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A NOTE ON GRAVITATIONAL BROWNIAN MOTION

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ABSTRACT

Chandrasekhar's theory of stellar encounters predicts a dependence of the Brownian motion of a massive particle on the velocity distribution of the perturbing stars. One consequence is that the expectation value of the massive object's kinetic energy can be different from that of the perturbers. This effect is shown to be modest however, and substantially smaller than claimed in a recent study based on an approximate treatment of the encounter equations.

Subject headings: black hole physics — gravitation — gravitational waves — galaxies: nuclei

1. INTRODUCTION

A massive object at the center of a stellar system undergoes a random walk in momentum space as its motion is perturbed by gravitational encounters with nearby stars. After the decay of transients, the steady-state velocity distribution $f(v)$ of the massive object is a balance between the accelerating forces of random perturbations and the decelerating force of dynamical friction. Chandrasekhar's (1942,1943) theory of gravitational encounters provides expressions for both forces as functions of the mass and velocity of the test particle, and the masses and velocity distribution of the field stars. In the case that the latter have a Maxwellian velocity distribution, the steady-state $f(v)$ describing the motion of the massive object is also a Maxwellian, with rms velocity given by the equipartition relation

$$v_{rms}^2 = \frac{m}{M} u_{rms}^2, \tag{1}$$

where M and v refer to the massive object and m and u to the perturbers.

One difference between Brownian motion in gravitating and non-gravitating systems is that the velocity distribution $f_f(u)$ of the perturbers in a galaxy or star cluster can be substantially non-Maxwellian. The steady-state velocity distribution of the massive object is still predicted to be Maxwellian (Merritt 2001), but there is no reason for v_{rms} to satisfy equation (1). In this note, the steady-state $f(v)$ is derived for a massive particle that is subject to gravitational perturbations from field stars with an arbitrary distribution of velocities. Departures from equipartition are found to be small, even when the background is strongly non-Maxwellian. The stronger dependence of v_{rms} on $f_f(u)$ claimed in a recent study (Chatterjee, Hernquist & Loeb 2002, hereafter CHL) is shown to result from approximations made in the application of Chandrasekhar's theory.

2. STEADY-STATE VELOCITY DISTRIBUTIONS

The steady-state velocity distribution $f(v)$ of a massive object ("black hole") that interacts via gravitational scattering with stars is given by the time-independent solution of the Fokker-Planck equation. Following the treatment in Merritt (2001) (hereafter Paper I), we can write

$$0 = f\left[\langle\Delta v_\parallel\rangle + \frac{1}{2v}\left(\langle\Delta v_\perp^2\rangle - 2\langle\Delta v_\parallel^2\rangle\right)\right] - \frac{1}{2}\frac{\partial}{\partial v}\left(f\langle\Delta v_\parallel^2\rangle\right) \tag{2}$$

where $\langle\Delta v_\parallel\rangle$, $\langle\Delta v_\parallel^2\rangle$ and $\langle\Delta v_\perp^2\rangle$ are the standard diffusion coefficients describing the motion of the black hole. In a steady state, we expect that v will be of order $\sqrt{m/M} \ll 1$ times the typical stellar velocity. Expanding the diffusion coefficients about $v = 0$,

$$\langle\Delta v_\parallel\rangle = -Av + Bv^3 \ldots, \tag{3a}$$

$$\langle\Delta v_\parallel^2\rangle = C + Dv^2 \ldots, \tag{3b}$$

$$\langle\Delta v_\perp^2\rangle = 2(C + Fv^2) \ldots, \tag{3c}$$

substituting into equation (2), and retaining the lowest order terms in v gives

$$0 \approx (A + 2D - F)f + \frac{C}{2}\frac{\partial f}{\partial v}\frac{1}{v} \tag{4a}$$

$$\approx Af + \frac{C}{2}\frac{\partial f}{\partial v}\frac{1}{v} \tag{4b}$$

since A is of order M/m times D and F. This has solution

$$f(v) = f_0 e^{-v^2/2\sigma^2}, \tag{5a}$$
$$\sigma^2 = \frac{C}{2A}, \tag{5b}$$

i.e. the black hole's velocity follows a Maxwellian distribution with 1D velocity dispersion σ.

If the background stellar distribution is assumed to be infinite and homogeneous, the diffusion coefficients are (Paper I)

$$\langle \Delta v_{\|} \rangle = -16\pi^2 G^2 M m n \int_0^\infty du \left(\frac{u}{v}\right)^2 f_f(u) H_1(v, u, p_{max}), \tag{6a}$$
$$\langle \Delta v_{\|}^2 \rangle = \frac{32}{3}\pi^2 G^2 m^2 n v \int_0^\infty du \left(\frac{u}{v}\right)^2 f_f(u) H_2(v, u, p_{max}), \tag{6b}$$

where $f_f(u)$ is the velocity distribution of the field stars, normalized to unit number; n is the field star density; p_{max} is the maximum impact parameter at which incoming stars are assumed to perturb the black hole; and

$$H_1(v, u, p_{max}) = \frac{1}{8u}\int_{|v-u|}^{v+u} dV \left(1 + \frac{v^2 - u^2}{V^2}\right) \ln\left(1 + \frac{p_{max}^2 V^4}{G^2 M^2}\right), \tag{7a}$$
$$H_2(v, u, p_{max}) = \frac{3}{8u}\int_{|v-u|}^{v+u} dV \left\{\left[1 - \frac{V^2}{4v^2}\left(1 + \frac{v^2 - u^2}{V^2}\right)^2\right] \ln\left(1 + \frac{p_{max}^2 V^4}{G^2 M^2}\right) + \left[\frac{3}{4}\frac{V^2}{v^2}\left(1 + \frac{v^2 - u^2}{V^2}\right) - 1\right] \frac{p_{max}^2 V^4/G^2 M^2}{1 + p_{max}^2 V^4/G^2 M^2}\right\} \tag{7b}$$

Equation (7b) includes the "non-dominant" terms which are of order unity when p_{max} is large. The integration variable V is the relative velocity of star and black hole when the star is at infinity.

In the standard approximation (e.g. Rosenbluth, MacDonald & Judd 1957; Spitzer 1987), the non-dominant terms are neglected, and the logarithmic terms are taken outside the integrals, since they are slowly varying with respect to V. One writes

$$\ln\left(1 + \frac{p_{max}^2 V^4}{G^2 M^2}\right) \approx 2 \ln \Lambda, \tag{8}$$

a constant, and

$$\langle \Delta v_{\|} \rangle = -16\pi^2 G^2 M m n \ln\Lambda \int_0^v du \left(\frac{u}{v}\right)^2 f_f(u), \tag{9a}$$
$$\langle \Delta v_{\|}^2 \rangle = \frac{32}{3}\pi^2 G^2 m^2 n \ln\Lambda v \left[\int_0^v du \left(\frac{u}{v}\right)^4 f_f(u) + \int_v^\infty du \left(\frac{u}{v}\right) f_f(u)\right], \tag{9b}$$

with leading terms

$$A_\Lambda = \frac{16}{3}\pi^2 G^2 M m n \ln\Lambda f_f(0), \tag{10a}$$
$$C_\Lambda = \frac{32}{3}\pi^2 G^2 m^2 n \ln\Lambda \int_0^\infty du u f_f(u). \tag{10b}$$

The predicted 1D velocity dispersion of the black hole is then (CHL)

$$\sigma_\Lambda^2 = \frac{C_\Lambda}{2A_\Lambda} = \left(\frac{m}{M}\right) f_f(0)^{-1} \int_0^\infty du u f_f(u). \tag{11}$$

However when the velocity of the massive particle is very low, the logarithms in equations (7a-b) will be close to zero for *all* stars with $u < v$, i.e. all stars that enter into the dynamical friction integral (9a). It is unclear in this case whether removing the logarithm from the integrands is a reasonable approximation (Chandrasekhar 1943; White 1949).

Returning to the exact expressions (6a-7b) and expanding to lowest order in v, we find after some algebra:

$$A = \frac{32}{3}\pi^2 G^2 M m n \int_0^\infty \frac{du}{u} f_f(u) \frac{p_{max}^2 u^4/G^2 M^2}{1 + p_{max}^2 u^4/G^2 M^2}, \tag{12a}$$
$$C = \frac{16}{3}\pi^2 G^2 m^2 n \int_0^\infty du u f_f(u) \log\left(1 + \frac{p_{max}^2 u^4}{G^2 M^2}\right). \tag{12b}$$

The non-dominant terms may be shown not to contribute to this order. Note that both diffusion coefficients now depend on the field-star velocity distribution at *all* values of u, and in fact the field stars that produce the dynamical friction force can be shown to have roughly the same velocity distribution as $f_f(u)$ (Paper I). In the standard treatment with "$\log\Lambda$'s removed," the dynamical friction force comes entirely from stars with $u < v$ (as in equation 9a), a physically unreasonable result.

The black hole's velocity dispersion is now

$$\sigma^2 = \frac{C}{2A} = \frac{1}{4}\frac{m}{M}\frac{\int_0^\infty du u f_f(u)\log\left(1+\frac{p_{max}^2u^4}{G^2M^2}\right)}{\int_0^\infty du u^{-1} f_f(u)\frac{p_{max}^2u^4/G^2M^2}{1+p_{max}^2u^4/G^2M^2}}. \tag{13}$$

If $f_f(u)$ is differentiable at low and high u, the denominator can be integrated by parts yielding

$$\sigma^2 = \frac{m}{M}\frac{\int_0^\infty du u f_f(u)\log\left(1+\frac{p_{max}^2u^4}{G^2M^2}\right)}{\int_0^\infty du\left(-df_f/du\right)f_f(u)\log\left(1+\frac{p_{max}^2u^4}{G^2M^2}\right)}. \tag{14}$$

This reduces to the approximate expression, equation (11), if the logarithmic terms are "taken out of the integrals."

In the case of a Maxwellian field star velocity distribution, $f_f(u) = f_M(u) = f_0e^{-u^2/2\sigma_f^2}$, the exact coefficients are

$$A_M = \frac{4\sqrt{2\pi}}{3}\frac{G^2Mmn}{\sigma_f^3}G(R_M), \qquad C_M = \frac{8\sqrt{2\pi}}{3}\frac{G^2m^2n}{\sigma_f}G(R_M), \tag{15a}$$

$$G(R_M) \equiv \frac{1}{2}\int_0^\infty dz\, e^{-z}\ln\left(1+4R_M^2z^2\right), \qquad R_M \equiv \frac{p_{max}\sigma_f^2}{GM}. \tag{15b}$$

and

$$\sigma^2 = \frac{C_M}{2A_M} = \left(\frac{m}{M}\right)\sigma_f^2, \tag{16}$$

independent of p_{max} (Paper I). Remarkably, this is also the value of σ^2 given by the approximate expression (11) when $f_f(u) = f_M(u)$.

For all other – non-Maxwellian – f_f's, however, equations (13) and (11) yield different results for σ^2, and each is different from the Maxwellian value. Furthermore σ^2 as predicted by the exact expression (13) is in general a function of the maximum impact parameter, since C and A depend differently on p_{max}.

Without loss of generality, we can express the dependence of σ^2 on p_{max} via a dimensionless variable R, where $R \equiv p_{max}\tilde{\sigma}^2/GM$ and $\tilde{\sigma}$ is a 1D velocity dispersion that characterizes the field star velocity distribution. Equating $G(R)$ in equation (15b) with $\ln\Lambda \equiv \ln(p_{max}/p_{min})$ reveals that $p_{min} \approx GM/\sqrt{2}\sigma_f$ (Paper I), hence $R \approx p_{max}/p_{min}$; note that p_{min} is also approximately equal to the "radius of influence" of the black hole. When p_{max} is large, i.e. $R \gg 1$, the logarithmic terms in equation (14) are nearly constant and σ^2 reduces to the value (11) derived by "taking out $\ln\Lambda$." But for a supermassive black hole in a galactic nucleus, p_{max} is of order the core radius r_c of the stellar distribution (Paper I), and observed core radii are of order a few times GM/σ_f^2 at most (e.g. Poon & Merritt 2001); hence $p_{max} \gtrsim p_{min}$ and R is of order unity. Thus we expect the exact and approximate expressions for σ^2 to be substantially different in many astrophysically interesting situations.

3. EXAMPLES

The equations derived above can be applied to various field-star distributions and the results for σ compared with the equipartition value. Following CHL, we quantify the departure of σ from equipartition via the parameter η, defined as

$$\eta \equiv \frac{M}{m}\frac{\sigma^2}{\sigma_\star^2} \tag{17}$$

with $\sigma_\star$ the stellar velocity dispersion at the location of the black hole; η is unity in the case of equipartition. Both σ and $\sigma_\star$ are computed under the assumption that the black hole sits at the center of the stellar system. We emphasize that the effect of the black hole on the stellar background is ignored.

King's (1966) family of models

$$f_K(E) = \begin{cases} (2\pi\sigma_K^2)^{-3/2}\left(e^{E/\sigma_K^2}-1\right), & \text{if } E \geq 0; \\ 0, & \text{otherwise} \end{cases} \tag{18}$$

have nearly Maxwellian central velocity distributions. Here E is the binding energy, $E = -u^2/2 + \psi(r)$, and $\psi(r) = -\Phi(r)$ with $\Phi(r)$ the gravitational potential due to the stars, equal to zero at the model's edge. The parameter $W \equiv \psi(0)/\sigma_K^2$ is a dimensionless measure of the central potential. For large W, the King model parameter σ_K is nearly equal to the true central velocity dispersion $\sigma_\star$, or

$$\sigma_\star^2 = \frac{2}{3}\frac{\int_0^W dE\,(W-E)^{3/2} f_K(E)}{\int_0^W dE\sqrt{W-E}f_K(E)}. \tag{19}$$

A natural definition for R is

$$R_K \equiv \frac{p_{max}\sigma_K^2}{GM}. \tag{20}$$

King models having $W \gtrsim 10$ fit the observed brightness profiles of some elliptical galaxies moderately well. Setting M to $\sim 0.1\%$ of the total stellar mass and equating p_{max} to the core radius gives R_K of order unity when $W \gtrsim 10$. Figure 1 shows that η is extremely close to one when R and W have these values. King models with smaller W do not represent galaxies well – their cores are too large – but are good fits to some globular clusters. The appropriate value of R_K for globular clusters is unclear however since it is not known whether such systems harbor massive central objects or what their likely masses are. If we suppose that the black hole has a mass equal to a fraction F of the stellar mass in the core, then $R_K \approx F^{-1}$. An upper limit on R comes from assuming that the black hole has a minimum mass $\sim 10M_\odot$ and that the core mass is $\sim 5\times 10^4 M_\odot$, its value in a massive, large-cored globular cluster like ω Centauri (e.g. Merritt, Meylan & Mayor 1997). Then $F \approx 2\times 10^{-4}$ and $R_K \approx 5000$. In such an extreme case, Figure 1 shows that η can approach its asymptotic limit of 1.75 (CHL), but only in highly unphysical models with $W \lesssim 2$.

Next we consider the polytropic models, some of which have strongly non-Maxwellian velocity distributions. The Plummer (1911) polytrope has $f_f = f_5(E)$, where

$$f_n(E) = \begin{cases} f_0 E^{n-\frac{3}{2}}, & \text{if } E>0;\\ 0, & \text{otherwise}\end{cases} \tag{21}$$

and the gravitational potential is again defined to be zero at the model's edge, which in this case lies at infinity. The density profile is

$$\rho(r) = \frac{3M_{gal}a^2}{4\pi}\frac{1}{\left(r^2+a^2\right)^{5/2}} \tag{22}$$

and the central velocity dispersion is $\sigma_\star^2(0) = GM_{gal}/6a$. We define

$$R_P \equiv \frac{p_{max}\sigma_\star^2(0)}{GM} \tag{23}$$

which gives

$$\eta = \frac{6}{7}\frac{\int_0^2 dx\,(2-x)^{7/2}\log\left(1+36R^2x^2\right)}{\int_0^2 dx\,(2-x)^{5/2}\log\left(1+36R^2x^2\right)}. \tag{24}$$

As $R\to\infty$, $\eta\to 4/3$ (CHL). Figure 2 shows η as a function of R. Values of R appropriate to supermassive black holes in galactic nuclei again give η close to one and the asymptotic value is approached only slowly as R is increased.

The third example considered by CHL was the $n=1$ polytrope. The structure of this model makes it a poor representation of *any* stellar system, since the density profile:

$$\rho(r) = \frac{M_{gal}}{4\pi^2a^2}r^{-1}\sin(r/a), \qquad r<\pi a \tag{25}$$

is "all core." Furthermore the distribution function is singular at the model's edge. For this reason we must use the first of the two expressions derived above for σ^2, equation (13), since the second assumes differentiability of f_f. We define R in terms of the central velocity dispersion as in equation (23), where $\sigma_\star^2(0) = GM_{gal}/2\pi a$. The result for η is

$$\eta = \frac{1}{16\pi R^2}\frac{\int_0^{1/\pi} dx\left(\frac{1}{\pi}-x\right)^{-1/2}\log\left(1+16\pi^2R^2\right)}{\int_0^{1/\pi} dxx\left(\frac{1}{\pi}-x\right)^{-1/2}\left(1+16\pi^2R^2\right)^{-1}}. \tag{26}$$

Figure 3 plots this function. The asymptotic value $\eta = 4$ (CHL) is again reached only very gradually as R increases.

4. DISCUSSION

The foregoing should not be taken to imply that a massive object at the center of a stellar system should always be in or near a state of equipartition with respect to its less-massive perturbers. Indeed it is not even clear what Chandrasekhar's theory would predict in many physically interesting situations. Chandrasekhar's theory is local in the sense that the density and velocity distribution of the field stars are assumed to everywhere be the same as they are at the location of the test particle. The effect of a massive particle on the structure of the background stellar system is also ignored. Both of these assumptions make it difficult to apply the theory to the case of Brownian motion of a supermassive black hole at the center of a galactic nucleus. The density of stars falls steeply away from the black hole in most galaxies; indeed a density falloff as steep as $\rho \sim r^{-1/2}$ is required for self-consistency if the stellar velocities are isotropic near the black hole (as assumed here). Furthermore the presence of the black hole implies a steeply-rising stellar velocity dispersion within its sphere of influence, $r \lesssim GM/\sigma_\star^2$, and the black hole's Brownian motion would be influenced by these higher velocities.

The likely importance of such effects is suggested by a recent numerical study of Brownian motion in a galaxy with a steep, $\rho \sim r^{-3/2}$ density cusp (Dorband, Hemsendorf & Merritt 2003). The stellar velocity dispersion in this model has a central value of zero in the absence of the black hole, yet the numerical integrations reveal a significant Brownian motion of the "black hole." Evidently the massive particles in these simulations are responding to stars far from the center, or to stars whose motions have themselves been influenced by the presence of the black hole, or both.

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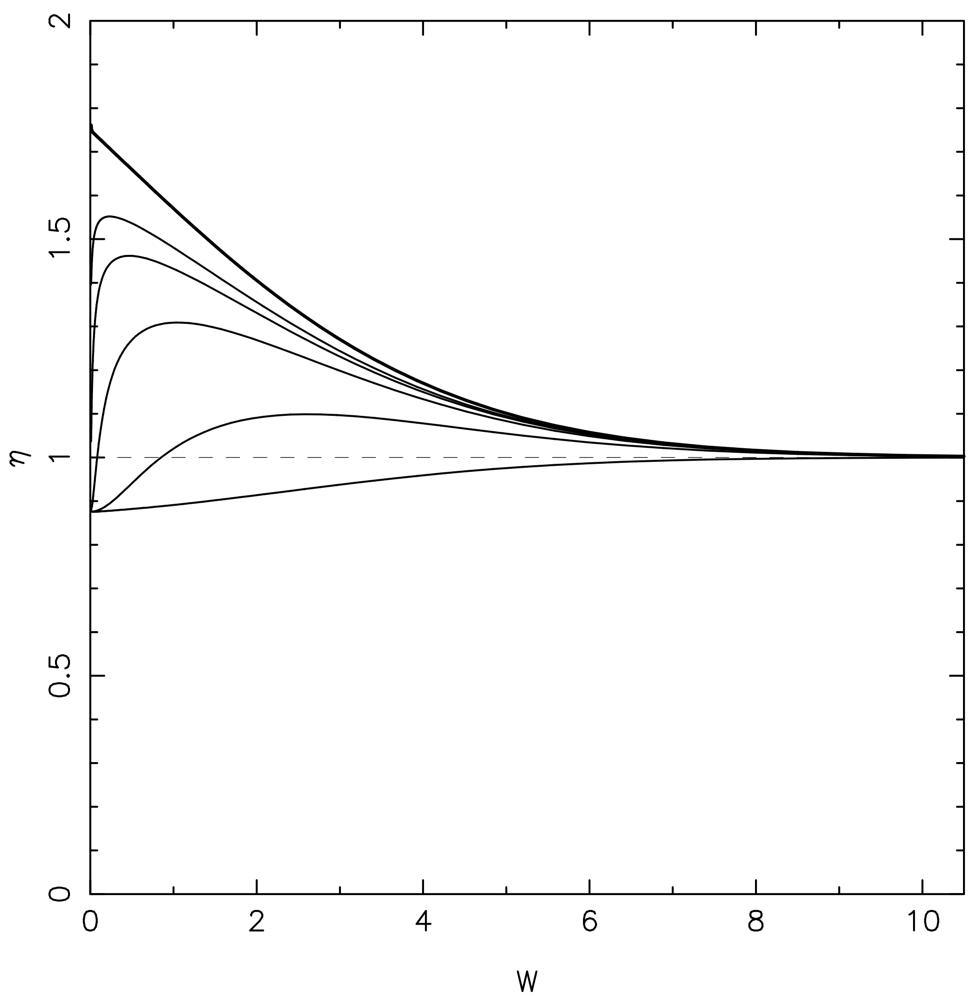

FIG. 1.— Departure from equipartition of a massive object at the center of a King-model galaxy. Thin curves are for $R = 0.1, 1, 10, 100, 1000$ moving upward, where R is the dimensionless maximum impact parameter defined by equation (20). Thick curve is the asymptotic form as $R \to \infty$. R is expected to be of order unity for a supermassive black hole at the center of a galaxy.

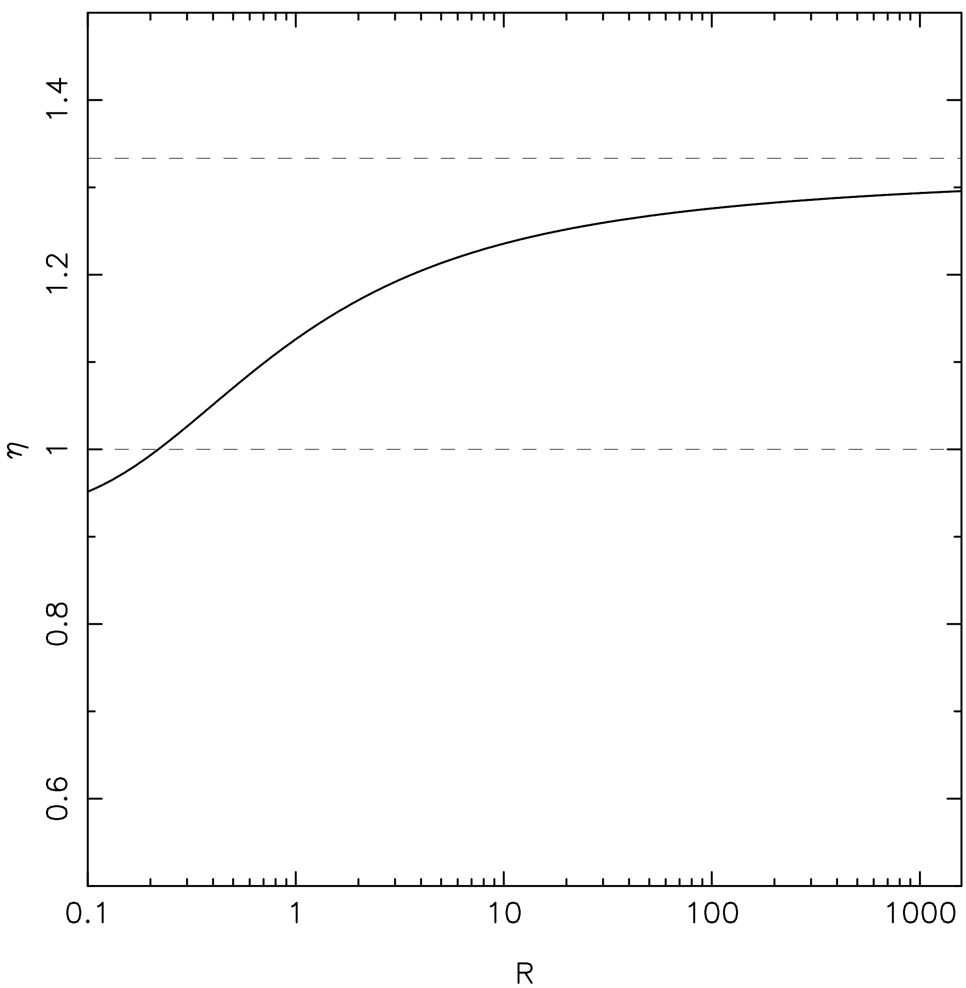

FIG. 2.— Values of η for a massive object at the center of a Plummer-model galaxy, as a function of the dimensionless maximum impact parameter R of equation (23). Upper dashed line shows asymptotic value for large R.

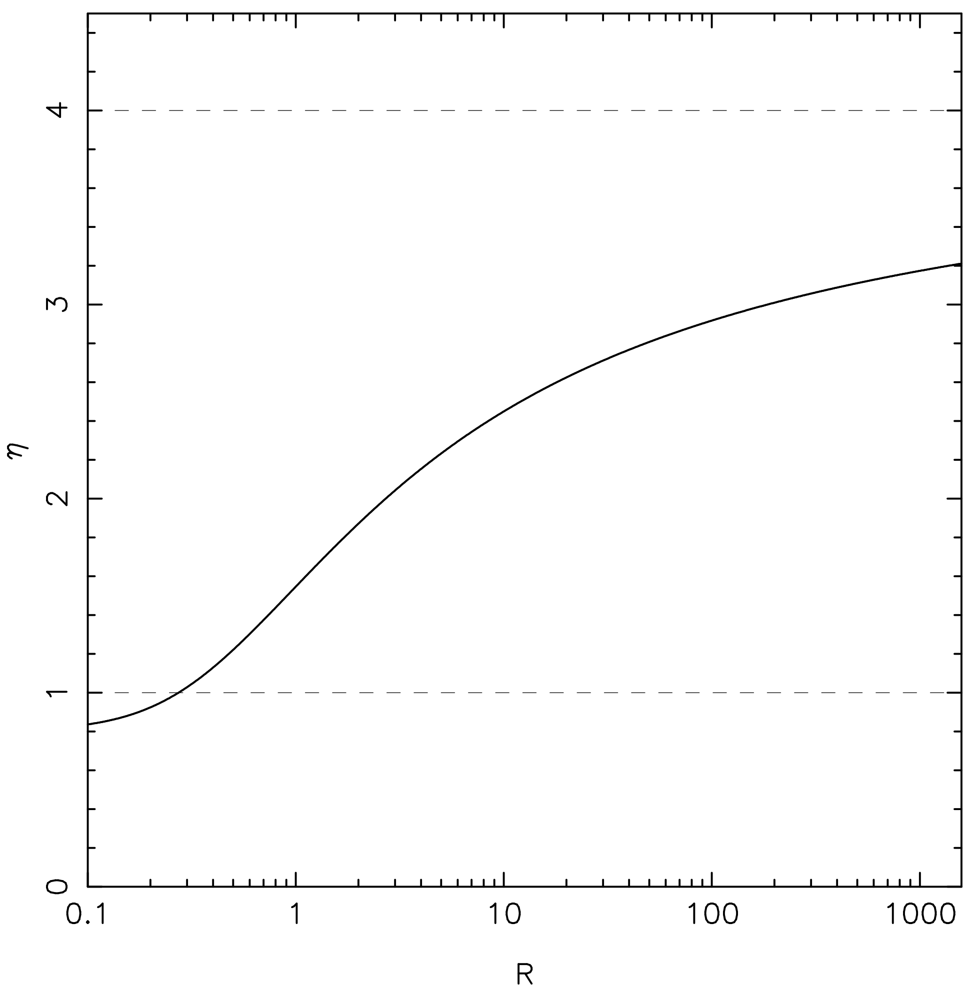

FIG. 3.— Like Figure 2, for the $n = 1$ polytrope.