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PREFACE

Welcome to the thirty-second annual conference of the Northeast Decision Sciences Institute. The 2003 meeting is being held at the Westin Hotel in Providence, Rhode Island. The Program includes 47 sessions comprising 147 competitively judged papers and nine special sessions in the form of symposia, panels, tutorials and workshops. Authors of the papers and special session participants include academics as well as practitioners from both the private and public sectors.

The quality of the meeting depends on the quality of the submitted papers and special session proposals as well as on the review and selection process. All papers were competitively double-blind reviewed by at least two reviewers. Many people committed significant amounts of time reviewing papers and providing constructive feedback to authors.

This *Proceedings* has several features designed to aid conference participants locate papers of interest to them. The papers are arranged alphabetically within their respective tracks. The index at the front of the book lists each paper and special session by track and the index at the back of the book lists authors.

We wish to thank everyone who volunteered time in preparation for this year's meeting. We understand the demands on your time and appreciate the commitment all of you have made to the Northeast Decision Sciences Institute.

We hope this year's meeting provides an opportunity for you present and discuss your work, to network with colleagues who have similar research and teaching interests.

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TESTING RATIONALITY OF SUBGROUPS IN MULTIVARIATE CONTROL CHARTS

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ABSTRACT

In this study a method will be developed to test whether the subgroups formed for T^2 control charts, a multivariate process control tool, are rational. In an earlier work [7] we suggested that a comparison of the mean square successive difference variance to the usual variance could be used to test the rationality of the subgroups in the univariate control charts. The method proposed in this paper is a multivariate extension of testing the equality of two variance estimators.

INTRODUCTION

Statistical Process Control (SPC) is a key component of the total quality philosophy in the sense that it is process-oriented, preventive and helps us to identify types of variation in the process so we know who is responsible for controlling and reducing the variation to improve the process. This is accomplished through the use of control charts, i.e., using either charts for variable data (univariate or multivariate) or attribute data. For more detail on the control charts and SPC, refer to, e.g., Grant and Leavenworth [3], Duncan [2], and Burr [1].

Multivariate SPC is an extension of univariate SPC where more than one (and sometimes correlated) quality characteristics (variables) exist. If that is the case, monitoring these variables jointly on a single chart would be preferred, as opposed to having a separate chart for each variable. T^2 Control charts are common tools for this purpose (see, for example, references [5] and [6]). The T^2 statistic was first introduced by Hotelling [8]. One of the issues in statistical process control is forming "rational subgroups." Rational subgroups are defined as those displaying only random variation within the subgroups. Shewhart [10], in his quote, "The engineer who is successful in dividing his data initially into rational subgroups based upon rational hypotheses is therefore inherently better off in the long run than the one who is not thus successful," emphasized the fact that the key to the successful use of control charts for the purpose of process control is to form rational subgroups for the charts. Subgroups which are not rationally defined may cause the control charts to give misleading signals about the status of the process, i.e., an out-of-control process may be declared in-control or vice versa.

The issue of rational subgroups is mostly ignored in the multivariate process control. In one of our earlier papers we developed a rationality test for the subgroups used in the univariate control charts [7]. In this study, we will extend this test to the multivariate case.

PROPOSED METHOD

The purpose of this paper is to develop a method to test whether the subgroups formed for T^2 control charts are rational. The method uses the test for equality of two covariance matrices. The first covariance matrix is the one normally described in statistical literature. The second matrix used in the test is the multivariate Mean Square Successive Difference (MSSD) covariance matrix [5]. This method is a multivariate extension of testing the equality of two variance estimators. In an earlier work [7] we suggested that a comparison of the mean square successive difference variance to the usual variance can be used to test the rationality of the subgroups. The MSSD estimator of variance, which is unbiased, is given by Hald [4] as follows.

$$q^2 = \frac{MSSD}{2} \qquad (1)$$

where MSSD is defined as

$$MSSD = \frac{1}{n-1}\sum_{i=1}^{n-1}(X_{i+1} - X_i)^2 \qquad (2)$$

The multivariate equivalents to the average and variance are, respectively, a vector of variable means and the covariance matrix. The multivariate equivalent to the mean square successive difference variance is the mean square successive difference covariance matrix, which is discussed in [5]. The elements of the MSSD covariance matrix $Cov(X_j, X_k)$ are

$$= \frac{\sum_{i=2}^{n}(X_{ji} - X_{ji-1})(X_{ki} - X_{ki-1})}{2(n-1)} \qquad (3)$$

In this paper we test for the equality of the average of the regular subgroup covariance matrices and the average of the MSSD subgroup covariance matrices to establish whether or not the subgroups are rational. The statistical equality of the matrices will imply that the subgroups formed are rational. The test is based on Wilk's [11] idea that the determinant of the covariance matrix is the multivariate analog of the variance. The test proposed, which is described in Kramer and Jensen's work [9], is as follows:

1.Calculate two covariance matrices S_1 and S_2 for each subgroup of size n (Let S_1 be the regular subgroup covariance matrix, and S_2 be the MSSD subgroup covariance matrix).

2. Calculate the average of S_1 and S_2 ($\bar{S}_1$ and $\bar{S}_2$) obtained for all the subgroups.

3. Calculate the weighted average of $\bar{S}_1$ and $\bar{S}_2$ as:

$$S = \frac{(n-1)\bar{S}_1 + (n-1)\bar{S}_2}{n+n-2} = \frac{(\bar{S}_1 + \bar{S}_2)}{2} \tag{4}$$

4. Calculate M as:

$$\begin{aligned} M &= (n+n-2)\ln|S| - (n-1)\ln|\bar{S}_1| - (n-1)\ln|\bar{S}_2| \\ &= 2(n-1)\ln|S| - (n-1)\ln|\bar{S}_1| - (n-1)\ln|\bar{S}_2| \\ &= \ln\left[\frac{|S|^{2(n-1)}}{|\bar{S}_1|^{n-1}|\bar{S}_2|^{n-1}}\right] \end{aligned} \tag{5}$$

5. Calculate m as:

$$= 1 - \left[\frac{1}{n-1} + \frac{1}{n-1} - \frac{1}{2(n-1)}\right]\left[\frac{2p^2+3p-1}{6(p+1)}\right] \tag{6}$$

where p is the number of variables.

6. Calculate G as

$$G = M \times m \tag{7}$$

The value of G will then be compared to the critical value of Chi-Square with $p(p+1)/2$ degrees of freedom at a selected significance level and a decision will be made about the rationality of the subgroups formed on the multivariate control chart. If the value of G exceeds the critical value of Chi-Square, then the subgroups are declared as not rational.

EXAMPLE

The example data is on the percentage by weight of a series of five screens of a particle size determination of which we use the data on the first three screens (the data set is given in the Appendix). Thus there are three variables (i.e., screens) in the data. The covariance matrix revealed the fact that the first screen is correlated negatively with the second screen and positively with the third screen.

When the proposed test is applied to this data set for an example of subgroups of size ten to check the rationality of subgroups to be used in the T^2 control chart, we get the G value of 4.750165. This value is less than the Chi Square value with a $3(3+1)/2=6$ degrees of freedom at a significance level of 99.5%, which is 18.55. Even at 90% significance level, the conclusion is the same. Thus we declare that the subgroups of size ten are rational. Different subgroup sizes can be tested the same way to find the ones which are rational (The computer program used to make the calculations is available from the authors).

CONCLUSION

In this paper we proposed a method to test the rationality of the subgroups used in T^2 control charts. This issue, i.e., rationality of the subgroups, has been neglected to a large extent in multivariate quality control. The proposed method could be a helpful tool for practioners to determine the proper subgroup size in multivariate control charts to get the most benefit from the SPC applications. Of course, knowledge and experience about the process should always be an input in determining the proper subgroup size, along with the tools such as the one proposed in this paper.

APPENDIX

Screen 1	Screen 2	Screen 3
8.07	19.60	24.21
6.43	15.62	19.26
6.95	16.89	20.86
6.80	16.51	20.40
6.96	16.91	20.89
6.57	15.96	19.72
6.57	15.96	19.72
7.26	17.63	21.78
7.87	19.13	23.63
6.98	16.96	20.95
7.45	18.11	22.37
6.68	16.24	20.06
7.36	17.88	22.09
6.55	15.91	19.65
6.58	15.99	19.75
6.50	10.79	20.51
7.07	12.17	22.21
6.33	10.37	19.99

6.59	11.00	20.77
7.36	12.88	23.09
6.62	11.09	20.87
6.42	10.59	20.26
6.72	11.32	21.16
6.92	11.81	21.76
7.37	12.90	23.11
9.19	17.33	27.08
7.38	12.93	23.15
7.30	12.74	22.91
6.30	10.30	19.90
6.71	11.30	21.13
6.07	9.75	19.22
8.44	15.51	24.33
5.63	8.67	17.89
6.87	11.70	21.63
6.61	11.06	20.85

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