-axis in all instances and accelerometer #1’s z-axis lies along the device negative x-axis and accelerometer #13’s z-axis lies along the device positive x-axis. Also of note is the fact only the misaligned z-axis ($Z_{\text{device}}$) acceleration is needed in the imposed loading equations in Section 3.3 which are used for the attitude estimates. A similar approach can be done to find the biases for the x-direction but was not considered in this work. Using this convention for the accelerometer orientations, two accelerometers on opposite sides of the array can be subtracted in order to sum their biases. The reason for subtraction is because, due to their respective rotations, the positive z-axes of the left half plane accelerometers coincide with the negative z-axes of the right half plane accelerometers which will result in bias addition instead of subtraction when two accelerometers are subtracted. This can be more clearly seen in the below equations. The following equations were developed in order to estimate the additive biases for the pitch and roll arrays, where $\theta_i$ and $\phi_i$, are defined in Appendix A.
\[ \Delta_1 = b_{13} \frac{1}{\sin \theta_{13}} - b_{i1} \frac{1}{\sin \theta_1} \]
\[ \Delta_2 = b_{12} \frac{1}{\sin \theta_{12}} - b_{i1} \frac{1}{\sin \theta_1} \]
\[ \Delta_3 = b_{12} \frac{1}{\sin \theta_{12}} - b_{21} \frac{1}{\sin \theta_2} \]  \hspace{1cm} (6.6)

where \( b_i \) refers to the bias of the \( i \)th accelerometer. Since accelerometers #1 through #6 are on the left-half plane and their orientation angles, \( \theta_i \), are negative, Equation (6.6) can be rewritten as,
\[ \Delta_1 = b_{13} \frac{1}{\sin \theta_{13}} + b_{i1} \frac{1}{\sin |\theta_1|} \]
\[ \Delta_2 = b_{12} \frac{1}{\sin \theta_{12}} + b_{i1} \frac{1}{\sin |\theta_1|} \]
\[ \Delta_3 = b_{12} \frac{1}{\sin \theta_{12}} + b_{21} \frac{1}{\sin |\theta_2|} \]  \hspace{1cm} (6.7)

which, as stated above, results in the accelerometer biases being added instead of subtracted.

The corresponding roll array equations are as follows:
\[ \Delta_1 = b_{1} \frac{1}{\cos \phi_1} - b_{13} \frac{1}{\cos \phi_{13}} \]
\[ \Delta_2 = b_{1} \frac{1}{\cos \phi_1} - b_{12} \frac{1}{\cos \phi_{12}} \]
\[ \Delta_3 = b_{2} \frac{1}{\cos \phi_2} - b_{12} \frac{1}{\cos \phi_{12}} \]  \hspace{1cm} (6.8)

Since accelerometers #8 through #13 are on the left-half plane and their orientation angles, \( \phi_i \), result in \( \cos \phi_i \) being negative, Equation (6.8) can be rewritten as,
\[ \Delta_1 = b_{1} \frac{1}{\cos \phi_1} + b_{13} \frac{1}{\cos (\pi - \phi_{13})} \]
\[ \Delta_2 = b_{1} \frac{1}{\cos \phi_1} + b_{12} \frac{1}{\cos (\pi - \phi_{12})} \]
\[ \Delta_3 = b_{2} \frac{1}{\cos \phi_2} + b_{12} \frac{1}{\cos (\pi - \phi_{12})} \]  \hspace{1cm} (6.9)

The \( \Delta' \)'s in the above equations are estimated through the algorithm in the previous section. Knowing the \( \Delta' \)'s and estimating \( b_2 \) through the algorithm in the previous section, a system of 3 equations with 3 unknowns is developed which can be solved for the remaining 3 biases. The sets of equations for the remaining accelerometers, as well as a more complete derivation, can be found in Appendix C. The full algorithm is presented in the following figure.
In Figure 6.2, the orange blocks represent the final accelerometer bias estimates. The algorithm given above is for the pitch array but is the same as applied for the roll array. The figure also shows the estimation algorithm for one set of four accelerometers but, using the equations in Appendix C, the figure can be slightly altered for the remaining accelerometers.

6.1.2 Particle Swarm Optimization for Determining PID Controller Gains

In Section 2.4.1, the equations for Particle Swarm Optimization were given. Using the equations of that section and the following parameters:

<table>
<thead>
<tr>
<th>Parameter/Variable</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_p$</td>
<td>1</td>
</tr>
<tr>
<td>$y_g$</td>
<td>1</td>
</tr>
<tr>
<td>$W$</td>
<td>1.2</td>
</tr>
<tr>
<td>$Q$</td>
<td>120</td>
</tr>
<tr>
<td>$b_{low}$</td>
<td>-10</td>
</tr>
<tr>
<td>$b_{up}$</td>
<td>10</td>
</tr>
</tbody>
</table>

Table 6.1: Particle Swarm Optimization Parameters for PID Gains

The PSO was carried out twice, once for the pitch array, and once for the roll array. The following cost function(s) were used for the pitch and roll arrays, respectively.

$$ J = \int_0^t (\theta_{true} - \dot{\theta})^2 dt $$

$$ J = \int_0^t (\phi_{true} - \dot{\phi})^2 dt \quad (6.10) $$
where the subscript "true" denotes the true value obtained from the nonlinear aircraft model and $\hat{\theta}$ and $\hat{\phi}$ are the attitude estimates from the dual-arc array. The optimization was carried out using Matlab and Simulink with a convergence tolerance of $10^{-5}$ meaning that the cost function value had to be less than the tolerance for convergence. Also, for the optimization routine, the particles were allowed outside of the bounds defined by $b_{low}$ and $b_{up}$. During preliminary runs it was found that the gains for estimating the $\Delta'$s were nearly identical in all instances. Because of this, the gains for the various $\Delta'$s were set to be the same within the optimization routine. The following table gives the resultant PID gains from the optimization runs.

<table>
<thead>
<tr>
<th>Parameter for Estimation</th>
<th>$K_p$</th>
<th>$K_i$</th>
<th>$K_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_2$</td>
<td>0.0054855</td>
<td>1.51288</td>
<td>0.0000226</td>
</tr>
<tr>
<td>$\Delta'$s</td>
<td>-0.0050188</td>
<td>1.5295</td>
<td>0.0000802</td>
</tr>
<tr>
<td>$b_4$</td>
<td>0.11733</td>
<td>0.69779</td>
<td>-0.0000175</td>
</tr>
<tr>
<td>$\Delta'$s</td>
<td>0.01218</td>
<td>1.0343</td>
<td>-0.0000286</td>
</tr>
<tr>
<td>$b_6$</td>
<td>-0.0032348</td>
<td>0.26027</td>
<td>0.000003</td>
</tr>
<tr>
<td>$\Delta'$s</td>
<td>0.0060491</td>
<td>0.55327</td>
<td>0.0000002</td>
</tr>
</tbody>
</table>

Table 6.2: PID Gains for Accelerometer Bias Estimation

### 6.2 Accelerometer Bias Estimation Results

The above method was chosen due to its robustness as well as its accuracy. This section shows the accuracy, and robustness, of the accelerometer bias estimation algorithm developed in the above section. The figures in this section show attitude estimates both with and without the algorithm enabled.

For this study, the simulation time was set to 10 seconds with a sampling frequency of 100 Hz. Biases and noise effects were placed on the rate accelerometers but rate gyroscope noise and biases were set to zero in order to isolate on the accelerometer bias estimation algorithm. The accelerometer biases were each set to 0.25 gees for this run and the noise variance was set to 0.00015 gees$^2$. Drift effects results are displayed for the Phase III maneuver(s) here but not for Phase I and II due to identical results. A more complete analysis with varying biases and drift effects can be found.
in Section 7. The following sections show the accuracy of the accelerometer bias estimation routine developed. For the comparison, the final attitude estimates were looked at in order to ascertain the effectiveness of the bias estimation algorithm.

In the following sections, Figure (a) shows the attitude estimate results when no accelerometer measurement error algorithm is present while Figure (b) shows the attitude estimate results in the presence of the aforementioned algorithm. The first 2 plots in each set show the pitch attitude error for the given phase. The second 2 plots show the pitch estimate compared to the true value of pitch. The third plot shows the accelerometer bias estimate for the pitch array. The fourth set of 2 plots shows the attitude estimation error for the roll. The fifth set of plots shows the roll estimate compared to the true value of roll. The final plot shows the accelerometer bias estimate for the roll array.

### 6.2.1 Phase I - Longitudinal Aircraft Maneuver

**Phase I - Longitudinal Maneuver: No Turbulence**

The following figures are the result of the accelerometer bias estimation algorithm during a longitudinal maneuver with no turbulence.

![Figure 6.3: Accelerometer Bias Estimation - Pitch Error, Phase I No Turbulence](image)
As seen in the above figures, the accelerometer bias estimation algorithm greatly improves on the attitude estimation for this case. The maximum absolute error when the bias estimation algorithm is applied is less than 0.17 degrees while the error when the bias is left unestimated is greater than 17 degrees, which is nearly 200 times less accurate. The same trend can be seen for the roll estimate in the following figures. As seen in Figure 6.5, the pitch bias estimate converges to values close to the true accelerometer bias.
Figure 6.6: Accelerometer Bias Estimation - Roll Error, Phase I No Turbulence

(a) No Bias Estimation

(b) Bias Estimation

Figure 6.7: Accelerometer Bias Estimation - Roll Estimate, Phase I No Turbulence

(a) No Bias Estimation

(b) Bias Estimation
As shown in the above figures, the roll estimate in the presence of estimating accelerometer biases is highly more accurate than without estimating the biases. As seen in Figure 6.8, the roll bias estimate converges to values close to the true accelerometer bias. For this case, the estimation of accelerometer biases greatly increases the accuracy of the attitude estimates.

**Phase I - Longitudinal Maneuver: Turbulence**

The following figures are the result of the accelerometer bias estimation algorithm during a longitudinal maneuver in the presence of turbulence.
Figure 6.9: Accelerometer Bias Estimation - Pitch Error, Phase I with Turbulence

Figure 6.10: Accelerometer Bias Estimation - Pitch Estimate, Phase I with Turbulence
As seen in the above figures, the accelerometer bias estimation algorithm greatly improves on the attitude estimation for this case. The maximum absolute error when the bias estimation algorithm is applied is less than 0.4 degrees while the error when the bias is left unestimated is greater than 17 degrees, which is nearly 50 times less accurate. The same trend can be seen for the roll estimate in the following figures. As seen in Figure 6.5, the pitch bias estimate converges to values close to the true accelerometer bias.
As shown in the above figures, the roll estimate in the presence of estimating accelerometer biases is highly more accurate than without estimating the biases. As seen in Figure 6.8, the roll bias estimate converges to values close to the true accelerometer bias. For this case, the estimation of accelerometer biases greatly increases the accuracy of the attitude estimates.
6.2.2 Phase II - Transverse Aircraft Maneuver

Phase II - Transverse Maneuver: No Turbulence

The following figures are the result of the accelerometer bias estimation algorithm during a transverse maneuver with no turbulence.

Figure 6.15: Accelerometer Bias Estimation - Pitch Error, Phase II No Turbulence

Figure 6.16: Accelerometer Bias Estimation - Pitch Estimate, Phase II No Turbulence
As seen in the above figures, the accelerometer bias estimation algorithm greatly improves on the attitude estimation. The maximum error for pitch estimate during a roll maneuver is less than 0.13 degrees. This is lower than the maximum error during a pure pitch maneuver. Again, the pitch array accelerometer bias estimate converges to values within 5% of the true value.
As shown in the above figures, the roll estimate in the presence of estimating accelerometer biases is highly more accurate than without estimating the biases. The roll estimate is more accurate during a Phase II maneuver due to sufficient excitation of the roll axis, which was not present during a pure pitch maneuver. Figure 6.20 shows that the roll array bias estimates do converge to the true bias value.
Phase II - Transverse Maneuver: Turbulence

The following figures are the result of the accelerometer bias estimation algorithm during a transverse maneuver in the presence of turbulence.

Figure 6.21: Accelerometer Bias Estimation - Pitch Error, Phase II with Turbulence

Figure 6.22: Accelerometer Bias Estimation - Pitch Estimate, Phase II with Turbulence
The above figures show that the pitch estimate error is less than 0.5 degrees but is not highly accurate due to the Phase II maneuver. Despite this, however, the attitude estimate when accelerometer biases are unestimated is highly inaccurate having an average error of approximately 15 degrees compared to the 0.2 degree average error when biases are estimated.

Figure 6.24: Accelerometer Bias Estimation - Roll Error, Phase II with Turbulence
As shown in the above figures, the roll estimate in the presence of estimating accelerometer biases is highly more accurate than without estimating the biases. The roll absolute error is maintained below 0.25 degrees after 4 seconds in the presence of turbulence. The roll error when no bias estimation is in place is greater than 14 degrees after 4 seconds. Again, the roll array accelerometer bias estimate is within 5% of the true value, as shown in Figure 6.26.
6.2.3 Phase III - Longitudinal/Transverse Aircraft Maneuver

Phase III - Longitudinal/Transverse Maneuver: No Turbulence

The following figures are the result of the accelerometer bias estimation algorithm during a longitudinal/transverse maneuver with no turbulence.

Figure 6.27: Accelerometer Bias Estimation - Pitch Error, Phase III No Turbulence

Figure 6.28: Accelerometer Bias Estimation - Pitch Estimate, Phase III No Turbulence
The pitch estimate, in the presence of estimated accelerometer biases, is highly accurate, maintaining an absolute estimate error of less than 0.25 degrees. Without bias estimation, however, the pitch error is maintained at greater than 15 degrees after 2 seconds, which is a highly inaccurate estimate.
The results of the roll estimate are comparable to those of the pitch estimate. The roll error is maintained at below 0.25 degrees and below 0.025 degrees after 6 seconds. The roll error with no bias estimate, however, is greater than 15 degrees after 2 seconds, which is undesirable. The bias estimate is also within 5% of the true value for this case as well.
Phase III - Longitudinal/Transverse Maneuver: Turbulence

The following figures are the result of the accelerometer bias estimation algorithm during a longitudinal/transverse maneuver in the presence of turbulence.

![Graphs showing accelerometer bias estimation - Pitch Error, Phase III with Turbulence](image)

(a) No Bias Estimation  
(b) Bias Estimation

Figure 6.33: Accelerometer Bias Estimation - Pitch Error, Phase III with Turbulence

![Graphs showing accelerometer bias estimation - Pitch Estimate, Phase III with Turbulence](image)

(a) No Bias Estimation  
(b) Bias Estimation

Figure 6.34: Accelerometer Bias Estimation - Pitch Estimate, Phase III with Turbulence
Despite a large degree of turbulence being present, the attitude error is very low during the full 10 seconds of simulation aside from a brief increase in error from 3.5 seconds to 4.5 seconds. This could be due to the complex dynamic maneuver as shown in Figure 6.34. The pitch array accelerometer bias estimates contain more noise than before due to the turbulence but are still accurate enough to give a highly accurate attitude estimate.

Figure 6.35: Pitch Bias Estimation - Phase III with Turbulence

Figure 6.36: Accelerometer Bias Estimation - Roll Error, Phase III with Turbulence
Similar to the pitch estimate, the roll estimate is highly accurate after a short while to within 0.25 degrees of the true attitude despite the presence of severe turbulence. The large attitude error at approximately 3.5 seconds is due to the complex dynamic maneuver, however, the error is only as large as 0.25 degrees which is still low.
6.2.4 Phase III - Longitudinal/Transverse Maneuver with Drift Effects

The following section shows the robustness of the accelerometer bias estimation algorithm in the presence of very severe sensor drift. For the following simulation results, the drift was set to 0.005 gees/second or 18 gees/hour.

Phase III - Longitudinal/Transverse Maneuver with Drift Effects: No Turbulence

The following figures are the result of the accelerometer bias estimation algorithm during a longitudinal/transverse maneuver with no turbulence in the presence of severe accelerometer drift errors.

Figure 6.39: Accelerometer Bias Estimation with Drift - Pitch Error, Phase III No Turbulence

(a) No Bias Estimation  (b) Bias Estimation
Figure 6.40: Accelerometer Bias Estimation with Drift - Pitch Estimate, Phase III No Turbulence

As shown in the above figures, the attitude estimation absolute error is only slightly higher, 0.02 gees, with severe drift than without. The pitch estimate error is still maintained at below 0.25 degrees. As seen in Figure 6.41, the bias estimate drifts with time due to the drift error applied to the accelerometers.
Figure 6.42: Accelerometer Bias Estimation with Drift - Roll Error, Phase III No Turbulence

Figure 6.43: Accelerometer Bias Estimation with Drift - Roll Estimate, Phase III No Turbulence
Similar to the pitch array results, the roll estimate in the presence of severe drift has a slightly higher absolute error than with no drift effects. In this case, the roll error is maintained below 0.25 degrees which is still an accurate estimate of the aircraft roll.

**Phase III - Longitudinal/Transverse Maneuver: Turbulence**

The following figures are the result of the accelerometer bias estimation algorithm during a longitudinal/transverse maneuver in the presence of turbulence and severe accelerometer drift.
Figure 6.45: Accelerometer Bias Estimation with Drift - Pitch Error, Phase III with Turbulence

Figure 6.46: Accelerometer Bias Estimation with Drift - Pitch Estimate, Phase III with Turbulence
The pitch estimate for a Phase III maneuver in the presence of turbulence with accelerometer drift effects is less accurate than with no drift effects, as it was with no turbulence. While less accurate, the absolute error is still maintained at less than 0.5 degrees. The accelerometer bias estimate is still accurate as it accounts for the drift effects of the accelerometers.
The same is seen in the roll array; the roll estimate is slightly less accurate in the presence of severe drift but is still an accurate attitude estimate with absolute error less than 0.25 degrees. Even in the presence of very severe drift, the attitude estimates for pitch and roll are still accurate due to the robustness of the accelerometer bias estimation algorithm and the assumption that the measurement errors are additive.
6.3 Accelerometer Bias Estimation Summary

As seen above, in all cases the estimation of accelerometer biases greatly improves the attitude estimate. Table 6.3 gives a summary of the results.

<table>
<thead>
<tr>
<th></th>
<th>Phase I</th>
<th>Phase II</th>
<th>Phase III</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>No Turb.</td>
<td>Turb.</td>
<td>No Turb.</td>
</tr>
<tr>
<td>θ (deg) Max absolute error after 4 secs</td>
<td>0.16974</td>
<td>0.09770</td>
<td>0.25328</td>
</tr>
<tr>
<td>θ (deg) Mean absolute error after 4 secs</td>
<td>0.08827</td>
<td>0.06850</td>
<td>0.09935</td>
</tr>
<tr>
<td>φ (deg) Max absolute error after 4 secs</td>
<td>0.07802</td>
<td>0.12955</td>
<td>0.15946</td>
</tr>
<tr>
<td>φ (deg) Mean absolute error after 4 secs</td>
<td>0.06995</td>
<td>0.07674</td>
<td>0.14532</td>
</tr>
</tbody>
</table>

Table 6.3: Accelerometer Bias Estimation Results

As shown in the above table, the mean pitch absolute errors in all cases are less than 0.25 degrees and the mean roll absolute errors are less than 0.15 degrees. For all instances, the roll estimates are more accurate than the pitch estimates which may be due to the amount of excitation for the given maneuver or the nature of the accelerometer array. Also, even in the presence of severe drift effects, the attitude estimation absolute errors are kept minimal and are very close to the errors in the presence of no drift effects. These errors could be further reduced given a more complex, nonlinear controller instead of the PID controller. A nonlinear controller would be difficult to implement properly but may yield more accurate results.
Chapter 7
Simulation Results

For the validation of the proposed algorithms and models, three simulation phases were performed, as they were in the previously shown results. Each phase was conducted twice; once with no turbulence applied and once with turbulence applied. Phase I consists of a purely longitudinal, pitch maneuver, while Phase II is a purely transverse, roll maneuver. The third phase is a combined longitudinal/transverse maneuver. The accelerometer and rate gyroscope biases were estimated during each phase and only the measurement error of drift was left absent in these runs.

The figures presented in this section are with a 0.25 gee accelerometer bias with noise variance of $0.00015gees^2$; rate gyroscope bias of $0.2deg/sec$ with noise variance of $0.15(deg/sec)^2$ was used. Tables 7.1 through 7.18 show the errors from simulation runs with differing biases and drift effects. In the previous work, [30], a complimentary filter was used to achieve a more accurate estimate of attitude. For this work, however, the complimentary filter was not used due to producing highly inaccurate results, due to it’s reliance on the accelerometer readings and being very sensitive to small errors in accelerometer measurements. Instead, the attitude estimates of the individual accelerometers were simply averaged to obtain an estimate of attitude. This was done due to the realization that some of the accelerometers had an attitude estimate which was too ”high” or too ”low” but averaging them together produced highly accurate attitude estimates.

For all simulation results, both the Unscented Kalman filter and accelerometer bias estimation algorithm were left ”on” as their results from not being used was given previously. Because the two algorithms work together to achieve an accurate estimate of attitude, errors in rate gyro bias estimation and accelerometer bias estimation were seen but any error in rate gyro bias was lessened through the accelerometer bias estimation algorithm and vice
versa producing accurate, reliable results as shown in the following sec-
tions. The algorithms used for rate gyro bias estimation and accelerometer
bias estimation were derived previously in Sections 5 and 6, respectively.
The parameters used for the Unscented Kalman filter are found in Table 5.1
and the gains for the accelerometer bias estimation algorithm are found in
Table 6.2.

7.1 Phase I Maneuver - No Turbulence

The following plots compare the true attitude from the full nonlinear air-
craft model to the attitude estimate from the dual-arc array using rate gyros
and accelerometers. For this section, a Phase I, longitudinal maneuver was
performed with no turbulence. Figures 7.1 through 7.5 show the attitude
estimates and bias estimates while Tables 7.1 through 7.3 show the attitude
estimate errors for varying biases and drift slopes for both the rate gyro-
scopes and accelerometers.

![Pitch Estimation Error](image1)

![Pitch Estimation](image2)

Figure 7.1: Pitch Estimation Results - Phase I No Turbulence
Figure 7.2: Pitch Bias Estimation Results - Phase I No Turbulence

Figure 7.3: Roll Estimation Results - Phase I No Turbulence
Figures 7.1 and 7.3 show that the attitude estimates for pitch and roll, respectively, are within 0.17 degrees of the true attitudes which shows the accuracy of the attitude estimation device and bias estimation algorithms. Figures 7.2 and 7.4 show that the bias estimates for both pitch and roll converge to the true bias values. The error reduces to nearly zero for all accelerometer bias estimates for both the pitch and roll arrays. This convergence shows that,
indeed, the bias estimation algorithm(s) is/are correctly estimating the accelerometer biases.

<table>
<thead>
<tr>
<th>Rate Gyro Drift Slope (deg/sec/hr)</th>
<th>Accel. Bias (gees)</th>
<th>Accel. Drift Slope (gees/hr)</th>
<th>θ Max Absolute Error (deg)</th>
<th>θ Mean Absolute Error (deg)</th>
<th>φ Max Absolute Error (deg)</th>
<th>φ Mean Absolute Error (deg)</th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>0.25</td>
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<td>0.17003</td>
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<td></td>
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</tr>
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<td>0.08211</td>
<td>0.07468</td>
</tr>
<tr>
<td></td>
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Table 7.1: Sensitivity of Bias Estimation Algorithms to Changes in Biases and Drift Slopes with Rate Gyro Bias of 0.2 deg/sec - Phase I No Turbulence
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Table 7.2: Sensitivity of Bias Estimation Algorithms to Changes in Biases and Drift Slopes with Rate Gyro Bias of 3 deg/sec - Phase I No Turbulence
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From the above tables, it can be seen that the higher the accelerometer bias, the higher the estimation error, which is expected. Despite this, however, the maximum absolute error for both the pitch and roll estimates remains below 0.25 deg and the average absolute error remains below 0.15 deg for all
cases. From the tables it can also be seen that rate gyroscope bias, rate gyroscope drift and accelerometer drift have very little effects on the accuracy of the attitude estimates. This shows the robustness of the bias estimation algorithms. While the higher accelerometer biases produce higher error in attitude estimates, a accelerometer bias of $1\,gee$ is very high and biases of that magnitude are not typically seen. The high bias was included in this study to show the robustness of the algorithm(s).

7.2 Phase I Maneuver - Turbulence

The following plots compare the true attitude from the full nonlinear aircraft model to the attitude estimate from the dual-arc array using rate gyros and accelerometers. For this section, a Phase I, longitudinal maneuver was performed with turbulence using the Dryden model from Section 4.2. Figures 7.6 through 7.10 show the attitude estimates and bias estimates while Tables 7.4 through 7.6 show the attitude estimate errors for varying biases and drift slopes for both the rate gyroscopes and accelerometers.

![Figure 7.6: Pitch Estimation Results - Phase I with Turbulence](image)

(a) Pitch Estimation Error  
(b) Pitch Estimation

Figure 7.6: Pitch Estimation Results - Phase I with Turbulence
Figure 7.7: Pitch Bias Estimation Results - Phase I with Turbulence

Figure 7.8: Roll Estimation Results - Phase I with Turbulence
Figures 7.6 and 7.8 show that the attitude estimates for pitch and roll, respectively, are within 0.4 degrees of the true attitudes which shows the accuracy of the attitude estimation device and bias estimation algorithms. The roll estimate, however, is more accurate than the pitch estimate in this case. Figures 7.7 and 7.9 show that the bias estimates for both pitch and roll converge to the true bias values. For this case, with turbulence, the bias estimations contain more error and noise due to the injection of severe turbulence to the
system. Despite this, the error still reduces to nearly zero for all accelerometer bias estimates for both the pitch and roll arrays. This convergence shows that, indeed, the bias estimation algorithm(s) is/are correctly estimating the accelerometer biases.

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Table 7.4: Sensitivity of Bias Estimation Algorithms to Changes in Biases and Drift Slopes with Rate Gyro Bias of 0.2 deg/sec - Phase I with Turbulence
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Table 7.5: Sensitivity of Bias Estimation Algorithms to Changes in Biases and Drift Slopes with Rate Gyro Bias of 3 deg/sec - Phase I with Turbulence
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Table 7.6: Sensitivity of Bias Estimation Algorithms to Changes in Biases and Drift Slopes with Rate Gyro Bias of 7 deg/sec - Phase I with Turbulence

From the above tables, it can be seen that the higher the accelerometer bias, the higher the estimation error for the roll estimation but for the pitch estimate the maximum error is lower, however, the mean error is higher. The mean error is more representative of the accuracy of the algorithms, however. The error for the roll estimate for nearly all cases is less than 0.35 deg.
The pitch mean absolute errors during this Phase I maneuver remain below 0.2\degree for all cases which is desirable as this is a pure pitching maneuver and a high degree of accuracy is needed. From the tables it can also be seen that rate gyroscope bias and rate gyroscope drift have very little effect on the accuracy of the attitude estimates. The accelerometer drift, however, effects the roll attitude estimate greatly but allows for more accurate roll estimates. The attitude estimation errors during turbulence are higher, as is expected, due to further corruption of the signals.

### 7.3 Phase II Maneuver - No Turbulence

The following plots compare the true attitude from the full nonlinear aircraft model to the attitude estimate from the dual-arc array using rate gyroscopes and accelerometers. For this section, a Phase II, transverse maneuver was performed with no turbulence. Figures 7.11 through 7.15 show the attitude estimates and bias estimates while Tables 7.7 through 7.9 show the attitude estimate errors for varying biases and drift slopes for both the rate gyroscopes and accelerometers.

![Figure 7.11: Pitch Estimation Results - Phase II No Turbulence](image)

(a) Pitch Estimation Error  
(b) Pitch Estimation

Figure 7.11: Pitch Estimation Results - Phase II No Turbulence
Figure 7.12: Pitch Bias Estimation Results - Phase II No Turbulence

Figure 7.13: Roll Estimation Results - Phase II No Turbulence
Figures 7.11 and 7.13 show that the attitude estimates for pitch and roll, respectively, are within 0.3 degrees of the true attitudes which shows the accuracy of the attitude estimation device and bias estimation algorithms. Figures 7.12 and 7.14 show that the bias estimates for both pitch and roll converge to the true bias values. The error reduces to nearly zero for all accelerometer bias estimates for both the pitch and roll arrays. This convergence shows that, indeed, the bias estimation algorithm(s) is/are correctly
estimating the accelerometer biases.

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<th>Rate Gyro Drift Slope (deg/sec/hr)</th>
<th>Accel. Bias (gees)</th>
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Table 7.7: Sensitivity of Bias Estimation Algorithms to Changes in Biases and Drift Slopes with Rate Gyro Bias of 0.2 deg/sec - Phase II No Turbulence
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Table 7.8: Sensitivity of Bias Estimation Algorithms to Changes in Biases and Drift Slopes with Rate Gyro Bias of 3 deg/sec - Phase II No Turbulence
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Table 7.9: Sensitivity of Bias Estimation Algorithms to Changes in Biases and Drift Slopes with Rate Gyro Bias of 7 deg/sec - Phase II No Turbulence

From the above tables, it can be seen that the higher the accelerometer bias, the higher the estimation error, which is expected. Despite this, however, the maximum absolute error for both the pitch and roll estimates remains below $0.16\text{deg}$ and the average absolute error remains below $0.15\text{deg}$ for all cases. From the tables it can also be seen that rate gyroscope bias, rate gyroscope...
drift and accelerometer drift have very little effect on the accuracy of the attitude estimates during this maneuver. This shows the robustness of the bias estimation algorithms. While the higher accelerometer biases produce higher error in attitude estimates, an accelerometer bias of $1 \text{ gee}$ is very high and biases of that magnitude are not typically seen. The high bias was included in this study to show the robustness of the algorithm(s). The results from this maneuver mirror the results from the Phase I maneuver.

### 7.4 Phase II Maneuver - Turbulence

The following plots compare the true attitude from the full nonlinear aircraft model to the attitude estimate from the dual-arc array using rate gyros and accelerometers. For this section, a Phase II, transverse maneuver was performed with turbulence using the Dryden model from Section 4.2. Figures 7.26 through 7.30 show the attitude estimates and bias estimates while Tables 7.10 through 7.12 show the attitude estimate errors for varying biases and drift slopes for both the rate gyroscopes and accelerometers.

![Pitch Estimation Results - Phase II with Turbulence](image)

(a) Pitch Estimation Error  
(b) Pitch Estimation

Figure 7.16: Pitch Estimation Results - Phase II with Turbulence
Figure 7.17: Pitch Bias Estimation Results - Phase II with Turbulence

Figure 7.18: Roll Estimation Results - Phase II with Turbulence
Figures 7.16 and 7.18 show that the attitude estimates for pitch and roll, respectively, are within 0.5 degrees of the true attitudes. The attitude estimates are less accurate than the previous case (Phase II maneuver with no turbulence) due to the severe turbulence present making it difficult to estimate parameters. The attitude estimates, however, are still accurate to within 0.5 degrees as evidenced by the aforementioned figures. Figures 7.17 and 7.19 show that the bias estimates for both pitch and roll converge to the true bias.
values. For this case, with turbulence, the bias estimations contain more error and noise due to the injection of severe turbulence to the system. Despite this, the error still reduces to nearly zero for all accelerometer bias estimates for both the pitch and roll arrays but contain higher error than the same maneuver with no turbulence. This convergence shows that, indeed, the bias estimation algorithm(s) is/are correctly estimating the accelerometer biases.
<table>
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<th>Rate Gyro Drift Slope (deg/sec/hr)</th>
<th>Accel. Bias (gees)</th>
<th>Accel. Drift Slope (gees/hr)</th>
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Table 7.10: Sensitivity of Bias Estimation Algorithms to Changes in Biases and Drift Slopes with Rate Gyro Bias of 0.2 deg/sec - Phase II with Turbulence
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Table 7.11: Sensitivity of Bias Estimation Algorithms to Changes in Biases and Drift Slopes with Rate Gyro Bias of 3 deg/sec - Phase II with Turbulence
### Phase II: Transverse Maneuver with Turbulence

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Table 7.12: Sensitivity of Bias Estimation Algorithms to Changes in Biases and Drift Slopes with Rate Gyro Bias of 7 deg/sec - Phase II with Turbulence

From the above tables, it can be seen that the higher the accelerometer bias, the higher the estimation error for the pitch roll estimations. The roll error for higher accelerometer biases becomes relatively inaccurate during this maneuver with mean errors as high as \(0.53\text{deg}\). While this is undesirable, accelerometer biases as high as \(1\text{gee}\) are rarely, if ever, seen. Although large
estimation errors are present at instances during the pure roll maneuver, the percent error between the attitude estimate and the true value remains below 5% error. From the tables it can also be seen that rate gyroscope bias and rate gyroscope drift have very little effects on the accuracy of the attitude estimates. The accelerometer drift, however, affects the roll attitude estimate but allows for more accurate roll estimates. The attitude estimation errors during turbulence are higher, as is expected, due to further corruption of the signals.

7.5 Phase III Maneuver - No Turbulence

The following plots compare the true attitude from the full nonlinear aircraft model to the attitude estimate from the dual-arc array using rate gyros and accelerometers. For this section, a Phase III, longitudinal/transverse maneuver was performed with no turbulence. Figures 7.21 through 7.25 show the attitude estimates and bias estimates while Tables 7.13 through 7.15 show the attitude estimate errors for varying biases and drift slopes for both the rate gyroscopes and accelerometers.

(a) Pitch Estimation Error
(b) Pitch Estimation

Figure 7.21: Pitch Estimation Results - Phase III No Turbulence
Figure 7.22: Pitch Bias Estimation Results - Phase III No Turbulence

Figure 7.23: Roll Estimation Results - Phase III No Turbulence
Figures 7.21 and 7.23 show that the attitude estimates for pitch and roll, respectively, are within 0.25 degrees of the true attitudes which shows the accuracy of the attitude estimation device and bias estimation algorithms. Both of the estimate errors are comparable due to the maneuver being of both pitch and roll. Because of this, both axes are sufficiently excited to allow for accurate parameter estimation. Figures 7.22 and 7.24 show that the bias estimates for both pitch and roll converge to the true bias values.

Figure 7.24: Roll Bias Estimation Results - Phase III No Turbulence

Figure 7.25: Rate Gyro Bias Estimation - Phase III No Turbulence
The error reduces to nearly zero for all accelerometer bias estimates for both the pitch and roll arrays. This convergence shows that, indeed, the bias estimation algorithm(s) is/are correctly estimating the accelerometer biases.

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Table 7.13: Sensitivity of Bias Estimation Algorithms to Changes in Biases and Drift Slopes with Rate Gyro Bias of 0.2 deg/sec - Phase III No Turbulence
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Table 7.14: Sensitivity of Bias Estimation Algorithms to Changes in Biases and Drift Slopes with Rate Gyro Bias of 3 deg/sec - Phase III No Turbulence
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Table 7.15: Sensitivity of Bias Estimation Algorithms to Changes in Biases and Drift Slopes with Rate Gyro Bias of 7 deg/sec - Phase III No Turbulence

From the above tables, it can be seen that the higher the accelerometer bias, the higher the estimation error, which is expected. Despite this, however, the maximum absolute error for both the pitch and roll estimates remains below 0.31 deg and the average absolute error remains below 0.16 deg for all cases. The roll errors for this maneuver remain more accurate than the
pitch estimates. From the tables it can also be seen that rate gyroscope bias, rate gyroscope drift and accelerometer drift have very little effects on the accuracy of the attitude estimates during this maneuver. This shows the robustness of the bias estimation algorithms. While the higher accelerometer biases produce higher error in attitude estimates, an accelerometer bias of 1\, g_{ee} is very high and biases of that magnitude are not typically seen. The high bias was included in this study to show the robustness of the algorithm(s). The results from this maneuver mirror the results from the Phase I and Phase II maneuvers.

7.6 Phase III Maneuver - Turbulence

The following plots compare the true attitude from the full nonlinear aircraft model to the attitude estimate from the dual-arc array using rate gyros and accelerometers. For this section, a Phase III, longitudinal/transverse maneuver was performed with turbulence using the Dryden model from Section 4.2. Figures 7.26 through 7.30 show the attitude estimates and bias estimates while Tables 7.16 through 7.18 show the attitude estimate errors for varying biases and drift slopes for both the rate gyros and accelerometers.

![Pitch Estimation Error](image1.png)

![Pitch Estimation](image2.png)

Figure 7.26: Pitch Estimation Results - Phase III with Turbulence
Figure 7.27: Pitch Bias Estimation Results - Phase III with Turbulence

Figure 7.28: Roll Estimation Results - Phase III with Turbulence
Figures 7.26 and 7.28 show that the attitude estimates for pitch and roll, respectively, are within 0.3 degrees of the true attitudes, aside from a small spike in the pitch estimate error, which shows the accuracy of the attitude estimation device and bias estimation algorithms. The attitude estimates in this case with turbulence are more accurate than in the previous phases due to the excitation of both axes allowing for accurate parameter estimation and, thus, accurate attitude estimates. Figures 7.27 and 7.29 show that the
bias estimates for both pitch and roll converge to the true bias values. For this case, with turbulence, the bias estimations contain more error and noise due to the injection of severe turbulence to the system. Despite this, the error still reduces to nearly zero for all accelerometer bias estimates for both the pitch and roll arrays. This convergence shows that, indeed, the bias estimation algorithm(s) is/are correctly estimating the accelerometer biases. This case, above the other cases, shows the high degree of accuracy of the algorithms developed within this work.
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<th>Rate Gyro Drift Slope (deg/sec/hr)</th>
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Table 7.16: Sensitivity of Bias Estimation Algorithms to Changes in Biases and Drift Slopes with Rate Gyro Bias of 0.2 deg/sec - Phase III with Turbulence
## Table 7.17: Sensitivity of Bias Estimation Algorithms to Changes in Biases and Drift Slopes with Rate Gyro Bias of 3 deg/sec - Phase III with Turbulence

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Table 7.18: Sensitivity of Bias Estimation Algorithms to Changes in Biases and Drift Slopes with Rate Gyro Bias of 7 deg/sec - Phase III with Turbulence

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<td>0.12322</td>
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<td>0.12346</td>
<td>0.59381</td>
<td>0.44229</td>
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<td>0.37794</td>
<td>0.12322</td>
<td>0.56994</td>
<td>0.42337</td>
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<tr>
<td></td>
<td></td>
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<td>0.37952</td>
<td>0.12190</td>
<td>0.49968</td>
<td>0.36676</td>
</tr>
</tbody>
</table>

From the above tables, it can be seen that the higher the accelerometer bias, the higher the estimation error, which is expected. The maximum errors for both pitch and roll are relatively high but remain below 0.6 deg for all cases. The mean absolute error for pitch is very accurate remaining at below 0.13 deg for all instances. The mean absolute error for roll, however, is
higher in nearly all cases. For accelerometer biases of 0.25\textit{gees} and 0.5\textit{gees} the mean error remains below 0.26\textit{deg} but becomes twice as high at twice the bias with a mean absolute error of 0.45\textit{deg}. As was the case in the Phase II maneuver, despite this higher mean absolute error the percent difference between the estimated roll and the true roll remains less than 5\%. The accuracy of the roll estimate for this Phase III longitudinal/transverse maneuver is higher than that of the pure roll, transverse maneuver. The attitude estimation errors during turbulence are higher, as is expected, due to further corruption of the signals.

7.7 Summary

As discussed in this section, the attitude estimation absolute errors remain within 0.7 degrees for all Phases both with and without turbulence present. The pitch estimation errors, in general, are lower than the roll estimation errors which may be due to the reliance of the pitch estimate on data from the pitch array as seen in the attitude equations (equations (3.3) and (3.4)). Because the roll estimate relies on information from the pitch array, the errors present in the pitch array are transferred to the roll array and are compounded with the errors from the transverse array.

This section also verifies the fact that attitude estimation is more accurate when no turbulence is present. The mean absolute errors for all three maneuvers with no turbulence are all less than 0.1\textit{deg} for all cases but can be as high as 0.57\textit{deg} when turbulence is present and a large accelerometer bias is seen. However, despite these few cases when larger error is seen in the attitude estimates, the algorithms still function as expected and produce attitude estimates that are within 5\%, and in many cases within 1\%, which is desirable.
Chapter 8

Conclusion and Future Work

8.1 Conclusion

In this work, the ability of an innovative, low-cost, two-dimensional accelerometer array, with rate gyroscope, to estimate rate gyroscope biases and accelerometer biases for an accurate and reliable attitude estimate was assessed and verified. The algorithms and methods derived and implemented in this study were applied to a device developed in a previous study. However, this work considered more real-world effects and improved on the accuracy of the attitude estimates. The development of this device provides various benefits over traditional Inertial Navigation Systems (INS). Typical INS systems rely on GPS and inclinometers as well as magnetometers to obtain an accurate and reliable attitude estimate. These sensors are highly susceptible to environmental effects. The dual-arc accelerometer array presented in this paper is more independent of these conditions as well as being lighter and cheaper than typical INS systems.

The developed algorithms and methods in this study expanded upon previous research conducted by applying a more accurate method, the Unscented Kalman filter, for the estimation of rate gyroscope biases, considering accelerometer biases, noise and drift effects and applying a method for the estimation of the aforementioned accelerometer measurement errors. The feasibility of these algorithms was accomplished using a fully nonlinear model of an aircraft during three different maneuvers both with and without a high degree of turbulence present. The largest errors seen in this study were during Phase II, transverse, and Phase III, longitudinal/transverse, maneuvers for the roll estimation. The maximum errors experienced here were ±0.6deg while the mean errors were ±0.45deg. The pitch estimates, however, remained below a maximum error of ±0.55deg and below a mean error of ±0.25deg. The roll estimate is less accurate as it depends on information
from the pitch array to obtain its attitude estimate. This information has error present and thus creates more error when carried through the roll array. These high errors were only seen when very high accelerometer biases were present on all accelerometers at 1g. This high bias is rare for sensors and, in real-world applications, the attitude estimates will be more accurate. For all cases, it was also shown that drift effects of both the rate gyroscope and the accelerometers have little to no effect on the accuracy of the attitude estimates and have error changes of less than 5%. This low change in accuracy shows that the algorithms and overall method developed is, in fact, highly robust. The method developed was made to allow for the system to be used for a variety of applications, not just aircraft. The end result of this work is the development and successful implementation of an accelerometer bias, drift and noise estimation algorithm coupled with an Unscented Kalman attitude filter for the estimation of rate gyroscope errors to provide an accurate and reliable estimate of attitude.

8.2 Future Work

This work focused on the Unscented Kalman filter for attitude estimation and an accelerometer bias estimation algorithm using PID controllers as applied to the previously developed dual-arc accelerometer array and nonlinear aircraft model. This study was entirely focused on simulation and no hardware was used in this study. A real-world test platform capable of imposing rotational and translational maneuvers, as seen in this work, is necessary for completely validating the method developed. The test platform would give a more complete analysis and verification of the method(s).

The Unscented Kalman filter is only one of many filters that can be used for attitude estimation. Other filters, such as an $\alpha - \beta$, filter should be looked at to possibly provide a more accurate estimation of rate gyroscope errors. The accelerometer bias estimation algorithm also has a great deal of improvement possible. The estimation of the additive bias terms can be improved through additional equations. Also, the PID controllers used to estimate the accelerometer measurement errors can be changed to another controller, perhaps a nonlinear controller, such as a sliding mode controller or adaptive controller. These controllers are far more difficult to implement
and would, thus, take more research and time to develop correctly.
Bibliography


[23] Jack B. Kuipers. Quaternions and rotation sequences : a primer with


Greg Welch and Gary Bishop. An introduction to the kalman filter.


Appendix A

Accelerometer Array Supplement

A.1 Longitudinal Array Accelerometer Offsets

<table>
<thead>
<tr>
<th>Accelerometer</th>
<th>Offset from Accelerometer # 7 - Degrees(Radians)</th>
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<tr>
<td>1</td>
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</tr>
<tr>
<td>2</td>
<td>-75(-1.3090)</td>
</tr>
<tr>
<td>3</td>
<td>-60(-1.0472)</td>
</tr>
<tr>
<td>4</td>
<td>-45(-0.78540)</td>
</tr>
<tr>
<td>5</td>
<td>-30(-0.52360)</td>
</tr>
<tr>
<td>6</td>
<td>-15(-0.26180)</td>
</tr>
<tr>
<td>7</td>
<td>0(-1.5708)</td>
</tr>
<tr>
<td>8</td>
<td>15(0.26180)</td>
</tr>
<tr>
<td>9</td>
<td>30(0.52360)</td>
</tr>
<tr>
<td>10</td>
<td>45(0.78540)</td>
</tr>
<tr>
<td>11</td>
<td>60(1.0472)</td>
</tr>
<tr>
<td>12</td>
<td>75(1.3090)</td>
</tr>
<tr>
<td>13</td>
<td>90(-1.5708)</td>
</tr>
</tbody>
</table>

Table A.1: Longitudinal Accelerometer Array Offsets [30]
### A.2 Transverse Array Accelerometer Offsets

<table>
<thead>
<tr>
<th>Accelerometer</th>
<th>Offset from Accelerometer # 1 - Degrees(Radians)</th>
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<tr>
<td>1</td>
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</tr>
<tr>
<td>2</td>
<td>15(0.26180)</td>
</tr>
<tr>
<td>3</td>
<td>30(0.52360)</td>
</tr>
<tr>
<td>4</td>
<td>45(0.78540)</td>
</tr>
<tr>
<td>5</td>
<td>60(1.0472)</td>
</tr>
<tr>
<td>6</td>
<td>75(1.3090)</td>
</tr>
<tr>
<td>7</td>
<td>90(1.5708)</td>
</tr>
<tr>
<td>8</td>
<td>105(1.8326)</td>
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<tr>
<td>9</td>
<td>120(2.0944)</td>
</tr>
<tr>
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<td>135(2.3562)</td>
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<tr>
<td>11</td>
<td>150(2.6180)</td>
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<tr>
<td>12</td>
<td>165(2.8798)</td>
</tr>
<tr>
<td>13</td>
<td>180(3.1416)</td>
</tr>
</tbody>
</table>

Table A.2: Transverse Accelerometer Array Offsets [30]

### A.3 Simulated Acceleration Measurements

The following are accelerometer measurements along the longitudinal array, $gA_{z,i}$ and accelerometer measurements along the transverse array, $gA_{y,i}$. The following measurements are for zero rate gyro and accelerometer biases with no drift effects. The noise variance for the rate gyroscopes was set to $0.15(\text{deg/sec})^2$ and the noise variance for the accelerometers was set to $0.00015\text{gees}^2$. The accelerometer measurements depicted here are the misaligned readings that are used in the imposed loading calculations as well as the measurements used for the determination of accelerometer biases.
Figure A.1: Accelerometer Array Misaligned Measurements - Phase I No Turbulence

Figure A.2: Accelerometer Array Misaligned Measurements - Phase I with Turbulence
Figure A.3: Accelerometer Array Misaligned Measurements - Phase II No Turbulence

Figure A.4: Accelerometer Array Misaligned Measurements - Phase II with Turbulence
A.4 Two-Dimensional Accelerometer Array Simulink Models

The figures presented here display the two-dimensional accelerometer array models as they were designed and implemented in Simulink.
Figure A.7: Longitudinal Array Measurement Model
Figure A.8: Transverse Array Measurement Model
Appendix B

Nonlinear Aircraft Simulation Model

This Appendix gives the modeling equations and coefficients necessary for the Simulation model developed in [30], which is used to validate the algorithms and methods discussed in this work.

B.1 Aircraft Modeling Equations and Coefficients [27]

The following sets of equations are necessary for the nonlinear, simulated aircraft model used in this work. The derivation of the equations, with the exception of the force equations, were given in Sections 2.1 and 2.2.

B.1.1 Force Equations

The force equations derived previously were for the body axis system while the force equations given here are for the stability axis system.

\[
\dot{\alpha} = q - (p \cos \alpha + r \sin \alpha) \tan \beta - \frac{LOM}{V_T \cos \beta} + \frac{g}{V_T \cos \beta} (\cos \theta \cos \phi \cos \alpha + \sin \theta \sin \alpha) \tag{B.1}
\]

\[
\dot{\beta} = p \sin \alpha - r \cos \alpha + \frac{1}{V_T} (YOM \cos \beta + DOM \sin \beta) + \frac{g}{V_T} (\cos \theta \sin \phi \cos \beta + \sin \theta \sin \beta \cos \alpha - \cos \theta \cos \phi \sin \beta \sin \alpha) \tag{B.2}
\]

\[
\dot{V}_T = YOM \sin \beta - DOM \cos \beta + g \left[ (\cos \theta \cos \phi \sin \alpha - \sin \theta \cos \alpha) \cos \beta + \cos \theta \sin \phi \sin \beta \right] \tag{B.3}
\]

where:

\[
DOM = \frac{D - T \cos \alpha}{m}, \quad YOM = \frac{Y}{m}, \quad LOM = \frac{L + T \sin \alpha}{m} \tag{B.4}
\]
B.1.2 Moment Equations

The moment equations were derived previously and given in Equation (2.23):

\[
\begin{bmatrix}
\dot{p} & \dot{I}_{xx} - \dot{q}I_{xy} - \dot{r}I_{xz} \\
-\dot{p} & \dot{I}_{xy} + \dot{q}I_{yy} - \dot{r}I_{yz} \\
-\dot{p} & \dot{I}_{xz} - \dot{q}I_{yz} + \dot{r}I_{zz}
\end{bmatrix} =
\begin{bmatrix}
qr(I_{yy} - I_{zz}) + (q^2 - r^2)I_{xy} - prI_{xy} + pqI_{xz} \\
pr(I_{zz} - I_{xx}) + (r^2 - p^2)I_{xz} - pqI_{yz} + qrI_{xy} \\
pq(I_{xx} - I_{yy}) + (p^2 - q^2)I_{xy} - qrI_{xz} + prI_{yz}
\end{bmatrix}
\] (B.5)

B.1.3 Kinematic Equations (Euler Model)

The kinematic equations, for the Euler Model, were derived previously and given in Equation (2.31):

\[
\begin{bmatrix}
\dot{\phi} \\
\dot{\theta} \\
\dot{\psi}
\end{bmatrix} =
\begin{bmatrix}
1 & s\phi\tan\theta & c\phi\tan\theta \\
0 & c\phi & -s\phi \\
0 & s\phi\sec\theta & c\phi\sec\theta
\end{bmatrix}
\begin{bmatrix}
p \\
q \\
r
\end{bmatrix}
\] (B.6)

B.1.4 Kinematic Equations (Quaternion Model)

The kinematic equations for the quaternion model were derived previously and given in Equation (2.54):

\[
\begin{bmatrix}
\dot{q}_0 \\
\dot{q}_1 \\
\dot{q}_2 \\
\dot{q}_3
\end{bmatrix} = \frac{1}{2}
\begin{bmatrix}
0 & -p & -q & -r \\
p & 0 & r & -q \\
q & -r & 0 & p \\
r & q & -p & 0
\end{bmatrix}
\begin{bmatrix}
q_0 \\
q_1 \\
q_2 \\
q_3
\end{bmatrix}
\] (B.7)

B.2 Aerodynamic Coefficients

B.2.1 Force Coefficients

\[
C_L = C_{L_0} + C_{L_\alpha}\alpha + \frac{\bar{c}}{2V_T}(C_{L_q}q + C_{L_\alpha}\dot{\alpha}) + C_{L_{\delta_e}}\delta_e + C_{L_{\delta_f}}\delta_f
\] (B.8)

\[
C_D = C_{D_0} + C_{D_\alpha}\alpha + \frac{\bar{c}}{2V_T}(C_{D_q}q + C_{D_\alpha}\dot{\alpha}) + C_{D_{\delta_e}}\delta_e + C_{D_{\delta_f}}\delta_f
\] (B.9)
\[ C_Y = C_{Y_0} + C_{Y_\beta} \beta + \frac{b}{2V_T} (C_{Y_p} p + C_{Y_r} r) + C_{Y_{\delta_a}} \delta_a + C_{Y_{\delta_r}} \delta_r \]  \hspace{1cm} (B.10)

**B.2.2 Moment Coefficients**

\[ C_l = C_{l_0} + C_{l_\beta} \beta + \frac{b}{2V_T} (C_{l_p} p + C_{l_r} r) + C_{l_{\delta_a}} \delta_a + C_{l_{\delta_r}} \delta_r \]  \hspace{1cm} (B.11)

\[ C_m = C_{m_0} + C_{m_\alpha} \alpha + \frac{\bar{c}}{2V_T} (C_{m_q} q + C_{m_{\dot{\alpha}}} \dot{\alpha}) + C_{m_{\delta_e}} \delta_e + C_{m_{\delta_f}} \delta_f \]  \hspace{1cm} (B.12)

\[ C_n = C_{n_0} + C_{n_\beta} \beta + \frac{b}{2V_T} (C_{n_p} p + C_{n_r} r) + C_{n_{\delta_a}} \delta_a + C_{n_{\delta_r}} \delta_r \]  \hspace{1cm} (B.13)

**B.2.3 Simulation Coefficient and Parameter Values**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
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<tr>
<td>( C_L )</td>
<td>Total Lift Coefficient</td>
<td>Equation (B.8)</td>
</tr>
<tr>
<td>( C_{L_0} )</td>
<td>Initial Lift Coefficient</td>
<td>0.4590697</td>
</tr>
<tr>
<td>( C_{L_\alpha} )</td>
<td>Lift Change with respect to ( \alpha )</td>
<td>4.5667216</td>
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<td>( C_{L_q} )</td>
<td>Lift Change with respect to pitch rate</td>
<td>8.0499999</td>
</tr>
<tr>
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<td>Lift Change with respect to ( \dot{\alpha} )</td>
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</tr>
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<td>Lift Change with respect to ( \delta_e )</td>
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<td>( C_{L_{\delta_f}} )</td>
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Table B.1: Lift Force Aerodynamic Coefficients
### Table B.2: Drag Force Aerodynamic Coefficients

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### Table B.3: Side Force Aerodynamic Coefficients

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<td>Side Force with respect to $\delta_a$</td>
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### Table B.4: Rolling Moment Aerodynamic Coefficients

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<td>$C_{m\delta_f}$</td>
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Table B.5: Pitching Moment Aerodynamic Coefficients

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<td>$C_{n\beta}$</td>
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Table B.6: Yawing Moment Aerodynamic Coefficients

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<td>$S$</td>
<td>Wing Planform Area</td>
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</tr>
<tr>
<td>$\bar{c}$</td>
<td>Average Wing Chord Length</td>
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</tr>
<tr>
<td>$b$</td>
<td>Wing Span Length</td>
<td>30.000 ft</td>
</tr>
<tr>
<td>$g$</td>
<td>Acceleration due to Gravity</td>
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</tr>
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<td>$\rho$</td>
<td>Air Density</td>
<td>0.001496 slugs/ft$^3$</td>
</tr>
<tr>
<td>$\bar{q}$</td>
<td>Dynamic Pressure</td>
<td>$\frac{1}{2}\rho V_T^2$</td>
</tr>
<tr>
<td>$m$</td>
<td>Aircraft Mass</td>
<td>756.526 slugs</td>
</tr>
<tr>
<td>$I_{xx}$</td>
<td>Primary Axis Moment of Inertia</td>
<td>8691.462 slug × ft$^2$</td>
</tr>
<tr>
<td>$I_{yy}$</td>
<td>Secondary Axis Moment of Inertia</td>
<td>70668.58 slug × ft$^2$</td>
</tr>
<tr>
<td>$I_{zz}$</td>
<td>Tertiary Axis Moment of Inertia</td>
<td>70418.67 slug × ft$^2$</td>
</tr>
<tr>
<td>$I_{xz}$</td>
<td>Cross Product Mass Moment of Inertia</td>
<td>8691.462 slug × ft$^2$</td>
</tr>
<tr>
<td>$I_{xy}$</td>
<td>Cross Product Mass Moment of Inertia</td>
<td>70668.58 slug × ft$^2$</td>
</tr>
<tr>
<td>$I_{yz}$</td>
<td>Cross Product Mass Moment of Inertia</td>
<td>70418.67 slug × ft$^2$</td>
</tr>
</tbody>
</table>

Table B.7: Aircraft Aerodynamic Constants
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{T_0}$</td>
<td>True Velocity</td>
<td>6752.679 ft/sec</td>
</tr>
<tr>
<td>$\alpha_0$</td>
<td>Angle-of-Attack</td>
<td>3.59697 deg</td>
</tr>
<tr>
<td>$\beta_0$</td>
<td>Sideslip Angle</td>
<td>0.0 deg</td>
</tr>
<tr>
<td>$p_0$</td>
<td>Roll Angular Rate</td>
<td>0.0 deg/sec</td>
</tr>
<tr>
<td>$q_0$</td>
<td>Pitch Angular Rate</td>
<td>0.0 deg/sec</td>
</tr>
<tr>
<td>$r_0$</td>
<td>Yaw Angular Rate</td>
<td>0.0 deg/sec</td>
</tr>
<tr>
<td>$\phi_0$</td>
<td>Roll Angle</td>
<td>0.0 deg</td>
</tr>
<tr>
<td>$\theta_0$</td>
<td>Pitch Angle</td>
<td>3.56967 deg</td>
</tr>
<tr>
<td>$\phi_0$</td>
<td>Yaw Angle</td>
<td>0.0 deg</td>
</tr>
<tr>
<td>$x_{e_0}$</td>
<td>Location relative to Primary Axis</td>
<td>0.0 ft</td>
</tr>
<tr>
<td>$y_{e_0}$</td>
<td>Location relative to Secondary Axis</td>
<td>0.0 ft</td>
</tr>
<tr>
<td>$z_{e_0}$</td>
<td>Location relative to Tertiary Axis</td>
<td>-20000 ft</td>
</tr>
</tbody>
</table>

Table B.8: Nonlinear Aircraft Simulation Initial Conditions
Figure B.1: Nonlinear Aircraft Model in Simulink
Appendix C

Accelerometer Biases Supplement

C.1 Accelerometer Bias Algorithm Derivation and Full Equations

As discussed in section 6, the equations for estimating accelerometer biases found in that section are only valid for accelerometers #1, #2, #12 and #13 for both the pitch and roll arrays. Presented here is the derivation for the equations found in that section as well as the equations for the remaining accelerometers, except for accelerometer #7.

Figure C.1: Pitch Array Accelerometer Orientations

From the figure, the following orientations for an accelerometer on the left half plane and one on the right half plane can be extracted:
In the above figure, the darker axes represent the coordinate axes for an accelerometer on the left-half plane and the lighter axes represent coordinate axes for an accelerometer on the right-half plane. These accelerometers do not need to be directly across from each other as this is a general derivation. In order to be able to add their biases together, and their measurements, the measured values from the left half plane accelerometer, \( b_{left} \), need to be resolved to the same set of axes as the right half plane accelerometer, \( b_{right} \). Equations (3.4) and (3.5) are used to determine the imposed loading of the accelerometers. Because those equations are used to determine the attitude estimates, they are used here to resolve the biases of the two accelerometers to the same set of axes. Equation (3.4), for the pitch array, can be written using Equation (6.1) as follows:

\[
A_{x,imp} = g[A_{z,i} + \beta_{z,i} - A_{z,cg} \cos \theta_i + (\sin \theta_{man} \sin \theta_i)] + r_d \left[ (2pr) \cos \phi_i \sin \phi_i - q^2 \sin^2 \phi_i - r^2 \cos^2 \phi_i - p^2 \right] / \sin \theta_i \quad \text{(C.1)}
\]

and Equation (3.5) can be written as:

\[
A_{y,imp} = g[A_{y,i} + \beta_{y,i} - A_{z,cg} \sin \phi_i - (\cos \theta_{man} \sin \phi_{man} \cos \phi_i)] - r_d \left[ (2pr) \cos \phi_i \sin \phi_i - q^2 \sin^2 \phi_i - r^2 \cos^2 \phi_i - p^2 \right] / \cos \phi_i \quad \text{(C.2)}
\]

Using the above equations in the attitude estimate equations:

\[
\theta_{est} = \sin^{-1}(A_{x,imp} - A_{x,cg}) \quad \text{(C.3)}
\]

\[
\phi_{est} = \tan^{-1} \left( \frac{A_{y,cg} - A_{y,imp}}{A_{z,cg} - A_{z,imp}} \right) \quad \text{(C.4)}
\]
Results in the true accelerometer measurement errors, $\beta_{z,i}$ and $\beta_{y,i}$, becoming transformed by $\frac{1}{\cos \phi_i}$ and $\frac{1}{\sin \theta_i}$, respectively. This results in the following equations for the bias estimation algorithm of the pitch array

$$
\Delta_1 = b_{13}\frac{\sin \theta_{13}}{1} - b_{1}\frac{\sin \theta_{1}}{1} \\
\Delta_2 = b_{12}\frac{\sin \theta_{12}}{1} - b_{1}\frac{\sin \theta_{1}}{1} \\
\Delta_3 = b_{12}\frac{\sin \theta_{12}}{1} - b_{2}\frac{\sin \theta_{2}}{1}
$$

(C.5)

or, because accelerometers #1 through #6 are on the left-half plane and their orientation angles, $\theta_i$, are negative:

$$
\Delta_1 = b_{13}\frac{\sin \theta_{13}}{1} + b_{1}\frac{\sin |\theta_{1}|}{1} \\
\Delta_2 = b_{12}\frac{\sin \theta_{12}}{1} + b_{1}\frac{\sin |\theta_{1}|}{1} \\
\Delta_3 = b_{12}\frac{\sin \theta_{12}}{1} + b_{2}\frac{\sin |\theta_{2}|}{1}
$$

(C.6)

Repeating the above for the next set of accelerometers for the pitch array:

$$
\Delta_1 = b_{11}\frac{\sin \theta_{11}}{1} - b_{3}\frac{\sin \theta_{3}}{1} \\
\Delta_2 = b_{10}\frac{\sin \theta_{10}}{1} - b_{3}\frac{\sin \theta_{3}}{1} \\
\Delta_3 = b_{10}\frac{\sin \theta_{10}}{1} - b_{4}\frac{\sin \theta_{4}}{1}
$$

(C.7)

or,

$$
\Delta_1 = b_{11}\frac{\sin \theta_{11}}{1} + b_{3}\frac{\sin \theta_{3}}{1} \\
\Delta_2 = b_{10}\frac{\sin \theta_{10}}{1} + b_{3}\frac{\sin \theta_{3}}{1} \\
\Delta_3 = b_{10}\frac{\sin \theta_{10}}{1} + b_{4}\frac{\sin |\theta_{4}|}{1}
$$

(C.8)

And the last set of accelerometers for the pitch array:

$$
\Delta_1 = b_{9}\frac{\sin \theta_{9}}{1} - b_{5}\frac{\sin \theta_{5}}{1} \\
\Delta_2 = b_{8}\frac{\sin \theta_{8}}{1} - b_{5}\frac{\sin \theta_{5}}{1} \\
\Delta_3 = b_{8}\frac{\sin \theta_{8}}{1} - b_{6}\frac{1}{\sin \theta_{6}}
$$

(C.9)

or,

$$
\Delta_1 = b_{9}\frac{\sin \theta_{9}}{1} + b_{5}\frac{\sin \theta_{5}}{1} \\
\Delta_2 = b_{8}\frac{\sin \theta_{8}}{1} + b_{5}\frac{\sin \theta_{5}}{1} \\
\Delta_3 = b_{8}\frac{\sin \theta_{8}}{1} + b_{6}\frac{1}{\sin \theta_{6}}
$$

(C.10)
Equations (C.6), (C.8) and (C.10) are used in the accelerometer bias estimation algorithm found in Chapter 6 for the estimation of the pitch array biases. The equations for the roll array are as follows:

\[
\begin{align*}
\Delta_1 &= b_1 \frac{1}{\cos \phi_1} - b_{13} \frac{1}{\cos \phi_{13}} \\
\Delta_2 &= b_1 \frac{1}{\cos \phi_1} - b_{12} \frac{1}{\cos \phi_{12}} \\
\Delta_3 &= b_2 \frac{1}{\cos \phi_2} - b_{12} \frac{1}{\cos \phi_{12}}
\end{align*}
\]

or, because accelerometers #8 through #13 are on the left-half plane and their orientation angles, \(\phi_i\), result in \(\cos \phi_i\) being negative:

\[
\begin{align*}
\Delta_1 &= b_1 \frac{1}{\cos \phi_1} + b_{13} \frac{1}{\cos \phi_{13}} \\
\Delta_2 &= b_1 \frac{1}{\cos \phi_1} + b_{12} \frac{1}{\cos \phi_{12}} \\
\Delta_3 &= b_2 \frac{1}{\cos \phi_2} + b_{12} \frac{1}{\cos \phi_{12}}
\end{align*}
\]

Repeating the above for the next set of accelerometers for the roll array:

\[
\begin{align*}
\Delta_1 &= b_3 \frac{1}{\cos \phi_3} - b_{11} \frac{1}{\cos \phi_{11}} \\
\Delta_2 &= b_3 \frac{1}{\cos \phi_3} - b_{10} \frac{1}{\cos \phi_{10}} \\
\Delta_3 &= b_4 \frac{1}{\cos \phi_4} - b_{10} \frac{1}{\cos \phi_{10}}
\end{align*}
\]

or,

\[
\begin{align*}
\Delta_1 &= b_3 \frac{1}{\cos \phi_3} + b_{11} \frac{1}{\cos \phi_{11}} \\
\Delta_2 &= b_3 \frac{1}{\cos \phi_3} + b_{10} \frac{1}{\cos \phi_{10}} \\
\Delta_3 &= b_4 \frac{1}{\cos \phi_4} + b_{10} \frac{1}{\cos \phi_{10}}
\end{align*}
\]

And the last set of accelerometers for the roll array:

\[
\begin{align*}
\Delta_1 &= b_5 \frac{1}{\cos \phi_5} - b_9 \frac{1}{\cos \phi_9} \\
\Delta_2 &= b_5 \frac{1}{\cos \phi_5} - b_8 \frac{1}{\cos \phi_8} \\
\Delta_3 &= b_6 \frac{1}{\cos \phi_6} - b_8 \frac{1}{\cos \phi_8}
\end{align*}
\]

or,

\[
\begin{align*}
\Delta_1 &= b_5 \frac{1}{\cos \phi_5} + b_9 \frac{1}{\cos \phi_9} \\
\Delta_2 &= b_5 \frac{1}{\cos \phi_5} + b_8 \frac{1}{\cos \phi_8} \\
\Delta_3 &= b_6 \frac{1}{\cos \phi_6} + b_8 \frac{1}{\cos \phi_8}
\end{align*}
\]
Equations (C.12), (C.14) and (C.16) are used in the accelerometer bias estimation algorithm found in Chapter 6 for the estimation of the roll array biases.

C.2 Acclerometer Bias Algorithm in Simulink

The following figures depict the various steps of the accelerometer bias estimation algorithm within Simulink.

Figure C.3: Pitch/Roll Bias Estimation in Simulink
The above 2 figures are applicable for both the longitudinal, pitch, and transverse, roll, arrays.
Appendix D

Supplemental Figures

This appendix section gives additional figures that were omitted from the main sections for conciseness. These figures give a more complete representation of what is happening for the cases presented.

D.1 Section 6.2 - Accelerometer Bias Estimation Results

This section gives supplemental figures for Section 6.2. These figures give a more complete representation of the nature of the accelerometer bias estimation algorithm(s).

D.1.1 Phase I - Longitudinal Aircraft Maneuver

Phase I - Longitudinal Maneuver: No Turbulence

![Figure D.1: Accelerometer Bias Estimation - Pitch Error, Phase I No Turbulence](image)

(a) No Bias Estimation  
(b) Bias Estimation

Figure D.1: Accelerometer Bias Estimation - Pitch Error, Phase I No Turbulence
(a) No Bias Estimation  
(b) Bias Estimation

Figure D.2: Accelerometer Bias Estimation - Pitch Estimate, Phase I No Turbulence

Figure D.3: Pitch Bias Estimation - Phase I No Turbulence
Figure D.4: Accelerometer Pitch Array Misaligned Measurements - Phase I No Turbulence

(a) No Bias Estimation
(b) Bias Estimation

Figure D.5: Accelerometer Bias Estimation - Roll Error, Phase I No Turbulence

(a) No Bias Estimation
(b) Bias Estimation
(a) No Bias Estimation  
(b) Bias Estimation

Figure D.6: Accelerometer Bias Estimation - Roll Estimate, Phase I No Turbulence

Figure D.7: Roll Bias Estimation - Phase I No Turbulence
Figure D.8: Accelerometer Roll Array Misaligned Measurements - Phase I No Turbulence

Figure D.9: Imposed Loads, Phase I No Turbulence
Phase I - Longitudinal Maneuver: Turbulence

(a) No Bias Estimation

(b) Bias Estimation

Figure D.10: Accelerometer Bias Estimation - Pitch Error, Phase I with Turbulence

(a) No Bias Estimation

(b) Bias Estimation

Figure D.11: Accelerometer Bias Estimation - Pitch Estimate, Phase I with Turbulence
Figure D.12: Pitch Bias Estimation - Phase I with Turbulence

(a) No Bias Estimation

(b) Bias Estimation

Figure D.13: Accelerometer Array Misaligned Measurements - Phase I with Turbulence
Figure D.14: Accelerometer Bias Estimation - Roll Error, Phase I with Turbulence

Figure D.15: Accelerometer Bias Estimation - Roll Estimate, Phase I with Turbulence
Figure D.16: Roll Bias Estimation - Phase I with Turbulence

(a) No Bias Estimation  
(b) Bias Estimation

Figure D.17: Accelerometer Roll Array Misaligned Measurements - Phase I with Turbulence
D.1.2 Phase II - Transverse Aircraft Maneuver

Phase II - Transverse Maneuver: No Turbulence

Figure D.19: Accelerometer Bias Estimation - Pitch Error, Phase II No Turbulence
Figure D.20: Accelerometer Bias Estimation - Pitch Estimate, Phase II No Turbulence

Figure D.21: Pitch Bias Estimation - Phase II No Turbulence
Figure D.22: Accelerometer Pitch Array Misaligned Measurements - Phase II No Turbulence

Figure D.23: Accelerometer Bias Estimation - Roll Error, Phase II No Turbulence
Figure D.24: Accelerometer Bias Estimation - Roll Estimate, Phase II No Turbulence

Figure D.25: Roll Bias Estimation - Phase II No Turbulence
(a) No Bias Estimation  
(b) Bias Estimation

Figure D.26: Accelerometer Roll Array Misaligned Measurements - Phase II No Turbulence

(a) No Bias Estimation  
(b) Bias Estimation

Figure D.27: Imposed Loads, Phase II No Turbulence
Phase II - Transverse Maneuver: Turbulence

Figure D.28: Accelerometer Bias Estimation - Pitch Error, Phase II with Turbulence

Figure D.29: Accelerometer Bias Estimation - Pitch Estimate, Phase II with Turbulence
Figure D.30: Pitch Bias Estimation - Phase II with Turbulence

Figure D.31: Accelerometer Array Misaligned Measurements - Phase II with Turbulence
Figure D.32: Accelerometer Bias Estimation - Roll Error, Phase II with Turbulence

Figure D.33: Accelerometer Bias Estimation - Roll Estimate, Phase II with Turbulence
Figure D.34: Roll Bias Estimation - Phase II with Turbulence

Figure D.35: Accelerometer Roll Array Misaligned Measurements - Phase II with Turbulence
D.1.3 Phase III - Longitudinal/Transverse Aircraft Maneuver

Phase III - Longitudinal/Transverse Maneuver: No Turbulence

(a) No Bias Estimation  
(b) Bias Estimation

Figure D.36: Imposed Loads, Phase II with Turbulence

(a) No Bias Estimation  
(b) Bias Estimation

Figure D.37: Accelerometer Bias Estimation - Pitch Error, Phase III No Turbulence
Figure D.38: Accelerometer Bias Estimation - Pitch Estimate, Phase III No Turbulence

Figure D.39: Pitch Bias Estimation - Phase III No Turbulence
Figure D.40: Accelerometer Pitch Array Misaligned Measurements - Phase III No Turbulence

Figure D.41: Accelerometer Bias Estimation - Roll Error, Phase III No Turbulence
(a) No Bias Estimation

(b) Bias Estimation

Figure D.42: Accelerometer Bias Estimation - Roll Estimate, Phase III No Turbulence

Figure D.43: Roll Bias Estimation - Phase III No Turbulence
Figure D.44: Accelerometer Roll Array Misaligned Measurements - Phase III No Turbulence

Figure D.45: Imposed Loads, Phase III No Turbulence
Phase III - Longitudinal/Transverse Maneuver: Turbulence

(a) No Bias Estimation

(b) Bias Estimation

Figure D.46: Accelerometer Bias Estimation - Pitch Error, Phase III with Turbulence

(a) No Bias Estimation

(b) Bias Estimation

Figure D.47: Accelerometer Bias Estimation - Pitch Estimate, Phase III with Turbulence
Figure D.48: Pitch Bias Estimation - Phase III with Turbulence

(a) No Bias Estimation
(b) Bias Estimation

Figure D.49: Accelerometer Array Misaligned Measurements - Phase III with Turbulence
Figure D.50: Accelerometer Bias Estimation - Roll Error, Phase III with Turbulence

(a) No Bias Estimation  
(b) Bias Estimation

Figure D.51: Accelerometer Bias Estimation - Roll Estimate, Phase III with Turbulence

(a) No Bias Estimation  
(b) Bias Estimation
Figure D.52: Roll Bias Estimation - Phase III with Turbulence

Figure D.53: Accelerometer Roll Array Misaligned Measurements - Phase III with Turbulence
D.1.4 Phase I - Longitudinal Maneuver with Drift Effects

Phase I - Longitudinal Maneuver with Drift Effects: No Turbulence

Figure D.55: Accelerometer Bias Estimation with Drift Effects - Pitch Error, Phase I No Turbulence
Figure D.56: Accelerometer Bias Estimation with Drift Effects - Pitch Estimate, Phase I No Turbulence

Figure D.57: Pitch Bias Estimation with Drift Effects - Phase I No Turbulence
Figure D.58: Accelerometer Pitch Array Misaligned Measurements with Drift Effects - Phase I No Turbulence

(a) No Bias Estimation  (b) Bias Estimation

Figure D.59: Accelerometer Bias Estimation with Drift Effects - Roll Error, Phase I No Turbulence

(a) No Bias Estimation  (b) Bias Estimation
Figure D.60: Accelerometer Bias Estimation with Drift Effects - Roll Estimate, Phase I No Turbulence

Figure D.61: Roll Bias Estimation with Drift Effects - Phase I No Turbulence
Figure D.62: Accelerometer Roll Array Misaligned Measurements with Drift Effects - Phase I No Turbulence

Figure D.63: Imposed Loads with Drift Effects, Phase I No Turbulence
Phase I - Longitudinal Maneuver with Drift Effects: Turbulence

Figure D.64: Accelerometer Bias Estimation with Drift Effects - Pitch Error, Phase I with Turbulence

(a) No Bias Estimation

(b) Bias Estimation

Figure D.65: Accelerometer Bias Estimation with Drift Effects - Pitch Estimate, Phase I with Turbulence

(a) No Bias Estimation

(b) Bias Estimation
Figure D.66: Pitch Bias Estimation with Drift Effects - Phase I with Turbulence

(a) No Bias Estimation  
(b) Bias Estimation

Figure D.67: Accelerometer Array Misaligned Measurements with Drift Effects - Phase I with Turbulence
Figure D.68: Accelerometer Bias Estimation with Drift Effects - Roll Error, Phase I with Turbulence

Figure D.69: Accelerometer Bias Estimation with Drift Effects - Roll Estimate, Phase I with Turbulence
Figure D.70: Roll Bias Estimation with Drift Effects - Phase I with Turbulence

Figure D.71: Accelerometer Roll Array Misaligned Measurements with Drift Effects - Phase I with Turbulence
D.1.5 Phase II - Transverse Maneuver with Drift Effects

Phase II - Transverse Maneuver with Drift Effects: No Turbulence

Figure D.73: Accelerometer Bias Estimation with Drift Effects - Pitch Error, Phase II No Turbulence
Figure D.74: Accelerometer Bias Estimation with Drift Effects - Pitch Estimate, Phase II No Turbulence

Figure D.75: Pitch Bias Estimation with Drift Effects - Phase II No Turbulence
Figure D.76: Accelerometer Pitch Array Misaligned Measurements with Drift Effects - Phase II No Turbulence

Figure D.77: Accelerometer Bias Estimation with Drift Effects - Roll Error, Phase II No Turbulence
Figure D.78: Accelerometer Bias Estimation with Drift Effects - Roll Estimate, Phase II No Turbulence

Figure D.79: Roll Bias Estimation with Drift Effects - Phase II No Turbulence
Figure D.80: Accelerometer Roll Array Misaligned Measurements with Drift Effects - Phase II No Turbulence

(a) No Bias Estimation
(b) Bias Estimation

Figure D.81: Imposed Loads with Drift Effects, Phase II No Turbulence

(a) No Bias Estimation
(b) Bias Estimation
Phase II - Transverse Maneuver with Drift Effects: Turbulence

Figure D.82: Accelerometer Bias Estimation with Drift Effects - Pitch Error, Phase II with Turbulence

(a) No Bias Estimation  
(b) Bias Estimation

Figure D.83: Accelerometer Bias Estimation with Drift Effects - Pitch Estimate, Phase II with Turbulence

(a) No Bias Estimation  
(b) Bias Estimation
Figure D.84: Pitch Bias Estimation with Drift Effects - Phase II with Turbulence

Figure D.85: Accelerometer Array Misaligned Measurements with Drift Effects - Phase II with Turbulence
Figure D.86: Accelerometer Bias Estimation with Drift Effects - Roll Error, Phase II with Turbulence

(a) No Bias Estimation  
(b) Bias Estimation

Figure D.87: Accelerometer Bias Estimation with Drift Effects - Roll Estimate, Phase II with Turbulence

(a) No Bias Estimation  
(b) Bias Estimation
Figure D.88: Roll Bias Estimation with Drift Effects - Phase II with Turbulence

Figure D.89: Accelerometer Roll Array Misaligned Measurements with Drift Effects - Phase II with Turbulence
D.1.6 Phase III - Longitudinal/Transverse Maneuver with Drift Effects

Phase III - Longitudinal/Transverse Maneuver with Drift Effects: No Turbulence

Figure D.90: Imposed Loads with Drift Effects, Phase II with Turbulence

Figure D.91: Accelerometer Bias Estimation with Drift Effects - Pitch Error, Phase III No Turbulence
Figure D.92: Accelerometer Bias Estimation with Drift Effects - Pitch Estimate, Phase III No Turbulence

Figure D.93: Pitch Bias Estimation with Drift Effects - Phase III No Turbulence
Figure D.94: Accelerometer Pitch Array Misaligned Measurements with Drift Effects - Phase III No Turbulence

Figure D.95: Accelerometer Bias Estimation with Drift Effects - Roll Error, Phase III No Turbulence
Figure D.96: Accelerometer Bias Estimation with Drift Effects - Roll Estimate, Phase III No Turbulence

Figure D.97: Roll Bias Estimation with Drift Effects - Phase III No Turbulence
Figure D.98: Accelerometer Roll Array Misaligned Measurements with Drift Effects - Phase III No Turbulence

Figure D.99: Imposed Loads with Drift Effects, Phase III No Turbulence
Phase III - Longitudinal/Transverse Maneuver with Drift Effects: Turbulence

Figure D.100: Accelerometer Bias Estimation with Drift Effects - Pitch Error, Phase III with Turbulence

(a) No Bias Estimation

(b) Bias Estimation

Figure D.101: Accelerometer Bias Estimation with Drift Effects - Pitch Estimate, Phase III with Turbulence

(a) No Bias Estimation

(b) Bias Estimation
Figure D.102: Pitch Bias Estimation with Drift Effects - Phase III with Turbulence

Figure D.103: Accelerometer Array Misaligned Measurements with Drift Effects - Phase III with Turbulence
Figure D.104: Accelerometer Bias Estimation with Drift Effects - Roll Error, Phase III with Turbulence

Figure D.105: Accelerometer Bias Estimation with Drift Effects - Roll Estimate, Phase III with Turbulence
Figure D.106: Roll Bias Estimation with Drift Effects - Phase III with Turbulence

Figure D.107: Accelerometer Roll Array Misaligned Measurements with Drift Effects - Phase III with Turbulence
D.2 Chapter 7 - Simulation Results

This section gives supplemental figures for Chapter 7. These figures give a more complete representation of the complete method’s simulation results. What is not presented in this paper are the plots for the various different rate gyro biases, rate gyro drift slopes, accelerometer biases and accelerometer drift slopes.

D.2.1 Phase I Maneuver - No Turbulence

Figure D.109: Pitch Estimation Results - Phase I No Turbulence
(a) Pitch Bias Estimation Error  
(b) Pitch Bias Estimation  

Figure D.110: Pitch Bias Estimation Results - Phase I No Turbulence

(a) Roll Estimation Error  
(b) Roll Estimation  

Figure D.111: Roll Estimation Results - Phase I No Turbulence
(a) Roll Bias Estimation Error  
(b) Roll Bias Estimation  

Figure D.112: Roll Bias Estimation Results - Phase I No Turbulence  

Figure D.113: Rate Gyro Bias Estimation - Phase I No Turbulence
Figure D.114: Accelerometer Roll Array Misaligned Measurements - Phase I no Turbulence

Figure D.115: Imposed Loads, Phase I with Turbulence
D.2.2 Phase I Maneuver - Turbulence

(a) Pitch Estimation Error

(b) Pitch Estimation

Figure D.116: Pitch Estimation Results - Phase I with Turbulence

(a) Pitch Bias Estimation Error

(b) Pitch Bias Estimation

Figure D.117: Pitch Bias Estimation Results - Phase I with Turbulence
Figure D.118: Roll Estimation Results - Phase I No Turbulence

Figure D.119: Roll Bias Estimation Results - Phase I with Turbulence
Figure D.120: Rate Gyro Bias Estimation - Phase I with Turbulence

Figure D.121: Accelerometer Roll Array Misaligned Measurements - Phase I with Turbulence
D.2.3 Phase II Maneuver - No Turbulence

Figure D.123: Pitch Estimation Results - Phase II No Turbulence
Figure D.124: Pitch Bias Estimation Results - Phase II No Turbulence

Figure D.125: Roll Estimation Results - Phase II No Turbulence
Figure D.126: Roll Bias Estimation Results - Phase II No Turbulence

Figure D.127: Rate Gyro Bias Estimation - Phase II No Turbulence
Figure D.128: Accelerometer Roll Array Misaligned Measurements - Phase II no Turbulence

Figure D.129: Imposed Loads, Phase II with Turbulence
D.2.4 Phase II Maneuver - Turbulence

![Pitch Estimation Error](image1)

(a) Pitch Estimation Error

![Pitch Estimation](image2)

(b) Pitch Estimation

Figure D.130: Pitch Estimation Results - Phase II with Turbulence

![Pitch Bias Estimation Error](image3)

(a) Pitch Bias Estimation Error

![Pitch Bias Estimation](image4)

(b) Pitch Bias Estimation

Figure D.131: Pitch Bias Estimation Results - Phase II with Turbulence
Figure D.132: Roll Estimation Results - Phase II No Turbulence

(a) Roll Estimation Error  
(b) Roll Estimation

Figure D.133: Roll Bias Estimation Results - Phase II with Turbulence

(a) Roll Bias Estimation Error  
(b) Roll Bias Estimation
Figure D.134: Rate Gyro Bias Estimation - Phase II with Turbulence

Figure D.135: Accelerometer Roll Array Misaligned Measurements - Phase II with Turbulence
D.2.5 Phase III Maneuver - No Turbulence

(a) Pitch Estimation Error  (b) Pitch Estimation

Figure D.137: Pitch Estimation Results - Phase III No Turbulence
Figure D.138: Pitch Bias Estimation Results - Phase III No Turbulence

Figure D.139: Roll Estimation Results - Phase III No Turbulence
Figure D.140: Roll Bias Estimation Results - Phase III No Turbulence

Figure D.141: Rate Gyro Bias Estimation - Phase III No Turbulence
Figure D.142: Accelerometer Roll Array Misaligned Measurements - Phase III no Turbulence

Figure D.143: Imposed Loads, Phase III with Turbulence
D.2.6 Phase III Maneuver - Turbulence

Figure D.144: Pitch Estimation Results - Phase III with Turbulence

Figure D.145: Pitch Bias Estimation Results - Phase III with Turbulence
Figure D.146: Roll Estimation Results - Phase III No Turbulence

Figure D.147: Roll Bias Estimation Results - Phase III with Turbulence
Figure D.148: Rate Gyro Bias Estimation - Phase III with Turbulence

Figure D.149: Accelerometer Roll Array Misaligned Measurements - Phase III with Turbulence
Figure D.150: Imposed Loads, Phase III with Turbulence