Computational Design, Simulation of Meshing, and Stress Analysis of Strain Wave Gear Drives

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Computational Design, Simulation of Meshing, and Stress Analysis of Strain Wave Gear Drives

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A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Engineering in the Department of Mechanical Engineering of the Kate Gleason College of Engineering

March 17, 2021
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“If knowledge can cause problems, it is not through ignorance that we can solve them.”

Isaac Asimov
Abstract

Computational Design, Simulation of Meshing, and Stress Analysis of Strain Wave Gear Drives

by Eloy Yague-Spaude

Strain wave gear (SWG) drives were patented in 1959 by C. Walton Musser as a coaxial, compact, and lightweight gear drive providing remarkably large gear ratios without backlash. This outstanding performance requires the use of a flexible gear, as well as a meshing process with two regions of tooth contact as opposed to traditional gear drives, which only mesh in a single region. The latter drives have been studied for centuries under the principles of solid mechanics whereas SWG drives lack strong bodies of literature, principles, and computational tools for their design.

SWG drives include three parts: wave generator, flexible spline, and ring gear. Typically, the wave generator is an elliptical cam surrounded by a flexible race ball bearing which, inserted inside the flexible spline, provides the input motion to the drive. The flexible spline is a cup-shaped spring with external teeth on the open-end which, deflected by the wave generator, mesh with the internal teeth of the ring gear in two regions along the major axis of the drive. This thesis dissertation focuses on understanding the influence of the geometries of the wave generator and tooth profiles of the flexible spline and the ring gear over stresses throughout the operation of SWG drives. In doing so, computational tools have been developed and design recommendations have been formulated. The wave generator geometries simplified, elliptical, and four roller have been implemented based on previous geometries, while an additional geometry called parabolic is newly proposed. For the tooth profiles, the involute, as a generated profile, and the double and quadruple circular arc geometries, as directly-defined profiles, have been implemented. Two- and three-dimensional finite element models have been developed in a custom-made software to generate fully parameterized models based on the design and manufacturing processes of SWG drives. The models are analyzed and the resulting stresses for each design are compared to determine which geometries and micro-geometry modifications are the most influential over the mechanical performance of this type of gear drive.
Significant improvement has been achieved by modifying the geometries of the wave generator and the tooth flanks of the flexible spline and the ring gear. Simplified and parabolic wave generator geometries proved similarly advantageous and the elliptical geometry resulted in the lowest compressive stresses, while the four roller geometry was discarded due to large stresses. The involute tooth profile was proven unsuitable while the directly-defined tooth profiles showed similarly beneficial outcomes when the root geometry was reinforced. The three-dimensional model evidenced the complex state of deflection of the flexible spline due to the cup-shaped spring and the need for micro-geometry modifications to further improve the behavior of SWG drives. Crowning and slope micro-geometry modifications on the wave generator and crowning on the tooth flanks of the flexible spline have been implemented. When combined, these modifications eliminated the areas of stress concentration due to the deflection of the flexible spline and allowed the contact pattern to move closer to the center of the teeth. These improvements resulted in remarkably lower stresses which serve to increase the overall mechanical performance of SWG drives.
Acknowledgements

• Dr. Alfonso Fuentes-Aznar for his long lasting relationship and trust

• Gleason Corporation for its financial support

• Dr. Hermann J. Stadtfeld and Dr. Haris Ligata for their interest and participation

• My parents and grandparents for inoculating me with a deep love for learning

• Last but not least, my extremely supporting wife for her patience and assistance
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<thead>
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<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>Length of the major axis of the ellipse</td>
</tr>
<tr>
<td>$a_p$</td>
<td>Parabola coefficient</td>
</tr>
<tr>
<td>$a_G$</td>
<td>Outwards deflection of Gleason’s experimental setup</td>
</tr>
<tr>
<td>$b$</td>
<td>Length of the minor axis of the ellipse</td>
</tr>
<tr>
<td>$B$</td>
<td>Circumferential backlash</td>
</tr>
<tr>
<td>$b_G$</td>
<td>Inwards deflection of Gleason’s experimental setup</td>
</tr>
<tr>
<td>$C$</td>
<td>Minor axis reduction coefficient</td>
</tr>
<tr>
<td>$c_c$</td>
<td>Flexible spline cup thickness coefficient</td>
</tr>
<tr>
<td>$c_p$</td>
<td>Angular tooth thickness reduction coefficient</td>
</tr>
<tr>
<td>$c_l$</td>
<td>Straight segment length reduction coefficient</td>
</tr>
<tr>
<td>$C_p$</td>
<td>Parabola length reduction coefficient</td>
</tr>
<tr>
<td>$d$</td>
<td>Deflection</td>
</tr>
<tr>
<td>$E$</td>
<td>Modulus of elasticity</td>
</tr>
<tr>
<td>$F_w$</td>
<td>Face width</td>
</tr>
<tr>
<td>$h_a$</td>
<td>Addendum coefficient</td>
</tr>
<tr>
<td>$h_d$</td>
<td>Dedendum coefficient</td>
</tr>
<tr>
<td>$j$</td>
<td>Unit vector along the y-axis</td>
</tr>
<tr>
<td>$k$</td>
<td>Unit vector along the z-axis</td>
</tr>
<tr>
<td>$l$</td>
<td>Length of straight segment</td>
</tr>
<tr>
<td>$l_c$</td>
<td>Flexible spline cup length</td>
</tr>
<tr>
<td>$l_{fs}$</td>
<td>Length of straight segment of flexible spline teeth</td>
</tr>
<tr>
<td>$l_{rg}$</td>
<td>Length of straight segment of ring gear teeth</td>
</tr>
<tr>
<td>$m$</td>
<td>Module</td>
</tr>
</tbody>
</table>
\( m_{fs} \)  Module of the flexible spline  
\( m_G \)  Gear/reduction ratio  
\( n \)  Number of lobes of the wave generator  
\( N \)  Number of teeth  
\( N \)  Normal vector  
\( N_D \)  Number of teeth of the driven element  
\( N_F \)  Number of teeth of the fixed element  
\( N_g \)  Number of teeth of shaper-cutter  
\( \mathbf{n} \)  Unit normal vector  
\( \mathbf{n}_a \)  Normal vector common to the tip and addendum circular arcs  
\( \mathbf{n}_c \)  Normal vector of a point of the lateral arc  
\( \mathbf{n}_d \)  Normal vector common to the dedendum and root circular arcs  
\( \mathbf{n}_D \)  Normal vector to the dedendum circle in point \( P_D \)  
\( \mathbf{n}_l \)  Normal vector to the straight segment \( l \)  
\( \mathbf{n}_p \)  Normal vector of a point of the parabola  
\( \mathbf{n}_p \)  Normal vector to the pitch circle in point \( P_p \)  
\( \mathbf{n}_r \)  Normal vector common to the root arc and the dedendum circle  
\( \mathbf{n}_t \)  Normal vector common to the tip arc and the addendum circle  
\( \mathbf{N}_r \)  Normal vector from the origin to the root circular arc center  
\( \mathbf{N}_t \)  Normal vector from the origin to the tip circular arc center  
\( N_{fs} \)  Number of teeth of the flexible spline  
\( N_{g} \)  Number of teeth of the ring gear  
\( O_a \)  Center of the addendum circular arc  
\( O_c \)  Center of the connector arc  
\( O_d \)  Center of the dedendum circular arc  
\( O_l \)  Center of the lateral arc  
\( O_r \)  Center of the root circular arc  
\( O_t \)  Center of the tip circular arc  
\( P_a \)  Point between straight segment \( l \) and addendum circular arc \( r_a \)
\( P_A \) Center point of tooth top land
\( P_d \) Point between straight segment \( l \) and dedendum circular arc \( r_d \)
\( P_D \) Center point of tooth slot
\( P_p \) Pitch point of tooth profile
\( P_r \) Point between the tooth profile and the dedendum circle
\( P_t \) Point between the tooth profile and the addendum circle
\( \mathbf{r} \) Position vector
\( r_4 \) Radius of a roller
\( r_a \) Radius of the addendum circular arc
\( r_A \) Radius of the addendum circle
\( r_b \) Radius of the base circle
\( r_c \) Radius of the connector arc
\( \mathbf{r}_c \) Position vector of the center of the connector arc
\( r_d \) Radius of the dedendum circular arc
\( r_D \) Radius of the dedendum circle
\( r_g \) Radius of shaper-cutter pitch circle
\( r_l \) Radius of the lateral arc
\( \mathbf{r}_l \) Position vector of the center of the lateral arc
\( r_p \) Radius of the pitch circle
\( \mathbf{r}_p \) Position vector of a point of the parabola
\( r_r \) Radius of the root circular arc
\( r_t \) Radius of the tip circular arc
\( r_{ci} \) Flexible spline cup inner radius
\( r_{cf} \) Flexible spline cup fillet radius
\( r_{fs} \) Undeformed inner radius of the flexible spline
\( r_{pp} \) Radius of the pushing pins
\( \mathbf{r}^{(O_a)} \) Position vector of the center \( O_a \)
\( \mathbf{r}^{(O_d)} \) Position vector of the center \( O_d \)
\( \mathbf{r}^{(O_r)} \) Position vector of the center \( O_r \)
\( \mathbf{r}(O_t) \)  Position vector of the center \( O_t \)
\( \mathbf{r}(P_a) \)  Position vector of point \( P_a \)
\( \mathbf{r}(P_A) \)  Position vector of point \( P_A \)
\( \mathbf{r}(P_d) \)  Position vector of point \( P_d \)
\( \mathbf{r}(P_D) \)  Position vector of point \( P_D \)
\( \mathbf{r}(P_p) \)  Position vector of point \( P_p \)
\( \mathbf{r}(P_r) \)  Position vector of point \( P_r \)
\( \mathbf{r}(P_t) \)  Position vector of point \( P_t \)

\( RP_{fs} \)  Reference point of the flexible spline
\( RP_{pp} \)  Reference point of the pushing pin
\( RP_{rg} \)  Reference point of the ring gear
\( RP_{wg} \)  Reference point of the wave generator

\( S \)  Global coordinate system
\( S_f \)  Fixed coordinate system
\( S_g \)  Coordinate system of generating element
\( S_{fs} \)  Coordinate system of the flexible spline
\( S_{pp} \)  Coordinate system of the pushing pin
\( S_{rg} \)  Coordinate system of the ring gear
\( S_{wg} \)  Coordinate system of the wave generator

\( T \)  Output torque
\( t_c \)  Flexible spline cup thickness
\( t_l \)  Tangent vector to the straight segment \( l \)
\( t_p \)  Tangent vector of a point of the parabola
\( t_P \)  Circumferential tooth thickness
\( t'_P \)  Actual circumferential tooth thickness
\( t_r \)  Rim thickness
\( u \)  Profile generating parameter
\( v_g \)  Translational speed of generating rack-cutter
\( x \)  Abscissa coordinate of point of the parabola
\( y_c \)  
Ordinate of the position vector of the center of the connector arc

\( \alpha \)  
Pressure angle

\( \alpha_a \)  
Aperture angle of the addendum circular arc

\( \alpha_c \)  
Aperture angle of the connector arc

\( \alpha_d \)  
Aperture angle of the dedendum circular arc

\( \alpha_l \)  
Aperture angle of the lateral arc

\( \alpha_p \)  
Angular pitch tooth thickness

\( \alpha_r \)  
Aperture angle of the root circular arc

\( \alpha_t \)  
Aperture angle of the tip circular arc

\( \alpha^* \)  
Corrected pressure angle

\( \alpha_{fs} \)  
Pressure angle of the flexible spline teeth

\( \alpha_{rg} \)  
Pressure angle of the ring gear teeth

\( \beta \)  
Aperture angle of roller

\( \gamma \)  
Flexible spline rotational angle

\( \lambda \)  
Parabolic region rotational angle

\( \nu \)  
Poisson’s ratio

\( \omega_g \)  
Rotational speed of generating shaper-cutter

\( \omega_{fs} \)  
Rotational speed of the flexible spline

\( \omega_{rg} \)  
Rotational speed of the ring gear

\( \sigma \)  
Stresses

\( \sigma_1 \)  
Maximum principal stress

\( \sigma_3 \)  
Minimum principal stress

\( \phi \)  
Generating angle

\( \phi_g \)  
Generating angle of shaper-cutter

\( \phi_{rg} \)  
Generating angle of the ring gear

\( \theta \)  
Longitudinal generating parameter
\( \theta_i \)  Input angular position
\( \theta_o \)  Output angular position
Chapter 1

Introduction

1.1 Chapter overview

This introductory chapter discloses the basic concepts, literature, and relevant information regarding the topic of this thesis dissertation. Section 1.2 provides an explanation of the concept "Strain Wave Gearing" which is the transmission mechanism object of study in throughout this work and called strain wave gear (SWG) drive. Then, their different elements are described, as well as how they operate. The material of SWG drives as compared to traditional gear drives is explained to finalize their brief introduction with their typical applications. Section 1.3 describes the advantages and disadvantages of SWG drives compared to traditional cylindrical gear drives. Section 1.4 provides a summary of the existing literature and research efforts done by other researchers to improve SWG drives. Important historical events since the release of SWG drives in 1959 frame their research. Then, Section 1.5 describes the societal context of this type of gear drive. The actions to answer the research question that motivates this work are exposed in Section 1.6. The last section of this chapter, Section 1.7, summarizes the proposed approach of this thesis to accomplish its objective.

1.2 "Strain Wave Gearing"

The concept of "Strain Wave Gearing" was first introduced in 1959 by Clarence Walton Musser [1]. He obtained a patent for a motion transmitting device working by the principles of "Strain Wave Gearing." The particularity of this device is that it employs a flexible gear as one
Chapter 1. Introduction

of its components. This device is commonly called by the name "harmonic drive", which is a trademark in certain countries [2,3]. Therefore, this device is going to be referred to as a strain wave gear (SWG) drive throughout this document.

In Musser’s patent, SWG drives were explained and defined, including their advantages, different lay-out possibilities, and limitations. However, SWG drives have not been fully understood yet, as they did not come into use until the end of the 20th century when the development of aerospace and robotic applications required a huge gear reduction in a considerably limited amount of space and weight [4–11]. Therefore, the study of the working mechanisms of SWG drives has been carried out for only a few decades, which has not been enough to fully comprehend their complex functioning.

1.2.1 Elements of SWG drives

Figure 1.1 shows the three elements of a typical SWG drive. It is composed of an elliptical cam with a roller bearing with a flexible outer race called the wave generator, a thin-rimmed external teeth gear located on the open end of a cup-shaped spring called the flexible spline, and an internal teeth gear called the ring gear.

![Figure 1.1: Elements of a typical SWG drive.](image)

The first component is a strain inducing element or wave generator, which is typically of an elliptical shape. This allows the wave generator to induce two strain waves per revolution in the SWG drive due to the two lobes that an ellipse has. However, with different shapes it may include more than two lobes to increment the number of strain waves induced per revolution. In the most common lay-out of SWG drives, the wave generator connects the input shaft to the drive [1,12].
The second part is a strain gear, most frequently called flexible spline or "flexspline." A strain gear, as defined by Musser, is a thin rimmed external cylindrical gear located on the open end of a cup-shaped spring. This cup is of a reduced thickness in order to deflect when strain is applied from its inside by the wave generator. The flexible spline can incorporate any type of tooth profile applicable to spur gears [1,13].

One of the differentiating features of SWG drives as compared to traditional solid gear drives is the flexible spline, which is a flexible gear. Besides, several pairs of teeth are in contact between the flexible spline and the ring gear on each meshing region of SWG drives. These meshing regions have teeth contacting on different flanks depending on the area of the meshing region under study. This translates into one of the most important contributions of SWG drives to the gearing field: there is no backlash during the meshing of SWG drives [1,14,15].

The last element of a SWG drive is a ring gear or circular spline. This is an internal cylindrical gear that meshes with the flexible spline teeth. Depending on the application, the output from the SWG drive can be connected to the flexible spline or to the ring gear. However, most applications employ a fixed ring gear, and the flexible spline serves as the output.

1.2.2 Operation of SWG drives

SWG drives work differently than other types of gears. Traditional gears are studied from the point of view of solid mechanics, in which each tooth and gear is considered as a rigid body that transmits the input rotation, modifying its speed, torque, and direction of rotation if necessary [13]. However, SWG drives take advantage of the elasticity provided by the flexible spline to deflect towards the ring gear. The wave generator is inserted inside the flexible spline to deflect the latter along the major axis of the wave generator. This forces the flexible spline into meshing with the ring gear, creating two meshing regions between the teeth of the flexible spline and those of the ring gear, as shown in Figure 1.2. This figure shows a section of the face width of the teeth of the flexible spline and the ring gear of a SWG drive in meshing position. Along the major axis of the wave generator there are two meshing regions, whereas along the minor axis, the teeth of the flexible spline are far apart from the teeth of the ring gear, creating
two clearance regions. The lobes of the wave generator produce the deflection needed by the flexible spline to mesh with the ring gear, which relates to their number of teeth.

![Diagram of meshing position of the teeth of the flexible spline and the ring gear.]

**Figure 1.2:** Meshing position of the teeth of the flexible spline and the ring gear.

The ring gear and the flexible spline have different numbers of teeth. Their difference is a multiple of the number of lobes of the wave generator. For instance, if a SWG drive using an elliptical wave generator has a flexible spline with 120 teeth, the ring gear may have 122 teeth, 124 teeth, 126 teeth, etc. This is because an elliptical wave generator has two lobes; therefore, the difference in number of teeth between the flexible spline and ring gear must be a multiple of 2. If the wave generator had three lobes, the difference in number of teeth between the flexible spline and ring gear would be a multiple of 3.

The total deflection that the wave generator imposes on the flexible spline is equal to the difference between the pitch diameter of the ring gear and the pitch diameter of the flexible spline. This guarantees that when the teeth of the flexible spline are deflected into meshing
position, their pitch circle along the major axis of the wave generator coincides with the pitch circle of the ring gear. This makes the operation of SWG drives significantly different than that of traditional gear drives. In SWG drives, there are two meshing regions as compared to a single meshing region as in most traditional gear drives [13].

The input or output to the SWG drive can be through any of its members. Besides, since it is a mechanism with two degrees of freedom, one element must be fixed during operation. The most typical arrangements employ a fixed ring gear or flexible spline with input through the wave generator and output though the remaining free element.

![Operation diagram of a SWG drive with (a) fixed ring gear and (b) fixed flexible spline.](image)

Figure 1.3: Operation diagram of a SWG drive with (a) fixed ring gear and (b) fixed flexible spline.

Figure 1.3 a) shows the most common operating layout of SWG drives. The wave generator serves as the input member of the drive, making the flexible spline deflect to its meshing position with the ring gear. The ring gear is fixed in this set up, and the output goes through the closed end of the flexible spline cup. Figure 1.3 b) shows a different operating layout of SWG drives. Here, the wave generator still serves as the input member of the gear drive, but the flexible spline is fixed. The imposed deflection by the wave generator makes the flexible spline teeth mesh with the ring gear that the output goes through. In case a) (Figure 1.3), a single rotation of the wave generator makes the flexible spline rotate in the opposite direction by however many teeth it is different than the ring gear. On the other hand, in case b) (Figure 1.3), a single rotation of the wave generator makes the ring gear rotate in the same direction by however many teeth it is different than the flexible spline.
1.2.3 Material of SWG drives

The material employed in traditional gear drives is typically case carburized steel, similar to AISI 8620 and AISI 9310. This type of steel is used to ensure that the surfaces of the teeth are hard enough to resist the tooth-to-tooth contact pressures while the inside of the teeth is kept soft to allow them to more easily resist bending stresses [16,17].

SWG drives, however, are typically made of stainless steel similar to grade 15-5 or UNS S15500 stainless steel. This is a type of stainless steel also designated as AMS 5659 and intended for aerospace applications [18]. Table 1.1 shows the approximate chemical compositions in percentage of each element of the steels UNS S15500 of SWG drives and AISI 8620 and 9310 of traditional case carburized gear drives for comparison. UNS S15500 employs less carbon than the other steels and, as a stainless steel, it includes a considerably larger amount of chromium. Nickel is also high in both UNS S15500 and AISI 9310, while copper is only present in UNS S15500. Silicon is present in a larger amount in the stainless steel, together with manganese. The latter is also present in the case carburized steels, although in a lesser amount. Certain niobium and tantalum is added to UNS S15500 as opposed to AISI 8620 and AISI 9310. These steels, on the other hand, include some molybdenum. Finally, the phosphorus and sulfur amounts included in the stainless steel of SWG drives are similar to those of the case carburized steels of traditional gear drives, although their amounts are not large in either case.

| Table 1.1: Comparison of chemical composition of steels S15500, AISI 8620, and AISI 9310. |
|---------------------------------|----------------|----------------|----------------|
| Element                        | Percentage per steel |
|                                | S15500 | 8620 | 9310 |
| Carbon                         | C      | 0.07 | 0.18 - 0.23 | 0.08 - 0.13 |
| Chromium                       | Cr     | 14.8 | 0.4 - 0.6 | 1.0 - 1.4 |
| Nickel                         | Ni     | 4.5  | 0.4 - 0.7 | 3.0 - 3.5 |
| Copper                         | Cu     | 3.5  | -         | -         |
| Silicon                        | Si     | 1.0  | 0.15 - 0.35 | 0.15 - 0.30 |
| Manganese                      | Mn     | 1.0  | 0.7 - 0.9 | 0.45 - 0.65 |
| Niobium                        | Nb     | 0.3  | -         | -         |
| Tantalum                       | Ta     | -    | -         | -         |
| Molybdenum                     | Mo     | -    | 0.15 - 0.25 | 0.08 - 0.15 |
| Phosphorus                     | P      | 0.04 | ≤ 0.035   | ≤ 0.025   |
| Sulfur                         | S      | 0.03 | ≤ 0.04    | ≤ 0.025   |

Table 1.2 shows the mechanical properties of the steels UNS S15500, AISI 8620, and AISI 9310
for comparison. The mechanical properties shown are tensile strength, yield strength, modulus of elasticity, and Poisson’s ratio. These properties characterize the strength of materials as well as their deformation characteristics under loading.

**Table 1.2: Mechanical properties comparison of steels S15500, AISI 8620, and AISI 9310.**

<table>
<thead>
<tr>
<th>Property</th>
<th>S15500</th>
<th>8620</th>
<th>9310</th>
<th>[units]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tensile strength</td>
<td>1380</td>
<td>530</td>
<td>1234</td>
<td>MPa</td>
</tr>
<tr>
<td>Yield strength</td>
<td>1275</td>
<td>385</td>
<td>986</td>
<td>MPa</td>
</tr>
<tr>
<td>Modulus of elasticity</td>
<td>196</td>
<td>190</td>
<td>210</td>
<td>GPa</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>0.27 - 0.30</td>
<td>0.33</td>
<td>-</td>
<td></td>
</tr>
</tbody>
</table>

Stainless steel UNS S15500 shows a tensile strength of 1380 MPa, similar to case carburized steel AISI 9310 of 1234 MPa. However, due to its intended use in aerospace applications, the yield strength of UNS S1550 of 1275 MPa is larger than that of AISI 9310 and considerably larger than that of AISI 8620, which is approximately 385 MPa. The higher yield strength of UNS S15500 means it can sustain considerably larger deformations before fracture than traditional case carburized steel gears. The steel AISI 8620 shows the lowest tensile and yield strengths among these steels. However, the moduli of elasticity of these steels are similar and close to 200 GPa, whereas the Poisson’s ratio of AISI 9310 is slightly larger than the similar Poisson’s ratios of UNS S15500 and AISI 8620 [18].

### 1.2.4 Applications of SWG drives

Due to the particular operating characteristics of SWG drives as compared to traditional gear drives, their main applications are:

- **Robots:** any type of industrial, service, assistance robot, etc. benefits from the use of SWG drives because they can be coupled directly to an electric motor, thus providing a large gear reduction in a considerably reduced amount of space. These are the main requirements of robot joints and actuators, so that the joint itself can use the least possible amount of space. An electric motor coupled with a SWG drive provides high torque output with low rotational speed, which is helpful for moving the limbs of the robot and their loads in a slow and controlled manner.
• Aerospace industry: SWG drives are considerably useful in aerospace applications due to producing a huge gear reduction in a limited space and weight. They have very few parts which are relatively small and arranged in a compact mechanism, characteristics that are requirements in aerospace applications in which every gram counts towards the final operation and payload that can be transported.

• Medical equipment: since SWG drives are small mechanisms with considerable accuracy given their lack of backlash, they are optimal devices for actuators in medical equipment requiring large gear reductions. Working with virtually no backlash makes SWG drives more accurate than traditional gear drives, regardless of the direction of rotation.

• Machine tools: similarly to medical equipment applications, machine tools that require large gear reductions with high accuracy and control in a reduced amount of space take advantage of SWG drives. These gear drives considerably reduce the amount of space that a compound gear train would require to achieve the same gear reduction.

To sum up, SWG drives are applicable in any situation that requires high gear reduction with considerable accuracy while being constrained in space and weight.

1.3 Advantages and disadvantages of SWG drives

The main advantages that SWG drives provide are:

• Large number of teeth in contact: depending on the number of lobes of the wave generator, SWG drives develop two or more regions of meshing between the teeth of the flexible spline and those of the ring gear. Apart from having more than just two or three pairs of teeth in contact as a traditional gear drive does, each of these regions of meshing has several pairs of teeth in contact. This is caused by the similitude in number of teeth between the flexible spline and the ring gear. The large number of teeth in contact increases the power density of this type of gear drive as compared to traditional gear drives with similar requirements.
1.3. Advantages and disadvantages of SWG drives

- Large gear reduction: the particular mechanism that a SWG drive constitutes provides this advantage by making the output toothed element rotate just the number of teeth it is different that the other toothed element during a single rotation of the input shaft. The achievable gear ratios with this type of gear drive are from 30 to 320 in a single stage [2], which would require several stages using traditional gear drives [13].

- Small size and light weight: since SWG drives are arranged coaxially and their few elements are inserted one inside the other as a single unit, they require a considerably smaller amount of space and weight than traditional gear drives. This is remarkably important in applications where the allowable space and weight of the transmission are constrained. Traditional gear drives would require considerably more complex arrangements to be used in the same applications a SWG drive would be suitable for.

- Ease of lubrication: this is due to the low sliding velocities and short travelling distances between the teeth of the flexible spline and the ring gear. The contact between the teeth is also developed on one side of each tooth to then switch to the opposite side of the same tooth, which distributes the lubricant around the drive.

- Able to work with virtually no backlash: this does not mean that there is no backlash between a tooth and its mating tooth slot. However, due to the particular mechanism of SWG drives and the remarkably large amount of deflected teeth in contact, the flexible spline element is unable to rotate if the wave generator and the ring gear are fixed. There is no clearance for rotation of the flexible spline due to the fact that along the major axis of the wave generator the right side of the teeth of the flexible spline mesh on their right
side and the left side teeth mesh on their left side with the teeth of the ring gear, as shown in Figure 1.4.

Although considerably advantageous in precise, lightweight, and space constrained applications, SWG drives are also affected by the following disadvantages:

- **Unable to provide low gear reduction ratios**: the gear ratio in SWG drives depends on the number of teeth of the flexible spline and the ring gear. The lower the number of teeth, the lower the gear reduction ratio is. However, low numbers of teeth increase the amount of deflection that has to be applied to the flexible spline. To provide low gear reduction ratios, it would be necessary to manufacture the flexible spline out of a soft material to be able to deflect it, considerably reducing their load-transmission capabilities [19].

- **Complex manufacturing required**: the manufacturing of SWG drives is more complex than traditional gear drives due to the use of a flexible gear. This gear, the flexible spline, is also a thin-rimmed gear required to deflect twice per revolution of the wave generator which forces the flexible spline to be made of a material with larger tensile strength than traditional gear drives in order to improve the fatigue life of SWG drives [20].

- **Wind-up tendency with higher loads**: the flexible spline design process is dependent on the transmitted load by the drive. The rim thickness of the flexible spline has to be large enough to transmit the desired torque. When a SWG drive transmits larger loads, regardless of the final design of its flexible spline, the teeth of the flexible spline may wind-up and jump a tooth slot of the flexible spline due to the large load imposed. This increases the torsional stresses of the flexible spline throughout the meshing process [21].

- **Lower efficiency than traditional gear drives**: the motion of the teeth of the flexible spline during the meshing of a SWG drive consists of each tooth moving into a tooth slot of the ring gear by sliding on one tooth flank to then depart the tooth slot of the ring gear by sliding on the opposite tooth flank. This sliding motion reduces the efficiency of SWG drives as compared to traditional gear drives. The continuous deflection of the flexible spline imposed by the wave generator and its ball bearing reduces the efficiency of SWG drives further. However, in order to provide the same gear ratio as SWG drives, several
1.4. Summary of existing literature

Extensive research has been done to understand the behavior of SWG drives and improve their performance. However, these research efforts mostly started two to three decades ago because, since the release of SWG drives with Musser’s patent in 1959, SWG drives were not

stages of traditional gear drives must be assembled, resulting in a lower overall efficiency than a single SWG drive providing that gear ratio [22].

- Non-linear transmission of torque: the flexibility of the flexible spline makes the load-transmission capabilities of SWG drives dependent on the torsional stiffness of this member. The load transmission of SWG drives also depends on the instantaneous number of pairs of teeth in contact between the flexible spline and the ring gear. This number changes as the wave generator rotates, leading to a variable transmission of torque by the output member of the drive [5].

- Non-constant instantaneous transmission ratio: due to the dependency on the torsional stiffness and flexibility of the flexible spline, SWG drives cannot provide a constant transmission of motion. Their motion transmission slightly varies throughout an input rotation of the wave generator into the drive, similarly to their torque transmission [4].

- Resonance vibration at certain operating conditions: utilizing a flexible element being deflected twice per input rotation leads to resonance vibration at high operational speeds and certain conditions. Resonance vibration is also influenced by the magnitude of the transmitted load [23].

- Low insight knowledge and literature about operating mechanisms of SWG drives: the recent introduction of SWG drives, as well as their prompt utilization in robotic and aerospace applications, has not allowed researchers and gear designers to fully understand their behavior. Compared to traditional gear drives, which have been used for centuries, SWG drives lack a strong body of literature providing insights about their operation to guide their design process.
being utilized until the 1980s and 90s for robotic and aerospace applications. As shown in Figure 1.5, they were not used during the 1960s and 70s given the lack of research efforts immediately after Musser’s patent [1]. In this figure, a summary of the development of SWG drives and their research efforts shows how this particular type of gear drive evolved through history as compared to remarkable historical events during the second half of the 20th century.

Figure 1.5: History-line of research efforts on SWG drives.

Although Musser introduced the concept of “Strain Wave Gearing” with his patent, a Russian researcher obtained a patent for an electric motor that employed a working principle similar to that of SWG drives in 1944, before the end of World War II. This electrical motor uses a thin-walled circular element to deflect by electromagnetic forces towards an outer element and produce a low-speed rotation output [21]. Interestingly, there is still a line of research focused on improving the performance of this so called electrostatic harmonic drive [24], which is beyond the scope of this thesis dissertation.

As shown in Figure 1.5, there was not much research towards developing SWG drives after Musser’s patent until 1968 when Ishikawa obtained a patent for a flexible spline using teeth of a different size than those of the ring gear [25]. In the midst of the Cold War, this patent started a line of research centered on improving the performance of SWG drives by developing their
tooth profiles and improving their geometrical features. This research includes several different publications in which different tooth profiles have been developed for the teeth of the flexible spline and the ring gear. These tooth profiles have mostly been based on the kinematics of the flexible spline element and the locus of movements of the flexible spline due to the imposed deflection by the wave generator. Several geometries have also been proposed for the wave generator, as well as different arrangements of the elements of a SWG drive \[6, 14, 15, 22, 25–38\].

Before the end of the Cold War, during the late 1980s, researchers started to develop kinematic models to ensure the positioning and response of SWG drives was as accurate as required by robots and other applications \[4, 12, 15, 38–41\]. Shortly after, researchers started to put a huge effort towards understanding the torque transmission and measurement mechanisms in SWG drives and their control \[3, 5, 7, 10, 23, 42–51\]. Due to the inherent elasticity of the drive having a flexible gear element, the torque transmission is non-linear. Other mechanical properties, such as compliance, structural damping, Coulomb friction, and deflections that have to be accounted for during the operation of SWG drives are also non-linear. In fact, torque sensors and control has been one of the main research efforts of this type of gear drive. For this reason, in the 1990s researchers started to develop dynamic models to explain the highly non-linear dynamic behavior of SWG drives \[8, 9, 11, 35, 43, 44, 47, 52–54\].

Before the change of century, some research was done about utilizing composite materials instead of steel for improvement of SWG drives \[55, 56\]. Composites were proposed for the flexible spline to take advantage of the freedom that using them provided in terms of mechanical properties in different spatial directions, as well as weight reduction. This allowed the authors to obtain a composite flexible spline able to transmit as much torque as a steel one. However, this research effort has not been continued due to the price difference and the lack of suitable results using composite materials as opposed to steel-made flexible splines.

In 2007, before the financial crisis, researchers started to develop finite element models and perform stress analysis on SWG drives. This research effort, however, is considerably limited as compared to other types of gear drives. This has not allowed the experts to fully comprehend the behavior of SWG drives due to the lack of a strong body of literature about them \[20, 30, 31, 36–38, 57–59\]. Besides, researchers have just started to study how the meshing
develops between the flexible spline and ring gear with techniques such as computer vision and simulation of meshing [59, 60].

Due to the short existence of SWG drives, there are still gaps in the existing literature and the comprehension of their behavior. These gaps motivate this thesis dissertation, which aims to develop design tools for design of SWG drives, as well as expand their understanding and available literature.

1.5 Societal context

"In 2017, (industrial) robot sales increased by 30 percent [...] a new peak for the fifth year in a row." [61] This is the starting line of the Executive Summary from the World Robotics 2018 report of the International Federation of Robotics (IFR) about industrial robots. The metal industry experienced the largest increase in robot sales, with a 55 percent rise, while the electronics industry rose by 33 percent and the automotive industry by 22 percent. This occurred due to the current trend towards automation and improving the manufacturing sector using innovative solutions. This ties into the fact that the stock market of industrial robots grew 15 percent in 2017, while the global market rose by 21 percent, or US 16.2 billion dollars. The estimated value of the global market of industrial robots was US 48 billion dollars. Additionally, robot sales were estimated to rise 10 percent during 2018.

As the summary states, there is a "tremendous, accelerating rise in demand for industrial robots worldwide." [61] This tendency is dominated by five countries owning 73 percent of the global sales of industrial robots. From largest to lowest sales, they are: China, Japan, the Republic of Korea, the United States, and Germany. Other countries are also increasing their investment in industrial robot deployment.

The manufacturing sector with the highest market share of industrial robots was the automotive industry. However, the electronics industry has been catching up and in 2017 each of these industries took approximately 32 percent of the total supply of industrial robots. The number of industrial robot installations in the remaining manufacturing industries keeps growing and the summary emphasizes: [61] "(The) potential for robot installations in the general industry is tremendous."
1.5. Societal context

The main advantage that using industrial robots provides over human beings is safety. Industrial robots are capable of working continuously in significantly worse conditions than an employee should ever be allowed to. For this reason, it is beneficial to improve the production processes of industrial robots so that more reliable and cost efficient designs can be developed and deployed in the field. Although this might reduce the need for operators in manufacturing cells, these operators can switch tasks to work on maintenance, programming, or improvement of the robots themselves; or they could focus on improving the company’s processes. Operators could spend more time training and adding value to themselves and the company while the robots perform the dirty work.

Each industrial robot has several joints that allow a section of the robot to rotate around a particular axis. Each joint has an electric motor as source of power to perform the rotation. However, the electric motors can be connected to the joint in different ways. The most simplistic transmission solution is to employ a direct drive from the electric motor to the joint. This directly transmits the output velocity from the electric motor to the joint without changing speed or torque. However, this is not the most efficient solution due to the fact that most electric motors operate more efficiently at specific high rotational speeds. For this reason, it is usually preferable to employ a reduction stage between the electric motor and the robot joint. The most common transmission solution is to employ a SWG drive or a cycloidal drive [28,30,38,51,57,58]. Both of these types of gear drive provide a large gear reduction in a reduced amount of space while maintaining a reasonable positional accuracy.

Although SWG drives were patented in the late 1950s, their operation has not been fully understood yet. This is because they employ a flexible toothed element as opposed to traditional gear drives, in which every toothed element is considered solid. Besides, SWG drives were not in use much until the end of the 20th century, which justifies the lack of research and literature about them. Most research efforts have been focused on developing kinematic models to increase their positional accuracy and torque measuring and carrying capacity [3–5,7,10,39,42–46,51]. However, the geometry of the different elements of a SWG drive and the applicability of micro-geometry modifications to their tooth geometries have not been fully studied.

The rising trend in industrial robot installations is seen within basically all manufacturing
sectors, which indicates the need to develop better tools for their design. Industrial robots will eventually be a significant element in the manufacturing of everything surrounding humanity. Consequently, their societal impact is remarkably large and the value of this thesis resides on improving the design capabilities of critical elements in industrial robots. These elements are the SWG drives installed on robot joints, and the aim is to increase their power density by modifying their geometry to achieve huge reduction ratios with precise motion control in limited space [6, 15, 29, 30, 33, 36–38, 57].

1.6 Research question and objective

The research question that this thesis addresses is: "What are the most suitable geometries of the wave generator, tooth profiles for the teeth of the flexible spline and the ring gear, and micro-geometry modifications for SWG drives to achieve their highest power density?" For this reason, the main objective is to develop suitable tools for design of SWG drives, with the ultimate goal to make them achieve their highest power density, as well as understand which elements of SWG drives are the most influential towards enhancing their mechanical performance.

In order to achieve the research objective, the following supporting actions are proposed:

- Develop a two-dimensional finite element model of SWG drives to perform simulation of meshing and stress analysis on this type of gear drive. This planar model will be based on the mid-section of the face width of the different elements of the modelled drive.

- Obtain the distribution of stresses during the meshing process of the different elements of a SWG drive, which serves to study how changes in the parameters of the design affect its strength.

- Implement the two-dimensional model in a custom-made virtual gear generator software to automatically generate the two-dimensional finite element models from the input parameters of the SWG drive. This would significantly increase the possibilities of analysis of all the influential parameters in their behavior.

- Verify and improve the geometrical features of each of the elements of a SWG drive in a planar section. These features include but are not limited to:
1.7 Summary of the proposed approach

- External geometry of the wave generator
- Rim thickness of flexible spline and ring gear
- Tooth profiles of the ring gear and the flexible spline

- Develop a three-dimensional finite element model of SWG drives to perform simulation of meshing and stress analysis on this type of gear drive. The main difference with the previous model is the addition of the cup-shaped spring of the flexible spline.

- Obtain the loaded distribution of both contact and bending stresses and implement the three-dimensional model in a custom-made virtual gear generator software to automatically generate the three-dimensional finite element models.

- Verify and improve the geometrical features of each of the elements of a SWG drive, considering the effect of the flexible spline cup and micro-geometry modifications of the gear tooth surfaces and the wave generator.

The main benefit of developing a three-dimensional model as a final tool for analysis of SWG drives is that it allows micro-geometry modifications of the gear tooth surfaces to be used to compensate the deflection imposed on the flexible spline cup along the longitudinal direction of the flexible spline teeth. The two-dimensional model only studies a planar section of the teeth.

1.7 Summary of the proposed approach

The proposed approach consists of developing the necessary tools to facilitate the design of SWG drives for the highest achievable power density. These tools permit the analysis of several geometrical features of SWG drives. Figure 1.6 shows the computational design algorithm of gear drives using a custom-made virtual gear generator software and a finite element solver software to evaluate their mechanical behavior. Firstly, the virtual gear generator is used to generate the gear geometries based on their design parameters and manufacturing processes. Then, it generates finite element models to evaluate the strength of the gears. The finite element solver provides the solution of these finite element models, which are post-processed using
the gear generator. The proposed tools of this thesis consist of two finite element models for analysis of SWG drives in two and three dimensions.

The first tool to be developed is a parameterized two-dimensional finite element model for plane stress analysis and simulation of meshing of a planar section of the teeth of a SWG drive considering the wave generator as a rigid element. The different geometries that can be improved with this model include different tooth profile geometries, rim thicknesses of the flexible spline and ring gear, as well as several different geometries of the wave generator.
The initial state of the model considers the flexible spline in its non-meshing position. Then, the model incorporates a simplified version of the particular wave generator geometry to deflect the flexible spline into contact with the ring gear. This allows the model to deflect the flexible spline into contact using the same geometry that produces strain waves while meshing. This model can be used to improve each geometrical feature of a SWG drive planar section, which helps towards understanding the contact problem in this type of gear drive.

The second and final tool to be developed is a three-dimensional finite element model which is used to analyze and improve different wave generator geometries, tooth profiles, and micro-geometry modifications. The main advantage of this model is that it includes the full geometry of the flexible spline, which incorporates the cup-shaped spring, as shown in Figure 1.7. This causes the deflection imposed on the teeth of the flexible spline to be different on their front and back transverse sections.

These tools help to design and improve each and every one of the geometrical features of a SWG drive and determine which design parameters and micro-geometry modifications are more influential towards achieving the highest possible power density during their operation.
Chapter 2

State of the Art

2.1 Chapter overview

The second chapter of this thesis provides the background concepts related to the development of SWG drives since their invention, as well as an in-depth literature review to show their current state of the art. Section 2.2 lays out key concepts and findings that Musser, the inventor of SWG drives, already disclosed in his patent. Within Section 2.3, the review of the existing literature begins with the research efforts focused on the development of tooth profiles and different geometries of SWG drives. In Section 2.4, the kinematic models developed to understand the motion of the different parts of a SWG drive are explained, as is how they initiated the torque measuring and mechanisms, as shown in Section 2.5. Later, Section 2.6 explains the dynamic models used to predict the loaded behavior of this type of gear drive. Section 2.7 shows the research efforts related to finite element models and analysis of SWG drives, due to their close relationship with the topic of this thesis. Finally, the limited research performed toward the study of meshing of SWG drives is exposed in Section 2.8.

2.2 SWG drives

When Musser introduced the concept of "Strain Wave Gearing" in 1959, he proposed the use of SWG drives with involute tooth profile for the teeth of the flexible spline and the ring gear, but using a slightly larger pressure angle on the flexible spline teeth to account for their
deformation when deflected into meshing [1]. This also avoids interference in the undeformed position between the flexible spline and the ring gear teeth.

Musser proposed controlling the backlash by adjusting the deflection imposed by the wave generator to produce the meshing between the teeth of the flexible spline and the ring gear. The deflection $d$ is defined by the following formula,

$$
   d = m(N_{rg} - N_{fs})
$$

(2.1)

where $m$ is the module of the toothed elements of the drive, $N_i$ represents number of teeth, and the subscripts $rg$ and $fs$ refer to the ring gear and the flexible spline, respectively. The applied deflection makes the flexible spline pitch circle deflect and coincide with the pitch circle of the ring gear in the areas around the major axis of the wave generator. This deflection constitutes the height of the strain wave that the wave generator imposes in the SWG drive [1].

Musser also defined the circular pitch of a SWG drive as a function of the deflection $d$ and the number of lobes $n$ of the wave generator. He determined that there was no need to use wave generators with more than three lobes due to the correspondingly larger deflections needed to deform the flexible spline into meshing position [1].

For the two arrangements of a SWG drive shown in Figure 1.3, the gear ratio $m_G$ of the SWG drive is given by the following formula,

$$
   m_G = \frac{N_D}{N_D - N_F}
$$

(2.2)

where $N_D$ and $N_F$ represent the number of teeth of the driven element and the fixed element of the SWG drive, respectively. For the case of a flexible spline with 120 teeth and a ring gear with 122, the gear ratio is -60 with the ring gear fixed and 61 with the flexible spline fixed. The toothed element with the largest number of teeth rotates in the same direction as the wave generator, whereas the toothed element with the lowest number of teeth rotates in the opposite direction than the wave generator [1].

Musser realized that the gear ratio is a function of the deflection $d$ in SWG drives and that the gear ratio is independent of the tooth profile geometry. Considering the same diameter of
the drive, the smaller the deflection $d$, the higher the gear ratio. Musser stated that gear ratios from 10 to 1 million can be achieved using SWG drives. However, lower gear ratios require the use of materials with a lower module of elasticity than steel, such as nylon for ratios between 25 and 75 and rubber or neoprene for ratios between 10 to 25. On the other hand, gear ratios beyond 200 require the use of ball or roller type wave generators [1].

The number of teeth in contact is influenced by the tooth profile, dimensions of the SWG drive, and difference in number of teeth of the flexible spline and ring gear. The number of teeth in contact decreases when reducing the height of the teeth and the pressure angle. Musser defined the contact ratio as the percentage of teeth in contact [1]. Due to the fact that this percentage can be close to fifty percent in SWG drives, manufacturing inaccuracies get corrected as consecutive teeth wear out during operation.

Musser realized that contact pressure, shear and bending stresses in the flexible spline, and the transmitted load constitute the most influential characteristics towards the behavior of SWG drives. He developed formulas for strength calculation and determined that the highest stresses were experienced by the flexible spline near the major axis of the wave generator. In his patent, Musser also disclosed several different layouts and arrangements of single or multiple SWG drives [1]. Due to their small size and weight, heat diffusion constitutes a limiting characteristic in SWG drives, sometimes requiring forced cooling for continuous operation.

Musser performed several experiments where the pressure angle was selected to avoid interference between the teeth of the flexible spline and the ring gear. The experiments showed no backlash on SWG drives and the noise produced by the drives was lower than the motor noise. Testing showed that the efficiency of SWG drives can be between 69 to 96 percent, depending on manufacturing accuracy and friction between the different elements. Additionally, wear of the SWG drives was insignificant given that the kinematics of SWG drives help distribute lubricant to all the teeth.
2.3 Tooth profiles and geometries

When designing any type of gear set, the geometry of different features and the particular tooth profile employed on its tooth flanks are critical parameters for its performance. In cylindrical gears, the most used tooth profile is the involute tooth profile due to its advantages [13]. However, the involute tooth profile is not the most suitable in SWG drives [38]. In order to avoid interference between the tip of the teeth of the flexible spline and those of the ring gear, the pressure angle must be different for the teeth of the flexible spline and the ring gear. For this reason, Musser proposed the use of a slightly larger pressure angle for the teeth of the flexible spline.

Alternatively, Ishikawa proposed the modification of the circular pitch of the flexible spline teeth [25]. If the circular pitch is increased, the contact occurs in the vicinity of the major axis of the wave generator. However, if the circular pitch of the flexible spline teeth is equal to the circular pitch of the ring gear, the contact will be found in pairs of teeth further from the major axis. Finally, if the circular pitch of the flexible spline is smaller than that of the ring gear, the contact will be found in still further pairs of teeth from the major axis of the wave generator.

In 1970, Carlson and Robinson [14] obtained a patent for actuators using SWG drives with preloaded wave generators. This was a variation of the original elliptical wave generator as proposed by Musser [1]. The preloaded wave generator consisted of four rollers that deflect the inner diameter of the flexible spline towards its meshing position with the ring gear. The objective was to ensure many pairs of teeth were meshing with no backlash around the circumference of the drive.

Brighton obtained a patent in 1976 for a particular type of SWG drive which, instead of using the involute tooth profile, employed a modified profile with a reduced pressure angle and a remarkably large tip relief for either or both of the flexible spline and ring gear teeth, as shown in Figure 2.1. Furthermore, this type of SWG drive used a crank shaft with eccentricity to locate three disks that constitute the wave generator [26]. The tip relief was applied so that the region between the pitch circle and the top land of the flexible spline was a smooth continuous circular arc. Brighton’s motivation was to increase efficiency and fatigue behavior in SWG drives to make them usable in gas turbines for helicopters or other aircraft. This approach was able to
reduce the reaction stresses on the flexible spline to less than half of those obtained using the involute profile. Besides, Brighton was able to reduce torque stresses on the flexible spline to a third of the involute profile torque stresses and enlarge the meshing region and number of teeth in contact between the flexible spline and the ring gear. He also proposed reducing the tooth depth on the flexible spline by flattening the top land after the rounding tip relief is applied, which significantly increases the clearance on the non-deflected region of the flexible spline.

Ishikawa also proposed the use of a different tooth profile [22, 27]. This profile consists of two circular arcs with opposite curvature that join at the pitch point, as shown in Figure 2.2. It provides a continuous transmission of movement and strengthens the teeth of both flexible
spline and ring gear. This is because the transition from root to flank follows a smooth curve on the circular arc’s profile. On the other hand, the traditional spur profile has a sharp indent in the transition between the flank to the root of the teeth, promoting a higher concentration of stresses in this area than that of Ishikawa’s proposed tooth profile.

Kondo and Takada published an article about tooth profiles in SWG drives [6]. They analyzed the kinematics of their meshing process. Their main assumptions were considering that the teeth are rigid and that there exists a neutral line in the flexible spline where the pitch between teeth is constant throughout the meshing process. Besides, they assumed that there is no friction between wave generator and flexible spline in a SWG drive. The article shows that the use of profiles developed by experience motivated the development of methods to optimize the tooth profile used in SWG drives. The authors employed the meshing equation to generate tooth profiles using a given generating profile. They realized that their proposed neutral line is located around the central area of the flexible spline rim. They generated three tooth profiles by rolling contact plates, pin wheel, and ring gear with involute teeth.

The difference between the number of teeth of the flexible spline and ring gear was proposed by Musser as a multiple of the number of lobes of the wave generator [1]. However, Maiti and Roy published an article focused on the reduction of the difference between number of teeth in gear drives which require interactions between external and internal gears, such as epicyclic gear drives and SWG drives [28]. They were trying to reduce the amount of tip interference in these drives when using the involute tooth profile. For epicyclic gear drives, the minimum tooth difference was found to be five teeth by applying addendum and center distance modifications. For the case of SWG drives, they realized that the tooth difference could not be lower than two teeth to avoid interference, but required the use of addendum coefficients smaller than one.

Maiti proposed a new type of wave generator to be used with conjugate involute tooth profiles [29]. The outer profile of this wave generator consists of two circular arcs, symmetrically opposite, that form the two lobes of a traditional elliptical wave generator. The results show that the root stresses using this wave generator and conjugated involute profiles are lower than those of commercially available SWG drives. However, the author realized that in order to obtain a SWG drive with no backlash, traditional design methodologies must be employed.
Kayabasi and Erzincanli performed a shape optimization study on SWG drives using finite element analysis [30]. Their objective was to calculate the stresses on the flexible spline teeth and obtain an optimal involute profile. They realized that numerical analysis by finite elements results in better estimates of stresses than experimental methods. When load is applied, different regions of the flexible spline deflect following particular geometric curves. Besides, the stress distribution along the flexible spline circumference is not continuous. The highest stresses on the flexible spline are located in the following quadrant towards where the wave generator is rotating.

Dong and Wang studied the elastic deformation produced in flexible splines due to the shape of the wave generator and the transmitted load [31]. Their method represents the elastic deformation of the cup-shaped flexible spline under the assumptions that the closed-end of the cup does not experience any deformation and that the deformation of the neutral surface of the flexible spline can be defined by neutral curves perpendicular to the axis of rotation. They defined three deformations on the flexible spline: circumferential, radial, and angular displacements. Dong and Wang studied the deformation imposed by three different wave generator geometries, such as the traditional elliptical wave generator, a disk type wave generator, and a wave generator with four rollers, similar to the wave generator proposed in [14]. Figure 2.3 shows their studied wave generator geometries. When load is applied, the deformation turns to be non-linear and requires the application of finite element analysis. For this, the authors only used the traditional elliptical cam wave generator.

The results of their research under no load showed that the radial displacements obtained by
finite element analysis in the middle section of the face width of the flexible spline coincide with those obtained by the authors’ analytical model. The shape of the neutral surface of the flexible spline coincides with that of the wave generator in the middle section of the flexible spline teeth. They realized that the deformation imposed on the flexible spline is linear and only a function of the geometry of the wave generator, when no load is applied. However, the results under load showed that the flexible spline cup distorts non-linearly. The flexible spline buckles proportionally to the amount of load transmitted by the SWG drive, which means that the elastic deformation of the flexible spline is not only a function of the shape of the wave generator employed, but also a function of the applied load. This deformation can be approximated using double cubic spline functions.

Chen et al. published an article about the parameterized design of SWG drives with tooth profiles consisting of two circular arcs joined by a straight line, as shown in Figure 2.4 [33]. The authors also studied the effect of the tooth profile on backlash. They determined that this profile provided wide meshing regions with several pairs of teeth in contact, as well as uniform backlash. The authors defined the parameters that generate the profile and a different profile using a conjugate of their proposed tooth profile. These were used to generate a finite element model to compute backlash, displacement, and rotational angle of the flexible spline.

Regarding other tooth profiles, Stadtfeld and Saewe published an article which compared
2.3. Tooth profiles and geometries

different non-conventional tooth profiles based on profile design and optimization, manufacturability, accuracy, and performance [34]. They studied several tooth profiles employed in different types of gear drives, such as cycloidal, Wildhaber-Novikov, convoloid, S-shaped, toroidal, globoidal, cosine, and asymmetric tooth profiles. In SWG drives, the convoloid tooth profile has been proposed as an alternative by their developers, Berlinger and Colbourne [32]. It consists of a convex circular arc addendum region which joins with a concave circular arc dedendum region by a straight section. It provides some of the same advantages offered by the involute profile in cylindrical gears, such as constant gear ratio and resistance to gearbox deflections. However, this geometry is unable to handle center distance changes and is vulnerable around the pitch region. Due to the complexity of the final tooth profile, its manufacturing requires the manufacturing of the cutting tool based on the kinematic relationship with the geometry of the desired convoloid tooth profile geometry. This makes the convoloid profile significantly more expensive to manufacture than the involute profile.

![Figure 2.5: Caterpillar effect concept by Gravagno et al. [15].](image)

Gravagno et al. published an article explaining how the wave generator geometry influences the kinematic error output of SWG drives [15]. The article studied six different wave generator geometries based on the geometry used for a disk type wave generator: an ellipsoidal shape,
two conjugate arc shapes, cycloidal curves, and 3-4-5 and 8th degree polynomial curves. The article shows the kinematic error produced by each different wave generator geometry. It introduces the concept of "caterpillar effect," where the major axis of the wave generator creates a region of full deflection of the flexible spline. This region allows the teeth of the flexible spline to fully mesh with the teeth of the ring gear. It is a region where the pitch of several teeth of the flexible spline coincide with the pitch of the teeth of the ring gear, as shown in Figure 2.5. As this region increases, the contact ratio increases, allowing a higher load to be shared by a higher number of pairs of teeth in contact. However, this also increases the bending stresses the flexible spline is subjected to. The article also studies the distribution of centrodes in a SWG drive by considering the movement of a tooth of the flexible spline to the fixed coordinate system of the gear drive.

The article by Gravagno et al. is interesting for studying the influence of the wave generator profile in the kinematic error. However, it should be extended to also study this parameter’s influence over stresses during the meshing of SWG drives. This requires the development of an at least two-dimensional model to compare the effect of each possible wave generator profile over the stresses on the flexible spline while meshing with the ring gear. The "caterpillar effect" region provides an interesting design feature towards increasing the power density of SWG drives. The largest region for the lowest stresses should be found so that the largest number of pairs of teeth in contact share the load in an application.

Zou et al. proposed a model to analyze SWG drives considering different geometries and internal interactions that take place during their operation [35]. They considered the geometry of a tooth pair in contact to compute the stiffness, the torque generated due to friction, and the kinematic error of the gear drive. They realized that the cup-shaped flexible spline cannot be homogeneously deformed by the wave generator. The sections of the flexible spline teeth closer to the closed-end of the cup will be deflected by the wave generator fully in contact with the inner surface of the flexible spline. On the other hand, the sections of the flexible spline teeth near the open-end of the cup will be deflected beyond the nominal deflection, causing the teeth to be deeply inserted in the tooth slots of the ring gear. This phenomenon justifies the development of a three-dimensional model for analyzing SWG drives.
Sahoo and Maiti studied the load sharing capabilities of the teeth of SWG drives [36, 37]. They used SWG drives employing the involute tooth profile and verified their experimental results with a three-dimensional finite element model. The authors realized that the stress distribution on the flexible spline is concentrated in the quadrants towards where the wave generator is rotating when load is applied. Additionally, the alternate quadrants from where the wave generator is coming experience no stresses when load is applied. On the other hand, when load is not applied, the stresses on the flexible spline are evenly distributed around its circumference. Sahoo and Maiti utilized different profile shift coefficients for the flexible spline teeth so that the number of pairs of teeth in contact sharing the load can be controlled. They also performed strain analysis with strain gauges located outside the cylindrical section of the flexible spline cup.

Pacana et al. used finite element analysis to study the influence of several geometries of the wave generator on the stresses of the flexible spline [58]. The geometries of the wave generator

**Figure 2.6:** Wave generator geometries studied by Pacana et al. [58].
studied were the two-roller, four-roller, cam, and disk wave generators, as shown in Figure 2.6. The authors realized that the most suitable wave generator geometry to minimize the stresses in the flexible spline was the elliptical cam type wave generator, originally proposed by Musser [1].

In 2018, Yu and Ting published an AGMA technical paper employing finite element analysis to design and modify the SWG tooth surfaces to obtain an optimized profile [38]. The authors employed a kinematic model to obtain conjugate tooth surfaces for the ring gear teeth from a double circular arc tooth profile on the flexible spline teeth. This was done to guarantee that both tooth profiles will be in tangency at all times during the meshing cycle. They used a plane stress model to verify the conjugation of the tooth profiles, which satisfied their assumptions. The article shows how the bending stresses increase in the flexible spline when it is deflected into contact, although there is no load applied. It also demonstrates how some teeth pairs of a SWG drive have no backlash by having contact with both flanks at the same time.

The flexible spline cup behaves as a three-dimensional compliant mechanism, whose deflection the authors divided in three components, one in the longitudinal direction, one in the circumferential direction, and one oscillating angular deflection. The authors’ analytical model assumes that the deflection of the flexible spline depends solely on the wave generator geometry without application of load. The authors used finite elements to compute the deformations when load is applied. The three-dimensional deformation of the flexible spline cup causes different deflections along its longitudinal axis. Therefore, the design parameters of the SWG drive may not be similarly applicable to the heel or the toe of its teeth.

The authors found four contact regions between the wave generator and the flexible spline. This is because the major axis of the wave generator contacts the inside of the flexible spline teeth underneath the heel, near the closed end of the flexible spline and the minor axis of the wave generator contacts the inside of the flexible spline teeth underneath the toe, near the open end of the flexible spline. Consequently, every cross section of the flexible spline teeth has a different geometry along its longitudinal direction. To overcome this phenomenon, the authors proposed the use of longitudinal micro-geometry modifications, such as crowning [13]. The authors also performed loaded tooth contact analysis to obtain the contact patterns when load is applied. The results showed that the contact pattern in SWG drives with longitudinal crowning
spreads over three different regions of the teeth. The main contact region, where the highest contact pressure is located, lies in the center of the face width. The second region has the lowest contact pressure and is located closer to the heel in the quadrant from where the wave generator is rotating. The third region has edge contact in the toe of the flexible spline due to its buckling when load is applied. It is located ahead of the rotation of the wave generator. The authors performed testing by running a SWG drive for several hours and verifying that the same contact regions develop as those obtained by tooth contact analysis. Their future work focuses on designing a wave generator that guarantees that the contact is going to be homogeneous with the flexible spline, verifying their results using Hertz theory, and studying different load and crowning scenarios.

### 2.4 Kinematic models

When the utilization of SWG drives started to spread, they found applications in situations where weight and space were constrains although high-performance was still required. Researchers then had to develop kinematic models to understand and correct the positional error output produced in SWG drives. The positional error is the difference between the theoretical output position and the actual output position of the output element of the drive [1]. The theoretical output position is computed with the following formula,

\[
\theta_o = \frac{\theta_i}{m_G}
\]

where \(\theta_i\) is the input angular position of the wave generator, \(\theta_o\) is the angular position of the output element of the drive, and \(m_G\) is the gear ratio of the drive as previously defined. The actual output position is measured in the output element of the particular SWG drive under study. The positional error has been a major concern for researchers working on achieving an accurate positioning method for SWG drives. Although SWG drives operate with virtually no backlash, they still suffer from some kinematical error, which leads to missed locations of the output.
In 1988, Hsia developed a kinematic model for analysis of SWG drives and provided design guidelines for robotic arms [4]. The motivation was to increase the positional accuracy of this type of gear drive operating at high speeds and to establish a kinematic foundation for a future dynamic model. The kinematic model takes into account the geometry of the wave generator and computes the actual output angular position as a function of the input angular position, assuming pure rolling between the flexible spline and the ring gear. This serves to compute the real gear ratio for each input position and compare it to the theoretical gear ratio, as shown in Figure 2.7. When the positional error between the theoretical output and the actual output is caused by the particular geometry of the drive, it is called structural error, which also exists as a function of the input position.

The kinematic model represents the relative motion of the flexible spline and the ring gear as the pure rolling rotation of the deformed pitch circle of the flexible spline around the inside of
the fixed pitch circle of the ring gear. Using an elliptical wave generator, the ratio $a/b$ between
the length of the major axis $a$ and the length of the minor axis $b$ of the ellipse constitutes the most
influential parameter towards positional error. Hsia realized that the peak-to-peak position
error decreases and the actual gear ratio of the drive increases when the ratio $a/b$ decreases.
The author also realized that there are certain input positions of the wave generator which
produce no positional error. The model can incorporate any wave generator geometry, as well
as take into account manufacturing and assembly deviations.

In 1991, Nye and Kraml performed a study to characterize and compensate positional error
in SWG drives [7]. They experimented with several drives of different sizes by varying working
conditions and measuring the difference between the angular position output and the theoreti-
cal position. The authors realized that SWG drives produce a considerably high first harmonic.
However, even with similar drives, the higher harmonics vary considerably. They also realized
that positional error reduces as the size of the drive increases. Positional error is slightly affected
by use over time and temperature. On the other hand, assembly positioning has to be accurate
to reduce the errors. The authors proposed measuring the output position and modifying the
input position to compensate for positional error.

Nye and Kraml also studied the effect of ratcheting and having a SWG drive working in a
dedoidaled condition. Ratcheting consists of a shift of the flexible spline and wave generator of
one or more teeth due to sudden high loading of the drive. This causes the parts of the drive to
not be aligned and concentric, which is called dedoidaled condition. This is mostly found due
to incorrect assembly and it reduces the wear life of the drive due to variations in initial friction
torque. However, this does not significantly affect positional error.

Ghorbel et al. studied the kinematic behavior of SWG drives [39]. The focus was on un-
derstanding the periodic kinematic error that SWG drives experience, which leads to lower
performance over time. The authors obtained experimental and analytical results showing that
the kinematic error consists of two components, a kinematic error component and a torque flex-
ibility component, due to the flexible spline cup elasticity. These errors produce a source of
vibration, dominant in higher speeds, that constitute an energy sink on a SWG drive. Besides,
the vibration produced by seemingly identical SWG drives is different.
In 2011, Dong et al. published an article in which the authors explained the kinematics of a SWG drive plane section [40]. The objective was to develop a kinematic model for SWG drives to represent the motion of a tooth of the flexible spline. Using conjugated tooth profiles in the flexible spline and ring gear, the authors considered that the deformation of the flexible spline was only geometrically produced by the wave generator. The kinematic model operates in the no load region and a disk type wave generator was considered. They modeled the motion of a tooth of the flexible spline due to the wave generator as a cam-follower mechanism, where the follower translates and oscillates at the same time. The authors realized that the displacement of a tooth is caused by the geometry of the wave generator and the gear ratio of the SWG drive. They also discovered that the teeth of the flexible spline experience different instantaneous gear ratios throughout the meshing process.

Later that year, Dong et al. also published about the kinematic effect of the flexible spline cup over the SWG drive operation [41]. They assumed that the teeth of the flexible spline are rigid, there is a neutral layer which represents the deformation of the flexible spline, and the closed-end of the flexible spline cup does not deform. They realized that the tooth profile of a flexible spline must be generated considering several characteristics of SWG drives, such as the deformation of the flexible spline, the behavior of the flexible spline cup, the conjugation with the teeth of the ring gear, and the applied load.

Pennestri and Valentini published an article about the kinematics of combined SWG drive mechanisms, which consist of SWG drives combined with traditional gears [12]. Throughout their article, the authors analyzed the kinematics of several different combinations of SWG drives and traditional gears by using a graphical method. They also developed an enumeration system to classify differently combined SWG drives.

In 2016, Gravagno et al. published an article about the influence that particular wave generator geometries have on the kinematic error of SWG drives [15]. The authors developed a kinematic model which computes the output kinematic error taking into account the geometry of the wave generator. Later, Yu and Ting used a planar kinematic model to develop a particular geometry for the teeth of the flexible spline and the ring gear in order to avoid interference between them [38], as shown in Figure 2.8.
2.5 Torque measurement and control

Torque measurement and control constitutes a major concern in the operation of SWG drives. This is because their behavior encompasses several non-linearities, causing the torque output to be variable throughout the meshing process. These non-linearities are sudden variations of torque output produced by the mechanical imperfections and dimensional inaccuracies of a SWG drive. Given the need for accurate control and the ability to determine the output torque of a SWG drive, researchers started to develop different torque measuring systems which focused on the use of strain gauges to take advantage of the inherent elasticity of the flexible spline cup. Besides, these systems intend to compensate for the torque ripples and improve the control of SWG drives.

In 1989, Hashimoto proposed the use of a torque sensor to provide robust, real-time motion control of robot actuators using SWG drives [5]. The motivation came from the fact that the dynamic behavior of SWG drives constitutes a non-linear time-dependent system when operating at high loads or speeds. Therefore, using a real time torque feedback system avoids computation of the torque output in real time. This system proved useful to measure the output torque of SWG drives with stability and accuracy, as well as being consistent despite external disturbances.

Hogan et al. published an article about motion control of SWG drives by impedance control [42]. The objective was to improve the control system of a robotic actuator in NASA’s space
station. Due to internal friction, SWG drives are not easily driven on the coast direction for which impedance control is proposed. Using a simplified model of the actuator, the open-loop response showed the dominant effect of the friction in this type of gear drive. The closed-loop response reduces the non-linear behavior caused by friction using torque feedback. The proposed control system improved the resolution of the previous system but requires dampening of resonant vibrations when the environment is taken into account.

Hashimoto et al. proposed the use of four strain gauges forming a Wheatstone bridge on the closed-end of the flexible spline cup to measure torque output in real time [43], as shown in Figure 2.9. These gauges measure the radial deflection of the flexible spline to control the output of the drive. The authors used finite elements to verify the most suitable locations for the strain gauges to accurately measure the strains caused by the torque output of the drive. Their specific locations are sensitive to circumferential deviations but provide the same response as conventional methods. The motivation for this method came from the widespread use of SWG drives as speed reducers in robot actuators, as well as the development of a dynamic model. With this method, the authors realized that the larger the flexible spline is, the least influenced it is by the torque applied, which proved useful in a closed-loop control system. However, the results were not acceptable for an open-loop control system.

Kosse published a paper about the phase angle change that occurs during the operation of SWG drives [10]. The paper shows a study of the change in phase angle between the input from the wave generator to the meshing area between the flexible spline and the ring gear. This is
caused by the transmission of torque in the drive. The results show that there is a limit in the phase angle change that will not vary with further increase on the input torque.

Hashimoto and Kiyosawa proposed the use of a built-in torque sensor to control the torque output of a SWG drive [44]. The authors installed several strain gauges on the closed end of the flexible spline cup to compute the torque in real time during the meshing process, taking advantage of the inherent elasticity of the cup. Then, they developed a dynamic model of the system and applied three different types of controllers based on either torque, friction, or disturbance. Through experiments, the authors realized that the loss of torque due to stiffening/hysteresis could reach 25 percent of the rated torque of a SWG drive. They also studied the effect of resonant vibration with the torque sensor attached. They were able to obtain accurate measurements of torque during operation, which significantly improved torque control, trajectory tracking, and vibrational behavior.

![Strain gauges location by Taghirad and Belanger [45].](image)

Later, Taghirad and Belanger employed the method proposed by Hashimoto et al. to correct the output torque variation in SWG drives [23, 45]. They installed four Rosette gauges on the closed end of a flexible spline cup to measure the output torque of a SWG drive in real
time, as shown in Figure 2.10. Hashimoto’s approach showed to be ineffective when misalignments were present in the gauges’ installation. Taghirad and Belanger proposed the use of a thin transparent film template to install the gauges with the most accuracy, which reduced their misalignments to a minimum. This approach allowed the torque measuring to be linear and independent of the rotation of the flexible spline. The authors wanted to reduce the high frequency torque oscillations generated during the meshing of SWG drives. These are called torque ripples and they originate from friction during the meshing process. The use of the fourth-order harmonic oscillator error model proved to be capable of characterizing the torque ripples. The latter were then corrected using a Kalman filter, which proved to be accurate, rapid, and capable of correcting the torque output component due to misalignments of the elements of the drive, which has a sinusoidal signature.

Taghirad and Belanger also applied an $H_\infty$ model for torque control [46]. They developed a model for SWG drives using frequency responses obtained through experiments. Multiplicative uncertainty was used to encapsulate the deviations from actual operation. The model was evaluated with a closed-loop system using both constrained and free motion situations. For the constrained motion case, the model was able to retain robust stability and improve tracking performance. On the other hand, the free motion case required a feed forward algorithm to compensate friction effects at low frequency. This algorithm significantly improved the behavior of the $H_\infty$ model at low frequency.

Godler et al. proposed the extension of Hashimoto’s original built-in torque sensor by including an odd number of strain gauges attached to the closed end of the flexible spline cup [3]. The gain of each of these strain gauges is modified by a heuristic approach or by a mathematical model to reduce the torque ripple that operating SWG drives generate. Without needing accurate positioning, the authors examined the proposed method with different torques and rotational speeds. Additionally, they performed a durability test on their torque sensing device, realizing that it does not deteriorate over time, which showed the method to be simple and effective.

Lu et al. published an article in 2013 about compensation of torque ripples in SWG drives
2.5. Torque measurement and control

[47]. Torque ripples have a frequency equal to twice that of the input frequency, whose dominating component repeats every half-turn of the input. As a scheme to control the output torque of a SWG drive with a built-in torque sensor, the authors proposed the use of a feedback control system and a feed-forward control system which directly learns from a disturbance observer control system. This control system evaluates the compensation error to compensate the induced torque ripples from the SWG drive. It uses successive cycles to iteratively build the control signal. This system is designed for lightweight service robot manipulators, as well as certain haptic devices and steer-by-wire systems, and provides robust stability verified with experiments using a hollow-shaft actuator.

Jung et al. developed a method to compensate torque ripples in the output signal of SWG drives by using an order analysis approach [48]. This method uses both a half and a full Wheatstone bridge which are less sensitive to misalignments as compared to other methods. The authors identified the periodic tangential signal on the gauges as the torque ripple and compensated for it with order analysis, which takes into account the angular position response rather than the time response. They were able to reduce the torque ripple signal amplitude to near zero.

Zhang et al. proposed the use of position measurements of input and output members of SWG drives to estimate the output torque with a compliance model of the drive [49]. The model takes into account the input torque from the voltage of the motor, as well as the kinematic error of the SWG drive. This approach is intended to improve positional accuracy and reduce cost of torque sensing mechanisms in robot joints. The authors compared the estimated torque with the torque measured by a commercial sensor to verify their results. The proposed approach showed to be immune to noise signals from the drive.

In 2017, Jung et al. used the order tracking method to develop a more economic joint torque measurement system [50]. This method uses order tracking to compensate the torque ripple signal obtained from the flexible spline elastic deformations. The torque ripple signal is rearranged into order signal to then be cancelled with a notch filter. The alignment of the gauges does not affect the accuracy of the results. The model is independent of speed variations and has lower error compared to torque measurements based on voltage of the motor.
Ma et al. studied the frictional effects on SWG drives operating at low speed [19]. Using an experimental set up to measure the behavior of a SWG drive in slow operation, the authors developed a frictional model to analytically reproduce the behavior of the drive. They discovered that there is a torque component generated during the operation of SWG drives due to friction at low speed. The authors proposed the use of improved lubricants and surface coatings in the teeth and the bearing around the wave generator cam to reduce the influence of low speed friction.

2.6 Dynamic models

Due to the non-linearities in the operation of SWG drives, dynamic models became necessary to understand their behavior. For this reason, researchers made use of the introduced torque measuring and controlling systems to develop dynamic models able to take into account the non-linearities of SWG drives and represent their behavior. These models consider the behavior of SWG drives in different ways.

In 1993, Hashimoto et al. proposed the use of strain gauges to measure the strain in the flexible spline cup during the operation of SWG drives [43]. This served to develop a dynamic model to understand the influence of torque transmission in this type of gear drive. Tuttle and Seering proposed a dynamic model of a SWG drive consisting of several massless planes with different interacting slopes that represent the friction between the different elements of a SWG drive [8], as shown in Figure 2.11. Their objective was to study certain characteristics of SWG drives, such as vibrations over the operating range and unpredictable changes in velocity. However, they realized an experimental analysis should be used instead of a dynamic model if accurate operation of SWG drives is desired.

Later, Tuttle and Seering proposed a non-linear dynamic model developed through experimental observation to describe the behavior of a SWG drive [52]. This is because SWG drives experience several non-linear dynamic characteristics. A decent portion of their operating range is contaminated by vibration, which dissipates energy, hindering the increase on speed. There are significant speed jumps throughout their operation, and these variations in speed and friction cause huge variations in performance. The model considered transmission compliance and
2.6. Dynamic models

The authors also developed a mathematical friction model for the contact between the teeth. They realized that the kinematic error changes several times per rotation of the wave generator and determined that friction in SWG drives is influenced by several components of constant, speed-dependent, position-dependent, and resonance vibration friction. Besides, the torque created due to friction is a uniform offset value. However, the model required the characterization of stiffness and friction to be accurate.

In 1997, Kircanski and Goldenberg published an article focused on developing a dynamic model to represent non-linear stiffness, friction, and hysteresis in SWG drives [9]. They performed an experimental study and developed a systematic procedure to estimate the model parameters. Their motivation came from the fact that SWG drives do not always purely multiply the input torque by the gear ratio to obtain the output torque. In this type of gear drive,
there may be a remarkably large loss of torque after the load reaches a certain value. However, when no load is applied, the only losses occur due to friction.

![Figure 2.12: Output torque limit by Kircanski and Goldenberg [9].](image)

The authors’ objectives were to show experimental evidence of these phenomena, develop an experimental strategy to represent the behavior of SWG drives, and determine a simple representation of the physical effects that dominate their behavior. The authors realized that the relationship between input and output torque follows a straight line around the middle region of the SWG drive. However, when the load is increased beyond a particular value, the output torque does not change anymore, as shown in Figure 2.12. This is due to the input torque saturating the SWG drive. The critical torque at which the drive saturates depends on the time from the start of application of load until the maximum value is reached. There is a point where the teeth are twisted to such an extreme that the wave generator cannot continue rotating. The authors realized that the loss of torque due to friction depends on the direction of rotation, rather than the rotational speed itself. Furthermore, deviations from the ideal shape of a tooth can cause torque ripples. For these reasons, the dynamic behavior of SWG drives is remarkably influenced by the operating conditions.

Taghirad and Belanger developed a linear model to control the torque transmission mechanism in motion constrained SWG drives [11]. They justified the use of a linear model for the
purpose of control after creating it based on experimental frequency response patterns. However, the authors implemented the inherent non-linearities of SWG drives by encapsulating multiplicative uncertainty in a first order weighted function in the model. This model allows for the development of robust torque controllers for SWG drives. The authors showed that a second or third order control transfer function is enough to model the behavior of SWG drives with the uncertainty of the non-linear aspects included. Finally, they developed a controller that proved to be robustly controlling the torque output in the closed loop system scenario.

In order to simplify the determination of the complex behavior of SWG drives, Taghirad and Belanger also proposed a systematic approach to determine the dynamic behavior of SWG drives [53]. Their goal was to obtain the simplest model that accurately represents the dynamic behavior of a SWG drive and implement a torque control algorithm. They proposed the use of simple models for hysteresis, compliance, and friction. They estimated the parameters of the model using approximation by least-squares. The authors employed two testing stations to perform experiments for both cases of having the flexible spline as the output element or the ring gear as the output element of the drive. They realized that using a linear stiffness model and relating the damping to a power of the speed with static model is the best approach to capture the hysteresis in SWG drives. Study of different performance characteristics allowed the authors to determine that a Stribeck model represents the dynamic behavior of a SWG drive in low speed operation. They conclude that stiffness and damping are critical to the development of a model to represent the dynamics of a SWG drive.

In 1998, Hashimoto and Kiyosawa proposed the use of a built-in torque sensor to develop a dynamic model to control the torque output in SWG drives in real time, without having to compute the real time torque output with a complicated controller [44]. This method was also used by Taghirad and Belanger to control the torque output with a dynamic model of SWG drives [23].

Tjahjowidodo et al. proposed a dynamic model that considers the SWG drive as a grey box which can be calibrated by means of non-linear regression [54]. This model was developed to represent the torsional compliance in SWG drives. The authors considered the contact between
teeth pairs averaged for simplicity. However, they implemented the sliding friction on engaging teeth and the effect of friction on the wave generator. The model represents the different interactions in a SWG drive as frictional planes in contact with different slopes. These slopes are proportional to the influence of friction in each interaction. The flexible spline non-linear stiffness is represented by a non-linear spring. However, their model cannot accurately estimate the torque at low operating speeds. Lu et al. focused on developing a dynamic model to compensate torque ripples in SWG drives with a built-in torque sensor [47].

2.7 Finite elements models and analysis

Due to the complicated behavior of SWG drives, researchers started using finite element analysis to evaluate their behavior. These finite element models have just started to be developed to simplify the operation of this type of gear drive and improve it. Some models consider a two-dimensional section of the drive, while others consider a simplified three-dimensional representation. The authors of these models made considerable assumptions for simplicity, which limits their applicability. These research efforts evidence the need to develop accurate and easy-to-use tools for finite element analysis of SWG drives to understand their behavior and improve their performance.

In 2007, Kayabasi and Erzincanli used finite element analysis to develop a suitable tooth geometry for the teeth of the flexible spline and those of the ring gear [30]. They were able to obtain the same results using finite element analysis and experiments, realizing that the load transmission in the flexible spline concentrates on the quadrant towards where the wave generator is rotating, as shown in Figure 2.13.

Dong and Wang performed finite element analysis to evaluate the influence of the shape of the wave generator and the transmitted load over stresses in the flexible spline [31]. They realized that the deformation of the flexible spline is heavily non-linear, which makes evaluating the behavior of SWG drives a required application of finite element analysis.

In 2016, Li performed stress analysis and evaluated how the flexible spline behaves with fatigue loads [57]. The author proposed a method to perform contact analysis in SWG drives by using finite elements. The model computes the bending and shear stresses on the flexible spline
cup element. Using this model, Li realized that the stresses along the radial and circumferential directions of the flexible spline do not change when the torque is modified. However, the shear stresses change as a function of the transmitted torque, having an average value of zero when the torque is zero. This means that the shear stress fluctuation on the flexible spline during the rotation of the wave generator has a constant amplitude regardless of the applied load, but
its average value is proportional to the transmitted torque, as shown in Figure 2.14. For these reasons, Li concluded that fatigue failure was caused by the deflection imposed by the wave generator and the transmitted torque.

Chen et al. performed a study of the influence of torque and pressure angle over torsional stiffness, number of teeth in contact, and stresses in SWG drives [20]. The authors employed a two-dimensional finite element model of a SWG drive in their analysis. The model considers the involute profile for the teeth of the flexible spline. Once this is computed, the ring gear is generated as a conjugate of the flexible spline tooth profile. Throughout the analysis, the wave generator and the ring gear are considered rigid. The contacts are defined frictionless but with finite sliding. In order to assemble the different parts of a SWG drive, the two-dimensional model considers a homogeneous pressure applied inside the flexible spline for this element to deflect towards the meshing position with the ring gear and become engaged by the wave generator. For their study of torsional stiffness and number of teeth in contact as a function of the torque applied, the authors fixed both the ring gear and the wave generator and applied three levels of torque on the flexible spline. They realized that the higher the applied torque is, the larger the number of teeth in contact and the torsional stiffness are. In order to compute the stresses generated during the operation of a SWG drive, the authors simulated the meshing of a drive by rotating the wave generator two revolutions while a single level of torque was applied to the flexible spline. They obtained contact and bending stresses lower than the yield and fatigue limits of the flexible spline.

On the other hand, the authors studied the influence of varying the pressure angle of the teeth of the flexible spline. They realized that the torsional stiffness and the bending stresses increase with the pressure angle. However, the number of teeth of the flexible spline in contact is reduced. They also realized that lowering the pressure angle of the flexible spline beyond a certain point leads to interference between the teeth of the flexible spline and the ring gear.

Pacana et al. used finite element analysis to study the influence of different wave generator geometries over the stress distribution on the flexible spline [58]. Their analyses consider half of the flexible spline as a shell element with its teeth simplified as a ring and that it is in contact with the wave generator. The model implements the load application as torque near the
2.7. Finite elements models and analysis

major axis region of the flexible spline. However, their model showed limitations due to the simplifications of not considering the contact with the teeth of the ring gear.

![Diagram of parabolic application of load](image)

**Figure 2.15:** Parabolic application of load by Sahoo and Maiti [37].

Sahoo and Maiti performed several studies using finite element analysis on SWG drives with involute profile teeth [36, 37]. They realized that there are several pairs of teeth in contact as a function of the load applied and the geometry of the wave generator employed. Their model uses a three-dimensional representation of the SWG drive where the load is applied as a parabola near the major axis region of the teeth of the flexible spline, as shown in Figure 2.15. Ma et al. performed simulation of meshing in SWG drives and verified their results using experimental validation and finite element analysis [59].

In 2018, Yu and Ting used finite element analysis over a three-dimensional model of a SWG drive, including micro-geometry modifications on the flank surfaces of the teeth of the flexible spline and the ring gear [38]. This model was used to find localized contact between the teeth of both members which, due to the complicated behavior of SWG drives, can only be studied using finite element analysis.

Wang et al. developed a considerably realistic finite element model of SWG drives in [62]. Their aim was to compute circumferential stresses on the flexible spline as a function of the length of the cup and compare them to their proposed analytical method. The authors realized that circumferential stress on the flexible spline causes frequent fatigue failures at the root of its
teeth. The obtained results, however, are not considerably accurate.

2.8 Study of the meshing process

Over the last few years, researchers have started to study and closely observe the meshing process of SWG drives. This is because they operate in a significantly different manner than more common gear drives. In a traditional gear drive, the meshing occurs between one side flank of a few teeth of the pinion and the same side flank of a few teeth of the gear, creating a single region of meshing [13]. However, in SWG drives, there are two regions of meshing where deflected external teeth of the flexible spline mesh with the considerably more rigid internal teeth of the ring gear [1]. This mechanism establishes contact between a significantly larger number of pairs of teeth than traditional gears would both due to the imposed deflection and doubling the number of meshing regions. Besides, the contact may occur with both flanks of a tooth, as opposed to single flank contact. For these reasons, it is necessary to develop models able to simulate the actual meshing process of SWG drives for it to be understood and improved. This research effort has just been started by a few researchers and it is still lacking a deep background of knowledge and understanding.

Ma et al. developed a method to determine the meshing characteristics of SWG drives [60]. This method employs computer vision to identify how the teeth engagement and disengagement process is performed, as shown in Figure 2.16. The authors developed a mathematical model that calculates the location of teeth pairs during the engagement. They realized that the tooth engagement phase in SWG drives constitutes a smooth, uniform process. On the other hand, the tooth disengagement phase is a variable process affected by the torque applied. By closely examining a considerably slowed down meshing process, the proposed model helps to understand how friction develops on the contact between teeth of the flexible spline and ring gear.

In 2018, Ma et al. studied the meshing process in SWG drives with experiments and finite element analysis [59]. They focused on the variation of the meshing stiffness and the number of teeth in contact as a function of time and torque applied. The authors realized that the meshing stiffness is directly influenced by the number of teeth in contact, which itself is highly
2.8. Study of the meshing process

**Figure 2.16:** Meshing images used by Ma et al. [60].

**Figure 2.17:** Number of meshing teeth as a function of load [59].
influenced by the torque applied. When using small values of torque, the number of teeth in contact increase significantly while the torque increases, as shown in Figure 2.17. This is due to the teeth deformation with the applied load. Although they obtained the meshing stiffness as a geometric relationship between torsion and wire springs stiffness, the stiffness rises as the number of teeth in contact increases. Throughout their simulation, the authors computed the meshing force distribution as a function of the pairs of teeth in contact, which shows a distribution similar to an inverted parabola with its maximum meshing force value near the central meshing tooth. However, this maximum value of meshing force is slightly displaced towards the region where the wave generator is moving. They obtained the number of pairs of teeth in contact and meshing stiffness as a function of the torque applied both in their simulation and experiments. They also proposed a method to calculate torque based on stiffness of the SWG drive.
Chapter 3

Approach

3.1 Chapter overview

The third chapter of this thesis dissertation provides a description of the implemented geometrical features, as well as an in-depth explanation of the finite element models developed for simulation of meshing and stress analysis of SWG drives.

Section 3.2 shows the different wave generator geometries included, while the different tooth profiles for the teeth of the flexible spline and the ring gear are explained in Section 3.3. Then, Section 3.4 defines the two-dimensional finite element model for stress analysis and simulation of meshing of SWG drives. As a consequence of the limitations of using a two-dimensional model, the developed three-dimensional finite element model is explained in Section 3.5.

3.2 Wave generator geometries

Traditionally, wave generators are manufactured as an elliptical shape cam with a flexible race ball bearing around it [1]. This bearing takes the shape of the elliptical cam to become a wave generator, which can be inserted in a flexible spline to deflect it into meshing. For this reason, one can assume that the contact between the wave generator and flexible spline is frictionless.

Over time, other geometries for the wave generator have been proposed, including wave
Chapter 3. Approach

generators composed of two and four rolling elements [10, 14, 15, 31, 38, 58]. The following geometries of the wave generator are considered in the finite element models, and their influence on the meshing and stresses of SWG drives studied:

- Simplified: based on two rolling elements [10].
- Elliptical: based on the traditional wave generator [1].
- Parabolic: proposed in this thesis dissertation.
- Four-roller: based on four rolling elements [14].

Each wave generator geometry requires the use of their corresponding pushing pins towards applying finite element analysis. These pushing pins incorporate the geometry of the wave generator to initially deflect the flexible spline into meshing with the ring gear. Every pushing pin geometry is generated by dividing the selected wave generator geometry into two halves. These halves are modified accordingly to become a pushing pin.

3.2.1 Simplified wave generator

The proposed simplified wave generator to be used in the finite element models of SWG drives consists of two semicircles and two straight sections with a reference point RP on its center, as shown in Figure 3.1. The straight sections connect both semicircles. The radius of the semicircles \( r_{pp} \) is equal to the radius of the circular pushing pins that this wave generator requires to deflect the flexible spline into the meshing position. The longer dimension of the wave generator or major axis \( a \) is long enough for the flexible spline to reach the deflected position in this direction.

The following formula defines the undeformed inner radius of the flexible spline \( r_{fs} \),

\[
r_{fs} = \frac{N_{fs}m_{fs}}{2} - h_{d}m_{fs} - t_{r}
\]  

(3.1)

where \( N_{fs} \) is the number of teeth of the flexible spline, \( m_{fs} \) is the module of the flexible spline, \( h_{d} \) is the dedendum coefficient of the flexible spline, and \( t_{r} \) is the rim thickness of the flexible spline.
3.2. Wave generator geometries

It can be determined that the radius of the pushing pins $r_{pp}$ must be smaller than the undeformed inner radius of the flexible spline $r_{fs}$ since the pushing pins and the wave generator need to fit inside the flexible spline. The following formula relates the radius of the pushing pins $r_{pp}$ with the undeformed inner radius of the flexible spline $r_{fs}$,

$$r_{pp} = C \left( r_{fs} - \frac{d}{2} \right)$$  \hspace{1cm} (3.2)

where $C$ is the minor axis reduction coefficient, a user defined coefficient equal or smaller than 1.0. This ensures that the maximum radius of the pushing pins $r_{pp}$ is equal to the minimum inside radius of the flexible spline after being deflected, which is equal to the inner radius of the flexible spline $r_{fs}$ minus half the deflection $d$. This creates some clearance underneath the flexible spline for it to deflect inward along the shorter dimension of the wave generator when the latter deflects the flexible spline outward along the major axis $a$. Parameter $a$ is defined as,

$$a = r_{fs} + \frac{d}{2}$$  \hspace{1cm} (3.3)
Simplified pushing pins

Due to the simplicity of this type of wave generator, the pushing pins associated with it are considerably simple as well. They constitute two circles with the same radius as that of the semicircles of the wave generator $r_{pp}$. Figure 3.2 shows the definition of one pushing pin. The reference point of each pushing pin is located in its center. This geometry of the pushing pins allows them to mimic the geometry of the simplified wave generator along the major axis, so that, after deflecting the flexible spline, the contact can be switched from the pushing pins to the wave generator.

![Figure 3.2: Simplified pushing pin.](image)

3.2.2 Elliptical wave generator

Elliptical and traditional wave generator geometries are similarly defined given the fact that both are in the shape of an ellipse. Figure 3.3 shows the elliptical wave generator in its coordinate system $S_{wg}$ with a reference point $RP$ on its center.

Similarly to the simplified wave generator, the major axis $a$ of the elliptical wave generator is calculated with Equation (3.3) where $r_{fs}$ is the undeformed inner radius of the flexible spline and $d$ is the deflection to be applied to the flexible spline as defined by Musser [1]. The minor
3.2. Wave generator geometries

The length of the minor axis of the ellipse is defined by subtracting half of the deflection,

\[ b = C \left( r_{fs} - \frac{d}{2} \right) \]  

(3.4)

where \( b \) is the length of the minor axis of the ellipse and \( C \) is the minor axis reduction coefficient, as previously defined. It is clear that Equations (3.2) and (3.4) lead to the same result. However, the simplified wave generator and the elliptical wave generator differ on their usage of this value. Here, the minor axis reduction coefficient \( C \) allows the minor axis \( b \) to be shorter than its maximum theoretical value.

**Elliptical pushing pins**

Figure 3.4 shows the definition of an elliptical pushing pin. The geometry of each pushing pin is created by dividing the geometry of the elliptical wave generator in half. Each open half is then closed by a straight line. This straight line locates the reference point for each pushing pin on its center so the boundary conditions can be applied there.
3.2.3 Parabolic wave generator

The parabolic wave generator makes use of a parabola as the portion of the geometry of the wave generator that makes contact with the inner surface of the flexible spline. It consists of two parabolic regions on the top and bottom sections of the wave generator connected by means of three circular arcs on each side and a reference point $RP$ on its center, as shown in Figure 3.5. The three connecting circular arcs on each side are two connector arcs and a lateral arc. The connector arcs are tangent to both their adjacent parabolic regions and the lateral arc. In Figure 3.5, $O_c$ and $O_l$ represent the centers of the connector arcs and the lateral arc, respectively. The minor and major axes of the wave generator $a$ and $b$ are computed similarly to those of the elliptical wave generator (Equations (3.3) and (3.4)).

Figure 3.6 shows the definition of the parabolic wave generator geometry. The points of the parabolic region are defined in the coordinate system of the wave generator $S_{wg}$ with the following equation,

$$r_p = \begin{bmatrix} x \\ a - a_p x^2 \\ 0 \\ 1 \end{bmatrix} \quad (3.5)$$

where $a_p$ is the parabola coefficient and $x$ is the abscissa coordinate of each point. The parabola coefficient $a_p$ is a user-defined parameter that is proportional to the curvature of the parabolic
3.2. Wave generator geometries

**Figure 3.5:** Parabolic wave generator.

**Figure 3.6:** Parabolic wave generator definition.
region. The use of the major axis $a$ in this formula makes the topmost point of the parabolic region equal to that of a traditional elliptical wave generator. Component $x$ is limited between $-C_p b$ and $C_p b$. Here, $C_p$ is a user-defined parameter that must be equal or smaller than 1.0, and $b$ is equal to the minor axis of the wave generator as previously defined.

In Figure 3.6, $r_c$ is a user-defined parameter equal to the radius of the circular arc that serves as a connector between the parabolic region of the wave generator and its lateral circular arc region. This radius serves as an input parameter in order to obtain the radius of the lateral circular arc so its middle point coincides with the minor axis $b$ of the wave generator.

The tangent to the parabolic region of the wave generator is obtained by deriving Equation (3.5) as a function of $x$ and normalizing the resulting vector, leading to

$$t_p = \begin{bmatrix} \frac{1}{\sqrt{1+4a_p^2x^2}} \\ \frac{-2a_p x}{\sqrt{1+4a_p^2x^2}} \\ \frac{1}{\sqrt{1+4a_p^2x^2}} \\ 0 \end{bmatrix}$$ \hspace{1cm} (3.6)

By performing the cross product between the z-axis unit vector $k$ and the tangent vector $t_p$, the normal to any point of the parabola is obtained as,

$$n_p = \begin{bmatrix} \frac{2a_p x}{\sqrt{1+4a_p^2x^2}} \\ \frac{-2a_p x}{\sqrt{1+4a_p^2x^2}} \\ \frac{1}{\sqrt{1+4a_p^2x^2}} \\ 0 \end{bmatrix}$$ \hspace{1cm} (3.7)

Using the vectors $r_p$ and $n_p$ for the last point of the parabolic region, the position of the center of the connector arc of the wave generator is defined as,

$$r_c = r_p - r_c n_p$$ \hspace{1cm} (3.8)
A normal to the lateral circular arc is

\[ \mathbf{n}_c = \begin{bmatrix} \cos \alpha_l \\ \sin \alpha_l \\ 0 \end{bmatrix} \] (3.9)

where \( \alpha_l \) is the aperture angle of the lateral circular arc from the horizontal axis, as shown in Figure 3.6. The position of the center of the lateral circular arc is defined as,

\[ \mathbf{r}_l = \mathbf{r}_c - (r_l - r_c) \mathbf{n}_c \] (3.10)

where \( r_c \) is the radius of the connector arc and \( r_l \) is the radius of the lateral circular arc, which is computed with the following formula,

\[ r_l = r_c + \frac{y_c}{\sin \alpha_l} \] (3.11)

Here, \( \alpha_l \) is the aperture angle of the lateral circular arc. This is the parameter for which a non-linear equation is solved to obtain the center of the lateral circular arc located in the x-axis. The aperture angle \( \alpha_c \) is equal to the angle of the normal to the parabola \( \mathbf{n}_p \) on the connecting point with the connector arc minus the aperture angle of the lateral circular arc \( \alpha_l \). The y-coordinate \( y_c \) of vector \( \mathbf{r}_c \) is used to compute \( r_l \), which is the resulting radius of the lateral circular arc.

Consequently, once the input parameters for the SWG gear drive and for the parabolic wave generator \( C_p, a_p \) and \( r_c \) are set up, the only parameter left to compute is the aperture angle of the connector arc \( \alpha_c \). This serves to obtain the geometry of the parabolic wave generator.

**Parabolic pushing pins**

Figure 3.7 shows the definition of a parabolic pushing pin. Similarly to an elliptical pushing pin, the geometry of a parabolic pushing pin is created by halving the geometry of its respective parabolic wave generator. Each parabolic wave generator half is then closed with a straight line that locates the reference point of the pushing pin \( RP_{pp} \) on its center.
3.2.4 Four roller wave generator

Figure 3.8 shows the definition of the four roller wave generator based on [14, 31, 58]. In this figure, \( a \) is the major axis of an elliptical wave generator as previously defined in Equation (3.3), \( r_4 \) is the radius of the rollers, and \( \beta \) is the aperture angle of each roller from the vertical axis. It can be seen that the four roller wave generator geometry deflects into meshing the inner diameter of the flexible spline along the whole angle \( \beta \) in both the top and bottom of the major axis of the wave generator. This wave generator geometry tries to ensure that a large number of teeth of the flexible spline will be in meshing position with the ring gear, further increasing the load transmission capability of the SWG drive.

Four roller pushing pins

Since the geometry of the four roller wave generator is divided in four circles of radius \( r_4 \) that fit inside the flexible spline, the geometry of a four roller pushing pin would be two of the four rollers for the top pushing pin and the same for the bottom pushing pin. These would be assembled with the flexible spline below their final position in the wave generator. During the pushing step, the top two rollers will move upwards and the bottom two downwards until reaching their final position, as shown in Figure 3.8.
3.3 Tooth profiles

While the involute profile is extensively used for cylindrical gears, it is not commonly used in SWG drives. For instance, even in his patent in 1959, Musser proposed the use of the involute profile with different pressure angles for the flexible spline and the ring gear [1]. This is because the kinematics of teeth engagement in SWG drives are different from those of traditional gear drives. In a SWG drive, a tooth of the flexible spline moves into a tooth slot of the ring gear and then moves out of it.

The following tooth profiles will be considered in the finite element models, and their influence on the meshing and stresses of the SWG drive studied:

- Involute: based on the original tooth profile for SWG drives [1].
- Double circular arc: based on [22] and [33].
- Quadruple circular arc: based on [38].
3.3.1 Involute tooth profile

The involute tooth profile is a generated tooth profile traditionally used in several types of gear drives due to its advantageous performance. Its advantages include ease of manufacturing, independence from center distance variations, and constant transmission of motion [13]. The involute tooth profile uses the involute curve for the geometry of the flank surface of each side of the teeth. The involute curve can be understood as the unwinding of a string from a circumference of radius $r_b$, called base circle. The radius of the base circle $r_b$ is defined by

$$r_b = r_p \cos \alpha$$

where $r_p$ is the pitch radius and $\alpha$ is the pressure angle of the to-be-generated gear. The pitch radius $r_p$ is defined by

$$r_p = \frac{N m}{2}$$

where $N$ is the number of teeth and $m$ is the module of the gear.

Figure 3.9 shows the involute curve definition in the global coordinate system $S$, where the first point of the unwinding string defines the involute curve. Throughout the unwinding process, the string is tangent to the base circle $r_b$ and perpendicular to the involute curve.
Flexible spline tooth generation

When the involute tooth profile is used in external gears, such as the flexible spline in SWG drives, this profile is manufactured using a straight profile rack-cutter. Figure 3.10 shows the generation process of the flexible spline with involute tooth profile using a rack-cutter. Here, the global coordinate system \( S \) is fixed to the frame while coordinate system \( S_{fs} \) is attached to the flexible spline and coordinate system \( S_g \) is attached to the rack-cutter.

The family of rack-cutter surfaces in the coordinate system \( S_{fs} \) fixed to the flexible spline is represented by \( r(u, \theta, \varphi) \). The parameters \( u \) and \( \theta \) define the surfaces in profile and longitudinal direction, respectively. Then, the meshing equation is solved to obtain each point of the flexible
spline tooth profile. With these parameters, the meshing equation is defined as,

\[ f(u, \theta, \varphi) = \left( \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial \theta} \right) \cdot \frac{\partial r}{\partial \varphi} = 0 \]  

where \( \varphi \) is the generating angle shown in Figure 3.10, which relates the translation of the rack-cutter \( v_s \) with the rotation of the flexible spline blank \( \omega_{fs} \). The solution to the meshing equation is found when the sliding velocity \( \frac{\partial r}{\partial \varphi} \) is perpendicular to the normal of the rack-cutter surface on the considered point.

### Ring gear tooth generation

On the other hand, the manufacturing of internal gears, such as the ring gear in SWG drives, requires the use of a shaper-cutter. A shaper-cutter is a cutting tool with the geometry of an externally toothed gear and is used for the cutting of internal teeth gears \[13\]. Figure 3.11 shows the generating process of the ring gear with involute tooth profile using a shaper-cutter in the global coordinate system \( S \) fixed to the frame. Coordinate systems \( S_{rg} \) and \( S_g \) are attached to the ring gear and the shaper-cutter, respectively.

The generation of the ring gear is performed similarly to that of the flexible spline. The meshing equation is applied considering the cutting surfaces of the shaper-cutter. However, the generating motion of the cutting tool is a rotation in this case. The shaper-cutter meshes with the ring gear blank and cuts teeth by rotating both elements with the following relation,

\[ \frac{\varphi_g}{\varphi_{rg}} = \frac{\omega_g}{\omega_{rg}} = \frac{N_{rg}}{N_g} \]  

where \( \varphi_i, \omega_i, \) and \( N_i \) represent rotational angle, rotational speed, and number of teeth, respectively. The subscripts \( g \) and \( rg \) represent the shaper-cutter and the ring gear, respectively.

### 3.3.2 Double circular arc

The double circular arc (DCA) profile is a directly defined tooth profile. The geometry of each flank consists of two circular arcs and a straight segment connected tangentially. Figure 3.12 shows a gear tooth employing the DCA profile on both tooth flanks. The symbols \( r_a \) and
3.3. Tooth profiles

\[ r_d \] represent the radius of the addendum and the dedendum circular arcs, respectively, while the straight segment is represented by its length \( l \). The active region of the DCA tooth profile consists of part of the addendum arc \( r_a \), the straight segment \( l \), and part of the dedendum arc \( r_d \). The dedendum circular arc \( r_d \) also connects tangentially to the root region of the gear, serving as the fillet area of each tooth.

Figure 3.13 shows the critical points for the definition of the DCA tooth profile in the global coordinate system \( S \) fixed to the gear. The y-axis is truncated for the purpose of readability. The radii \( r_p \), \( r_a \), and \( r_D \) represent the pitch, addendum, and dedendum radii of the gear, respectively. The pitch radius \( r_p \) is defined similarly to the involute profile as in Equation (3.13). The addendum radius \( r_A \) and the dedendum radius \( r_D \) are defined with the following formulas,

\[
r_A = r_p \pm h_a m
\]  

(3.16)
Figure 3.12: Gear tooth using the DCA tooth profile.

Figure 3.13: Points and parameters for definition of the DCA tooth profile.
3.3. Tooth profiles

\[ r_D = r_p \mp h_d m \]  \hspace{1cm} (3.17)

where \( h_a \) is the addendum coefficient and \( h_d \) is the dedendum coefficient. The upper signs on the previous equations are used to compute the critical points of the flank profile geometry of a tooth of the flexible spline (external teeth), while the bottom signs are used for a tooth of the ring gear (internal teeth).

The topmost point of the DCA profile \( P_A \) is the center of the top land of a tooth. Its coordinates are computed with the following formula,

\[
\begin{align*}
\mathbf{r}^{(P_A)} &= r_A \mathbf{j} = \\
&= \begin{bmatrix} 0 \\ r_A \\ 0 \\ 1 \end{bmatrix}
\end{align*}
\]  \hspace{1cm} (3.18)

where \( \mathbf{r}^{(P_A)} \) is the position vector of point \( P_A \) and \( \mathbf{j} \) is the unit vector along the \( y \)-axis of the gear coordinate system \( S \). As Figure 3.13 shows, point \( P_A \) is located on the addendum circle of the gear with radius equal to \( r_A \).

The bottommost point of the DCA profile \( P_D \) is the center of the root land of a tooth slot. Its position vector \( \mathbf{r}^{(P_D)} \) is computed with the following formula,

\[
\mathbf{r}^{(P_D)} = r_D \mathbf{n}_D 
\]  \hspace{1cm} (3.19)

where \( \mathbf{n}_D \) is the normal to the dedendum circle \( r_D \) on point \( P_D \) computed as,

\[
\mathbf{n}_D = \begin{bmatrix} \sin \alpha_T \\ \cos \alpha_T \\ 0 \end{bmatrix}
\]  \hspace{1cm} (3.20)

Here, \( \alpha_T \), also shown in Figure 3.13, is defined as half the angular thickness of a tooth of the gear employing the DCA profile, that is

\[
\alpha_T = \frac{\pi}{N} 
\]  \hspace{1cm} (3.21)
The angle $\alpha$ in Figure 3.13 is the pressure angle of the tooth whereas the angle $\alpha_P$ is defined as the angular pitch thickness of the tooth as follows,

$$\alpha_P = \frac{c_P \alpha_T}{2}$$  \hspace{1cm} (3.22)

where $c_P$ is a user-defined coefficient that modifies the angular tooth thickness $\alpha_T$ to become the angular pitch thickness $\alpha_P$. By default, coefficient $c_P$ is set to 1.0. This angle $\alpha_P$ and the pitch radius $r_P$ define the position vector of the pitch point of the DCA profile $r^{(P_P)}$ with the formula,

$$r^{(P_P)} = r_P n_P$$  \hspace{1cm} (3.23)

where $n_P$ is the normal to the pitch circle of radius $r_P$ on point $P_P$ as,

$$n_P = \begin{bmatrix} \sin \alpha_P \\ \cos \alpha_P \\ 0 \end{bmatrix}$$  \hspace{1cm} (3.24)

The next critical points needed to define the DCA profile are the initial and final points of the straight segment $l$. In Figure 3.13, these two points are labeled as $P_a$ and $P_d$, respectively. These are the points of tangency between the addendum arc $r_a$ and the straight segment $l$ and the dedendum arc $r_d$ and the straight segment $l$. The distance between the pitch point $P_P$ and the point $P_a$ is defined as a function of the length of the straight segment $l$ with the following equation,

$$P_P P_a = c_I l$$  \hspace{1cm} (3.25)

where $c_I$ is a user-defined coefficient. By default, coefficient $c_I$ is set to 0.5. Consequently, the distance between the pitch point $P_P$ and the point $P_d$ is

$$P_P P_d = l - c_I l = (1 - c_I) l$$  \hspace{1cm} (3.26)
Figure 3.13 shows the corrected pressure angle $\alpha^*$ as,

$$\alpha^* = \alpha - \alpha_P \tag{3.27}$$

which defines the normal $n_l$ to the straight segment $l$ as,

$$n_l = \begin{bmatrix} \cos \alpha^* \\ \sin \alpha^* \\ 0 \end{bmatrix} \tag{3.28}$$

Considering the corrected pressure angle $\alpha^*$, a tangent to the straight segment $l$ is given by,

$$t_l = \begin{bmatrix} -\sin \alpha^* \\ \cos \alpha^* \\ 0 \end{bmatrix} \tag{3.29}$$

which is perpendicular to the normal to the straight segment $n_l$ as shown in Figure 3.13.

From the pitch point $P_P$, the position vector of the point of tangency with the addendum arc $r(P_a)$ is computed with the following formula,

$$r(P_a) = r(P_P) + c_l t_l \tag{3.30}$$

and the position vector of the point of tangency with the dedendum arc $r(P_d)$ is,

$$r(P_d) = r(P_P) - (1 - c_l) t_l \tag{3.31}$$

The final points to define the DCA profile are the centers of the addendum and the dedendum circular arcs, $O_a$ and $O_d$, respectively, shown in Figure 3.13. These centers extend from the initial and final points of the straight segment $l$ in the same direction as the normal to the straight segment $n_l$. This ensures that both the addendum and the dedendum arcs are tangent to the straight segment $l$ in their connecting points $P_a$ and $P_d$. The position vector of the center
of the addendum arc $\mathbf{r}^{(O_a)}$ is determined as

$$\mathbf{r}^{(O_a)} = \mathbf{r}^{(P_a)} - r_a \mathbf{n}_l$$  \hspace{1cm} (3.32)

whereas the position vector of the center of the dedendum arc $\mathbf{r}^{(O_d)}$ is given by

$$\mathbf{r}^{(O_d)} = \mathbf{r}^{(P_d)} + r_d \mathbf{n}_l$$  \hspace{1cm} (3.33)

Once these points are computed, all the necessary information to create a flank with the DCA tooth profile has been obtained.

Figure 3.14: Towards the solution of equations to compute the DCA tooth profile.

Figure 3.14 shows the solution of the DCA profile in the global coordinate system $S$ fixed to the gear. The angles $\alpha_a$ and $\alpha_d$ represent the aperture angles of the addendum and the dedendum circular arcs, respectively. Using a Hybrid Newton nonlinear solver [63], these angles are computed so that the final point of the addendum arc $P_t$ is located on the addendum circle $r_A$ of the gear, the final point of the dedendum arc $P_r$ is located on the dedendum circle $r_D$ of the gear, and the dedendum circular arc $r_d$ connects tangentially to the root region.
In order to do so, the aperture angles $\alpha_a$ and $\alpha_d$ are first given an initial value lower than 45 degrees. The aperture angle $\alpha_a$ serves to compute the unit normal $n_a$ as,

$$n_a = \begin{bmatrix} \cos (\alpha^* + \alpha_a) \\ \sin (\alpha^* + \alpha_a) \\ 0 \end{bmatrix}$$  \hfill (3.34)

where $\alpha^*$ is the corrected pressure angle as obtained before. Similarly, the aperture angle $\alpha_d$ is used to compute the unit normal $n_d$ with the following formula,

$$n_d = \begin{bmatrix} \cos (\alpha^* + \alpha_d) \\ \sin (\alpha^* + \alpha_d) \\ 0 \end{bmatrix}$$  \hfill (3.35)

The unit normals $n_a$ and $n_d$ provide the last points of the addendum and dedendum circular arcs, as shown in Figure 3.14. The position vector of point $P_t$ is obtained with the following formula,

$$r^{(P_t)} = r^{(O_a)} + r_a n_a$$  \hfill (3.36)

and the position vector of point $P_r$ with,

$$r^{(P_r)} = r^{(O_d)} - r_d n_d$$  \hfill (3.37)

where $O_a$ and $r_a$ are the center and radius of the addendum arc, and $O_d$ and $r_d$ are the center and radius of the dedendum arc.

The next step is to compute a unit normal using the position vector $r^{(P_r)}$, which is converted to the normal $N$ and normalized with the equation

$$n = \frac{N}{|N|}$$  \hfill (3.38)

to obtain the unit normal $n$. 
Finally, the nonlinear equation solver modifies the value of the aperture angles $\alpha_a$ and $\alpha_d$, as well as the length of the straight segment $l$, until the following conditions are satisfied

$$\begin{align*}
    r_A &= |r^{(P_t)}| \\
    r_D &= |r^{(P_r)}| \\
    n &= n_d
\end{align*}$$

(3.39)

as shown in Figure 3.14. The final values of the aperture angles $\alpha_a$ and $\alpha_d$ and the length of the straight segment $l$ provided by the nonlinear equation solver are the solution to this system of equations. As a result, points $P_t$ and $P_r$ are located on the addendum circle $r_A$ and the dedendum circle $r_D$, respectively, and the dedendum circular arc $r_d$ connects tangentially to the root region of the gear.

### 3.3.3 Quadruple circular arc

The quadruple circular arc (QCA) profile is a directly defined tooth profile. The geometry of each flank consists of four circular arcs and a straight segment connected tangentially. Figure 3.15 shows a gear tooth employing the QCA profile on both tooth flanks. The circular arcs are represented by their radii $r_i$. The subscripts $t$, $a$, $d$, and $r$ refer to the tip, the addendum, the dedendum, and the root circular arcs, respectively. The straight segment is denoted by its length $l$. The active region of the QCA profile consists of the addendum arc $r_a$, the straight segment $l$, and the dedendum arc $r_d$. The addendum arc $r_a$ connects with the top land of the tooth using the tip arc $r_t$, which is tangent to both the top land and the addendum arc $r_a$. Similarly, the dedendum arc $r_d$ connects to the root land of the gear using the root arc $r_r$, which is tangent to both the root land and the dedendum arc $r_d$.

Figure 3.16 shows the critical points for the definition of the QCA tooth profile in the global coordinate system $S$ fixed to the gear. The $y$-axis is truncated for the purpose of readability and the radii $r_P$, $r_A$, and $r_D$ are computed similarly to the DCA tooth profile (Equations (3.13), (3.16), and (3.17), respectively).

The position vectors of the topmost and the bottommost points of the QCA profile, $r^{(P_A)}$ and
3.3. Tooth profiles

Figure 3.15: Gear tooth using the QCA tooth profile.

Figure 3.16: Points and parameters for definition of the QCA tooth profile.
The remaining parameters are also computed similarly to those of the DCA tooth profile. Respectively, parameters $\alpha_T$ and $\alpha_P$ computed with Equations (3.21) and (3.22) are half the angular tooth thickness and the angular pitch thickness of a gear tooth whose pressure angle is $\alpha$.

In Figure 3.16, the pitch point of the QCA tooth profile is $P_P$ and the limiting points of the straight segment $l$ are $P_a$ and $P_d$, computed as mentioned above using the corrected pressure angle $\alpha^*$. This angle defines the normal vector $n_{l}$ and the tangent vector $t_{l}$. The tangent vector $t_{l}$ is used first to compute the position of points $P_a$ and $P_d$ from the pitch point $P_P$ as a function of the length of the straight segment $l$ and $c_l$, a user-defined coefficient. Then, the normal vector $n_{l}$ is used to compute the position of the center of the addendum circular arc $O_a$ from point $P_a$ and that of the center of the dedendum circular arc $O_d$ from point $P_d$ (Equations (3.30) and (3.31)).

Figure 3.17 shows the solution of the QCA profile in the global coordinate system fixed to the gear $S$. The angles labeled as $\alpha_i$ represent the aperture angles of the four circular arcs of the QCA tooth profile. Similarly to the DCA tooth profile, using a Hybrid Newton nonlinear equation solver [63], these angles are computed so that the final point of the tip arc $P_t$ is located on the addendum circle $r_A$ of the gear, and the final point of the root arc $P_r$ is located on the dedendum circle $r_D$ of the gear. In this case, however, the length of the straight segment $l$ is an input parameter.

Similarly to the previous tooth profile, the aperture angles $\alpha_a$ and $\alpha_d$ are given an initial value lower than 45 degrees. The aperture angle $\alpha_a$ serves to compute the common unit normal to the tip and the addendum arcs $n_a$ with the following formula,

$$n_a = \begin{bmatrix} \cos (\alpha^* + \alpha_a) \\ \sin (\alpha^* + \alpha_a) \\ 0 \end{bmatrix}$$  \hspace{1cm} (3.40)
3.3. Tooth profiles

The aperture angle $\alpha_d$ is used to compute the common unit normal to the root and the dedendum arcs $n_d$ given by

$$n_d = \begin{bmatrix} \cos (\alpha^* + \alpha_d) \\ \sin (\alpha^* + \alpha_d) \\ 0 \end{bmatrix} \quad (3.41)$$

From the center of the addendum arc $O_a$, the center of the tip circular arc $O_t$ is then computed as

$$r^{(O_t)} = r^{(O_a)} + (r_a - r_t)n_d \quad (3.42)$$
where $r_a$ and $r_t$ are the radii of the addendum and tip arcs, respectively. From the center of the dedendum arc $O_d$, the center of the root circular arc $O_r$ is computed as,

$$r^{(O_r)} = r^{(O_d)} - (r_d - r_r)n_d \quad (3.43)$$

where $r_d$ and $r_r$ are the radii of the dedendum and tip arcs, respectively.

The next step is to compute the normals $N_t$ and $N_r$ using the coordinates of the center $O_t$ of the tip arc and the center $O_r$ of the root arc and the origin of coordinates in system $S$, as shown in Figure 3.17. These normals are then normalized with the equation

$$n_i = \frac{N_i}{|N_i|} \quad (3.44)$$

to obtain the unit normals $n_t$ and $n_r$. These are the common normals between the tip arc and the addendum circle $r_A$ and between the root arc and the dedendum circle $r_D$, respectively.

The unit normal $n_t$ is used to compute the point $P_t$ with the following formula,

$$r^{(P_t)} = r^{(O_t)} + r_t n_t \quad (3.45)$$

and the unit normal $n_r$ obtains the point $P_r$ as,

$$r^{(P_r)} = r^{(O_r)} - r_r n_r \quad (3.46)$$

where $O_t$ and $r_t$ are the center and radius of the tip arc, and $O_r$ and $r_r$ are the center and radius of the root arc. Finally, the nonlinear equation solver modifies the value of the aperture angles $\alpha_a$ and $\alpha_d$ until the following conditions are satisfied

$$r_A = |r^{(P_t)}|$$

$$r_D = |r^{(P_r)}| \quad (3.47)$$

as shown in Figure 3.17. The final values of the aperture angles $\alpha_a$ and $\alpha_d$ provided by the nonlinear equation solver are the solution to this system of equations. As a result, the normal
N_t is aligned with the segment O_tP_t and the normal N_r is aligned with the segment O_rP_r. These solutions provide point P_t on the addendum circle and point P_r on the dedendum circle, as well as a smooth transition between the tip circular arc r_t and the addendum circle r_A and the root circular arc r_r and the dedendum circle r_D.

To compute all the points of the QCA profile, the aperture angle \( \alpha_t \) is obtained with the following formula,

\[
\alpha_t = \alpha^t - \alpha_a - \alpha^* 
\]  

(3.48)

where \( \alpha^t \) is the angle between the normal N_t and the horizontal axis, as shown in Figure 3.17. Similarly, the aperture angle \( \alpha_r \) is obtained with the following formula,

\[
\alpha_r = \alpha^r - \alpha_d - \alpha^* 
\]  

(3.49)

where \( \alpha^r \) is the angle between the normal N_r and the horizontal axis.

### 3.4 Two-dimensional finite element model of SWG drives

Each finite element model is created with an input file for the finite element solver software and contains one SWG drive design. This design and its corresponding input file is generated from the parameters of the design with the Integrated Gear Design (IGD) computer program. IGD is a custom-made virtual gear generator which takes into account the particular manufacturing process of the gears to computationally generate their geometry [13]. The two-dimensional model includes five parts, with two being deformable bodies and the remaining three being rigid bodies.

#### 3.4.1 Deformable parts of the two-dimensional model

The deformable parts are the ring gear and the flexible spline. Each two-dimensional model contains a planar slice of the flexible spline and the ring gear. Their mesh containing their finite elements is a parameterized mapped mesh generated with IGD. The number of elements for each section of the tooth geometry of the deformable parts is selected.
Figure 3.18 shows the different sections that divide the geometry of each tooth in the two-dimensional model. In IGD, the number of elements is specified for the active profile, the under profile, the inner/outer diameter, and the lower rim. The active profile is defined on the entire flank of the tooth and the center point of the root between two teeth. Between the center point of the top land to the first point of the active profile is the under profile section. It defines half of the elements of the top land and the number of layers directly below the active profile. These layers of elements extend from the top land to the upper rim section. The section underneath the rim of each tooth is called either the inner diameter, for the flexible spline, or the outer diameter, for the ring gear. The lower rim section includes the elements located between the middle point of the rim of each tooth and the inner or outer diameter of the gear, as shown in Figure 3.18.

The element type employed for the deformable parts is CPS4I, which is a plane stress element with four-nodes and incompatible modes. This type of element is designed to work with linear elements subjected to bending, as with the deformation of the flexible spline [64].
Since the flexible spline and the ring gear are the only deformable elements of the model, their material properties are specified in IGD. These are particular values of the modulus of elasticity $E$ and Poisson’s ratio $\nu$ for the material of the flexible spline and the ring gear.

### 3.4.2 Rigid parts of the two-dimensional model

The rigid parts are the wave generator and the pushing pins for the selected wave generator design. The wave generator is a wire rigid part with the shape of the selected design. The pushing pins are two or four identical rigid wire parts that, as explained, resemble the regions of the wave generator that will be in contact with the inner diameter of the flexible spline during the meshing process. The pushing pins are inserted into the flexible spline to deflect it towards the meshing position with the ring gear.

Both the wave generator and pushing pins are meshed with R2D2 elements. This element type is a two-dimensional rigid element with two nodes connected by a linear link and designed for plane strain or stress analyses [64]. For this reason, the meshes of the wave generator and the pushing pins are defined as lines of elements coinciding with their geometry. These elements are rigidly connected to a reference point ($RP$) and all their degrees of freedom are related to the movement of their reference point.

### 3.4.3 Assembly of the elements of the drive

The assembly of the different parts of the model is generated with IGD so that the model looks like Figure 3.19. The points labeled with $RP_i$ are reference points of the different elements of the two-dimensional model, where the subscripts $pp$, $wg$, $fs$, and $rg$ refer to pushing pin, wave generator, flexible spline, and ring gear, respectively. This figure does not show the reference points of the flexible spline $RP_{fs}$ and the ring gear $RP_{rg}$ because they coincide with the reference point of the wave generator. It can be seen how the flexible spline is not deflected at the beginning of the analysis and it overlaps with the ring gear. Since the longitudinal dimension of the wave generator incorporates the deflection, it also overlaps with the undeformed flexible spline.
Figure 3.19: Two-dimensional finite element model of SWG drives.

Figure 3.20 shows a close view of the top and side regions of the ring gear and the flexible spline without deflection. A simulation considering contact between the flexible spline and the ring gear at this stage would not be able to run due to the large amount of interference between them. For this reason, the model requires different interactions depending on the stage of the simulation of meshing.

### 3.4.4 Interactions between the elements of the drive

There are several interactions that take place when running the two-dimensional model of SWG drives. The contact interactions considered are between:

- The pushing pins and the inner surface or diameter of the flexible spline.
- The wave generator and the inner diameter of the flexible spline.
3.4. Two-dimensional finite element model of SWG drives

These interactions are based on hard contact between surfaces. The contact between the pushing pins and the flexible spline and between the wave generator and the flexible spline is considered frictionless. This is because the wave generator is traditionally of an elliptical shape surrounded by a ball bearing with flexible race. This ball bearing contacts the inner surface of the flexible spline, and while deflected, rotates around the wave generator. On the other hand, the last contact considers the presence of static friction between the teeth of the flexible spline and those of the ring gear.

However, these contacts are not active throughout all steps of the model. It would be troublesome to have them active at the beginning of the analysis because, at this stage, the wave
generator interferes with the flexible spline which itself interferes with the ring gear. Consequently, there must be different events in the two-dimensional model which organize the engagement and disengagement of the different contacts, as well as the application of boundary conditions and loads.

### 3.4.5 Boundary conditions and steps to perform simulation of meshing

The boundary conditions that the model implements are critical towards the simulation of meshing. They define the fixtures and movements that take place during the meshing simulation of SWG drives. The boundary conditions are applied on the reference points (RP) of each element, shown in Figure 3.19. The boundary conditions are divided between boundary conditions restraining motion and boundary conditions defining motion, where they are used to define fixtures and movements, respectively. Considering SWG drives where the input occurs through the wave generator and the output is either through the flexible spline or the ring gear, the boundary conditions restraining motion are:

- **Ring gear**: fixed displacement and rotation on $RP_{rg}$ or fixed displacement of any node of its outer diameter.
- **Flexible spline**: fixed displacement and rotation on $RP_{fs}$.
- **Pushing pins**: fixed displacement on the x-direction and fixed rotation on $RP_{pp}$.
- **Wave generator**: fixed displacement and rotation on $RP_{wg}$.

The boundary conditions defining motion of the elements of the SWG drive to simulate its meshing are:

- **Pushing pins**: displacement on the y-direction on $RP_{pp}$.
- **Wave generator**: rotation around $RP_{wg}$.

In order to represent a realistic model, this two-dimensional model incorporates several steps that bring the aforementioned assembly, interactions, and boundary conditions to operational status in SWG drives. The first step brings the pushing pins into contact with the flexible
spline, which in turn is deformed into the meshing position with the ring gear. During this step, the contacts between the wave generator and the flexible spline and between the teeth of the flexible spline and those of the ring gear are deactivated. This eliminates any possible conflicts that the interference of the flexible spline with the ring gear and the wave generator may cause. The boundary conditions restraining motion in effect during this step are the ones applied to the flexible spline and the pushing pins. The only boundary condition defining motion in effect is the displacement on the y-direction of the pushing pins, in order to find the contact with the flexible spline and deflect the latter into meshing with the ring gear. The final point of the pushing pins is such that each pushing pin forces the flexible spline to deflect half the total deflection that it experiences. Considering that there are two pushing pins, their final position sets the deformed condition of the flexible spline to operate in SWG drives.

The next step consists of changing the contact interaction from occurring between the flexible spline and the pushing pins to instead occur between the wave generator and the flexible spline. Because the final position of the pushing pins after the first step matches the geometry of the wave generator, the change of contact can happen. Also, in this second step, the contact between the teeth of the flexible spline and those of the ring gear is activated. These two contacts are the main interactions that will continue taking place until the end of the model to simulate the operation of SWG drives and the meshing between the flexible spline and ring gear. All the boundary conditions restraining motion are in effect during this step. On the other hand, the boundary condition defining motion for the pushing pins is active again, but in the opposite direction, so that the pushing pins come back to their original position at the end of their task in this model. From now on, the pushing pins do not participate in the model. Consequently, the contact between the pushing pins and the flexible spline is deactivated and will remain deactivated throughout the analysis.

The remaining steps of the two-dimensional model consist of small rotations of the wave generator serving as input until half a rotation of the wave generator is completed, so that the meshing of the SWG drive can be simulated. Half a rotation of the wave generator is considered enough because it allows the output toothed element, the flexible spline or the ring gear, to switch positions one tooth slot, which would be repeated during the second half of the rotation,
making the analysis redundant. During these steps, the only interactions taking place are the contact between the wave generator and the flexible spline and the contact between the flexible spline and the ring gear.

The applied boundary conditions restraining motion depend on the selected output toothed element of the SWG drive. If the output is through the flexible spline, the displacement and rotation of the ring gear and the displacement of the wave generator are fixed. On the other hand, if the output is through the ring gear, the displacement and rotation of the flexible spline reference point $R_{f s}$ are fixed while the outer circumference of the ring gear is not allowed to deform. This would allow the flexible spline to deform towards meshing without rotating throughout the simulation. The wave generator is rotated by applying its boundary condition defining motion. To make the meshing process smooth, the half rotation that the wave generator performs is divided in several reduced rotation steps that the user selects with IGD. Each new small rotation that the wave generator performs increments its rotation angle counterclockwise until half a rotation is completed.

**Load application**

In order to make the model more realistic, a constant torque $T$ is applied on the output element to simulate the effect of torque transmission by the drive. At a later stage, this torque can be modified to represent the actual non-linear output torque that SWG drives transmit.

The application of torque in the output toothed element of the SWG drive requires the creation of a constraint of the continuum distributing type [64]. When the flexible spline is the output, this constraint links the points on the inner surface of the flexible spline with its reference point $R_{f s}$. On the other hand, when the ring gear is the output, the constraint links the points on the outer surface of the ring gear with its reference point $R_{r g}$. The constraint allows for any boundary condition or load applied on a reference point $R_{i}$ to be transmitted to linked points. For this reason, it is used to transmit torque during the simulation of meshing.

The torque is applied to the reference point of the output element during the third step when the wave generator is engaged with the flexible spline. While the wave generator rotates to simulate the meshing, the torque is held constant.
3.4.6 Advantages and disadvantages of the two-dimensional finite element model

After the implementation of the two-dimensional finite element model for the analysis of SWG drives, several studies can be performed. These studies focus on the influence that different geometries of each component of a SWG drive have on the distribution of stresses of the flexible spline and ring gear.

The distributions of stresses provide information about where the higher stresses are located and if a geometrical feature of an element of SWG drives is causing a large concentration of stresses somewhere in the drive. These higher stresses must be targeted for reduction, which may be accomplished by modifying the geometry of the element of the drive that is causing them.

The main advantage of the two-dimensional finite element model is the lack of complexity and time to generate models of SWG drives with different geometries. Several geometries of the wave generator and tooth profiles for the flexible spline and the ring gear teeth were defined earlier in this chapter and their performance as a function of their parameters will be evaluated in later chapters of this thesis.

Using a two-dimensional finite element model to analyze SWG drives does not provide information about how the flexible spline cup influences the transmission of motion and torque. The two-dimensional model does not show how the contact develops in longitudinal direction between the different elements of the drive. The fact that the two-dimensional model only considers a planar state of stress leads to this lack of information.

As explained in Chapter 1, a typical SWG drive is an asymmetric mechanism in the longitudinal direction due to the flexible spline cup protruding beyond the toothed region of the drive. Therefore, it is necessary to develop a three-dimensional finite element model to study further the influence of the geometrical features of SWG drives over their distributions of bending and contact stresses considering the cup-shaped spring of the flexible spline.
3.5 Three-dimensional finite element model of SWG drives

The three-dimensional finite element model of SWG drives differs from the two-dimensional model by considering the cup-spring of the flexible spline, as well as by providing depth to the two-dimensional model. The three-dimensional model includes finite elements along the face width of the different elements of the SWG drive. Consequently, it permits the consideration and evaluation of micro-geometry modifications on the flank surfaces of the teeth of the flexible spline and the ring gear, and also on the outer geometry of the wave generator. These micro-geometry modifications are intended to counteract the negative effect of particular geometrical features of SWG drives over bending and contact stresses.

In the three-dimensional model of SWG drives and similarly to the two-dimensional model, an input file is created with IGD which directly includes all the design parameters for a particular SWG drive design [13]. Each design corresponds to a unique finite element input file to be analyzed with the finite element software solver.

The three-dimensional model also includes five parts where two are deformable bodies and the rest are rigid bodies. These are mainly generated as an extrusion of the elements of the two-dimensional model in the z-axis. This extrusion is performed by expanding the parts of the two-dimensional model half a face width $F_w$ towards the positive z-axis and the other half toward the negative z-axis. However, the main difference between the two- and three-dimensional models is the inclusion of the cup-shaped spring of the flexible spline. This serves to evaluate the influence of the full geometry of the flexible spline during the meshing process when it deflects twice per revolution of the wave generator.

3.5.1 Cup-shaped spring of the flexible spline definition

The cup-shaped spring of the flexible spline is defined in IGD with the following parameters:

- Length of the cup, $l_c$.
- Thickness coefficient of the cup, $c_c$.
- Fillet radius of the cup, $r_{cf}$. 

3.5. Three-dimensional finite element model of SWG drives

- Inner radius of the cup, \( r_{ci} \).

These parameters determine the geometry of the flexible spline beyond its teeth as shown in Figure 3.21. The length of the cup \( l_c \) is the axial dimension (along the \( z_{fs} \) axis) between the back of the teeth of the flexible spline and the closed-end of its cup-shaped spring. Therefore, the total axial length of the flexible spline in the three-dimensional model consists of the sum of the face width \( F_w \) of the flexible spline and the length of its cup \( l_c \).

![Figure 3.21: Definition of the cup-spring of the flexible spline.](image)

The thickness coefficient of the cup \( c_c \) defines the thickness of the cup as a function of the rim thickness \( t_r \) of the teeth of the flexible spline with the following formula,

\[
t_c = c_c t_r
\]  

(3.50)

where \( t_c \) is the resulting thickness of the flexible spline cup. The thickness coefficient of the cup \( c_c \) is of a value between zero and one in order for the thickness of the cup to be equal or smaller than the rim thickness \( t_r \) of the flexible spline. This creates a gap between the root region of the teeth of the flexible spline and the outer radius of its cup as shown in Figure 3.21.
The geometry of the cup-shaped spring of the flexible spline is further defined by the radius of its fillet $r_{cf}$. This parameter defines the radius of the outer side of the fillet between the cylinder and closed-end regions of the cup. Finally, since the flexible spline is typically the output member of the drive, the closed-end of the cup includes a bore of radius equal to $r_{ci}$ named as the inner radius of the cup. This would serve as connection to the output shaft by defining the boundary conditions of the flexible spline on its reference point $RP_{fs}$. This point is located on the center of the bore of the closed-end of the flexible spline in the three-dimensional model, instead of the center of the flexible spline as in the two-dimensional model of SWG drives.

3.5.2 Deformable parts of the three-dimensional model

Similarly to the two-dimensional model, the deformable parts of the three-dimensional model of SWG drives are the flexible spline and the ring gear. Here, the original mesh for each tooth of the flexible spline and the ring gear is similar to that shown in Figure 3.18. However, the expanded mesh of the flexible spline in the three-dimensional model is as shown in Figure 3.22, where the rim section of the flexible spline has been adapted to connect to the elements of its cup-shaped spring. The number of elements on each of the meshed regions is also selected in IGD for the tooth mesh of both the flexible spline and the ring gear.

![Figure 3.22: Three-dimensional parameterized finite element mesh on a flexible spline tooth.](image)

In regards to the tooth meshes of the flexible spline and the ring gear, an additional number of elements has to be specified in IGD for the three-dimensional model as compared to the
two-dimensional model; this is the number of elements in the longitudinal direction. This is typically chosen as one fewer element than the number of elements in the longitudinal direction selected for the wave generator to improve convergency once the flexible spline is deformed and contacting the wave generator.

The cup-shaped spring of the flexible spline is meshed following the number of elements of the rim section of the flexible spline towards the closed-end of the cup of the flexible spline. Figure 3.23 shows the finite element mesh of a tooth of the flexible spline including its corresponding meshed section of the cup. The cup mesh is connected to the inner two elements of the rim region of the flexible spline mesh while the bore of the closed-end of the flexible spline is rigidly connected to the reference point of the flexible spline $RP_{fs}$. This point is located on the center of the bore at the closed-end of the flexible spline cup. The number of elements on
the mesh of the cup of the flexible spline is selected in IGD for the cup cylinder, fillet, and closed-end regions of the cup shown in Figure 3.23.

![Three-dimensional parameterized finite element mesh on a ring gear tooth.](image)

**Figure 3.24:** Three-dimensional parameterized finite element mesh on a ring gear tooth.

Figure 3.24 shows the mesh of a tooth of the ring gear where the mesh is identically defined to that of Figure 3.18, but extruded along the longitudinal direction. As aforementioned, the only additional number of elements to be specified is that of the longitudinal direction of the teeth of the ring gear.

Figure 3.25 shows the entire geometries of the flexible spline a) and the ring gear b) included in the three-dimensional finite element model of SWG drives. The definition of these geometries is parameterized in IGD and automatically generated into a finite element input file. The flexible spline and the ring gear are shown in their respective coordinate systems $S_{fs}$ and $S_{rg}$ which coincide as the global coordinate system $S$ in the finite element model.

The deformable parts of the three-dimensional model differ from their previous definition in the two-dimensional model by being expanded into three dimensions, including the cup-spring of the flexible spline, and changing the location of the reference point of the flexible spline $RP_{fs}$. In the two-dimensional model, this point $RP_{fs}$ is located on the center of the entire geometry of the flexible spline whereas here, the reference point of the flexible spline $RP_{fs}$ is located on the center of the bore at the closed-end of the flexible spline. This is because the reference node $RP_{fs}$ is where the boundary conditions of the flexible spline are applied. The reference point of the ring gear $RP_{rg}$, however, is located on the center of its entire geometry, similarly to the
3.5. Three-dimensional finite element model of SWG drives

Figure 3.25: Deformable parts of the three-dimensional finite element model of SWG drives: (a) Flexible spline, (b) Ring gear.

two-dimensional model of SWG drives.

The element type employed for the deformable parts of the three-dimensional model is C3D8I, which includes eight nodes and incompatible modes. This type of element is suitable for bending, which is the loading that the flexible spline is mostly subjected to due to its imposed deflection by the wave generator [64].

Similarly to the two-dimensional model of SWG drives, the material properties of the flexible spline and the ring gear are specified in IGD. These are the mechanical properties of modulus of elasticity $E$ and Poisson’s ratio $\nu$.

3.5.3 Rigid parts of the three-dimensional model

The three-dimensional finite element model of SWG drives includes the same rigid parts as the two-dimensional model: the pushing pins (2) and the wave generator. Here, the elliptical geometry of the wave generator is selected for reference as it is the most typically used geometry of the wave generator [1]. Figure 3.26 shows the complete geometry of the pushing pins a) and the wave generator b) in their initial position at the beginning of an analysis. These parts are
shown in their coordinate systems \(S_{pp}\) and \(S_{wg}\) for the pushing pins and the wave generator, respectively. Once assembled in the finite element analysis, these coordinate systems coincide in the global coordinate system \(S\). The major axis of both the pushing pins and the wave generator is located along the y-axis, while the minor axis runs along the x-axis.

a) Pushing pins

b) Wave generator

![Diagram of coordinate systems for pushing pins and wave generator](image)

**Figure 3.26:** Rigid parts of the three-dimensional finite element model of SWG drives: (a) Pushing pins, (b) Wave generator.

The main difference of the rigid parts between the two- and three-dimensional models is the extrusion of the pushing pins and the wave generator along the longitudinal direction. Similarly to the two-dimensional model of SWG drives, the reference points of the pushing pins \(RP_{pp}\) are located on the center of the geometry that does not contact the flexible spline. The reference point of the wave generator \(RP_{wg}\) is located on the center of its entire geometry.

The pushing pins and the wave generator are meshed with elements of type R3D4, which are four-noded bilinear elements with four sides [64]. These form the rigid three-dimensional geometries of the pushing pins and the wave generator in the three-dimensional model of SWG drives. The elements of the pushing pins and the wave generator are connected to their reference points \(RP_{pp}\) and \(RP_{wg}\), respectively, where the boundary conditions are applied throughout the analysis.
3.5.4 Assembly of the elements of the drive

The assembly of deformable and rigid parts in the three-dimensional model of SWG drives is parameterized in IGD and provided to the finite element software solver as shown in Figure 3.27 in the global coordinate system $S$. The reference points of the ring gear $R_{P_{rg}}$ and the wave generator $R_{P_{wg}}$ coincide in the assembly. However, the reference point $R_{P_{pp}}$ of the upper pushing pin is slightly below the center of coordinates $O$ and the reference point $R_{P_{pp}}$ of the lower pushing pin is slightly above the center of coordinates $O$. Their distance from the center of coordinates $O$ is equal to the to-be-imposed deflection $d$ on the flexible spline by the wave generator. Finally, the reference point of the flexible spline $R_{P_{fs}}$ is located on the center of the bore in the closed-end as the output member of the SWG drive.

Figure 3.27: Three-dimensional finite element model of SWG drives.

At the beginning of the analysis, similarly to the two-dimensional model, the flexible spline is not deformed in the three-dimensional model as shown in Figure 3.27. This causes the flexible spline to overlap with the wave generator near the major axis regions of the drive in this initial
position. The teeth of the flexible spline also interfere with several teeth of the ring gear near the minor axis regions of the SWG drive at this stage, as shown in Figure 3.20. This is not a concern at the beginning of the analysis, as the contacts between the flexible spline and the wave generator and between the flexible spline and the ring gear are not activated until the deflection $d$ has been fully applied.

The interactions between the different parts of the three-dimensional model of SWG drives are identical to those of the two-dimensional model.

### 3.5.5 Boundary conditions and steps to perform simulation of meshing

The boundary conditions applied in the three-dimensional model of SWG drives differ from the two-dimensional model in the number of degrees of freedom that have to be restricted, as well as the location of the application of the boundary conditions. However, the boundary conditions are mostly the same as in the two-dimensional model, either defining a fixture or a particular motion of an element of the drive. The boundary conditions restraining motion include the degree of freedom along the z-axis so that the SWG drive does not slip out of assembly once deflection is applied and load transmitted. On the other hand, the boundary conditions applied to the flexible spline are applied on its reference point $RP_{fs}$ on the center of the bore in the closed-end. In the two-dimensional model this reference point is in the center of the flexible spline.

The steps organizing the application of the boundary conditions in the three-dimensional finite element model are similar to those of the two-dimensional model. The first step includes the contact between the flexible spline and the pushing pins, which serve to deflect the flexible spline into meshing position by moving the pushing pins along the major axis the deflection $d$. During this step, the ring gear and the wave generator are fixed and not interacting with the flexible spline or the pushing pins.

The second step constitutes the end of the usefulness of the pushing pins by releasing their contact with the flexible spline. The flexible spline engages with the wave generator for this to be the input member throughout the rest of the analysis. This is done by activating the contact interaction between the inner diameter of the flexible spline and the outer geometry of the
wave generator. The interactions of the pushing pins are deactivated from this step on. The full meshing position between the flexible spline and the ring gear is also achieved in this step by activating the contact interaction between all the teeth of the flexible spline and the ring gear. This contact and the contact between the flexible spline and the wave generator are possible now due to the applied deflection $d$ on the flexible spline. After being deflected, the teeth of the flexible spline near the minor axis move inward and away from interfering with the teeth of the ring gear, whereas the teeth of the flexible spline near the major axis move outward and away from the major axis region of the wave generator previously overlapping with the flexible spline during the first step.

Similarly to the two-dimensional model, the remaining steps of the three-dimensional model simulate the meshing between the teeth of the flexible spline and those of the ring gear by rotating the wave generator in small increments as the input member of the drive. During these steps, the wave generator is rotated by applying its boundary condition defining motion while the wave generator contacts the flexible spline and the teeth of the flexible spline contact those of the ring gear.

The simulation of half an input rotation of the wave generator leads the three-dimensional model to require a huge computational effort to analyze, as compared to the two-dimensional model. For this reason, only 45 degrees of input rotation of the wave generator will be considered to evaluate the stresses that the different elements of the SWG drive are subjected to.

**Load application**

In terms of load application, a constant torque $T$ is also applied here to the output element of the drive to simulate the effect of the converted and transmitted torque in SWG drives. Similarly to the two-dimensional model, a constraint is required on the output element to impose the torque $T$ [64]. When the flexible spline is the output element, this constraint links the points on the bore of the close-end of the cup of the flexible spline with its reference point $RP_{fs}$. On the other hand, when the ring gear is the output element, the constraint links the nodes on the outer surface of the ring gear to its reference point $RP_{rg}$. 
Here, the output torque $T$ is also applied during the third step after the wave generator has been engaged with the flexible spline and the input rotation of the wave generator begins.

### 3.5.6 Advantages of the three-dimensional model

After the implementation of the three-dimensional finite element model of SWG drives, multiple studies can be performed to evaluate the resulting distributions of stresses on the flexible spline and the ring gear. These analyses differ from those performed with the two-dimensional model on the fact that the three-dimensional model includes the cup-shaped spring of the flexible spline, which considerably affects its performance. In this case, instead of deflecting the flexible spline on a planar surface, the deflection is affected by the cup of the flexible spline so the teeth are deflected outwards near the major axis of the drive and inwards near the minor axis while the cup sectors bend around the bore of its closed-end.

The analyses with the three-dimensional model show where concentrations of stresses are located on SWG drives with the consideration of the cup, as well as what the maximum stresses are. Large maximum stresses and their concentration areas can be targeted for modification. These modifications can be of the design parameters of the SWG drive under study. However, the modification of the parameters of the drive has a limit on the reduction of stresses that can be achieved. In order to further reduce the stresses experienced by the SWG drive throughout the simulation of meshing with the three-dimensional model, micro-geometry modification of the tooth flank surfaces of the flexible spline and the outer surface of the wave generator is necessary.

The main advantage of the three-dimensional finite element model of SWG drives is the inclusion of the cup-shaped spring of the flexible spline, which serves to appropriately evaluate the geometry of the different features of each element of SWG drives.
Chapter 4

Two-dimensional simulation of meshing and stress analysis

4.1 Chapter overview

This chapter provides the initial results and discussion about the use of the two-dimensional model for stress analysis and simulation of meshing of SWG drives.

Section 4.2 explains the context from which the results have to be interpreted, which include the parameters of the SWG drive reference design, the characteristics of this design, the system of units, and the computational performance required for each model. Then, the initial state and the required steps of the two-dimensional model are explained in Section 4.3, leading to the application of deflection to achieve contact in Section 4.4.

Section 4.5 explains the simulation of meshing of SWG drives and Section 4.6 shows the differences between tensile and compressive stresses obtained with the two-dimensional model. The approach for evaluation of different designs follows the explanation of stresses in Section 4.7.

Section 4.8 illustrates the effect of torque transmission on the stresses experienced by the different elements of the SWG drive. The stresses of the deformable elements are then compared in Section 4.9.

Finally, Sections 4.10 and 4.11 show the development of stresses in the root of the flexible spline and the influence of its rim thickness over stresses obtained with the two-dimensional model of SWG drives, respectively.
4.2 Context of the results

Unless otherwise specified, the results are obtained for a reference SWG drive design with the parameters shown in Table 4.1. The layout for this case is the traditional SWG drive layout, where the wave generator serves as the input member of the drive while the ring gear is fixed. The flexible spline is the output member in this layout.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Flexible spline</th>
<th>Ring gear</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of teeth, $N$</td>
<td>120</td>
<td>122</td>
<td>-</td>
</tr>
<tr>
<td>Module, $m$</td>
<td>1</td>
<td>mm</td>
<td></td>
</tr>
<tr>
<td>Pressure angle, $\alpha$</td>
<td>20</td>
<td>deg</td>
<td></td>
</tr>
<tr>
<td>Addendum coef., $h_a$</td>
<td>0.6</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>Dedendum coef., $h_d$</td>
<td>0.8</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>Face width, $F_w$</td>
<td>10</td>
<td>mm</td>
<td></td>
</tr>
<tr>
<td>Rim thickness, $t_r$</td>
<td>1</td>
<td>2</td>
<td>mm</td>
</tr>
<tr>
<td>Major axis, $a$</td>
<td>59.2</td>
<td>mm</td>
<td></td>
</tr>
<tr>
<td>Type of tooth profile</td>
<td>QCA</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>Radius of tip circular arc, $r_t$</td>
<td>0.3</td>
<td>mm</td>
<td></td>
</tr>
<tr>
<td>Radius of addendum circular arc, $r_a$</td>
<td>1.0</td>
<td>mm</td>
<td></td>
</tr>
<tr>
<td>Length of straight segment, $l$</td>
<td>0.2</td>
<td>mm</td>
<td></td>
</tr>
<tr>
<td>Radius of dedendum circular arc, $r_d$</td>
<td>2.0</td>
<td>mm</td>
<td></td>
</tr>
<tr>
<td>Radius of root circular arc, $r_r$</td>
<td>0.3</td>
<td>mm</td>
<td></td>
</tr>
<tr>
<td>Angular tooth thickness reduction coef., $c_P$</td>
<td>1.0</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>Straight segment length reduction coef., $c_l$</td>
<td>0.5</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>Modulus of elasticity, $E$</td>
<td>210</td>
<td>GPa</td>
<td></td>
</tr>
<tr>
<td>Poisson’s ratio, $\nu$</td>
<td>0.29</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>Output torque, $T$</td>
<td>20</td>
<td>Nm</td>
<td></td>
</tr>
</tbody>
</table>

The flexible spline of the reference SWG drive case has 120 teeth and the ring gear has 122 teeth. Using Equation (2.2), the gear reduction ratio $m_G$ is -60. To obtain a full rotation of the flexible spline (the output member of the drive) in a particular direction, the wave generator has to rotate 60 revolutions in the opposite direction.

The tooth profile employed for the teeth of the flexible spline and the ring gear is the QCA tooth profile, using the radii and length shown in Table 4.1 for the circular arcs and the straight segment, respectively. Figure 4.1 shows the geometry of a tooth of the flexible spline and a tooth of the ring gear with the defined QCA profile.

The considered angular tooth thickness reduction coefficient $c_P$ equal to 1.0 provides QCA profile teeth without backlash. This means that the thicknesses of a tooth of the flexible spline
and a tooth of the ring gear are the same in the undeformed condition. The pitch point of the teeth of the flexible spline and the ring gear is located on the center of the straight segment of their QCA profile, due to the straight segment length reduction coefficient $c_l$ being equal to 0.5.

The material selected for the flexible spline and the ring gear elements of the SWG drive is steel, with mechanical properties shown in Table 4.1. In order to reproduce the torque transmission, 20 Nm of output torque $T$ are applied on the flexible spline considering the most common layout of SWG drives, as shown in Figure 1.3 $a$). The wave generator and the pushing pins are modeled as rigid elements, which means that their mechanical properties are not needed [64].

Throughout the results, the unit system employed is the International System of Units (SI), where distances are measured in millimeters (mm), forces are measured in Newton (N), and stresses are measured in Mega Pascals (MPa). This is because computing the results using distances as millimeters and forces as Newton provides Mega Pascals automatically for stress variables. These stresses can be divided in tensile and compressive stresses. Tensile stress is obtained from the maximum principal stress $\sigma_1$ computed by the finite element software solver using the input files from the models. On the other hand, compressive stress is obtained from the minimum principal stress $\sigma_3$ also computed by the finite element software solver. The minimum principal stress $\sigma_3$ is converted to absolute values to be shown as compressive stress [64].
In order to evaluate the mechanical strength of ductile materials, von Mises stress is generally used. Von Mises stress is a scalar value computed from the Cauchy stress tensor, which takes into account the principal stresses. When a particular state of stresses leads to von Mises stress higher than the yield strength of the material, the material will yield beyond its elastic limit. Von Mises stress will be used for comparison of different SWG drive designs because the absolute values of tensile and compressive stresses in SWG drives vary significantly, yet sometimes have comparable averages.

A module for the automatic generation of completely parameterized two-dimensional finite element models has been developed in the custom-made software IGD. After generating a model, all the parameters of the SWG drive design included in the model have been introduced. There is no need to introduce any additional parameters after this point. The finite element software solver ABAQUS is then used to perform the analysis of the finite elements models [64]. The finite element model of the considered SWG drive employs 35 steps to carry out half a rotation of the wave generator. This model includes 53,815 nodes and 46,550 finite elements, which constitute the flexible spline, the ring gear, the wave generator, and two pushing pins. The analysis of this finite element model takes approximately two hours using four cores of a processor Intel(R) Xeon(R) CPU E3-1240 v6 of a desktop computer of the Gear Research Laboratory at RIT.

4.3 Initial state and steps of the two-dimensional model

This section explains several aspects of the behavior and results obtained using the two-dimensional finite element model for stress analysis and simulation of meshing of SWG drives.

The SWG drive reference design uses a simplified wave generator. The geometry of the wave generator is defined with a minor axis reduction coefficient $C$ equal to 0.8. This leaves some clearance between the wave generator and the flexible spline along the minor axis direction, as shown in Figure 4.2.

The left of Figure 4.2 shows the initial state of the two-dimensional finite element model of the selected SWG drive design. The pushing pins are located inside the flexible spline together with the wave generator, which interferes with the flexible spline until the latter has been fully
4.3. Initial state and steps of the two-dimensional model

The ring gear is fixed outside the flexible spline with some interference between their teeth, especially along the minor axis direction of the wave generator. The detail in the right of Figure 4.2 shows the mapped mesh on the flexible spline and the ring gear for the case under study. The mesh divides each tooth vertically in two halves, with finer elements near the tooth flanks. The rim regions of the flexible spline and the ring gear implement coarser elements.

Figure 4.3 shows the variation of maximum tensile stress on the flexible spline element throughout a full analysis using the two-dimensional finite element model. The horizontal axis shows the different steps that the model includes. During Step 1, the pushing pins impose the deflection on the flexible spline of the SWG drive. The wave generator engages into contact with the flexible spline inner surface in Step 2 so that the pushing pins can be released. In this step, the contact between the teeth of the flexible spline and the ring gear is also activated. In Step 3, the wave generator is in its initial position for the analysis and then the torque is applied on the output element of the drive. It is applied until it reaches its full value, which is Step 4. This is the step when the setup for the process of simulation of meshing is finalized.

From Step 4 and on, the wave generator rotates in small increments of its rotation angle for each step to perform the steady-state simulation of meshing, until half a rotation of the wave generator is complete. The maximum tensile stress during the first two steps is considerably
Chapter 4. Two-dimensional simulation of meshing and stress analysis

Two-dimensional finite element analysis step [number]

Step 1: Pins deflect flexible spline
Step 2: Wave generator engages and pins are released
Step 3: Torque applied with wave generator in initial position

**FIGURE 4.3:** Variation of maximum tensile stress on the flexible spline throughout the steps of the two-dimensional finite element analysis.

lower than that obtained during the simulation of meshing beyond Step 3. The tensile stress at Step 3 is also lower due to the beginning of the application of torque. From Step 4 and onward, the torque is fully applied.

For stress analysis of SWG drives, Steps 3 and onward are considered. This serves to show the effect of the application of torque on the stresses of the drive at the beginning of the analysis, as well as the steady-state simulation of meshing. The simulation of meshing is performed by rotating the wave generator beginning in Step 4, after the torque reaches its full amount with the wave generator fixed.

4.4 Application of deflection to achieve contact

At the beginning of each analysis, the teeth of the flexible spline interfere with those of the ring gear due to the flexible spline not being deformed yet. During Step 1 of the model, the
4.4. Application of deflection to achieve contact

pushing pins deflect the flexible spline into meshing position with the ring gear. Figure 4.4 shows the distribution of compressive stresses on the flexible spline and the ring gear around the point of contact between the flexible spline and one of the pushing pins. This is the region, along the major axis of the drive, where the contact will be located between a lobe of the wave generator and the inner surface of the flexible spline. Figure 4.4a) shows the compressive stress at the beginning of the application of deflection by the pushing pins. Since the ring gear is not contacting any element, its compressive stress is zero. Conversely, the inner diameter of the flexible spline experiences compressive stresses due to being in contact with the pushing pins at the beginning of the deflection.

Figure 4.4b) shows the final position of the flexible spline teeth once the maximum deflection d has been imposed by the pushing pins. Here, the outer geometry of the pushing pins contacting the flexible spline coincides with the geometry of the wave generator, as shown in Figure 4.5. This allows to disable the contact between the flexible spline and the pushing pins and, at the same time, enable the contact between the flexible spline and the wave generator during the second step of the analysis. From this point onwards, the pushing pins are not used for the analysis. Due to the higher deflection imposed on the flexible spline, its compressive stresses on the rim region increase during the application of deflection while the ring gear remains without contact.

Then, in Step 2, the contact between the teeth of the flexible spline and those of the ring gear is activated, creating regions with higher compressive stress near the flank surfaces of the teeth, as shown in Figure 4.4c). However, the highest compressive stresses are still concentrated near the inner diameter of the flexible spline and the overall compressive stresses slightly reduce. The regions of higher compressive stresses on the tooth flanks indicate the location of the tooth-to-tooth contact between the teeth of the flexible spline and the ring gear. This constitutes one of the most challenging problems towards obtaining convergence in the two-dimensional model, due to the large deformation imposed on the flexible spline element and the difference in tooth thicknesses between the teeth of the flexible spline and the ring gear.

Figure 4.6 shows the distribution of tensile stresses on the flexible spline and the ring gear, during the same steps as shown in Figure 4.4. At the beginning of the application of deflection
Chapter 4. Two-dimensional simulation of meshing and stress analysis

Figure 4.4: Distribution of compressive stresses on the flexible spline and the ring gear: (a) beginning of deflection, (b) maximum deflection, and (c) tooth contact activation (Step 2).
by the pushing pins, the only tensile stresses are located around the root region of the teeth of the flexible spline, as shown in Figure 4.6 a). When the deflection $d$ is fully applied but the contact between the teeth of the flexible spline and the ring gear is not active, the tensile stresses on the flexible spline are still located on the root region of its teeth. However, the tensile stresses reach a maximum value of 318.288 MPa, as shown in Figure 4.6 b). Figure 4.6 c) shows the distribution of tensile stresses when the contact is localized between the teeth of the flexible spline and those of the ring gear in Step 2. Here, due to the contact with the teeth of the ring gear, the maximum tensile stresses still located on the root region of the flexible spline teeth are slightly reduced to become 295.453 MPa.

During the deflection of the flexible spline into contact with the ring gear, which occurs from the beginning of the analysis until the localization of tooth contact in the model, the tensile stresses are mostly located on the flexible spline and the ring gear does not experience any significant tensile stresses, as shown in Figure 4.6.
### Chapter 4. Two-dimensional simulation of meshing and stress analysis

#### Figure 4.6: Distribution of tensile stresses on the flexible spline and the ring gear: (a) beginning of deflection, (b) maximum deflection, and (c) tooth contact activation (Step 2).

<table>
<thead>
<tr>
<th>Max. Principal Stress $\sigma_1$ (MPa)</th>
<th>Value</th>
<th>Color</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) Ring gear</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7.227</td>
<td>Red</td>
<td></td>
</tr>
<tr>
<td>6.624</td>
<td>Orange</td>
<td></td>
</tr>
<tr>
<td>6.022</td>
<td>Yellow</td>
<td></td>
</tr>
<tr>
<td>5.420</td>
<td>Green</td>
<td></td>
</tr>
<tr>
<td>4.818</td>
<td>Cyan</td>
<td></td>
</tr>
<tr>
<td>4.216</td>
<td>Blue</td>
<td></td>
</tr>
<tr>
<td>3.613</td>
<td>Dark Blue</td>
<td></td>
</tr>
<tr>
<td>3.011</td>
<td>Dark Green</td>
<td></td>
</tr>
<tr>
<td>2.409</td>
<td>Light Green</td>
<td></td>
</tr>
<tr>
<td>1.807</td>
<td>Light Blue</td>
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</tr>
<tr>
<td>1.204</td>
<td>Purple</td>
<td></td>
</tr>
<tr>
<td>0.602</td>
<td>Pink</td>
<td></td>
</tr>
<tr>
<td>0.000</td>
<td>White</td>
<td></td>
</tr>
</tbody>
</table>

| b) Ring gear                         |       |       |
| 318.288                              | Red   |       |
| 291.764                              | Orange|       |
| 265.240                              | Yellow|       |
| 238.716                              | Green |       |
| 212.192                              | Cyan  |       |
| 185.668                              | Blue  |       |
| 159.144                              | Dark Blue|     |
| 132.620                              | Dark Green|    |
| 106.096                              | Light Green|  |
| 79.572                               | Light Blue|   |
| 53.048                               | Purple|       |
| 26.524                               | Pink  |       |
| 0.000                                | White |       |

| c) Ring gear                         |       |       |
| 295.453                              | Red   |       |
| 270.832                              | Orange|       |
| 246.211                              | Yellow|       |
| 221.590                              | Green |       |
| 196.969                              | Cyan  |       |
| 172.348                              | Blue  |       |
| 147.727                              | Dark Blue|     |
| 123.106                              | Dark Green|    |
| 98.484                               | Light Green|  |
| 73.863                               | Light Blue|   |
| 49.242                               | Purple|       |
| 24.621                               | Pink  |       |
| 0.000                                | White |       |
4.5 Simulation of meshing by rotating the wave generator

In order to perform the simulation of meshing of SWG drives, the wave generator is rotated through half a rotation in the two-dimensional model. This is because when using a wave generator with two lobes, a full rotation of the wave generator simply doubles the results obtained with just half a rotation. Throughout the analysis, the wave generator is slowly rotated in small increments of its angle of rotation until completing a rotation of 180 degrees.

**Figure 4.7:** Distribution of compressive stresses on the flexible spline and the ring gear throughout 2 degrees of rotation of the wave generator: (a) two-teeth contact, (b) beginning of three-teeth contact, and (c) end of three-teeth contact.
Figure 4.7 shows the distribution of compressive stress around one of meshing regions of the flexible spline and the ring gear during a small rotation of the wave generator. The tooth contact is only located between the outside flanks of two teeth of the flexible spline and the inside flanks of two teeth of the ring gear, which means there are only two pairs of teeth in contact in each meshing region, as shown in Figure 4.7 \textit{a}).

When the wave generator is further rotated, one more pair of teeth engages into contact towards where the lobe of the wave generator is moving, establishing contact between three pairs of teeth, as shown in Figure 4.7 \textit{b}). This pair of teeth in contact also makes contact on the outside flank of a tooth of the flexible spline and the inside flank of a tooth of the ring gear from the point of view of the lobe of the wave generator. Figure 4.7 \textit{c}) shows how the center pair of teeth in contact switches contact from one flank to the opposite flank by further rotating the wave generator. Continuing the rotation of the wave generator, the pair of teeth in contact located farthest from where the lobe of the wave generator is moving to disengages, leading to the meshing state shown in Figure 4.7 \textit{a}), but rotated one tooth further.

Due to the lack of backlash between the teeth of the flexible spline and the ring gear of this SWG drive design, the contact only occurs between two to three pairs of teeth in each meshing region. This can be further increased by modifying the angular tooth thickness reduction coefficient $c_p$ to create backlash between the teeth of the flexible spline and the ring gear [36].

4.6 Understanding tensile and compressive stresses on the two-dimensional model

Due to their different behavior as compared to traditional gear drives, SWG drives experience comparable values of absolute compressive and tensile stresses. This is because of the deflection imposed on the flexible spline by the wave generator at all times.

Figure 4.8 shows the distribution of tensile stresses around the major and minor axes regions of the SWG drive with torque applied. Near the major axis of the drive, the maximum tensile stress is located in the root of the teeth of the flexible spline. On the other hand, near the minor axis, the highest tensile stresses are located on the inner diameter of the flexible spline. The
4.6. Understanding tensile and compressive stresses on the two-dimensional model

**Figure 4.8:** Distribution of tensile stresses around the major and minor axes regions of the SWG drive.
deflection imposed to the flexible spline causes its inner diameter to be loaded in tension near the minor axis, whereas the roots of the flexible spline teeth experience tension near the major axis. At this stage of the simulation of meshing, the wave generator has rotated 45 degrees counterclockwise and the maximum tensile stress reaches a value of 377.846 MPa on the flexible spline.

Figure 4.9 shows the distribution of compressive stresses around the major and minor axes regions of the SWG drive with torque applied. This is shown at the same stage of the meshing simulation as Figure 4.8. Near the major axis of the drive, the maximum compressive stress is located on the inner diameter of the flexible spline where the contact occurs between a lobe of the wave generator and the flexible spline. Due to tooth contact between the flexible spline and the ring gear, the flank surfaces of the teeth also experience compressive stresses around the major axis region, as shown in Figure 4.9. At this stage, the absolute maximum compressive stress reaches a value of 249.032 MPa on the flexible spline.

On the other hand, the compressive stresses near the minor axis of SWG drives are located on the root of the teeth of the flexible spline. This is also caused by the deflection imposed on the flexible spline. The rim of the flexible spline near the minor axis region is bent so that the roots of the flexible spline teeth experience compressive stresses while the inner diameter of the flexible spline experiences tensile stresses, as shown in Figures 4.8 and 4.9.

4.7 Approach for evaluation of different designs

The comparable values of absolute compressive and tensile stresses in SWG drives evaluated with the two-dimensional model leads to conflict when evaluating different designs. For this reason, von Mises stress as computed by the finite element software solver provides a suitable method to compare different designs. This is because von Mises stress takes into account both tensile and compressive stresses while still showing the location of the maximum tensile and absolute compressive stresses [64].

Figure 4.10 shows the distribution of von Mises stresses obtained with the two-dimensional model of SWG drives with torque applied. Around the major axis region of the drive, the maximum von Mises stress is located on the root of the flexible spline teeth. On the other hand,
4.7. Approach for evaluation of different designs

**Figure 4.9:** Distribution of compressive stresses around the major and minor axes regions of the SWG drive.
Figure 4.10: Distribution of von Mises stresses around the major and minor axes regions of the SWG drive.
von Mises stresses near the minor axis are relatively lower than those experienced near the major axis. This coincides with the results obtained when studying tensile and compressive stresses because von Mises stress provides a representation of both stresses. On the other hand and depending on the wave generator geometry, the absolute value of the compressive stresses can reach higher values than those of the tensile stresses in this type of gear drive. For these reasons, von Mises stress has been considered as the reference stress parameter for comparison of different SWG drive designs.

The location of maximum von Mises stress in this case coincides with the location of the maximum tensile stress, as shown in Figures 4.8 and 4.10. If the absolute maximum compressive stress is higher than the maximum tensile stress, the maximum von Mises stress would be located on the same spot as the maximum compressive stress. Consequently, von Mises stress provides a better representation of the overall mechanical behavior of SWG drives.

### 4.8 Effect of transmitted torque on stresses

In order to develop a realistic model of any type of gear drive, the transmitted torque has to be taken into account. For this reason, the two-dimensional model of SWG drives implements the application of torque on the output element of the drive. This is because the output torque is remarkably larger than the input torque in SWG drives due to their high reduction ratio [1].

Considering the flexible spline as the output member, Figure 4.11 shows the distribution of tensile stresses on the flexible spline during the rotation of the wave generator both when torque is not applied and when the output torque is 20 Nm. When torque is not applied, the average maximum tensile stress on the flexible spline during the rotation of the wave generator is equal to approximately 314.861 MPa, which rises to an average of 385.060 MPa when torque is applied.

The application of the output torque in the model satisfies the results provided in the existing literature [30, 36], as shown in Figure 4.11. The tensile stresses on the flexible spline are reduced in the region from where the wave generator is coming from, while the highest tensile stresses on the flexible spline are concentrated on the region towards where the wave generator is going to rotate.
**Chapter 4. Two-dimensional simulation of meshing and stress analysis**

**Figure 4.11:** Change of the distribution of tensile stresses on the flexible spline when torque is applied.
4.8. Effect of transmitted torque on stresses

4.8.1 Torque influence on the variations of maximum tensile, absolute compressive, and von Mises stresses

Figure 4.12 shows the variation of the maximum tensile stresses on the flexible spline element of the SWG drive throughout the process of simulation of meshing, including both cases with and without the application of torque. The tensile stresses fluctuate throughout the rotation of the wave generator due to the progressive engagement and disengagement of contact between different pairs of teeth. When torque is applied, the maximum tensile stresses on the flexible spline are considerably higher than when torque is not applied.

At the beginning of the rotation of the wave generator, the tensile stresses coincide for both cases with and without torque applied, as explained above. At the beginning of the simulation of meshing, the torque is progressively applied until reaching its maximum value.

Figure 4.13 shows the variation of maximum compressive stresses on the flexible spline during the simulation of meshing for both cases with or without torque applied. When torque is not applied, the maximum compressive stresses on the flexible spline are similar in value to the maximum compressive stresses with the wave generator fixed, as shown by the first point in
Figure 4.13: Variation of absolute maximum compressive stress on the flexible spline with and without application of torque.

Figure 4.13. On the other hand, when torque is applied, the maximum compressive stresses on the flexible spline considerably rise. The maximum compressive stresses fluctuate considerably more with torque applied. This phenomenon affects fatigue life on SWG drives, where the flexible spline is subjected to variable alternating stresses depending on the applied torque.

Figure 4.14 shows the variation of maximum von Mises stress on the flexible spline element of SWG drives for both cases with and without application of torque. When torque is applied, the maximum von Mises stress reaches considerably higher values than for the case when torque is not applied. The average maximum von Mises stress with torque applied is equal to 372.963 MPa, whereas, when torque is not applied, the average maximum von Mises stress is equal to 304.984 MPa. This is similar to considering the maximum tensile and absolute maximum compressive stresses as a function of torque applied, demonstrating that the use of von Mises stress constitutes a realistic approach to evaluate the mechanical performance of SWG drives.
4.9 Stress comparison of deformable elements

The complexity of SWG drives makes stress analysis more complicated than that of traditional gear drives. For this reason, it is advantageous to properly interpret the results obtained with the two-dimensional model so that accurate conclusions can be drawn from them.

Figure 4.15 shows the differences between the distributions of tensile stresses of the flexible spline and the ring gear elements on the region around a lobe of the wave generator. The tensile stresses experienced by the flexible spline are significantly larger than those experienced by the ring gear. Besides, as shown in Figure 4.7, the compressive stresses on the flexible spline caused by the deflection imposed by the wave generator are considerably larger than the compressive stresses created due to teeth contact with the ring gear. Consequently, the flexible spline appears to be the critical element towards stress analysis of SWG drives due to being subjected to deformation at all times and experiencing considerably higher tensile and compressive stresses than the ring gear.

The maximum tensile stresses on the flexible spline and the ring gear elements of SWG
Chapter 4. Two-dimensional simulation of meshing and stress analysis

Max. Principal Stress $\sigma_1$ (MPa)

**Flexible spline**

- 370.769
- 339.872
- 308.975
- 278.077
- 247.180
- 216.282
- 185.385
- 154.487
- 123.590
- 92.692
- 61.795
- 30.897
- 0.000

**Ring gear**

- 41.643
- 38.173
- 34.703
- 31.232
- 27.762
- 24.292
- 20.822
- 17.351
- 13.881
- 10.411
- 6.941
- 3.470
- 0.000

**Figure 4.15:** Comparison of distributions of tensile stresses between the flexible spline and the ring gear near a lobe of the wave generator.

**Figure 4.16:** Variation of maximum tensile stress on the flexible spline and the ring gear.
drives with torque applied are shown in Figure 4.16. The maximum tensile stresses during the meshing are remarkably lower on the ring gear than those experienced by the flexible spline during the simulation of meshing. This phenomenon indicates that studying the variation of maximum tensile stresses on the flexible spline provides crucial information for stress analysis of SWG drives.

![Figure 4.17: Variation of absolute maximum compressive stress on the flexible spline and the ring gear.](image)

Figure 4.17 shows the variations of absolute maximum compressive stress on the flexible spline and the ring gear when torque is applied. The maximum compressive stresses on the flexible spline element are considerably higher than those experienced by the ring gear. This is because the flexible spline is subjected to compressive stresses from both the interaction between the teeth of the flexible spline and the ring gear and between the inner diameter of the flexible spline and the wave generator. The latter is held while the teeth contact changes, as explained above.

Figures 4.16 and 4.17 provide evidence that the flexible spline is the critical element for mechanical performance of SWG drives. This justifies the use of the distribution and maximum value of stresses of the flexible spline for evaluation of the mechanical performance of SWG drives in two dimensions.
4.10 Development of stresses in the root of the flexible spline

The two-dimensional model of SWG drives has shown that the critical element for their mechanical performance is the flexible spline. This is caused by the rim of the flexible spline always being subjected to tensile or compressive stresses, which repeat twice per revolution of the wave generator. In traditional gear drives, the critical region towards transmitting higher loads is located on the root area of the pinion member, which is subjected to tensile stress due to the tooth-to-tooth contact with the ring gear, causing bending on the root of the tooth transmitting load. Conversely, in SWG drives, the roots of the teeth of the flexible spline are alternatively subjected to the maximum tensile and compressive stresses throughout its meshing. For this reason, it is necessary to study the stresses experienced by the roots of the flexible spline; this section focuses on showing them.

Figure 4.18 shows the location of the root element of the flexible spline that is under study. The figure shows the initial state of the analysis without the pushing pins. The stresses on the different nodes of this element are queried directly from the finite element solver and the node subjected to the highest stresses is considered for study, which is indicated as Point $P$ in Figure 4.18. The selected element on the root area of the flexible spline is located at the beginning of a
4.10. Development of stresses in the root of the flexible spline

tooth near the minor axis of the drive at the start of the process of simulation of meshing. This allows for the observation of stresses as a function of the position and movement of a lobe of the wave generator with respect to the root element under study.

![Graph showing evolution of tensile and compressive stresses on the root of the flexible spline as a function of the position of the wave generator.]

**Figure 4.19:** Evolution of tensile and compressive stresses on the root of the flexible spline as a function of the position of the wave generator.

Figure 4.19 shows the tensile and compressive stresses on the node of the root element of the flexible spline under study throughout the cycle of meshing as a function of the position of a lobe of the wave generator. This corresponds to a range of rotation of the wave generator from -90 degrees to 90 degrees with respect to the root studied. Consequently, when the wave generator reaches the zero degree position angle, a lobe is directly underneath the root of the flexible spline under study. The blue line shows the compressive stress while the green line shows the tensile stress experienced by the node of the flexible spline. At the beginning of the cycle of meshing, when the position angle of the lobe of the wave generator is between -90 degrees to approximately -35 degrees, the only stress that the root is subjected to is compressive stress. This is due to the flexible spline being deflected inwards near the minor axis region while the lobe of the wave generator is still far from contacting with the inner diameter of the flexible spline directly underneath the root.

As a lobe of the wave generator approximates the root of the flexible spline, the compressive
stress reduces in absolute value until becoming zero when the root starts to experience tensile stress due to the imposed deflection by the lobe of the wave generator. The sharp increase in tensile stress, shown in Figure 4.19 when the position angle of the wave generator is between -35 to around -5 degrees, is caused by the rotation of the lobe of the wave generator imposing deflection as well as the transmission of torque on the drive by the contacting pairs of teeth.

The tensile stress is suddenly reduced when the lobe of the wave generator contacts the flexible spline underneath the root under study, where the angular position of the wave generator is zero degrees. This is also caused by the pairs of teeth in contact between the flexible spline and the ring gear sharing the transmitted load, as well as the wedging tendency of the teeth of the flexible spline in the tooth slots of the ring gear. After this sudden reduction, when the position angle of the wave generator is approximately equal to 5 degrees, the tensile stress experienced by the root slightly increases to a lower value than that experienced before, as shown in Figure 4.19.

While the lobe of the wave generator gets further from the root under study, the tensile stress lowers until reaching zero when the root of the flexible spline is subjected again to compressive stress as shown in Figure 4.19 at an approximate position angle of the wave generator of 45 degrees. The absolute compressive stress keeps rising while the lobe of the wave generator moves away; this stress cycle is repeated two times for each rotation of the wave generator.

Figure 4.19 also shows both the increased compressive and tensile stresses experienced by the flexible spline when it is ahead of the movement of a lobe of the wave generator, as evidenced in the literature [10]. When the lobe of the wave generator has already passed underneath the root of the flexible spline under study, the stresses reach lower maximum values than before.

Figure 4.20 shows the von Mises stress on the node of the root element of the flexible spline under study. Von Mises stress clearly represents the previously shown tensile and compressive stresses in Figure 4.19. Von Mises stress is relatively high at the beginning of the cycle of meshing, corresponding to the compressive stress experienced by the root of the flexible spline when the lobe of the wave generator is far. Similarly, the initial von Mises stress reduces until reaching zero to then rise again corresponding with the tensile stress. Figure 4.20 shows the sharp
Development of stresses in the root of the flexible spline

Figure 4.20: Evolution of von Mises stress on the root of the flexible spline as a function of the position of the wave generator.

The von Mises stress on the root of the flexible spline initially increases as the lobe of the wave generator gets closer to the root under study, and then suddenly reduces when the lobe is directly underneath the root. Von Mises stress partially rises after this sudden decrease while the lobe of the wave generator is moving away from the point under study, similarly to Figure 4.19. Finally, von Mises stress reduces until becoming zero to then increase again, representing the compressive stress to which the root of the flexible spline is subjected when the lobe moves away from it.

As mentioned earlier, it can be seen in Figure 4.20 how von Mises stress reaches higher maximum absolute values when the lobe of the wave generator is coming towards the root of the flexible spline under study. This is caused by the transmission of torque and load, which leads to slightly reduced stresses when the lobe has passed the region under study and is moving away from it.
4.11 Influence of the rim thickness of the flexible spline over stresses

The flexible spline is the critical element for mechanical performance of SWG drives. For this reason, this section focuses on studying the influence of the rim thickness $t_r$ of the flexible spline over stresses using the two-dimensional model. The analyses for this section are generated using the elliptical wave generator with minor axis reduction coefficient $C$ equal to 1.0, the geometry of the wave generator proposed by Musser in 1959 [1].

Except for the rim thickness $t_r$ of the flexible spline, the remaining parameters of the SWG design used are those of Table 4.1. The use of the elliptical wave generator is intended to simplify the comparison between different rim thicknesses of the flexible spline, as well as to clearly show its influence over stresses.

4.11.1 Flexible spline rim thickness geometries

Figure 4.21 shows a tooth of the flexible spline with different rim thicknesses. This figure shows a tooth of the flexible spline with the QCA tooth profile of parameters shown in Table 4.1. The rim of a gear tooth is the section that extends beneath the root of the tooth [13]. The rim thickness $t_r$ is the radial dimension between the root and its inner or outer diameter for external or internal gears, respectively. For the case of the flexible spline, the rim thickness $t_r$ is the radial dimension between the root of the teeth and the inner diameter, as shown in Figure 4.21. The material included in the global geometry of a tooth increases with its rim thickness $t_r$ while the flank profile geometry is the same.

![Figure 4.21: Comparison of geometries of a tooth of the flexible spline as a function of its rim thickness $t_r$.](image)
4.11.1 Influence of the rim thickness of the flexible spline over stresses

For this SWG drive design with QCA tooth profile, the whole depth of a tooth is equal to 1.4 mm. The whole depth is computed as the product of the module times the sum of the addendum $h_a$ and dedendum $h_d$ coefficients \[13\]. Here, the module is 1 mm and the addendum $h_a$ and dedendum $h_d$ coefficients are 0.6 and 0.8, respectively. When the rim thickness $t_r$ of the flexible spline is 1 mm, as shown in Figure 4.21, the rim of the flexible spline is radially smaller than the whole depth of its teeth. However, when the rim thickness $t_r$ increases to 1.5 mm, it is slightly larger than the whole depth of a tooth. Figure 4.21 shows that employing a rim thickness $t_r$ equal to 2 mm leads to a rim section considerably larger than the whole depth of the teeth.

The preliminary analyses of this section are carried out considering rim thicknesses of the flexible spline between 0.5 and 3.0 mm in intervals of 0.5 mm. Beyond rim thicknesses equal to 3.0 mm, the obtained stresses are considerably beyond the strength limit of the material of the flexible spline. On the other hand, for rim thicknesses $t_r$ of the flexible spline below 0.5 mm, the stress concentrates on the rim excessively, leading to failure upon execution of the analyses due to the considerably small size of the deflected rim of the flexible spline as compared to its larger teeth.

4.11.2 Results obtained as a function of the rim thickness of the flexible spline

Figure 4.22 shows the variation of maximum tensile stress on the flexible spline for rim thicknesses $t_r$ between 0.5 and 3.0 mm. The maximum tensile stress on the flexible spline increases considerably with its rim thickness $t_r$. When the rim thickness $t_r$ of the flexible spline is beyond the whole depth of its teeth, the SWG drive design experiences larger tensile stresses which worsen the mechanical performance of the drive.

Figure 4.22 also shows the oscillatory behavior of the maximum tensile stress on the flexible spline. This stress oscillates especially when the rim thickness $t_r$ of the flexible spline is equal to 0.5, 2.5, and 3.0 mm. On the other hand, when the rim thickness $t_r$ of the flexible spline is between 1.0 and 2.0 mm, the oscillation of the maximum tensile stress is slightly reduced.

Figure 4.23 shows the variation of absolute maximum compressive stress on the flexible spline for the same cases shown in Figure 4.22. The absolute maximum compressive stress on
the flexible spline increases with the rim thickness $t_r$, similarly to the maximum tensile stress on the flexible spline. However, the maximum compressive stress on the flexible spline is equal to an almost constant value for each rim thickness $t_r$ of the flexible spline, as opposed to the
oscillatory behavior shown on the maximum tensile stress.

At the beginning of the analyses with rim thicknesses \( t_r \) of the flexible spline larger than 0.5 mm, the initial value of absolute maximum compressive stress on the flexible spline is larger than the later value developed, as shown in Figure 4.23. This is caused by the applied torque reaching its maximum and kept value after the wave generator starts rotating, as explained above. For the case of rim thickness \( t_r \) equal to 0.5 mm, the initial value of absolute maximum compressive stress is just slightly larger than the later developed value.

![Graph showing variation of maximum von Mises stress on the flexible spline as a function of its rim thickness \( t_r \).](image)

**Figure 4.24:** Variation of maximum von Mises stress on the flexible spline as a function of its rim thickness \( t_r \).

Figure 4.24 shows the variation of maximum von Mises stress on the flexible spline for rim thicknesses of the flexible spline between 0.5 and 3.0 mm. The maximum von Mises stress represents both the maximum tensile and absolute compressive stresses where the stress increases with the rim thickness \( t_r \) of the flexible spline, as shown in Figures 4.22 and 4.23. With rim thicknesses \( t_r \) of the flexible spline between 1.0 to 2.0 mm, the maximum von Mises stress shows the reduced oscillatory behavior of the maximum tensile stress on the flexible spline. Besides, the lower stress shown previously for a rim thickness \( t_r \) of the flexible spline equal to 0.5 mm are also represented in the maximum von Mises stress, as shown in Figure 4.24.
When the rim thickness $t_r$ increases, the flexible spline experiences higher stresses than employing reduced rim thicknesses, which worsens the mechanical performance of SWG drives. For this reason, emphasis should be put on employing the thinnest possible rim for the flexible spline when designing SWG drives. However, a designer might find it advantageous to select flexible splines with rim thicknesses similar or slightly smaller than the whole depth of the teeth of the flexible spline. This would result in maximum stresses oscillating less, which would assist to compute fatigue life on the flexible spline.

### 4.11.3 Further analyses of the rim thickness $t_r$ of the flexible spline

The previous results show that larger rim thicknesses result in considerably larger stresses on the flexible spline. These results may be influenced by the employed mesh for the teeth of the flexible spline and the ring gear. For this reason, several studies have been performed modifying the mesh in different ways. The objective is to verify that the initial results are robust and consistent regardless of the tooth mesh so that realistic conclusions can be obtained regarding the influence of the rim thickness $t_r$ of the flexible spline over stresses.

![Old and new meshes for rim thickness analyses with the two-dimensional model.](image)

The results shown below are obtained for the SWG drive with parameters given in Table 4.1 and the elliptical geometry of the wave generator with minor axis reduction coefficient $C$ equal to 1.0 as proposed by Musser and typically used in commercial SWG drives [1]. However, the
4.11. Influence of the rim thickness of the flexible spline over stresses

The original mesh employed on the teeth of the flexible spline and the ring gear is modified. The original mesh included elements on the rim region of the flexible spline that were not radially equal in size. This was especially significant below the roots of the teeth, as shown in Figure 4.25 left. The elements on the lower rim section of the mesh of each tooth are considerably larger than the elements located on the upper rim region. The right side of this figure shows the new mesh used in these analyses where the elements on the lower and upper rim regions of the teeth are of the same radial length beneath the roots. The remaining meshing regions on the tooth sections also incorporate homogeneously sized elements.

Comparison of stresses between meshes

Table 4.2 shows the number of elements on each meshing region of the teeth of the flexible spline (FS) and the ring gear (RG). Both flexible spline and ring gear include 21 elements in the active profile, 10 in the inner diameter, 3 in the top land, and 2 in the lower rim. The positioning factor, which defines where the center point of the mesh of each tooth is located with respect to the whole depth of the tooth, is 1.1 for both flexible spline and ring gear. This is the mesh arrangement employed on previous analyses of the rim thickness $t_r$ of the flexible spline.

<table>
<thead>
<tr>
<th>Mesh parameter</th>
<th>FS</th>
<th>RG</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elements in active profile</td>
<td>21</td>
<td>21</td>
</tr>
<tr>
<td>Elements in inner diameter</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Elements in top land</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Elements in lower rim</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Positioning factor</td>
<td>1.1</td>
<td>1.1</td>
</tr>
</tbody>
</table>

Three analyses have been performed to compare the influence of the previous mesh, the new mesh, and the new mesh with 20 elements in the inner diameter. The last case is used to determine if refining the mesh near the inner diameter region improves results. Figure 4.26 shows the new mesh on a tooth of the flexible spline with 20 elements in the inner diameter region. As compared to Figure 4.25 right, the rim region of the flexible spline teeth is considerably refined here.

Figure 4.27 shows the variation of maximum von Mises stress on the flexible spline throughout the cycle of meshing for the three different mesh cases under study. The previously utilized
**Chapter 4. Two-dimensional simulation of meshing and stress analysis**

**Figure 4.26:** Flexible spline tooth with the new mesh and 20 elements on the inner diameter section.

**Figure 4.27:** Variation of maximum von Mises stress on the flexible spline with the old, new, and refined meshes.
mesh shows the largest maximum von Mises stress on the flexible spline with considerable oscillatory behavior and an average stress equal to 286.461 MPa. On the other hand, while employing the new mesh, the maximum von Mises stress on the flexible spline reduces significantly to an average of 244.965 MPa. However, when the new mesh is used with 20 elements on the inner diameter as opposed to 10 elements, the maximum von Mises stress slightly rises to an average of 253.106 MPa. The new mesh shows approximately the same oscillatory behavior as the previous mesh.

The larger stresses when using 20 elements on the inner diameter indicate a slightly more detailed output from the two-dimensional model, which is beneficial in this section. For this reason, the following analyses to evaluate the influence of the rim thickness $t_r$ of the flexible spline over stresses employ the newly proposed mesh with 20 elements on the inner diameter region of the flexible spline teeth, although this increases the computational cost of the model.

**Stresses with the same number of elements on the rim region**

The novel mesh provides slightly reduced stress results. This is due to the homogeneously sized elements beneath the roots of the teeth. Several analyses have been performed with the new mesh and 8 elements on the lower rim region in order to further refine the mesh on the rim of the flexible spline as shown in Table 4.3.

<table>
<thead>
<tr>
<th>Mesh parameter</th>
<th>FS</th>
<th>RG</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elements in active profile</td>
<td>21</td>
<td>21</td>
</tr>
<tr>
<td>Elements in inner diameter</td>
<td>20</td>
<td>10</td>
</tr>
<tr>
<td>Elements in top land</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Elements in lower rim</td>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>Positioning factor</td>
<td>1.1</td>
<td>1.1</td>
</tr>
</tbody>
</table>

Figure 4.28 shows the mesh of a tooth of the flexible spline for four different values of the rim thickness $t_r$ equal to 0.2, 0.6, 1.0, and 1.4 mm. When the rim thickness $t_r$ of the flexible spline is equal to 0.2 mm, the elements on the rim region are considerably squeezed together and of a smaller size than in previous meshes. However, as the rim thickness $t_r$ increases with a constant number of elements on the lower rim (8), the elements in the rim region increase in
size up to when the rim thickness $t_r$ is equal to 1.4 mm and the elements in the rim region are of a similar size to those on the tooth region of the flexible spline.

These four rim thicknesses of the flexible spline have been analyzed leading to the results shown in Figure 4.29. This figure shows the variation of maximum von Mises stress on the flexible spline. Similarly to the previous results as a function of the rim thickness $t_r$ of the flexible spline, when the thickness increases the stresses experienced by the flexible spline proportionately increase as previously obtained. The lowest rim thickness $t_r$ studied (0.2 mm) results in an average maximum von Mises stress equal to 120.551 MPa, while the largest rim thickness $t_r$ studied (1.4 mm) reaches a considerably higher average maximum von Mises stress equal to 352.634 MPa.

On the other hand, the oscillatory behavior shown on the maximum von Mises stress in Figure 4.29 is similar for every value of the rim thickness $t_r$ of the flexible spline. There is only
4.11. Influence of the rim thickness of the flexible spline over stresses

Rotation angle of the wave generator [deg]

Maximum von Mises stress on the flexible spline ($\sigma$) [MPa]

$t_r = 0.2 \text{ mm}$
$t_r = 0.6 \text{ mm}$
$t_r = 1.0 \text{ mm}$
$t_r = 1.4 \text{ mm}$

0 30 60 90 120 150 180

100 150 200 250 300 350

FIGURE 4.29: Variation of maximum von Mises stress on the flexible spline as a function of its rim thickness $t_r$ with the same number of elements per section.

a slightly larger oscillatory behavior when the rim thickness $t_r$ is equal to 0.6 mm.

These results lead to the same conclusion as the previous results where the stresses on the flexible spline increase with its rim thickness $t_r$.

Stresses with similarly sized elements

The analyses in this section use a number of elements in the rim region proportional to the rim thickness $t_r$ of the flexible spline. Each element in the rim region is approximately 0.03 mm in radial length when measured on the center of a root between two adjacent teeth.

<table>
<thead>
<tr>
<th>Mesh parameter</th>
<th>FS</th>
<th>RG</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elements in active profile</td>
<td>21</td>
<td>21</td>
</tr>
<tr>
<td>Elements in inner diameter</td>
<td>20</td>
<td>10</td>
</tr>
<tr>
<td>Elements in top land</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Elements in lower rim</td>
<td>-</td>
<td>2</td>
</tr>
<tr>
<td>Positioning factor</td>
<td>1.1</td>
<td>1.1</td>
</tr>
</tbody>
</table>

Table 4.4 shows the parameters of the mesh for these analyses. The flexible spline is still refined with 20 elements in the inner diameter, as well as elements on the lower rim that depend
on its rim thickness $t_r$. The number of elements in the lower rim as a function of the rim thickness $t_r$ of the flexible spline is shown in Table 4.5. The rim thicknesses $t_r$ of the flexible spline under study in this section are 0.18, 0.42, 0.72, and 1.02 mm, so that the lower rim includes 2, 10, 20, and 30 elements, respectively. When the rim thickness $t_r$ is equal to 0.18 mm, the lower rim only includes 2 elements due to the upper rim including 4 elements between the last node of the active profile and the first node of the lower rim. Dividing the rim thickness $t_r$ of 0.18 mm over 6 elements on the rim region provides a radial length of 0.03 mm for each element below the root region.

<table>
<thead>
<tr>
<th>$t_r$ [mm]</th>
<th>Lower rim elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.18</td>
<td>2</td>
</tr>
<tr>
<td>0.42</td>
<td>10</td>
</tr>
<tr>
<td>0.72</td>
<td>20</td>
</tr>
<tr>
<td>1.02</td>
<td>30</td>
</tr>
</tbody>
</table>

Table 4.5: Number of elements on the flexible spline lower rim as a function of the rim thickness.

Figure 4.30 shows the mesh of a tooth of the flexible spline for the rim thicknesses $t_r$ under study in this section where the number of elements on the lower rim is dependent on the rim thickness $t_r$. When the rim thickness $t_r$ of the flexible spline is equal to 0.18 mm, the elements in the lower rim are remarkably constricted and squeezed together under the center point of the tooth mesh. This is caused by the positioning factor being equal to 1.1 as shown in Table 4.4. A smaller positioning factor raises the center point of the mesh, which would provide more space for the elements on the lower rim section below the center of each tooth.

For this reason, Table 4.6 shows the parameters of the mesh used in the analyses of this section where the positioning factor has been slightly reduced to 1.05 in order to accommodate elements in the lower rim. The remaining parameters remain the same, as well as the lower rim elements proportional to the rim thickness $t_r$ of the flexible spline as shown in Table 4.5.

Figure 4.31 shows the mesh of a tooth of the flexible spline for each rim thickness $t_r$ of the flexible spline under study (0.18, 0.42, 0.72, and 1.02 mm) with a positioning factor equal to 1.05. When the rim thickness $t_r$ is equal to 0.18 mm, the elements on the lower rim are not as distorted below the center of the tooth. Furthermore, using these parameters leads to considerably homogeneously sized elements in the rim region of the flexible spline regardless
4.11. Influence of the rim thickness of the flexible spline over stresses

When the rim thickness \( t_r \) of the flexible spline increases, the mesh is considerably refined due to the increase in elements on the lower rim to maintain a radial length of approximately 0.03 mm for each element below the root region of the teeth.

The results obtained for each rim thickness \( t_r \) of the flexible spline with different number of elements in the lower rim are shown in Figure 4.32. The lowest stresses on the flexible spline are achieved with the lowest rim thickness \( t_r \) equal to 0.18 mm and two elements on the lower rim. Similarly to the previous results, the maximum von Mises stress on the flexible spline increases of its rim thickness \( t_r \).
**Figure 4.31**: Mesh comparison as a function of the rim thickness $t_r$ of a tooth of the flexible spline with elements of the same size on the rim region and larger positioning factor.

proportionately with its rim thickness $t_r$.

**Torque influence with similarly sized elements**

Larger rim thicknesses $t_r$ lead to a stiffer flexible spline. This means that for the same amount of deflection imposed by the wave generator, larger rim thicknesses $t_r$ of the flexible spline result in larger stresses. However, there is an advantage provided by increasing the amount of material in the rim region of the flexible spline. Larger rim thicknesses $t_r$ should be capable of transmitting larger torques with lesser stresses than the same loads with thinner rims.

This section focuses on the influence of the transmitted torque (output torque $T$) over stresses as a function of the rim thickness $t_r$ of the flexible spline. The objective is to identify if it is actually advantageous to use larger rims for the flexible spline.
4.11. Influence of the rim thickness of the flexible spline over stresses

The results shown in this section employ the same mesh parameters and number of elements as shown in Tables 4.6 and 4.5. Therefore, the geometries of the teeth and their mesh under study is as shown in Figure 4.31, where there are homogeneously sized elements in the rim region of the flexible spline that depend on its rim thickness $t_r$ and are of a radial length approximately equal to 0.03 mm below the root region. Similarly to the previous analyses, the cases under study here include rim thicknesses $t_r$ of the flexible spline equal to 0.18, 0.42, 0.72, and 1.02 mm while the output torque $T$ of the flexible spline as output member of the drive is varied.

Figure 4.32 shows the variation of maximum von Mises stress on the flexible spline for the different rim thicknesses $t_r$ under study and 20 and 50 Nm of output torque $T$. The continuous lines show the results when the output torque $T$ is equal to 20 Nm, whereas the dashed lines indicate the results when the output torque $T$ is equal to 50 Nm.

The lowest stresses are provided by the smallest rim thickness $t_r$ of the flexible spline equal to 0.18 mm. This case results in an average maximum von Mises stress equal to 115.605 MPa when the output torque $T$ is 20 Nm. Oppositely, transmitting 50 Nm of torque leads to an average maximum von Mises stress of 136.393 MPa. There is a 20 MPa difference between the
average maximum von Mises stresses when transmitting 20 or 50 Nm while the rim thickness \( t_r \) of the flexible spline is equal to 0.18 mm.

When the rim thickness \( t_r \) of the flexible spline is equal to 0.42 mm, the difference between the average maximum von Mises stress when transmitting 20 or 50 Nm is equal to approximately 9 MPa, a considerably smaller difference than when the rim thickness \( t_r \) is 0.18 mm. This difference further reduces when the rim thickness \( t_r \) of the flexible spline increases. In the case when the rim thickness \( t_r \) of the flexible spline is equal to 1.02 mm, the difference of average maximum von Mises stress between transmitting 20 or 50 Nm of torque \( T \) is equal to 1 MPa.

These results indicate that the larger the rim thickness \( t_r \) of the flexible spline, the stiffer it is and the larger the load that can be transmitted without damaging the flexible spline.

Figure 4.34 shows additional results about the influence of the transmitted load as a function of the rim thickness \( t_r \) of the flexible spline with homogeneously sized elements in the rim region. This figure shows four line charts where each one shows the variation of maximum von Mises stress experienced by the flexible for a single rim thickness \( t_r \) and the same scale. The rim
4.11. Influence of the rim thickness of the flexible spline over stresses

thicknesses $t_r$ under study are the same as previous analyses equal to 0.18, 0.42, 0.72, and 1.02 mm, while output torques $T$ have been studied equal to 20, 50, 100, and 200 Nm, as shown in the legend at the bottom of Figure 4.34.

When the rim thickness $t_r$ of the flexible spline is equal to 0.18 mm, the increase in stresses provided by applying a torque of 200 Nm instead of a torque of 20 Nm, is considerably large and approximately equal to 110 MPa. This difference is remarkably reduced when the rim thickness $t_r$ of the flexible spline increases to 0.42 mm, which further reduces with larger rim thicknesses $t_r$. This emphasizes the increased stiffness provided by larger values of the rim thickness $t_r$ of the flexible spline. Finally, when the rim thickness $t_r$ is equal to 1.02 mm, the difference in average maximum von Mises stresses between transmitting 20 to 200 Nm is smaller than 20 MPa.

Regarding the oscillatory behavior of the maximum von Mises stress on the flexible spline,
it increases with the transmitted load \((T)\), regardless of the value of the rim thickness \(t_r\) of the flexible spline. This increased oscillatory behavior is illustrated in Figure 4.34 by the lines showing the maximum von Mises stress when transmitting 200 Nm, whereas the oscillatory behavior is slightly reduced when transmitting smaller loads.

Consequently, it can be assumed that the selection of the rim thickness \(t_r\) for the flexible spline during the design stages of SWG drives requires a trade-off between the resulting stresses on the flexible spline (the critical element for mechanical performance of SWG drives) and the amount of torque \(T\) that has to be transmitted with the drive. Large loads would require larger rim thicknesses on the flexible spline.
Chapter 5

Two-dimensional stress analysis of different wave generator geometries

5.1 Chapter overview

The two-dimensional model for stress analysis and simulation of meshing of SWG drives allows for the evaluation of several geometries of each of its elements on a planar section of the toothed region. The geometry of the wave generator constitutes a highly influential element due to continuously deflecting the flexible spline. For this reason, the results shown in this chapter evaluate the influence over stresses on the flexible spline of the different geometries of the wave generator explained in Chapter 3. The analyses with the two-dimensional model are performed considering the most common layout of SWG drives where the wave generator serves as the input member and the flexible spline serves as the output member of the drive while the ring gear is fixed. The results are obtained for the reference SWG drive design with parameters shown in Table 4.1, and varying parameters for each wave generator geometry.

The results obtained with the simplified, elliptical, parabolic, and four roller wave generator geometries are illustrated in Sections 5.2, 5.3, 5.4, and 5.5, respectively. The mechanical performance obtained from each geometry in the two-dimensional models is compared in Section 5.6. Section 5.7 shows two different approaches to modify the geometry of the parabolic wave generator with the intention to improve the mechanical performance of SWG drives.

Finally, Section 5.8 explains the results of the analyses considering the alternative layout of SWG drives where the flexible spline is fixed and the ring gear is the output member of
the drive. These analyses are performed with the improved geometry of each wave generator previously evaluated with the two-dimensional model.

5.2 Simplified wave generator geometries

Figure 5.1 shows different geometries of the simplified wave generator with minor axis reduction coefficient $C$ equal to 0.50, 0.65, 0.80, and 1.00 in the coordinate system of the wave generator $S_{wg}$. The minor axis reduction coefficient $C$ modifies the radius of the pushing pins $r_{pp}$ as computed in Equation (3.2). Increasing the minor axis reduction coefficient $C$ causes the simplified wave generator to increase its size along the minor axis. Since the parameters of these cases are those shown in Table 4.1, the major axis of the wave generator $a$ is kept constant at all times. This parameter defines the length of the wave generator along the major axis as computed with Equation (3.3). However, the portion of the wave generator geometry in contact with the flexible spline has a lower curvature when using higher values of the minor axis reduction coefficient $C$ due to the increase in the radius of the pushing pins $r_{pp}$.

After performing several analyses varying the minor axis reduction coefficient $C$, the results obtained with minor axis coefficients equal or below 0.90 are approximately the same. When the minor axis reduction coefficient $C$ is considered between 0.91 to 0.98, the obtained results provide useful insight for improvement of the mechanical performance of SWG drives. These results have been obtained by varying the minor axis reduction coefficient $C$ from 0.91 to 0.98 in increments of 0.01 of the coefficient. A minor axis reduction coefficient $C$ beyond 0.98 produces remarkably high stresses due to the deflection imposed on the flexible spline by the consequently large wave generator. For these reasons, cases with a minor axis reduction coefficient $C$ below 0.91 and above 0.98 are omitted in this section.

Figure 5.2 shows the variation of maximum tensile stress on the flexible spline using different geometries of the simplified wave generator with minor axis reduction coefficient $C$ between 0.91 to 0.98. Increasing the minor axis reduction coefficient $C$ considerably lowers the maximum tensile stress on the flexible spline.

On the other hand, Figure 5.3 shows the variation of absolute maximum compressive stress
on the flexible spline for the same cases shown in Figure 5.2. The absolute maximum compressive stress for the cases where the minor axis reduction coefficient $C$ is between 0.91 to 0.94 are also comparable and oscillate due to the engagement and disengagement of different pairs of teeth in contact. However, when the minor axis reduction coefficient $C$ further increases to 0.95...
and 0.96, the absolute maximum compressive stress lowers and tends to become less oscillating, as shown in Figure 5.3. This is because, as the simplified wave generator becomes larger, the influence of the compressive stresses caused by the contact between the wave generator
and the flexible spline reduces. Oppositely, the compressive stresses caused by the deflection of the flexible spline become more influential. When the minor axis reduction coefficient $C$ is equal to 0.97, the absolute maximum compressive stress on the flexible spline increases slightly and becomes a straight line with a magnitude similar to that of the cases where the minor axis reduction coefficient $C$ is between 0.91 to 0.94. However, when the minor axis reduction coefficient $C$ further increases to 0.98, the absolute maximum compressive stress rises considerably, as shown in Figure 5.3. This is caused by the deflection imposed on the flexible spline by an excessively large simplified wave generator geometry.

![Graph](image)

**Figure 5.4**: Variation of maximum von Mises stress on the flexible spline with the simplified wave generator geometries.

Figure 5.4 shows the variation of maximum von Mises stress on the flexible spline. Similarly to Figures 5.2 and 5.3, the maximum von Mises stress for the cases where the minor axis reduction coefficient $C$ is between 0.91 to 0.94 are comparable. When the minor axis reduction coefficient $C$ increases to 0.97, the maximum von Mises stress lowers considerably. This is because this case results on the lowest absolute maximum compressive stress, as well as a relatively lower maximum tensile stress, as shown in Figures 5.3 and 5.2, respectively. However, further increasing the minor axis reduction coefficient $C$ to 0.98, the maximum von Mises stress
rises due to the remarkably large absolute maximum compressive stress for this case. Due to the use of von Mises stress, which considers tensile and compressive stresses at the same time, the results shown in Figure 5.4 constitute a reasonable approach to evaluate the mechanical performance of SWG drives with the two-dimensional model and the simplified wave generator geometry.

**Improved simplified wave generator geometry**

The simplified wave generator geometry with minor axis reduction coefficient \( C \) equal to 0.97 is selected as the improved simplified wave generator geometry due to the low level of maximum von Mises stress. Figure 5.5 shows the variations of maximum stresses on the flexible spline using the selected improved geometry of the simplified wave generator. The tensile stress constitutes the highest maximum stress that the flexible spline experiences yielding an average maximum stress equal to 236.072 MPa. On the other hand, the absolute maximum compressive stress yields an average maximum value equal to 214.869 MPa. The absolute maximum compressive stress is considerably homogeneous. The maximum von Mises stress shows the oscillating behavior of the maximum tensile stress. Its average value is equal to 228.659 MPa.

Figure 5.6 shows the distribution of tensile stresses around the major and minor axes regions of the SWG drive with the improved simplified wave generator geometry with minor axis reduction coefficient \( C \) equal to 0.97. The distribution of tensile stresses is shown for the case when the wave generator has rotated 45 degrees counterclockwise. At this stage, the location of the maximum tensile stress is at the root of the teeth of the flexible spline near the major axis and slightly ahead of the direction of rotation of the lobe of the wave generator. On the other hand, the minor axis region only experiences tensile stresses on the inner diameter of the flexible spline with a lower value than those experienced near the major axis.

Figure 5.7 shows the distribution of compressive stresses around the major and minor axes regions of the SWG drive for the same case as that shown in Figure 5.6. The maximum compressive stress is located at the root of the teeth of the flexible spline near the minor axis, as opposed to the results obtained in Section 4.3 where the maximum compressive stress is located near the
5.2. Simplified wave generator geometries

![Graph showing variations of maximum stresses on the flexible spline with minor axis coefficient C equal to 0.97.](image)

**Figure 5.5:** Variations of maximum stresses on the flexible spline with minor axis coefficient $C$ equal to 0.97.

major axis using a simplified wave generator with minor axis reduction coefficient $C$ equal to 0.80, as shown in Figure 4.9. In the major axis region, the compressive stresses are not as high as they are in the minor axis region. The tensile and compressive stresses experienced by the flexible spline induce repeated alternating stresses on the root of the teeth of the flexible spline throughout the operation of SWG drives, which affects the fatigue life of the flexible spline.

Figure 5.8 shows the distribution of von Mises stresses around the major and minor axes regions of the wave generator when it has rotated 45 degrees counterclockwise. The maximum von Mises stress is located at the root of the teeth of the flexible spline near the major axis, similarly to the location of maximum tensile stress shown in Figure 5.6. This is because the maximum stress value reached by the tensile stress is higher than the maximum stress value reached by the absolute compressive stress, as shown in Figure 5.5.

Figures 5.6, 5.7, and 5.8 provide useful information towards understanding where the transmitted load is causing stresses on the SWG drive, which serves for improvement of its mechanical performance. The simplified wave generator geometry with minor axis reduction coefficient $C$ equal to 0.97 is still considered the improved simplified wave generator geometry for the reference SWG drive case design of parameters shown in Table 4.1, due to it resulting in the lowest
Chapter 5. Two-dimensional stress analysis of different wave generator geometries

Figure 5.6: Distribution of tensile stresses around the major and minor axes regions of the SWG drive with minor axis coefficient $C$ equal to 0.97.
5.2. *Simplified wave generator geometries*

**Figure 5.7:** Distribution of compressive stresses around the major and minor axes regions of the SWG drive with minor axis coefficient $C$ equal to 0.97.
FiguRe 5.8: Distribution of von Mises stresses around the major and minor axes regions of the SWG drive with minor axis coefficient $C$ equal to 0.97.
5.3 Elliptical wave generator geometries

Figure 5.9 shows different geometries of the elliptical wave generator with minor axis reduction coefficient $C$ equal to 0.50, 0.65, 0.80, and 1.00, in the coordinate system of the wave generator $S_{wg}$. The minor axis reduction coefficient $C$ modifies the minor axis $b$, as computed in Equation (3.4). Since the parameters of this case are those shown in Table 4.1, the length of the major axis of the wave generator, $a$, as computed with Equation (3.3) is kept constant at all times. The portion of the wave generator in contact with the flexible spline has a lower curvature when using higher values of the minor axis reduction coefficient $C$.

After performing several analyses varying the minor axis reduction coefficient $C$, the results obtained with minor axis coefficients below 0.970 are approximately the same. However, when the minor axis reduction coefficient $C$ is considered between 0.970 to 1.000, the results obtained provide useful insight for improvement of the mechanical performance of SWG drives. These results have been obtained by varying the minor axis reduction coefficient $C$ from 0.970 to 1.000 in increments of 0.005 of the coefficient. The cases with a minor axis reduction coefficient $C$ below 0.970 are omitted in this section.

Figure 5.10 shows the variation of maximum tensile stress on the flexible spline for different geometries of the elliptical wave generator with minor axis reduction coefficient $C$ between 0.970 to 1.000. Increasing the minor axis reduction coefficient $C$ considerably lowers the maximum tensile stress on the flexible spline.

On the other hand, Figure 5.11 shows the variation of absolute maximum compressive stress on the flexible spline for the same cases shown in Figure 5.10. When the minor axis reduction coefficient $C$ increases, the absolute maximum compressive stress is reduced and tends to become less oscillating, as shown in Figure 5.11. This is because, as the elliptical wave generator becomes larger, the influence of the compressive stresses caused by the contact between the wave generator and the flexible spline lowers, whereas the compressive stresses caused by the
Chapter 5. Two-dimensional stress analysis of different wave generator geometries

When the minor axis reduction coefficient $C$ is equal to $1.000$, the absolute maximum compressive stress on the flexible spline reaches its lowest value. Figure 5.12 shows the variation of maximum von Mises stress. When the minor axis reduction coefficient $C$ increases to $1.000$, the maximum von Mises stress reaches its lowest value. The deflection of the flexible spline becomes more influential.
5.3. Elliptical wave generator geometries

**Figure 5.10:** Variation of maximum tensile stress on the flexible spline with the elliptical wave generator geometries.

**Figure 5.11:** Variation of absolute maximum compressive stress on the flexible spline with the elliptical wave generator geometries.

Results shown in Figure 5.12 constitute a reasonable approach to evaluate the mechanical performance of SWG drives with the two-dimensional model using the elliptical wave generator geometry.
Improved elliptical wave generator geometry

The elliptical wave generator geometry with minor axis reduction coefficient $C$ equal to 1.00 is selected as the improved geometry due to the low level of maximum von Mises stress. Figure 5.13 shows the variations of maximum tensile, absolute compressive, and von Mises stresses on the flexible spline using the selected improved geometry. The tensile stress constitutes the highest maximum stress that the flexible spline experiences yielding an average value equal to 297.314 MPa. On the other hand, the absolute maximum compressive stress yields an average maximum value equal to 197.017 MPa. The maximum von Mises stress shows the oscillating behavior of the maximum tensile stress and yields an average value equal to 288.101 MPa.

Figure 5.14 shows the distribution of tensile stresses around the major and minor axes regions of the SWG drive using the improved elliptical wave generator geometry with minor axis reduction coefficient $C$ equal to 1.00. The distribution of tensile stresses is shown for the case when the wave generator has rotated 45 degrees counterclockwise. At this stage, the location of the maximum tensile stress is at the root of the teeth of the flexible spline near the major
5.3. Elliptical wave generator geometries

Figure 5.13: Variations of maximum stresses on the flexible spline with minor axis coefficient \( C \) equal to 1.00.

axis and slightly behind the direction of rotation of the wave generator. On the other hand, the minor axis region only experiences tensile stresses on the inner diameter of the flexible spline with a lower value than those experienced near the major axis.

Figure 5.15 shows the distribution of compressive stresses around the major and minor axes regions of the SWG drive for the same case as in Figure 5.14. The maximum compressive stress is located at the root of the teeth of the flexible spline near the minor axis. In the major axis region, the compressive stresses are not as high as they are in the minor axis region with the improved elliptical wave generator geometry.

Figure 5.16 shows the distribution of von Mises stresses. The maximum von Mises stress is located at the root of the teeth of the flexible spline near the major axis, similarly to the location of maximum tensile stress shown in Figure 5.14. Figures 5.14, 5.15, and 5.16 provide useful information towards understanding where the transmitted load is causing stresses on the SWG drive, which serves for improvement of its mechanical performance. The elliptical wave generator geometry with minor axis reduction coefficient \( C \) equal to 1.00 is considered the improved elliptical wave generator geometry for the reference case SWG drive design with parameters shown in Table 4.1, due to resulting in the lowest maximum von Mises stress.
Chapter 5. Two-dimensional stress analysis of different wave generator geometries

Figure 5.14: Distribution of tensile stresses around the major and minor axes regions of the SWG drive with minor axis coefficient $C$ equal to 1.00.
5.3. Elliptical wave generator geometries

**Figure 5.15:** Distribution of compressive stresses around the major and minor axes regions of the SWG drive with minor axis coefficient $C$ equal to 1.00.
Figure 5.16: Distribution of von Mises stresses around the major and minor axes regions of the SWG drive with minor axis coefficient $C$ equal to 1.00.
5.4 Parabolic wave generator geometries

The parabolic wave generator geometry requires four different parameters for its definition, namely

- Minor axis reduction coefficient \( C \)
- Parabola coefficient \( a_p \)
- Parabola length reduction coefficient \( C_p \)
- Connector arc radius \( r_c \)

as opposed to the simplified and elliptical wave generator geometries, which only require the minor axis reduction coefficient \( C \). This section shows the obtained results from modifying each defining parameter and their influence over stresses in SWG drives. Each parameter is modified separately until an improved value is found, which is then used when modifying the following parameter.

Table 5.1 shows the parameters of the geometry of the parabolic wave generator employed to evaluate the influence of the minor axis coefficient \( C \) on the stresses developed in the flexible spline. This geometry uses a parabola coefficient \( a_p \) equal to 0.01, which leads to significant curvature on the parabolic region. If the parabola coefficient \( a_p \) is smaller, the parabolic region has a considerably reduced curvature, resembling a straight section. The parabola length reduction coefficient \( C_p \) is selected as 0.5, which defines the parabolic region of the wave generator as a function of the length of the minor axis \( b \). This allows the parabolic region to extend up to half of the length of the minor axis \( b \). Finally, the radius of the connector arc \( r_c \) between the parabolic regions and the lateral circular arcs of the parabolic wave generator is equal to 10 mm. This provides a smooth transition from the parabolic regions to the lateral circular arcs for the SWG drive design employed with parameters shown in Table 4.1.

Table 5.1: Parameters of the initial geometry of the parabolic wave generator.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>[units]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parabola coefficient, ( a_p )</td>
<td>0.01</td>
</tr>
<tr>
<td>Parabola length reduction coef., ( C_p )</td>
<td>0.5</td>
</tr>
<tr>
<td>Connector arc radius, ( r_c )</td>
<td>10.0 mm</td>
</tr>
</tbody>
</table>
Chapter 5. Two-dimensional stress analysis of different wave generator geometries

The geometries evaluated in this study employ the same parameters for the parabolic region, regardless of the minor axis reduction coefficient $C$ employed. However, when this coefficient increases, the parabolic region of the wave generator increases due to it being a function of the length of the minor axis $b$ and the parabola length reduction coefficient $C_p$, as explained in Section 3.2.

5.4.1 Results obtained with different minor axis reduction coefficients $C$

Figure 5.17 shows different geometries of the parabolic wave generator with minor axis reduction coefficient $C$ equal to 0.50, 0.65, 0.80, and 1.00 in the coordinate system of the wave generator $S_{wg}$. The only changing parameter is the length of the minor axis $b$ computed with Equation (3.4), which leads to an increase in the parabolic region without affecting its curvature near the major axis of the drive. With the parameters employed for the parabolic wave generator and a minor axis coefficient $C$ equal to 1.00, the parabolic wave generator resembles a circumference, although the parabolic regions protrude on the top and bottom of the wave generator (major axis regions).

Several analyses have been performed varying the minor axis reduction coefficient $C$ for the parabolic wave generator employing the parameters shown in Table 5.1. However, due to the similarity between the parabolic regions of the different parabolic wave generators, the stress results are approximately equal for any value of the minor axis reduction coefficient $C$. This deems this parabolic wave generator design independent of the variation of the minor axis reduction coefficient $C$. For instance, when the minor axis reduction coefficient $C$ is equal to 0.50, the average maximum von Mises stress on the flexible spline is equal to 372.986 MPa, whereas when the minor axis reduction coefficient $C$ is equal to 1.00, the average maximum von Mises stress on the flexible spline is equal to 371.676 MPa. It can be concluded that the stresses on the flexible spline are not significantly affected by the minor axis reduction coefficient $C$.

5.4.2 Results obtained with different parabola coefficients $a_p$

This section focuses on the study of the influence of the parabola coefficient $a_p$ over stresses on the flexible spline. The SWG drive design selected employs the parameters shown in Table
4.1 except for the wave generator, which employs those parameters shown in Table 5.2. The parabolic wave generator uses a minor axis reduction coefficient $C$ equal to 1.0. Besides, it has been shown that there is a lack of influence of the minor axis reduction coefficient $C$ over stresses with the parabolic wave generator and therefore the largest possible minor axis reduction coefficient $C$ is selected here.
Chapter 5. Two-dimensional stress analysis of different wave generator geometries

Table 5.2: Partially-improved (1/3) parameters of the geometry of the parabolic wave generator.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minor axis reduction coef., $C$</td>
<td>1.0</td>
</tr>
<tr>
<td>Parabola length reduction coef., $C_p$</td>
<td>0.5</td>
</tr>
<tr>
<td>Connector arc radius, $r_c$</td>
<td>10.0 mm</td>
</tr>
</tbody>
</table>

Figure 5.18: Comparison of geometries of the parabolic wave generator for different values of the parabola coefficient $a_p$. 
Figure 5.18 shows several geometries of the parabolic wave generator as a function of the parabola coefficient $a_p$ while the remaining parameters are kept constant. The parabola coefficient $a_p$ is shown with values equal to 0.005, 0.007, 0.009, and 0.010, which only affect the geometry of the parabolic region of the wave generator. The curvature of the parabolic region increases with the parabola coefficient $a_p$, while lower values of the parabola coefficient $a_p$ lead to flatter parabolic regions and more protuberance by the connector arcs of radius $r_c$. Figure 5.18 also shows how the length of the parabolic region on the horizontal direction remains constant up to half the length of minor axis $b$ of the wave generator, while the parabola coefficient $a_p$ is modified.

![Figure 5.18: Geometries of the parabolic wave generator.](image)

Figure 5.19: Coordinates of the position vector $r_p$ of the parabolic region of the wave generator for different values of the parabola coefficient $a_p$.

Figure 5.19 shows a detail of the parabolic regions shown in Figure 5.18 for the same values of the parabola coefficient $a_p$. The parabolic region extends in the horizontal direction up until a value equal to the parabola length reduction coefficient $C_p$ times the minor axis $b$. With the employed parameters, the length of the major axis of the wave generator $a$ is equal to 59.2 mm while the length of the minor axis $b$ is equal to 57.2 mm. With the parabola length reduction coefficient $C_p$ equal to 0.5, the parabolic regions extend 28.6 mm on the horizontal direction. However, modifying the parabola coefficient $a_p$ changes the vertical location of the last point of
the parabolic region, as shown in Figure 5.19, as well as the curvature of the parabolic region. The influence of the parabola coefficient \(a_p\) on the geometry approximately accounts for less than 1 mm change on the vertical location of the last point of the parabolic region for a change of 0.001 of the parabola coefficient \(a_p\).

After performing several analyses with different values of the parabola coefficient \(a_p\), it has been observed that using the parabola coefficients \(a_p\) lower than 0.00960 lead to remarkably high stresses, rendering this range of the parabola coefficient \(a_p\) ineffective to improve the mechanical performance of SWG drives. On the other hand, when using parabola coefficients \(a_p\) equal or larger than 0.00990, the resulting stresses are the same. For these reasons, this section shows the results obtained for values of the parabola coefficient \(a_p\) between 0.00960 and 0.00990 in increments of 0.00005.

![Graph showing variation of maximum tensile stress on the flexible spline with different values of the parabola coefficient \(a_p\).](image)

**Figure 5.20**: Variation of maximum tensile stress on the flexible spline with different values of the parabola coefficient \(a_p\).

Figure 5.20 shows the variation of maximum tensile stress on the flexible spline with the parabola coefficient \(a_p\) between 0.00960 and 0.00990. As the parabola coefficient \(a_p\) increases from 0.00960, the maximum tensile stress experienced by the flexible spline reduces considerably until it reaches its minimum when the parabola coefficient \(a_p\) is equal to 0.00970 as shown
in Figure 5.20. For parabola coefficients larger than 0.00970, the maximum tensile stress increases again.

Figure 5.21 shows the variation of absolute maximum compressive stress on the flexible spline for the same values of the parabola coefficient $a_p$ shown in Figure 5.20. Increasing the parabola coefficient $a_p$ from 0.00960 reduces the absolute maximum compressive stress until its minimum value with parabola coefficient $a_p$ equal to 0.00980. When the parabola coefficient is further increased, the absolute maximum compressive stress increases. The oscillatory behavior of the compressive stress also increases and becomes more evident.

Figure 5.22 shows the variation of maximum von Mises stress. As the parabola coefficient $a_p$ increases from 0.00960, the maximum von Mises stress reduces until reaching the lowest value when the parabola coefficient $a_p$ equals 0.00970. Further increasing the parabola coefficient $a_p$ leads to higher variations of maximum von Mises stress similarly to the maximum tensile and absolute compressive stresses shown in Figures 5.20 and 5.21, respectively.

Figure 5.22 serves to evaluate the mechanical performance of SWG drives employing the parabolic wave generator by considering the value of the parabola coefficient $a_p$ leading to the lowest variation of maximum von Mises stress on the critical element, the flexible spline. Consequently, middle range values of the parabola coefficient $a_p$ are recommended for improved
Chapter 5. Two-dimensional stress analysis of different wave generator geometries

![Graph showing variation of maximum von Mises stress on the flexible spline with different values of the parabola coefficient $a_p$.](image)

**Figure 5.22:** Variation of maximum von Mises stress on the flexible spline with different values of the parabola coefficient $a_p$.

performance of SWG drives.

**Improved parabolic wave generator geometry considering $a_p$**

The parabolic wave generator with parabola coefficient $a_p$ equal to 0.0097 is selected as the improved parabolic wave generator geometry. Figure 5.23 shows the variations of maximum tensile, absolute compressive, and von Mises stresses on the flexible spline with the parabolic wave generator of parabola coefficient $a_p$ equal to 0.0097. The lowest maximum stress experienced by the flexible spline is the absolute compressive stress, yielding an average value of 207.910 MPa. The highest maximum stress is the tensile stress, yielding an average value of 258.500 MPa, while the maximum von Mises stress yields an average value equal to 250.383 MPa.

The maximum tensile and von Mises stresses show considerable oscillatory behavior due to the influence of the tooth-to-tooth contact. This contact loads the teeth in bending, which is shown as tensile stress on the roots of the flexible spline as a function of the number of pairs of teeth in contact sharing the transmitted torque. Since the number of pairs of teeth in contact
varies while the wave generator rotates, the maximum tensile and, consequently, von Mises stresses oscillate. On the other hand, the maximum absolute compressive stress is considerably more homogeneous due to the constant contact between the lobes of the wave generator and the inner diameter of the flexible spline.

Figure 5.24 shows the distribution of tensile stresses on both the minor and major axes regions of the SWG drive when the parabolic wave generator with parabola coefficient $a_p$ equal to 0.0097 has rotated 45 degrees counterclockwise. The maximum tensile stress is located on the root of the teeth of the flexible spline near the major axis region of the drive but slightly ahead of the movement of the lobe of the wave generator. The flexible spline is also subjected to tensile stress on its inner diameter near the minor axis region due to the inward deflection experienced, while the major axis regions are deflected outward by the lobes of the wave generator.

Figure 5.25 shows the distribution of compressive stresses for the same position of the parabolic wave generator as shown in Figure 5.24. As opposed to the distribution of tensile stress, the flexible spline experiences the highest compressive stress on the roots of its teeth near the minor axis region of the drive, whereas the inner diameter of the flexible spline experiences compressive stress near the major axis region due to the contact between the lobes of
Chapter 5. Two-dimensional stress analysis of different wave generator geometries

**Figure 5.24:** Distribution of tensile stresses around the major and minor axes regions of the SWG drive with parabola coefficient $a_p$ equal to 0.0097.
5.4. Parabolic wave generator geometries

Figure 5.25: Distribution of compressive stresses around the major and minor axes regions of the SWG drive with parabola coefficient $a_p$ equal to 0.0097.
the wave generator and the inner diameter of the flexible spline. It can also be seen the slight compressive stress located on the flanks of the teeth of the flexible spline and the ring gear due to their meshing near the major axis region. This justifies the less oscillatory behavior that the maximum absolute compressive stress shows due to the low magnitude of compressive stress created by the tooth-to-tooth contact in SWG drives, while the flexible spline is continually deformed by the lobes of the wave generator in contact with its inner diameter. This leads to much higher and constant compressive stress.

Figure 5.26 shows the distribution of von Mises stresses on the SWG drive when the parabolic wave generator has rotated 45 degrees counterclockwise. The location of the maximum von Mises stress experienced by the flexible spline is on one of its roots near the major axis of the drive, coinciding with the location of maximum tensile stress. This is because the magnitude of the tensile stress is considerably larger than the magnitude of the compressive stress experienced by the flexible spline. The generated stresses due to the tooth-to-tooth contact between the flexible spline and the ring gear are also shown in Figure 5.26, as well as how the rim of the flexible spline experiences both tensile and compressive stress around its entire circumference. This depends on the region of the flexible spline under observation, as explained above.

With the SWG drive design parameters shown in Table 4.1, the parabolic wave generator of parameters shown in Table 5.2 and parabola coefficient $a_p$ equal to 0.0097 constitutes an improved SWG drive design with parabolic wave generator, which enhances the mechanical performance of this type of gear drive.

### 5.4.3 Results obtained with different parabola length reduction coefficients $C_p$

The parabola length reduction coefficient $C_p$ defines the length of the parabolic regions of the parabolic wave generator as a function of the length of the minor axis $b$. For this reason, this section focuses on the study of the influence of the parabola length reduction coefficient $C_p$ over the stresses on the flexible spline.

Table 5.3 shows the parameters of the parabolic wave generator geometry employed in this section, which considers different values of the parabola length reduction coefficient $C_p$. The minor axis reduction coefficient $C$ is set equal to 1.0 due to its lack of influence over stresses in
5.4. Parabolic wave generator geometries

Figure 5.26: Distribution of von Mises stresses around the major and minor axes regions of the SWG drive with parabola coefficient $a_p$ equal to 0.0097.
this type of wave generator, whereas the parabola coefficient $a_p$ is equal to 0.0097 as it has been found to improve the mechanical performance of SWG drives, detailed in the previous section. The radius of the connector arc $r_c$ is again equal to 10 mm.

**Table 5.3:** Partially-improved (2/3) parameters of the geometry of the parabolic wave generator.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minor axis reduction coeff., $C$</td>
<td>1.0</td>
</tr>
<tr>
<td>Parabola coef., $a_p$</td>
<td>0.0097</td>
</tr>
<tr>
<td>Connector arc radius, $r_c$</td>
<td>10.0 mm</td>
</tr>
</tbody>
</table>

Figure 5.27 shows different geometries of the parabolic wave generator in the coordinate system of the wave generator $S_{wg}$ as a function of the parabola length reduction coefficient $C_p$. The values of the parabola length reduction coefficient $C_p$ for each geometry are equal to 0.30, 0.50, 0.70, and 0.90. The parabola length reduction coefficient $C_p$ limits the horizontal dimension of the parabolic region of this type of wave generator depending on the length of the minor axis $b$ which is kept constant and extending up to 57.2 mm.

When the parabola length reduction coefficient $C_p$ is equal to 0.30, the parabolic regions of both lobes of the parabolic wave generator extend up to 30 percent of the length of the minor axis $b$. This leads to a considerably short parabolic region as compared to the geometry of the entire wave generator, which resembles a round shape. On the other hand, when the parabola length reduction coefficient $C_p$ increases, the parabolic regions extend further out. When the parabola length reduction coefficient $C_p$ is equal to 0.90, the parabolic regions extend up to almost the entire length of the minor axis $b$, specifically 90 percent of it. When the parabola length reduction coefficient $C_p$ is so large, the connector arc regions of the parabolic wave generator become considerably pointed, which leads to high concentration of stresses in the contact between the wave generator and the inner diameter of the flexible spline.

Several studies have been performed varying the parabola length reduction coefficient $C_p$ along its entire range with the two-dimensional model of SWG drives. The results show that parabola length reduction coefficients $C_p$ equal or lower than 0.42 lead to similar stress results in SWG drives with parabolic wave generator of parameters shown in Tables 4.1 and 5.3. On the other hand, when using parabola length reduction coefficients $C_p$ equal or higher than 0.52, the
obtained stresses increase remarkably. For these reasons, this section shows results for values of the parabola length reduction coefficient $C_p$ between 0.42 and 0.52.

Figure 5.28 shows the variation of maximum tensile stress on the flexible spline for values of the parabola length reduction coefficient $C_p$ equal to 0.42, 0.45, 0.48, 0.50, 0.51, and 0.52. The values of the parabola length reduction coefficient $C_p$ between 0.42 and 0.45, and 0.45 and 0.48 are omitted for simplicity of the results. They follow the tendency shown in the results with parabola length reduction coefficients $C_p$ equal to 0.42, 0.45, and 0.48.
When the parabola length reduction coefficient $C_p$ is equal to 0.42 and 0.45, the initial values of maximum tensile stress on the flexible spline coincide. However, the maximum tensile stress for these values of the parabola length reduction coefficient $C_p$ diverge after the application of torque. When the parabola length reduction coefficient $C_p$ is increased to 0.48 and 0.50, the maximum tensile stress decreases considerably. The initial value of maximum tensile stress for parabola length reduction coefficients $C_p$ equal to 0.48 and 0.50 becomes similar to the average value of maximum tensile stress. For the parabola length reduction coefficient $C_p$ equal to 0.51, the initial maximum tensile stress is higher than the later developed maximum tensile stress. This value of the parabola length reduction coefficient $C_p$ leads to a considerably reduced variation of maximum tensile stress, as shown in Figure 5.28. If the parabola length reduction coefficient $C_p$ is increased to values equal to 0.52 or higher, the maximum tensile stress experienced by the flexible spline rises considerably. This is caused by the remarkably pointed connector arc regions of the parabolic wave generator with large values of the parabola length reduction coefficient $C_p$ while keeping a constant value for the radius of the connector arc $r_c$.

Similarly to when studying the influence over stresses of the parabola coefficient $a_p$, the maximum tensile stress shows considerable oscillatory behavior. This is due to the engagement
5.4. Parabolic wave generator geometries

and disengagement of different pairs of teeth in contact between the flexible spline and the ring gear while the wave generator rotates.

![Figure 5.29: Variation of absolute maximum compressive stress on the flexible spline with different values of the parabola length reduction coefficient $C_p$.](image)

Figure 5.29 shows the variation of absolute maximum compressive stress experienced by the flexible spline for the same values of the parabola length reduction coefficient $C_p$ shown in Figure 5.28. In this case, when the parabola length reduction coefficient $C_p$ is equal to 0.42, the absolute maximum compressive stress rises to a considerably high value after the torque has been applied. This variation of absolute maximum compressive stress also shows a remarkable oscillatory behavior due to the high influence of the engagement and disengagement of different pairs of teeth in contact when the parabolic region of the wave generator is small.

When the parabola length reduction coefficient $C_p$ increases to 0.45, the absolute maximum compressive stress reaches its lowest value. For values of the parabola length reduction coefficient $C_p$ equal to 0.42 and 0.45, the initial values of the absolute maximum compressive stress experienced by the flexible spline are the same. However, the absolute maximum compressive stress diverges after the application of torque. When the parabola length reduction coefficient $C_p$ is further increased to 0.48 and 0.50, the absolute maximum compressive stress increases due to the increase on the length of the parabolic region in contact with the flexible spline. On the other hand, when the parabola length reduction coefficient $C_p$ is equal to 0.51, the average
value of the absolute maximum compressive stress reaches a lower value with a considerably reduced oscillatory behavior. Finally, with parabola length reduction coefficient $C_p$ equal or higher than 0.52, the absolute maximum compressive stress experienced by the flexible spline rises considerably due to the pointed geometry of the connector arcs regions of the parabolic wave generator in contact with the inner diameter of the flexible spline.

![Graph showing variation of maximum von Mises stress on the flexible spline](image)

**FIGURE 5.30:** Variation of maximum von Mises stress on the flexible spline with different values of the parabola length reduction coefficient $C_p$.

Figure 5.30 shows the variation of maximum von Mises stress on the flexible spline for the values of the parabola length reduction coefficient $C_p$ studied in this section. Between parabola length reduction coefficients $C_p$ equal to 0.42 and 0.51, the maximum von Mises stress reduces, meaning that the maximum von Mises stress experienced by the flexible spline reaches its lowest value with a parabola length reduction coefficient $C_p$ equal to 0.51. This is caused by the enlargement of the parabolic region of the wave generator in contact with the inner diameter of the flexible spline. However, when the parabola length reduction coefficient $C_p$ is increased to values equal to 0.52 or more, the maximum von Mises stress experienced by the flexible spline increases considerably beyond the previously obtained maximum stress.

The maximum von Mises stress inherits the oscillatory behavior shown in Figures 5.28 and
5.29. For low values of the parabola length reduction coefficient $C_p$, the initial values of the maximum von Mises stress coincide before the application of torque. However, when the parabola length reduction coefficient $C_p$ is equal to 0.48 and 0.50, the initial value of maximum von Mises stress is similar to the later developed. For values of the parabola length reduction coefficient $C_p$ equal or higher than 0.51, the initial maximum von Mises stress is higher than the average value of the maximum von Mises stress experienced by the flexible spline. This is caused by the relatively large parabolic regions of the wave generator in contact with the inner diameter of the flexible spline.

**Improved parabolic wave generator geometry considering $C_p$**

The parabolic wave generator with parabola length reduction coefficient $C_p$ equal to 0.51 is selected as the improved parabolic wave generator geometry as a function of the parabola length reduction coefficient $C_p$. This is because it leads to the lowest variation of maximum von Mises stress on the flexible spline.

![Variations of maximum stresses on the flexible spline with parabola length reduction coefficient $C_p$ equal to 0.51.](image)

**Figure 5.31:** Variations of maximum stresses on the flexible spline with parabola length reduction coefficient $C_p$ equal to 0.51.

Figure 5.31 shows the variations of maximum tensile, absolute compressive, and von Mises stresses on the flexible spline with the parabolic wave generator of parabola length reduction...
coefficient $C_p$ equal to 0.51. Although not shown in the figure, the initial peaks of maximum tensile and von Mises stresses are equal to 276.400 and 267.800 MPa, respectively. The lowest maximum stress experienced by the flexible spline is the maximum absolute compressive stress in which the resulting average stress is equal to 204.529 MPa. The highest variation of maximum stress is the tensile stress reaching an average maximum stress equal to 236.282 MPa, while the maximum von Mises stress is slightly lower with an average value equal to 228.883 MPa.

The maximum von Mises stress constitutes a summary of both variations of maximum tensile and absolute compressive stresses where the maximum tensile stress is slightly reduced by the considerably lower variation of maximum absolute compressive stress, as shown in Figure 5.31. Similarly to when studying the influence of the parabola coefficient $a_p$, the maximum tensile and von Mises stresses show considerable oscillatory behavior due to the influence of the tooth-to-tooth contact between the teeth of the flexible spline and the ring gear. The maximum absolute compressive stress is considerably more homogeneous due to the constant contact between the lobes of the wave generator and the inner diameter of the flexible spline. In this case, however, the maximum absolute compressive stress shows a slight oscillatory behavior.

Figure 5.32 shows the distribution of tensile stresses on both the minor and major axes regions of the SWG drive when the parabolic wave generator with parabola length reduction coefficient $C_p$ equal to 0.51 has rotated 45 degrees counterclockwise. The maximum tensile stress is located on the root of the teeth of the flexible spline near the major axis region of the drive, slightly ahead of the movement of the lobe of the wave generator. The flexible spline is also subjected to tensile stress on its inner diameter near the minor axis region due to the inward deflection experienced, while the major axis regions are deflected outward by the lobes of the wave generator.

Figure 5.33 shows the distribution of compressive stresses for the same position of the parabolic wave generator as shown in Figure 5.32. As opposed to the distribution of tensile stress, the flexible spline experiences the highest compressive stress on the roots of its teeth close to the minor axis region of the drive, whereas the inner diameter of the flexible spline experiences compressive stress near the major axis region due to the contact with the lobes of the wave generator. The slight compressive stress located on the flanks of the teeth of the flexible
5.4. Parabolic wave generator geometries

**Figure 5.32**: Distribution of tensile stresses around the major and minor axes regions of the SWG drive with parabola length reduction coefficient $C_p$ equal to 0.51.
FIGURE 5.33: Distribution of compressive stresses around the major and minor axes regions of the SWG drive with parabola length reduction coefficient $C_p$ equal to 0.51.
spline and the ring gear due to their meshing near the major axis region, where the magnitude of compressive stress created by the tooth-to-tooth contact is considerably reduced can also be seen. The flexible spline is continually deflected by the lobes of the wave generator in contact with its inner diameter leading to much higher and constant compressive stress.

Figure 5.34 shows the distribution of von Mises stresses on the SWG drive when the parabolic wave generator has rotated 45 degrees counterclockwise. The location of the maximum von Mises stress experienced by the flexible spline is on the root of a tooth near the major axis of the drive, coinciding with the location of maximum tensile stress. This is because the magnitude of the tensile stress is considerably larger than the magnitude of the compressive stress experienced by the flexible spline. The generated stresses due to the tooth-to-tooth contact between the flexible spline and the ring gear are also shown in Figure 5.34, as well as how the rim of the flexible spline experiences both tensile and compressive stress around its entire circumference, depending on the region of the flexible spline under observation, as explained above.

With the SWG drive design of parameters shown in Table 4.1, the parabolic wave generator of parameters shown in Table 5.3 and parabola length reduction coefficient $C_p$ equal to 0.51 constitutes an improved design with the parabolic wave generator.

### 5.4.4 Results obtained with different radii of the connector arc $r_c$

The last parameter that defines the geometry of the parabolic wave generator is the radius of the connector arc $r_c$. This section studies its influence over the stresses obtained with the two-dimensional model of SWG drives. Table 5.4 shows the selected values of the remaining parameters that define the geometry of the parabolic wave generator. In this case, the previously found improved values of the minor axis reduction coefficient $C$, the parabola coefficient $a_p$, and the parabola length reduction coefficient $C_p$ are considered.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>[units]</th>
</tr>
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<tbody>
<tr>
<td>Minor axis reduction coef., $C$</td>
<td>1.0</td>
</tr>
<tr>
<td>Parabola coef., $a_p$</td>
<td>0.0097</td>
</tr>
<tr>
<td>Parabola length reduction coef., $C_p$</td>
<td>0.51</td>
</tr>
</tbody>
</table>

Table 5.4: Partially-improved (3/3) parameters of the geometry of the parabolic wave generator.
Figure 5.34: Distribution of von Mises stresses around the major and minor axes regions of the SWG drive with parabola length reduction coefficient $C_p$ equal to 0.51.
Several analyses have been performed varying the radius of the connector arc $r_c$ along a wide range. This section shows the results obtained for radius of the connector arc $r_c$ equal to 10, 15, 20, and 25 mm. This is because when employing the parabolic wave generator of parameters shown in Table 5.4 in a SWG drive design of parameters shown in Table 4.1, the lowest stresses are obtained for a radius of the connector arc $r_c$ equal to 10 mm. Besides, when employing values of the radius of the connector arc $r_c$ lower than 10 mm, the analyses with the two-dimensional model fail to continue executing due to the teeth of the flexible spline getting blocked in the tooth slots of the ring gear. This is caused by the excessively pointed connector arcs with low values of its radius $r_c$.

A figure showing the effect of different values of the radius of the connector arc $r_c$ on the geometry of the parabolic wave generator is omitted in this section due to the constant parabolic region provided the values of the parabola coefficient $a_p$ and the parabola length reduction coefficient $C_p$ on this type of wave generator.

![Graph showing the effect of different values of the radius of the connector arc $r_c$ on the geometry of the parabolic wave generator.](image)

**Figure 5.35:** Variation of maximum tensile stress on the flexible spline with different radii of the connector arc $r_c$.

Figure 5.35 shows the variation of maximum tensile stress on the flexible spline for values of the radius of the connector arc $r_c$ equal to 10, 15, 20, and 25 mm. For any value of the radius of the connector arc $r_c$, the initial maximum tensile stress experienced by the flexible spline before
the application of torque is considerably larger than the average maximum tensile stress. When the radius of the connector arc $r_c$ is equal to 10 mm, the maximum tensile stress on the flexible spline reaches its lowest variation, although still showing considerable oscillatory behavior. When the radius of the connector arc $r_c$ increases to 15 mm, the maximum tensile stress also increases. However, the increase in maximum tensile stress for greater values of the radius of the connector arc $r_c$ is considerably larger than when varying the radius of the connector arc $r_c$ from 10 mm to 15 mm. Consequently, larger values of the radius of the connector arc $r_c$ lead to higher maximum tensile stress on the flexible spline.

Figure 5.36 shows the variation of absolute maximum compressive stress on the flexible spline for the same values of the radius of the connector arc $r_c$ shown in Figure 5.35. When the radius of the connector arc $r_c$ is equal to 10 and 15 mm, the initial values of absolute maximum compressive stress on the flexible spline are higher than the developed average compressive stress after the torque is applied. Here, it can be observed how the absolute maximum compressive stress also increases with the value of the radius of the connector arc $r_c$. The radius of the connector arc $r_c$ equal to 10 mm leads to the lowest absolute maximum compressive stress, similarly to the maximum tensile stress.

Figure 5.37 shows the variation of maximum von Mises stress experienced by the flexible
spline for values of the radius of the connector arc $r_c$ equal to 10, 15, 20, and 25 mm. This figure illustrates the influence of the radius of the connector arc $r_c$ over stresses in SWG drives, similarly to Figures 5.35 and 5.36.

The lowest level of von Mises stress on the flexible spline is achieved by using a value of the radius of the connector arc $r_c$ equal to 10 mm. However, for any value of the radius of the connector arc $r_c$, the initial maximum von Mises stress is considerably larger than the later developed average maximum von Mises stress. This is caused by the considerably large parabolic wave generator constantly imposing deflection along the entire inner diameter of the flexible spline. The connector arc regions of the parabolic wave generator contact smoothly with the inner diameter of the flexible spline with the radius of the connector arc $r_c$ equal to 10 mm. The maximum von Mises stress experienced by the flexible spline considerably rises for higher values of the radius of the connector arc $r_c$.

Figure 5.37 also shows the oscillatory behavior of the maximum von Mises stress on the flexible spline caused by the engagement and disengagement of different pairs of teeth in contact.
between the flexible spline and the ring gear. These pairs of teeth in contact share the transmitted load in the two-dimensional model of SWG drives.

With the parameters of the parabolic wave generator shown in Table 5.4, the radius of the connector arc $r_c$ equal to 10 mm constitutes an improved value of this parameter due to resulting in the lowest von Mises stress as shown in Figure 5.37. However, the parabola coefficient $a_p$ and the parabola length reduction coefficient $C_p$ provide the final geometry of the lobes of the parabolic wave generator, which are in contact and deflecting the inner diameter of the flexible spline at all times. The parabolic regions constitute the critical regions for mechanical performance of SWG drives employing this type of wave generator.

5.4.5 Improved parabolic wave generator geometry

The parabolic wave generator geometry has higher complexity than the simplified and elliptical wave generator geometries. Apart from the basic parameters of SWG drives, the simplified and elliptical geometries make use of a single additional parameter for their definition, the minor axis reduction coefficient $C$. The four roller wave generator uses two parameters, the aperture angle $\beta$ and the radius of the roller $r_4$. On the other hand, the parabolic wave generator requires four different parameters to define its geometry while taking into account the parameters of the SWG drive itself.

Table 5.5 shows the parameters used to define the geometry of the parabolic wave generator, as well as their found values for improved mechanical performance of SWG drives. The minor axis reduction coefficient $C$ is set as 1.0 due to its lack of influence in stresses. The parabola coefficient $a_p$ and the parabola length reduction coefficient $C_p$ equal to 0.0097 and 0.51, respectively, lead to considerably reduced stresses on SWG drives. Finally, the radius of the connector arc $r_c$ equal to 10 mm provides reduced stresses as opposed to higher values which considerably increase stresses. These improved values have been found for a SWG drive case design of parameters shown in Table 4.1.

The study and variation of the parameters shown in Table 5.5 aiming to reduce the obtained stresses lead to a remarkably improved geometry of the parabolic wave generator for this SWG drive design. The improved geometry of the parabolic wave generator provides a smoother
Table 5.5: Parameters of the improved parabolic wave generator geometry.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Parameter [units]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minor axis reduction coef., $C$</td>
<td>1.0</td>
</tr>
<tr>
<td>Parabola coef., $a_p$</td>
<td>0.0097</td>
</tr>
<tr>
<td>Parabola length reduction coef., $C_p$</td>
<td>0.51</td>
</tr>
<tr>
<td>Connector arc radius, $r_c$</td>
<td>10.0 mm</td>
</tr>
</tbody>
</table>

Contact between its parabolic regions, or lobes of the wave generator, and the inner diameter of the flexible spline.

Figure 5.38 shows the distribution of von Mises stresses around the major axis region of the SWG drive with the improved geometry of the parabolic wave generator. This geometry of the parabolic wave generator contacts with the inner diameter of the flexible spline on three regions per lobe of the wave generator. The center of each parabolic region, as well as the connector arcs contact the inner diameter of the flexible spline. Although there are three regions of contact, the contact between the connector arcs and the flexible spline produces considerably less deflection of the flexible spline, which reduces its stresses.

The location of the maximum von Mises stress is on the root of a tooth of the flexible spline slightly behind the movement of the parabolic wave generator, as shown in Figure 5.38. This is
caused by the transmitted load between the teeth of the flexible spline and the ring gear creating bending stresses on the root of the teeth of the constantly deflected flexible spline.

Employing the parabolic wave generator in SWG drives is recommended, as it reduces the stresses obtained on the flexible spline, the critical element for mechanical performance of SWG drives. The parabolic wave generator, therefore, constitutes a competitive alternative for enhancing the mechanical performance of SWG drives.

5.5 Four roller wave generator geometry

The geometry of the four roller wave generator is inspired by the preloaded wave generator concept patented in 1970 [14]. The implemented geometry of the four roller wave generator uses the following parameters,

- Aperture angle of roller, $\beta$
- Radius of a roller, $r_4$

The aperture angle of the roller $\beta$ defines the angle between the vertical axis and a line extending from the center of coordinates $S_{wg}$ to the center of each of the four rollers, as shown in Figure 5.39. The radius of a roller $r_4$ specifies the size of each roller by the radius in mm.

The four roller wave generator is designed to provide the necessary deflection to the inner diameter of the flexible spline with four, instead of two, regions of contact. This allows the meshing between the teeth of the flexible spline and those of the ring gear to extend across a wider region while the wave generator rotates. This section focuses on evaluating the influence of this type of wave generator and its different parameters over the stresses obtained with the two-dimensional model of SWG drives. Several analyses have been performed considering multiple combinations of the defining parameters of the four roller wave generator.

5.5.1 Results obtained with different aperture angles $\beta$

Figure 5.39 shows several geometries of the four roller wave generator in the coordinate system of the wave generator $S_{wg}$ for different values of the aperture angle $\beta$. These geometries are shown for values of the aperture angle $\beta$ equal to 10, 15, 20, and 25 degrees with a radius
of the roller $r_4$ equal to 8 mm. The higher the value of the aperture angle $\beta$ used, the further the four rollers spread outwards in the horizontal direction while keeping constant their radii $r_4$ and the distance from the center of coordinates $O_{wg}$ to their outermost point. However, the minimum allowable aperture angle $\beta$ for any radii of the roller $r_4$ has to ensure that the rollers keep certain clearance between them, allowing a wave generator with four rollers to be manufactured.

This section studies the influence of the aperture angle $\beta$ over stresses obtained with the two-dimensional model. Since a radius of the roller $r_4$ equal to 8 mm provides significant clearance for an aperture angles $\beta$ equal to 10 degrees, it is selected for analysis of the SWG drive reference case design of parameters shown in Table 4.1. Several cases have been performed modifying the aperture angle $\beta$, however, this section shows results for aperture angles $\beta$ between 10 to 20 degrees in increments of 2 degrees. Showing further results seems unnecessary due to the considerably large stresses obtained with this type of wave generator. The minimum aperture angle $\beta$ equal to 10 degrees is selected to ensure the rollers of 8 mm of radius do not overlap, as shown in Figure 5.39.

Figure 5.40 shows the variation of maximum tensile stress on the flexible spline for values of the aperture angle $\beta$ between 10 to 20 degrees. While the aperture angle $\beta$ increases, the maximum tensile stress experienced by the flexible spline increases. The minimum level of maximum tensile stress on the flexible spline is found for an aperture angle $\beta$ equal to 10 degrees. Its average is considerably larger than the lowest average maximum tensile stress found with the simplified, elliptical, and parabolic geometries of the wave generator.

Figure 5.41 shows the variation of absolute maximum compressive stress on the flexible spline for the same values of the aperture angle $\beta$ shown in Figure 5.40. Similarly to the maximum tensile stress results, the absolute maximum compressive stress experienced by the flexible spline rises considerably with the aperture angle $\beta$.

Figure 5.42 shows the variation of maximum von Mises stress experienced by the flexible spline for values of the aperture angle $\beta$ between 10 and 20 degrees. As shown in Figures 5.40 and 5.41, the maximum stress experienced by the flexible spline considerably rises with higher values of the aperture angle $\beta$. This is caused by the considerably open four rollers imposing
The aperture angle $\beta$ significantly influences the stresses experienced by the flexible spline. This parameter should be reduced to the lowest possible value while keeping the rollers clear of contacting one another. Even though a minimum variation of maximum von Mises stress deflection on the inner diameter of the flexible spline at all times.

**Figure 5.39:** Comparison of geometries of the four roller wave generator for different values of the aperture angle $\beta$. 
5.5. Four roller wave generator geometry

is found as a function of the aperture angle $\beta$, this result is still considerably larger than any previously found improved geometry of the simplified, elliptical, and parabolic geometries of the wave generator. For this reason, these stress results deem the four roller wave generator unsuitable for the improvement of the mechanical performance of SWG drives. Four rollers in contact with the inner diameter of the flexible spline lead to much larger stresses than having...
only two lobes, as in the other geometries of the wave generator. Besides, the lack of a large contact area between the wave generator and the inner diameter of the flexible spline leads to large concentration of stresses.

5.5.2 Results obtained with different roller radii $r_4$

The second parameter that defines the geometry of the four roller wave generator is the radius of the roller $r_4$. This parameter depends on the aperture angle $\beta$ to keep the rollers clear of each other. For larger radii of the roller $r_4$, larger aperture angles $\beta$ would be necessary to ensure proper manufacturing of the four roller wave generator.

This section shows results obtained for different values of the radius of the roller $r_4$. Although several cases have been studied varying the radius of the roller $r_4$, this section focuses on the results obtained for radii of the roller $r_4$ equal to 2, 4, 6, and 8 mm while the aperture angle $\beta$ is kept constant and equal to 10 degrees. This aperture angle $\beta$ leads to the lowest maximum von Mises stress on the flexible spline with the four roller wave generator, as explained above. The results obtained for a radius of the roller $r_4$ equal to 10 mm have been omitted due to the resulting interference between the roller with the aperture angle $\beta$ equal to 10 degrees.
5.5. *Four roller wave generator geometry*

- **Figure 5.43**: Variation of maximum tensile stress on the flexible spline with different values of the roller radius $r_4$. 

Figure 5.43 shows the variation of maximum tensile stress on the flexible spline for values of the radius of the roller $r_4$ equal to 2, 4, 6, and 8 mm. For any value of the radius of the roller $r_4$, the maximum tensile stresses on the flexible spline are considerably similar, meaning that there is no significant influence over the maximum tensile stress on SWG drives as a function of the radius of the roller $r_4$.

Figure 5.44 shows the variation of absolute maximum compressive stress on the flexible spline for the same values of the radius of the roller $r_4$ shown in Figure 5.43. The maximum compressive stress experienced by the flexible spline provides similar insights to the maximum tensile stress results. Varying the radius of the roller $r_4$, there is no significant influence on compressive stress. With a radius of the roller $r_4$ equal to 2 mm, however, the average absolute maximum compressive stress reduces slightly. The amplitude of the absolute maximum compressive stress remains approximately the same.

Finally, Figure 5.45 shows the variation of maximum von Mises stress experienced by the flexible spline for values of the radius of the roller $r_4$ equal to 2, 4, 6, and 8 mm. Similarly to the results shown in Figures 5.43 and 5.44, there is no significant influence on the maximum von Mises stress experienced by the flexible spline while varying the radius of the roller $r_4$. With a
Chapter 5. Two-dimensional stress analysis of different wave generator geometries

**Figure 5.44:** Variation of absolute maximum compressive stress on the flexible spline with different values of the roller radius $r_4$.

**Figure 5.45:** Variation of maximum von Mises stress on the flexible spline with different values of the roller radius $r_4$. 
5.5. Four roller wave generator geometry

radius of the roller \( r_4 \) equal to 2 mm, the amplitude of the maximum von Mises stress is slightly reduced, although its average value developed is similar to that obtained when employing other values of the radius of the roller \( r_4 \).

The radius of the roller \( r_4 \) of the four roller wave generator geometry does not constitute an influential parameter towards improving the mechanical performance of SWG drives with this type of wave generator. Different values of the radius of the roller \( r_4 \) lead to comparable stress results on the flexible spline while keeping the aperture angle \( \beta \) constant.

5.5.3 Maximum stresses with the four roller wave generator and contact with the flexible spline

The four roller wave generator concept intends to reduce the stresses on the flexible spline by spreading the imposed deflection on its inner diameter, as well as increasing the number of contacting regions between the wave generator and the flexible spline. However, the obtained results show that this type of wave generator leads to considerably higher stresses than any improved geometry of the simplified, elliptical, and parabolic wave generator geometries.

This section shows the lowest maximum stress results obtained for the SWG drive design of parameters shown in Table 4.1 with the four roller wave generator and the distribution of stresses on the SWG drive resulting from the use of this type of strain inducing element.

Figure 5.46 shows the variations of maximum tensile, absolute compressive, and von Mises stresses experienced by the flexible spline and the four roller wave generator of 10 degrees of aperture angle \( \beta \) and 8 mm of radius of the roller \( r_4 \). The highest variation of stress is the maximum tensile stress with an average stress value equal to 509.190 MPa. This is of a considerably higher value than the average maximum tensile stress obtained with the improved simplified, elliptical, and parabolic wave generator geometries.

With the four roller wave generator, the lowest variation of stress is the maximum absolute compressive stress on the flexible spline with an average value equal to 291.980 MPa. This variation of maximum stress shows a reduced oscillatory behavior as compared to the maximum tensile and von Mises stresses. The maximum von Mises stress with an average equal to
493.208 MPa is slightly lower than the maximum tensile stress by the influence of the compressive stress. For both variations of maximum tensile and von Mises stresses, the results show certain oscillatory behavior. This is caused by the engagement and disengagement of different pairs of teeth in contact which share the transmitted load between the flexible spline and the ring gear while the wave generator rotates. However, the initial value of maximum tensile and von Mises stresses is slightly lower than the stress developed after the torque is applied.

Due to the considerably different geometry of the four roller wave generator as compared to the simplified, elliptical, and parabolic wave generator geometries, the study of the contact between the four rollers and the inner diameter of the flexible spline provides some insight about the higher stresses produced by this type of wave generator. Figure 5.47 shows the distribution of von Mises stresses around the major axis region of the SWG drive with the four roller wave generator of 10 degrees of aperture angle $\beta$ and 8 mm of radius of the roller $r_4$.

On each major axis region of the drive, the two rollers of the wave generator in contact with the flexible spline produce two zones of compression on the flexible spline inner diameter, which slightly reduce the stresses on the flexible spline around the center of the major axis regions of the SWG drive. However, these two deflection areas per meshing region lead to a
5.6. Comparison of improved wave generator geometries

The two-dimensional model proves convenient towards identifying the influence on mechanical performance of the geometry of their different components within a planar section.

considerably higher number of pairs of teeth in contact between the flexible spline, as well as excessive compression and deflection imposed on the inner diameter of the flexible spline and the flexible spline itself. Besides, due to the deflection imposed by four relatively small rollers instead of two considerably larger lobes of the wave generator, the maximum von Mises stress location is found near the compression region caused by one of the rollers in the area to where the wave generator is moving.

The results obtained with the four roller wave generator provide considerably higher stresses than those obtained with the improved simplified, elliptical, and parabolic wave generator geometries. Consequently, the four roller wave generator is considered unsuitable and should not be used in SWG drives if the aim is to increase their power density and enhance their mechanical performance.

5.6 Comparison of improved wave generator geometries

The two-dimensional model proves convenient towards identifying the influence on mechanical performance of the geometry of their different components within a planar section.
Chapter 5. Two-dimensional stress analysis of different wave generator geometries

This planar section is considered on the toothed region of the SWG drive design under study. In this chapter, the implemented wave generator geometries have been evaluated and the influence over stresses of their particular defining parameters studied. The objective to improve the mechanical performance of SWG drives is achieved by realizing which design parameters are the most influential in each particular geometry of the wave generator.

After several analyses have been performed, the improved geometry of each wave generator type result in the following parameters:

- Simplified: minor axis reduction coefficient \( C \) equal to 0.97.
- Elliptical: minor axis reduction coefficient \( C \) equal to 1.0.
- Parabolic: shown in Table 5.5.
- Four roller: aperture angle \( \beta \) and radius of the roller \( r_4 \) equal to 10 degrees and 8 mm, respectively.

These results are obtained for the SWG drive design of parameters shown in Table 4.1, and lead to the lowest von Mises stress on the flexible spline.

Table 5.6 shows the average value of the maximum tensile, absolute compressive, and von Mises stresses on the flexible spline for each improved wave generator geometry. The lowest average maximum von Mises stress is obtained with the simplified and parabolic wave generator geometries of a value approximately equal to 229 MPa. However, while their resulting average maximum tensile stress is also similar, these wave generator geometries differ in their resulting average maximum absolute compressive stress by approximately 10 MPa, where the lower average maximum compressive stress is found with the parabolic wave generator geometry.

| Wave generator | Stresses: \( \sigma_1 \) [MPa] | Compressive, \( |\sigma_3| \) [MPa] | Von Mises, \( \sigma_{\text{W}} \) [MPa] |
|----------------|----------------|----------------|----------------|
| Simplified     | 236.072        | 214.869        | 228.659        |
| Elliptical     | 297.310        | 197.020        | 288.101        |
| Parabolic      | 236.280        | 204.530        | 228.883        |
| Four roller    | 509.190        | 291.980        | 493.208        |
5.6. Comparison of improved wave generator geometries

On the other hand, the elliptical wave generator geometry as proposed by Musser [1] leads to larger average maximum tensile and von Mises stresses than the simplified and parabolic wave generator geometries. These average maximum stresses are 60 MPa higher, as shown in Table 5.6. However, the lowest average maximum absolute compressive stress is obtained with the elliptical wave generator. The improved four roller wave generator geometry results in considerably higher average maximum stresses than any other wave generator geometry. For instance, the average maximum von Mises stress obtained with the four roller wave generator more than doubles the average maximum von Mises stress obtained with the simplified wave generator geometry.

In order to enhance the mechanical performance of SWG drives, the use of the simplified or the parabolic wave generator geometries is recommended for a SWG drive design of parameters shown in Table 4.1. The differences between the simplified and the parabolic wave generator geometries consist of the more complex definition of the parabolic wave generator requiring four parameters, as opposed to a single parameter as required by the simplified wave generator. However, the simplified and parabolic wave generator geometries are comparable for enhanced mechanical performance of SWG drives. If the aim is to reduce the compressive stress imposed by the lobes of the wave generator on the inner diameter of the flexible spline, the elliptical wave generator geometry should be selected similar to the wave generator proposed by Musser in 1959 [1].

The considerably worse mechanical performance of SWG drives obtained with the four roller wave generator justifies its lack of use since the concept was patented in 1970 [14]. Due to the obtained results and the lack of literature and utilization of the four roller wave generator, it should not be used in SWG drives because of the remarkably large stresses obtained by imposing the deflection on the flexible spline in more locations than only two lobes of the wave generator.

In summation, the best mechanical performance in terms of lower stresses with the two-dimensional model for any geometry of the wave generator is obtained with the smoothest possible contact between the lobes of the wave generator and the inner diameter of the flexible spline. The performed analyses provide useful insight towards improving the mechanical
performance of SWG drives and understanding their behavior. However, a three-dimensional model of SWG drives is necessary to account for the effect of the deformation of the cup-shaped spring of the flexible spline.

5.7 Asymmetric parabolic wave generator geometry

The parabolic wave generator geometry proves advantageous to enhance the mechanical performance of SWG drives evaluated with the two-dimensional model. When load is transmitted in this type of gear drive, the stresses on the flexible spline concentrate in the quadrant where the wave generator is rotating towards. For this reason, modifying the geometry of a wave generator to make it asymmetric may serve to enhance the performance of SWG drives, although with the disadvantages of providing a single direction of motion drive.

This section shows two different approaches and their results over stresses by modifying the geometry of the parabolic wave generator. The resulting modified geometry is called asymmetric parabolic wave generator. Conclusions about the effectiveness of this type of wave generator are also shown. The two different approaches are an angle-based modification of the parabolic region and a bilateral definition of the input parameters. In the first, an angle is employed to slightly rotate the parabolic region of the parabolic wave generator while the remaining parameters are kept constant. The second approach uses the same parameters of the parabolic wave generator but defines them differently for each side of the wave generator.

5.7.1 Angle-based asymmetric parabolic wave generator geometry

The parabolic region of the original geometry of the parabolic wave generator is rotated slightly. To do this, rotational angle $\lambda$ is applied at the center of the parabolic region. Figure 5.48 shows two geometries of the asymmetric parabolic wave generator, one without any angle $\lambda$ applied and another with a slight angle $\lambda$ in the coordinate system of the wave generator $S_{wg}$. Positive values of angle $\lambda$ rotate the parabolic region counter-clockwise, raising its right side on the upper major axis region as illustrated. Negative values of angle $\lambda$ rotate the parabolic region clockwise.
5.7. Asymmetric parabolic wave generator geometry

Figure 5.48 also shows a circumference of radius equal to the length of the major axis $a$ of the wave generator computed with Equation (3.3). The length of the major axis $a$ should be the maximum radial dimension of the wave generator. This, however, is not possible when rotating the parabolic region around its center. The slightest rotation angle $\lambda$ applied on the center of the parabolic region translates part of the parabolic region beyond the circumference of radius $a$.

The analyses performed to evaluate the angle-based asymmetric parabolic wave generator use the parameters of the improved parabolic wave generator geometry shown in Table 5.5 and those of the reference SWG drive design in Table 4.1. Here, the results are shown for values of the rotational angle $\lambda$ equal to -0.5, 0.0, and 0.5 degrees.

Figure 5.49 shows the variation of maximum von Mises stress on the flexible spline for the selected rotational angles $\lambda$. When the rotation angle $\lambda$ is different than zero degrees, the maximum von Mises stress on the flexible spline rises remarkably due to the compression of the flexible spline teeth in the tooth slots of the ring gear. This is caused by the parabolic region extending beyond the circumference of radius equal to the length of the major axis $a$. 
Chapter 5. Two-dimensional stress analysis of different wave generator geometries

Although the oscillatory behavior of the maximum von Mises stress is relatively low with the rotation angle $\lambda$ equal to zero degrees, the oscillatory behavior considerably increases with other values of $\lambda$. The oscillatory behavior of the maximum stresses is typically dependent on the number of teeth in contact transmitting the load of the SWG drive. With the compression of the teeth in the tooth slots of the ring gear caused by rotational angles $\lambda$ different than zero, the oscillatory behavior is also influenced by the position of the segment of the parabolic region beyond the circumference of radius $a$. If it is underneath a root of the flexible spline, maximum stresses reduce due to the inherent flexibility of the flexible spline. However, when the raised portion of the parabolic region is underneath a tooth of the flexible spline, it compresses that tooth inside a tooth slot of the ring gear, raising stresses considerably.

Consequently, it can be assumed that no portion of any wave generator geometry can be beyond the length of the major axis $a$ computed with Equation (3.3). For this reason, this type of asymmetric parabolic wave generator is deemed unsuitable for enhancing the mechanical performance of SWG drives.
5.7.2 Bilaterally-defined asymmetric parabolic wave generator geometry

A different approach to obtain the asymmetric parabolic wave generator consists of using different input parameters for the left side (LS) and the right side (RS) of the geometry of the parabolic wave generator.

![Parabolic wave generator geometry](image)

**Figure 5.50**: Bilateral definition of the asymmetric parabolic wave generator geometry.

Figure 5.50 shows an example geometry of the asymmetric parabolic wave generator compared to the parabolic wave generator with the same parameters for both sides. The modified parameters in the asymmetric parabolic wave generator are those of the right side. This is shown in the coordinate system of the wave generator $S_{wg}$ with the circumference of radius $a$, similarly to the angle-based approach. In this case, the parameters for either side of the asymmetric parabolic wave generator are modified without obtaining segments of the wave generator beyond the circumference of radius $a$.

The parameters subject to modification are the parabola coefficient $a_p$, the parabola length reduction coefficient $C_p$, and the radius of the connector arc $r_c$ as the defining parameters of the original parabolic wave generator. Several analyses have been performed modifying the input
parameters separately for each side of the asymmetric parabolic wave generator.

Considering the improved parameters of the parabolic wave generator shown in Table 5.5, Table 5.7 shows the studied ranges of modification for each input parameter for each side of the bilaterally-defined asymmetric parabolic wave generator. The minor axis reduction coefficient $C$ is maintained equal to 1.0.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Modified between</th>
<th>[units]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parabola coef., $a_p$</td>
<td>[0.0095, 0.0099]</td>
<td>-</td>
</tr>
<tr>
<td>Parabola length reduction coef., $C_p$</td>
<td>[0.45, 0.55]</td>
<td>-</td>
</tr>
<tr>
<td>Connector arc radius, $r_c$</td>
<td>[8, 12]</td>
<td>mm</td>
</tr>
</tbody>
</table>

Regardless of the modified side of the parabolic wave generator, higher values of the parabola coefficient $a_p$ lead to failures in the execution of the two-dimensional model due to significant interference between the teeth of the flexible spline and the ring gear after the deflection has been applied. The interference is found between the opposite flank of the teeth from the side of the wave generator that has been modified. Higher values of the parabola length reduction coefficient $C_p$ also lead to failures in the execution of the model due to numerical errors during the application of the deflection.

On the other hand, lower values of the parabola coefficient $a_p$ and the parabola length reduction coefficient $C_p$ lead to higher stresses on the flexible spline, which constitute no improvement of the mechanical performance previously obtained with the improved geometry of the parabolic wave generator.

Modifying the radius of the connector arc $r_c$ leads to slightly higher stresses on the flexible spline but without excessive influence due to this parameter not significantly influencing the obtained geometry of the parabolic wave generator as shown in Section 5.4. Figure 5.51 shows the variation of maximum von Mises stress on the flexible spline for different values of the radius of the connector arc $r_c$ for each side of the parabolic region. LS and RS indicate which side of the asymmetric parabolic wave generator the radius of the connector arc $r_c$ has been modified. There is no significant change on stresses when separately modifying the radius of the connector arc $r_c$, regardless of the side where it is not equal to 10 mm as found previously.
5.8. Alternative layout of SWG drives

SWG drives are typically operated with the wave generator as the input member of the drive, while either the flexible spline or the ring gear serve as output and the remaining toothed element is fixed. Throughout the analyses shown in this thesis, the flexible spline is considered
as the output member of the drive while the ring gear is fixed, which constitutes the most common layout of SWG drives [2]. However, the second most common layout of SWG drives employs the flexible spline as the fixed element while the output member of the drive is the ring gear. The flexible spline is allowed to deflect radially, but is fixed from rotating around the z-axis.

The main differences between the analyses performed with the two-dimensional model using the traditional layout and the alternative layout of SWG drives are changes in the boundary conditions and the coupling for the application of the output torque. With the traditional layout of SWG drives, the flexible spline is set free to rotate once the deflection and torque are applied while the wave generator rotates as the input member. In the alternative layout, the ring gear is set free to rotate after the torque is applied. Furthermore, the torque is applied on the ring gear element instead of the flexible spline. This is done by adding a coupling on the reference node of the ring gear $RP_{rg}$ connected to the nodes on its outer diameter in order to apply the output torque at Step 3, similarly to when using the traditional layout. However, the output torque is applied in the opposite direction due to the fact that the ring gear, as the output element, rotates in the same direction as the wave generator, which remains as the input member of the drive.

![Diagram showing the input and output rotation direction as a function of the selected layout of SWG drives.](image)

**Figure 5.52:** Input and output rotation direction as a function of the selected layout of SWG drives.

Figure 5.52 shows the direction of rotation of the input and output of the drive for both
layouts of SWG drives where the traditional layout is shown to the left of the figure and the alternative layout studied in this section is shown to the right. WG, FS, and RG refer to the wave generator, the flexible spline, and the ring gear, respectively.

The inner diameter of the flexible spline is still connected to reference node $RP_{fs}$, although this coupling is now only used to impose a motion restraining boundary condition. This boundary condition restrains the flexible spline from rotating, although it is allowed to deflect and move radially. To allow the ring gear to rotate, the only boundary condition applied on the reference node of the ring gear $RP_{rg}$ restrains the nodes of its outer diameter from moving radially. The only degree of freedom of the ring gear is the rotation around the z-axis. The ring gear rotates two tooth slots per input revolution in the same direction the wave generator rotates.

This section shows results obtained with the two-dimensional model employing the alternative layout of SWG drives. This layout and its influence over stresses on SWG drives is evaluated considering four analyses and the parameters shown in Table 4.1, although the output torque $T$ of 20 Nm is applied on the ring gear. Each analysis employs a different improved geometry of the wave generator, making use of the previously found improved parameters with the traditional layout of SWG drives. These parameters for each improved geometry of the wave generator are shown in Section 5.6.

### 5.8.1 Results obtained with the simplified wave generator

Figure 5.53 shows the variations of maximum stresses on the flexible spline considering the alternative layout and the simplified wave generator geometry with minor axis reduction coefficient $C$ equal to 0.97. The obtained results with the alternative layout are considerably similar to those obtained with the traditional layout as shown in Figure 5.5. However, the stresses obtained with the alternative layout are slightly lower than those obtained with the traditional layout.

With the traditional layout of SWG drives, the flexible spline experiences an average maximum von Mises stress of 228.659 MPa with the improved simplified wave generator geometry. When employing the alternative layout of SWG drives, the average maximum von Mises stress on the flexible spline is equal to 228.649 MPa, which is similar to the average maximum von
Chapter 5. Two-dimensional stress analysis of different wave generator geometries

Figure 5.53: Variations of maximum stresses on the flexible spline as fixed element with the improved simplified wave generator.

Mises stress with the traditional layout. The oscillatory behavior results are also similar when considering the flexible spline either as the output or fixed element. However, the maximum tensile stress on the flexible spline shows slightly higher peak values in this case.

5.8.2 Results obtained with the elliptical wave generator

Figure 5.54 shows the variations of maximum stresses on the flexible spline as the fixed element of the drive using the improved geometry of the elliptical wave generator with minor axis reduction coefficient $C$ equal to 1.0. The obtained results show a remarkably reduced initial value of maximum tensile, absolute compressive, and von Mises stresses as compared to the results obtained employing the traditional layout of SWG drives, shown in Figure 5.3. However, the results employing the alternative layout are similar to those obtained with the traditional layout.

In terms of maximum average stresses, the traditional layout of SWG drives results on an average maximum von Mises on the flexible spline equal to 288.101 MPa. However, when using the alternative layout of SWG drives with the improved elliptical wave generator geometry, the
average maximum von Mises stress is equal to 288.626 MPa, which is just slightly higher than the average maximum von Mises stress obtained with the ring gear fixed.

5.8.3 Results obtained with the parabolic wave generator

Figure 5.55 shows the variations of maximum stresses on the flexible spline element employing the alternative layout of SWG drives with the improved parabolic wave generator geometry of parameters shown in Table 5.5. Compared to the results obtained with the traditional layout of SWG drives shown in Figure 5.31, the alternative layout of SWG drives significantly reduces the initial peaks of maximum stresses on the flexible spline. The oscillatory behavior is approximately equal for both layouts.

The improved parabolic wave generator geometry produces an average maximum von Mises stress of 228.883 MPa with the traditional layout, whereas this average stress is equal to 228.062 MPa when employing the alternative layout. This alternative layout results in a slight reduction of maximum average stresses with the improved parabolic wave generator geometry.
5.8.4 Results obtained with the four roller wave generator

Figure 5.56 shows the variations of maximum stresses on the flexible spline as the fixed element of the drive when employing the improved geometry of the four roller wave generator with aperture angle $\beta$ and roller radius $r_4$ equal to 10 degrees and 8 mm, respectively. Compared to the results obtained with the traditional layout of SWG drives shown in Figure 5.46, the stress results obtained with the alternative layout are slightly lower. The oscillatory behavior of the maximum stresses are similar for both cases, although the amplitude of the oscillation is smaller with the alternative layout.

The results obtained with the traditional layout of SWG drives provide an average maximum von Mises stress on the flexible spline equal to 493.208 MPa. On the other hand, the alternative layout leads to a slightly lower average maximum von Mises stress equal to 489.041 MPa. This 4 MPa difference constitutes the largest difference between the results obtained with the traditional and alternative layouts of SWG drives with the previously found improved wave generator geometries. However, the stresses obtained with the alternative layout and the improved four roller wave generator geometry are still considerably high to deem the four roller
wave generator suitable for SWG drives.

With the obtained results, one can assume that there is no significant stress difference between the traditional and alternative layouts of SWG drives. Regardless of using the flexible spline or the ring gear as the output member of the drive, the highest stresses are experienced by the flexible spline element and of a similar magnitude. For these reasons, this thesis omits further results employing the alternative layout due to the similarity of results.

### 5.8.5 Summary of results obtained with both layouts

The layout of SWG drives where the ring gear is used as the output member, as shown in Figure 1.3 b), provides a slightly larger gear ratio (Equation (2.2)) than the most common layout where the flexible spline is the output element. In both cases, the wave generator serves as the input member of the drive. Besides, with the ring gear as output, the direction of the output rotation is the same as the input rotation of the wave generator.

Several analyses with the ring gear as the output member have been performed considering the different geometries of the wave generator. These analyses used the improved parameters of each wave generator geometry. Table 5.8 shows the average maximum von Mises stress
on the flexible spline for each improved wave generator geometry. The average maximum von Mises stress is shown for both the results obtained before with the traditional and most common layout of SWG drives and the results obtained with the alternative layout of SWG drives.

<table>
<thead>
<tr>
<th>Wave generator</th>
<th>Stress</th>
<th>Von Mises, $\sigma$ [MPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simplified</td>
<td></td>
<td>228.659 228.649</td>
</tr>
<tr>
<td>Elliptical</td>
<td></td>
<td>288.101 288.626</td>
</tr>
<tr>
<td>Parabolic</td>
<td></td>
<td>228.883 228.062</td>
</tr>
<tr>
<td>Four roller</td>
<td></td>
<td>493.208 489.041</td>
</tr>
</tbody>
</table>

Employing the alternative layout of SWG drives, the average maximum von Mises stress on the flexible spline is similar to that obtained with the traditional layout. Comparing the average maximum von Mises stress obtained with both layouts, the highest difference is provided by the four roller wave generator of approximately 4 MPa less than with the traditional layout. This difference is not significant due to the large stresses obtained with this type of wave generator, as well as its previously found un-suitability in SWG drives. Considering the simplified, elliptical, and parabolic wave generator geometries, the largest difference in average maximum von Mises stress on the flexible spline between the traditional and alternative layouts of SWG drives is less than 1 MPa in the case of the parabolic wave generator.

Consequently, the mechanical performance of SWG drives obtained with either layout is comparable. The two-dimensional model serves to evaluate both layouts and improve the mechanical performance of SWG drives by modifying the input design parameters of their different components.
Chapter 6

Two-dimensional stress analysis of different tooth profile geometries

6.1 Chapter overview

The following region which can be studied with the two-dimensional model for stress analysis and simulation of meshing is the toothed region of SWG drives. The results shown in this chapter focus on the influence over stresses of different geometries for the teeth of the flexible spline and the ring gear. The aim is to reduce the obtained stresses by modifying the defining parameters of each geometry. The results are evaluated in terms of their stresses on the flexible spline, which was determined as the critical component for mechanical performance of SWG drives in Chapter 4.

Section 6.2 shows the influence of including backlash over stresses with the two-dimensional model. Then, Sections 6.3, 6.4, and 6.5 include the results obtained with different geometries of the involute, QCA, and DCA tooth profiles, for the teeth of the flexible spline and the ring gear. These sections conclude with an improved geometry for each of the tooth profiles which are compared in Section 6.6.

Unless otherwise specified, the results shown in this chapter are obtained for the SWG drive design of parameters shown in Table 4.1 and the traditional layout where the flexible spline is the output member. The wave generator geometry used for study of different tooth profiles is the elliptical geometry with minor axis reduction coefficient $C$ equal to 1.0 due to being the most commonly utilized wave generator in current applications of SWG drives [1, 2]. This geometry
of the wave generator avoids any clearance between the flexible spline and the wave generator
so that the effect of varying the parameters of the tooth profiles of the flexible spline and the
ring gear can be fully observed.

6.2 Influence of backlash over stresses

The backlash present on a gear drive is related to the thickness of the teeth of its elements. The
circumferential tooth thickness \( t_P \) of a single tooth is defined by the following formula,

\[
t_P = \frac{\pi m}{2}
\]

(6.1)

where \( m \) is the module of the teeth. In directly defined tooth profiles, this is obtained when the
angular tooth thickness reduction coefficient is equal to 1.0.

Tooth thickness and, consequently, backlash are influenced by the angular tooth thickness
reduction coefficient \( c_P \) in directly defined profiles, such as the DCA or QCA tooth profiles. For
a tooth manufactured with the involute profile, the tooth thickness is governed by the profile
shift coefficient [13]. This parameter establishes how deep the cutting tool penetrates the bulk
material to generate the teeth. The deeper the tool goes, the thinner the generated involute
profile teeth will be and the larger the resulting backlash.

The actual circumferential tooth thickness \( t'_P \) of a directly defined tooth profile is computed
by

\[
t'_P = c_P t_P
\]

(6.2)

where the angular tooth thickness reduction coefficient \( c_P \) is a user-defined parameter with
positive value equal or lower than one, while the resulting available circumferential backlash \( B \)
with teeth defined with DCA or QCA tooth profiles is obtained with the following formula,

\[
B = t_P (1 - c_P)
\]

(6.3)

resulting from subtracting the actual circumferential tooth thickness \( t'_P \) from the circumferential
tooth thickness \( t_P \). This means that, for a directly defined tooth profile, the lower the value of
the angular tooth thickness reduction coefficient $c_p$, the smaller its tooth thickness $t_p$ and the larger the available backlash $B$. This is considering that the tooth thickness of the mating member is kept constant and equal to its maximum value with angular tooth thickness reduction coefficient $c_p$ equal to 1.0. In case the angular tooth thickness reduction coefficient $c_p$ is modified for both members of the drive, the total available circumferential backlash would be the sum of the backlash of each gear.

Figure 6.1 shows the outer section of several overlapping QCA profile teeth with different angular tooth thickness reduction coefficients $c_p$ for comparison of the available backlash $B$. This is shown in the global coordinate system $S$, which is truncated for the purpose of readability. The resulting total circumferential backlash $B$ available as a function of the angular tooth thickness reduction coefficient $c_p$ is shown in Table 6.1.
Table 6.1: Influence of $c_p$ over circumferential backlash $B$.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$c_p$</th>
<th>$B$ [mm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>0.07854</td>
<td></td>
</tr>
<tr>
<td>0.90</td>
<td>0.15708</td>
<td></td>
</tr>
<tr>
<td>0.85</td>
<td>0.23562</td>
<td></td>
</tr>
<tr>
<td>0.80</td>
<td>0.31416</td>
<td></td>
</tr>
</tbody>
</table>

When the angular tooth thickness reduction coefficient $c_p$ is equal to 1.0, the tooth thickness $t_P$ is maximal, as computed with Equation (6.1). With angular tooth thickness reduction coefficient $c_p$ equal to 0.9, the tooth thickness $t'_P$ reduces to 90 percent of its maximum tooth thickness $t_P$ providing an available total circumferential backlash $B$ equal to 0.15708 mm. When the angular tooth thickness reduction coefficient $c_p$ further reduces to 0.8, the tooth thickness $t'_P$ is considerably smaller extending up to 80 percent of its maximum value $t_P$. This value of the angular tooth thickness reduction coefficient $c_p$ provides a backlash $B$ of 0.31416 mm.

This section focuses on studying the influence of backlash on the stresses obtained with the two-dimensional model of SWG drives. In this case, the study is performed for the SWG drive reference design of parameters shown in Table 4.1, which employs the QCA tooth profile. However, the angular tooth thickness reduction coefficient $c_p$ of the ring gear is varied to modify the backlash present during each analysis.

6.2.1 Results obtained with different angular tooth thickness reduction coefficients $c_p$

The results presented in this section show the stresses on the flexible spline with QCA profile teeth. The angular tooth thickness $c_p$ of the QCA profile teeth of the ring gear is selected to be modified in order to provide backlash. Due to being the stronger element, the ring gear and not the flexible spline is considered to operate with reduced angular tooth thickness reduction coefficient $c_p$. The flexible spline has been determined to be the critical element towards mechanical performance of SWG drives and reducing the tooth thickness of the flexible spline would further weaken it.
The angular tooth thickness reduction coefficient $c_p$ for the teeth of the ring gear is modified as 0.8, 0.9, and 1.0, while the remaining parameters for the teeth of the ring gear, the flexible spline, and the SWG drive are kept constant and equal to those shown in Table 4.1.

Figure 6.2 shows the variation of maximum tensile stress on the flexible spline for the mentioned angular tooth thicknesses $c_p$ for the teeth of the ring gear. When the angular tooth thickness reduction coefficient $c_p$ is equal to 1.0, the maximum tensile stress experienced by the flexible spline achieves its highest value during the meshing with backlash $B$ equal to zero. However, when the angular tooth thickness reduction coefficient $c_p$ is reduced even in the slightest, the maximum tensile stress considerably reduces to an average value equal to 272.77 MPa, whereas with an angular tooth thickness reduction coefficient $c_p$ equal to 1.0, the average maximum tensile stress is equal to 297.31 MPa.

Further lowering the angular tooth thickness reduction coefficient $c_p$ to 0.8, which considerably increases the available backlash $B$, causes the maximum tensile stress on the flexible spline to remain approximately the same. This occurs for any value of the angular tooth thickness reduction coefficient $c_p$ lower than 1.0 that is suitable for the teeth of the ring gear from the point of view of gear design.
For the case with angular tooth thickness reduction coefficient $c_p$ equal to 1.0, there is an initial peak of maximum tensile stress on the flexible spline which is considerably higher than the average maximum tensile stress. This is also observed when the angular tooth thickness reduction coefficient $c_p$ is equal to 0.8. On the other hand, when the angular tooth thickness reduction coefficient $c_p$ is equal to 0.9, there is no initial peak of maximum tensile stress.

The initial peak of tensile stress on the flexible spline is caused by the full application of the deflection moving the teeth of the flexible spline into double flank contact in the tooth slots of the ring gear. This is caused by the lack of backlash $B$ when the angular tooth thickness reduction coefficient $c_p$ is equal to 1.0.

The maximum tensile stress shown in Figure 6.2 shows a considerable oscillatory behavior, especially for the case with angular tooth thickness reduction coefficient $c_p$ equal to 1.0. This is caused by the tooth-to-tooth contact and the number of pairs of teeth in contact influencing the load sharing capabilities of flexible spline and ring gear. The highest oscillation is obtained when the backlash is equal to zero due to the different tooth-to-tooth contact combinations. These contacts may be between a single pair of teeth or two teeth of the ring gear contacting on both flanks of one tooth of the flexible spline. When the angular tooth thickness reduction coefficient $c_p$ is reduced (increasing the backlash in the drive), the oscillatory behavior of the maximum tensile stress reduces slightly due to the more space available for the flexible spline to move between the wave generator and the ring gear.

Figure 6.3 shows the variation of absolute maximum compressive stress experienced by the flexible spline for the same values of the angular tooth thickness reduction coefficient $c_p$ of the ring gear shown in Figure 6.2. The average absolute maximum compressive stress on the flexible spline during the meshing is similar for any value of the angular tooth thickness reduction coefficient $c_p$. This means that the angular tooth thickness reduction coefficient $c_p$ of the teeth of the ring gear has no influence over the maximum absolute compressive stress experienced by the flexible spline (which is located on its inner diameter, far from the tooth contact region). The average absolute maximum compressive stress experienced by the flexible spline for any value of the angular tooth thickness reduction coefficient $c_p$ is equal to approximately 197.5 MPa.
6.2. Influence of backlash over stresses

When the angular tooth thickness reduction coefficient $c_p$ is equal to 1.0, there is a considerably high initial peak of maximum absolute compressive stress. This situation relaxes once the torque is applied and the simulation of meshing begins. When no backlash exists, the flanks of the teeth of the flexible spline are forced into the tooth slots of the ring gear, which causes certain teeth of the flexible spline to contact on both flanks right after the application of the deflection. This considerably increases the initial maximum stress for an angular tooth thickness reduction coefficient $c_p$ equal to 1.0, as shown in Figures 6.2 and 6.3.

Double flank contact typically leads to tooth breakage during operation in traditional gear drives [13]. Some center distance misalignments when a gear drive is operating can lead to short-lived zero backlash states that can immediately break a tooth and cause substantial damage to the mechanism. For this reason, gear designers avoid developing a gear set without backlash. In SWG drives, it is also necessary to include certain backlash to avoid double-flank contact, especially at the beginning of the analysis.

Regarding the oscillatory behavior of the maximum absolute compressive stress shown in Figure 6.3, it is considerable for any value of the angular tooth thickness reduction coefficient $c_p$. This, however, increases slightly for rotation angles of the wave generator between 45 and
120 degrees. This oscillatory behavior is caused by the influence of the tooth contacts between the flexible spline and the ring gear over compressive stresses.

![Graph showing variation of maximum von Mises stress on the flexible spline with different values of the angular tooth thickness reduction coefficient $c_p$.](image)

**Figure 6.4**: Variation of maximum von Mises stress on the flexible spline with different values of the angular tooth thickness $c_p$ of the ring gear.

Figure 6.4 shows the variation of maximum von Mises stress experienced by the flexible spline for values of the angular tooth thickness reduction coefficient $c_p$ of the ring gear equal to 0.8, 0.9, and 1.0. Similarly to the maximum tensile stress shown in Figure 6.2, the maximum von Mises stress on the flexible spline reaches its highest value with an angular tooth thickness reduction coefficient $c_p$ equal to 1.0. However, when the angular tooth thickness reduction coefficient $c_p$ is lower than 1.0, the maximum von Mises stress on the flexible spline reduces significantly. The average maximum von Mises stress on the flexible spline is approximately equal to 264 MPa during the meshing process for any value of the angular tooth thickness reduction coefficient $c_p$ below 1.0, whereas for angular tooth thickness reduction coefficient $c_p$ equal to 1.0, the average maximum von Mises stress is equal to 288.1 MPa.

Although the magnitude of the absolute maximum compressive stress shown in Figure 6.3 is independent from the value of the angular tooth thickness reduction coefficient $c_p$ of the ring gear, the maximum von Mises stress is dominated by the angular tooth thickness reduction
6.2. Influence of backlash over stresses

coefficient $c_p$ in a similar manner than the maximum tensile stress shown in Figure 6.2. The maximum von Mises stress on the flexible spline also shows the initial peak of stresses for values of the angular tooth thickness reduction coefficient $c_p$ equal to 0.8 and, especially, 1.0. The latter is caused by forcing the zero-backlash teeth of the flexible spline into the tooth slots of the ring gear.

The maximum von Mises stress shows considerable oscillatory behavior, which is specially noticeable when the angular tooth thickness reduction coefficient $c_p$ is equal to 1.0. This is caused by the influence of the number of pairs of teeth in contact over the maximum tensile stress experienced by the flexible spline. With angular tooth thickness reduction coefficients $c_p$ below 1.0, the oscillatory behavior of the maximum von Mises stress is slightly reduced due to the larger clearance available to the teeth of the flexible spline.

For values of the angular tooth thickness reduction coefficient $c_p$ of the ring gear lower than 1.0 and the parameters shown in Table 4.1, the maximum tensile and, accordingly, von Mises stresses are significantly reduced. Besides, the average maximum stresses are approximately the same regardless of the backlash included in the drive. This means that the angular tooth thickness reduction coefficient $c_p$ of the teeth of the ring gear should be slightly smaller than 1.0 to include a minimal amount of backlash without weakening the particular SWG drive design.

With the obtained results, it can be assumed that, in order to obtain enhanced mechanical performance of SWG drives, it is recommended to slightly reduce the tooth thickness of the ring gear to ensure there is some backlash after the deflection has been applied. This will allow the teeth of the flexible spline to rearrange between the wave generator and the ring gear once deflected, even before the load is transmitted, taking advantage of the elasticity provided by the cup-shaped spring of the flexible spline.

6.2.2 Rotational angle for tooth contact convergency

The addition of a minimal amount of backlash between the teeth of the flexible spline and those of the ring gear enhances the mechanical performance of SWG drives. However, a sufficiently large amount of backlash leads the two-dimensional finite element model to not converge after the deflection has been applied. At this stage, the backlash impedes the model from
obtaining tooth contact between the flexible spline and the ring gear due to the gap between their teeth. This gap leads the finite element solver to interrupt the analysis because it interprets that the flexible spline is floating, requiring the addition of boundary conditions to hold it in place without tooth contact.

In order to solve this issue, a small rotational angle $\gamma$ is applied to the flexible spline to find the initial tooth contact between its teeth and those of the ring gear. This rotation is applied on the flexible spline reference node $RP_{fs}$ during Step 3, at the same time that the output torque $T$ is applied. Figure 6.5 shows a detail of the major axis region of the SWG drive when the deflection has been applied. The dashed line indicates the position of the teeth of the flexible spline after the rotation has been performed.

When employing a directly defined tooth profile, such as the DCA or QCA profiles, the rotational angle of the flexible spline $\gamma$ is calculated with the following formula,

$$\gamma = \frac{\pi}{2N_{fs}}(1 - c_p)$$  \hspace{1cm} (6.4)

where the lower angular pitch thickness reduction coefficient $c_p$ of the flexible spline or the ring
6.3. Involute tooth profile geometry

gear is considered together with the number of teeth of the flexible spline $N_{fs}$ to ensure that the rotational angle $\gamma$ is large enough to find the tooth contact.

This combination of parameters occasionally leads to an excessively large rotational angle $\gamma$. However, this excessive rotation of the flexible spline only creates a high peak of stresses during Step 3, which is still part of the setup steps of the model. The large stresses during Step 3 are caused by compression between the teeth of the flexible spline and the ring gear, which finishes by Step 4 when the torque is fully applied and the flexible spline is held in place by its contact with the wave generator and the ring gear.

When utilizing the involute tooth profile, the rotational angle $\gamma$ of the flexible spline is equal to the difference between the angular tooth thicknesses of the teeth of the ring gear and the flexible spline.

6.3 Involute tooth profile geometry

Clarence Walton Musser proposed the use of the involute tooth profile in his patent about the concept of “Strain Wave Gearing” [1]. This profile was selected because it has been the most advantageous and extensively used tooth profile in the history of gear theory and design. Musser proposed the use of a slightly larger pressure angle for the teeth of the flexible spline, intended to counteract the deformation that the teeth of the flexible spline experience due to being continuously deflected into meshing with the ring gear. However, since the numbers of teeth of the flexible spline and the ring gear are typically large, their generated involute profile tooth flanks are similar and considerably straight, as shown in Figure 6.6.

![Figure 6.6: Geometry of the involute profile teeth of flexible spline and ring gear.](image-url)
When using the two-dimensional finite element model of SWG drives with the involute tooth profile, there are certain convergency issues due to interference and overlapping of the flexible spline teeth with those of the ring gear after the deflection has been applied. This is usually solved by reducing the time increment during the initial steps of the analysis to assist the finite element solver software in finding convergency.

This section shows the results obtained with the two-dimensional model employing the involute tooth profile for both the teeth of the flexible spline and the ring gear. These analyses are performed for the SWG drive design of parameters shown in Table 4.1 except those related to the QCA tooth profile, which are unnecessary for generation of involute profile teeth.

6.3.1 Results obtained with different pressure angles for the teeth of the flexible spline $\alpha_{fs}$

Since Musser proposed the use of a slightly larger pressure angle for the involute profile teeth of the flexible spline [1], the results shown in this section are obtained with different pressure angles $\alpha$ for the teeth of the flexible spline. The addendum and dedendum coefficients $h_a$ and $h_d$, respectively, the number of teeth $N$, and the module $m$ are shown in Table 4.1 to define the cutting tool that will generate the involute profile teeth of the flexible spline and the ring gear.

The results in this section are obtained with varying pressure angles of the flexible spline $\alpha_{fs}$ between 19.8 to 20.4 degrees in increments of 0.2 degrees. The pressure angle of the ring gear $\alpha_{rg}$ is equal to 20.0 degrees. This is intended to illustrate the performance obtained with different pressure angles $\alpha$ for the teeth of the flexible spline in SWG drives with the involute tooth profile.

Figure 6.7 shows the variation of maximum tensile stress on the flexible spline for the selected pressure angles $\alpha_{fs}$. When the pressure angle of the flexible spline $\alpha_{fs}$ is equal to 19.8 degrees, the maximum tensile stress on the flexible spline is significantly larger than the lowest achieved when the pressure angle of the flexible spline $\alpha_{fs}$ is equal to 20.2 degrees. When the pressure angle of the flexible spline $\alpha_{fs}$ increases to 20.4 degrees, the maximum tensile stress rises considerably to a remarkably large value on the flexible spline.
The maximum tensile stress shows considerable oscillatory behavior. This oscillation is largest when the pressure angle of the flexible spline $\alpha_{fs}$ is equal to 19.8 degrees, where the tooth-to-tooth contact becomes the most critical interaction for performance. However, when the pressure angle of the flexible spline $\alpha_{fs}$ is equal to 20.2 degrees, the oscillatory behavior is markedly reduced; this is opposite to the remaining analyses, where the oscillation is considerably pronounced, but not as significantly as when the pressure angle of the flexible spline $\alpha_{fs}$ equals 19.8 degrees.

Figure 6.8 shows the variation of absolute maximum compressive stress on the flexible spline for the same pressure angles of the flexible spline $\alpha_{fs}$ shown in Figure 6.7. The lowest absolute maximum compressive stress is obtained when the pressure angle of the flexible spline $\alpha_{fs}$ is equal to 20.0 degrees, where the flanks of the flexible spline and the ring gear are parallel in the undeformed condition. When the pressure angle of the flexible spline $\alpha_{fs}$ is decreased to 19.8 degrees, the absolute maximum compressive stress increases significantly due to the consequent edge contact between the tip of the teeth of the flexible spline and the flanks of the teeth of the ring gear with this considerably reduced pressure angle. On the other hand, when the pressure angle of the flexible spline $\alpha_{fs}$ increases beyond 20.0 degrees, the absolute
Chapter 6. Two-dimensional stress analysis of different tooth profile geometries

Figure 6.8: Variation of absolute maximum compressive stress on the flexible spline with different pressure angle $\alpha_{fs}$.

Maximum compressive stress also increases.

Figure 6.9: Variation of maximum von Mises stress on the flexible spline with different pressure angle $\alpha_{fs}$.

Figure 6.9 shows the variation of maximum von Mises stress on the flexible spline for pressure angles of the flexible spline $\alpha_{fs}$ between 19.8 and 20.4 degrees. The lowest maximum von
Mises stress is obtained when the pressure angle of the flexible spline $\alpha_{fs}$ is equal to 20.0 degrees, similarly to the absolute maximum compressive stress shown in Figure 6.8. This is caused by the reduced edge contact when the pressure angle of the flexible spline $\alpha_{fs}$ is equal to that of the ring gear $\alpha_{rg}$. However, when the pressure angle of the flexible spline $\alpha_{fs}$ is equal to 19.8 degrees, the maximum von Mises stress on the flexible spline considerably increases due to the edge contact between the tip of the teeth of the flexible spline and the flanks of the teeth of the ring gear. When the pressure angle of the flexible spline $\alpha_{fs}$ is increased beyond 20.0 degrees, the maximum von Mises stress significantly increases because of the edge contact, in this case between tip of the teeth of the ring gear and the flanks of the teeth of the flexible spline.

These results deem both the lower and higher studied ranges of the pressure angle of the flexible spline $\alpha_{fs}$ unsuitable for mechanical performance of SWG drives. The best case would be a drive with involute profile teeth with a similar pressure angle $\alpha$ for both the teeth of the flexible spline and the ring gear, as opposed to the slightly larger pressure angle of the flexible spline $\alpha_{fs}$ proposed by Musser [1]. The same pressure angle $\alpha$ for both members of the drive leads to better mechanical performance of SWG drives in terms of the resulting von Mises stress.

**Performance of the involute tooth profile geometry and tooth contact**

Considering the obtained results, the improved geometry of the involute tooth profile employs the same pressure angle $\alpha$ for the teeth of the flexible spline and the ring gear. However, the mechanical performance obtained with this improved geometry is remarkably worse than the performance previously obtained with the QCA tooth profile in SWG drives in Chapter 5.

Figure 6.10 shows the variations of maximum tensile, absolute compressive, and von Mises stresses on the flexible spline for the case when the pressure angles $\alpha$ of the flexible spline and the ring gear are both equal to 20.0 degrees. The maximum tensile stress on the flexible spline achieves an average value of 373.672 MPa during the meshing, which is again remarkably higher than the results obtained with the QCA tooth profile.

Regarding the absolute maximum compressive stress, this geometry for the teeth leads to an average absolute maximum compressive stress equal to 370.947 MPa. This is significantly higher than any absolute maximum compressive stress obtained in Chapter 5, especially due to
Chapter 6. Two-dimensional stress analysis of different tooth profile geometries

The rotation angle of the wave generator [deg] versus the maximum stress on the flexible spline [MPa] is shown in Figure 6.10. The stresses are categorized into tensile ($\sigma_1$), von Mises ($\sigma$), and absolute compressive ($|\sigma_1|$).

The figure illustrates the variations of maximum stresses on the flexible spline with the pressure angle $\alpha$ equal to 20.0 degrees. In its proximity to the average maximum tensile stress in this case. In the previously shown results, the absolute maximum compressive stress is never close or comparable to the maximum tensile stress on the flexible spline. This is caused by the remarkable edge contact obtained with the involute tooth profile.

Finally, the maximum von Mises stress achieves an average value equal to 369.366 MPa when the pressure angle of the flexible spline $\alpha_{fs}$ is equal to that of the ring gear $\alpha_{rg}$. As shown in Figure 6.10, the maximum stresses are considerably similar, which reveals that they are highly influenced by the tooth-to-tooth contact between the flexible spline and the ring gear with the involute tooth profile. The oscillatory behavior is also remarkably similar and synchronized between the maximum stresses, emphasizing the negative influence of the tooth-to-tooth contact.

To illustrate the unsuitability of the involute tooth profile towards achieving enhanced mechanical performance of SWG drives, Figure 6.11 shows the distribution of von Mises stress around the major axis region of the SWG drive, considering the cases where the pressure angle of the flexible spline $\alpha_{fs}$ is equal to a) 20.0, b) 19.8, and c) 20.4 degrees. In case a), although there is certain edge contact between the tip of the teeth of the flexible spline and the flanks of the teeth of the ring gear, the compressive stress on the inner diameter of the flexible spline due to

![Figure 6.10: Variations of maximum stresses on the flexible spline with pressure angle $\alpha$ equal to 20.0 degrees.](image)
6.3. Involute tooth profile geometry

\[ \alpha_{fs} = 20.0 \text{ degrees} \]

Von Mises Stress \( \sigma \) (MPa)

\[
\begin{array}{c}
370.319 \\
339.459 \\
308.599 \\
277.739 \\
246.880 \\
216.020 \\
185.160 \\
154.300 \\
123.440 \\
92.580 \\
61.720 \\
30.860 \\
0.000
\end{array}
\]

a) \( \alpha_{fs} = 20.0 \) degrees

\[ \alpha_{fs} = 19.8 \text{ degrees} \]

Von Mises Stress \( \sigma \) (MPa)

\[
\begin{array}{c}
781.140 \\
716.045 \\
650.950 \\
585.855 \\
520.760 \\
455.665 \\
390.570 \\
325.475 \\
260.380 \\
195.285 \\
130.190 \\
65.095 \\
0.000
\end{array}
\]

b) \( \alpha_{fs} = 19.8 \) degrees

\[ \alpha_{fs} = 20.4 \text{ degrees} \]

Von Mises Stress \( \sigma \) (MPa)

\[
\begin{array}{c}
1440.837 \\
1320.768 \\
1200.698 \\
1080.629 \\
960.559 \\
840.490 \\
720.420 \\
600.350 \\
480.281 \\
360.211 \\
240.142 \\
120.072 \\
0.000
\end{array}
\]

c) \( \alpha_{fs} = 20.4 \) degrees

\textbf{FIGURE 6.11:} Distribution of von Mises stresses around the major axis region of the SWG drive with different pressure angle \( \alpha_{fs} \) of the flexible spline.
the contact with the wave generator imposing the deflection at all times is still considerable and shown in the distribution of von Mises stress around the major axis of the drive. Here, there are four pairs of teeth in contact between the flexible spline and the ring gear without including the center tooth of the flexible spline.

However, when the pressure angle of the flexible spline $\alpha_{fs}$ is equal to 19.8 degrees as in case b), the contact between the tip of the teeth of the flexible spline and the flanks of the ring gear increases considerably. Instead of having four pairs of teeth in contact in each meshing region of the drive as in case a), here there are five pairs of teeth in contact in each meshing region. Furthermore, the tip of the center tooth of the flexible spline is compressed between the tooth flanks of the ring gear as shown in Figure 6.11 b). This edge contact increases stresses remarkably, which reduces the coloring and importance of the compressive stress on the inner diameter of the flexible spline.

Finally, Figure 6.11 c) shows the distribution of von Mises stress around the major axis region when the pressure angle of the flexible spline $\alpha_{fs}$ is equal to 20.4 degrees. Considerable edge contact can be observed between the tip of the teeth of the ring gear and the flanks of the teeth of the flexible spline. Here, there are four pairs of teeth in contact per meshing region of the SWG drive, where the tips of two teeth of the ring gear are compressed between the flanks of the flexible spline. This leads to incredibly high stresses, which almost render the compressive stress on the inner diameter of the flexible spline nonexistent. Between case a) and case c), the compressive stress on the inner diameter of the flexible spline apparently disappears due to the high stresses around the edge contacts, although the compressive stress from the interaction between the wave generator and the flexible spline is occurring at all times due to the deflection imposed by the wave generator.

The use of the involute tooth profile for the flank geometry of both the flexible spline and the ring gear leads to remarkable edge contact between them. This phenomenon deems the involute tooth profile unsuitable for enhanced mechanical performance of SWG drives. Besides, having so many teeth on both the flexible spline and the ring gear leads to almost completely straight and equal tooth flanks when the teeth are generated with the involute tooth profile. The edge contacts are caused by the considerably similar flank surfaces of the flexible spline and the
ring gear, as well as the lack of roundness and reliefs that enhance the meshing performance of traditional gear drives. Consequently, the involute tooth profile is not recommended in SWG drives and further results will be omitted in this thesis.

6.4 Quadruple circular arc tooth profile geometry

The geometry of the flanks of teeth with the QCA tooth profile requires five different parameters for its definition,

- Straight segment length $l$
- Dedendum circular arc radius $r_d$
- Addendum circular arc radius $r_a$
- Root circular arc radius $r_r$
- Tip circular arc radius $t$

For this reason, different studies can be performed with the two-dimensional model to analyze the influence of each parameter over stresses. These analyses are performed considering the parameters of the QCA tooth profile shown in Table 4.1, as well as the elliptical wave generator geometry of minor axis reduction coefficient $C$ equal to 1.0.

This section shows the obtained results by modifying each defining parameter of the QCA tooth profile until an improved value is found, which is then used when modifying the following parameter. This provides a considerably improved mechanical performance of SWG drives as a function of the geometry of the QCA tooth profile.

Each defining parameter is varied for both the teeth of the flexible spline and the ring gear. The shown results begin with the study of the length of the straight segment $l$ due to its influence over stresses in SWG drives.
6.4.1 Results obtained with different lengths of the straight segment \( l \)

After performing analyses varying the radius of the dedendum arc \( r_d \) and the length of the straight segment \( l \), the variation of the length of the straight segment \( l \) appears more influential over the stresses. For this reason, the straight segment has been determined more critical towards improving the mechanical performance of SWG drives. This section shows the results obtained with different lengths of the straight segment \( l \). Table 6.2 shows the initial parameters of the QCA tooth profile for evaluation of the influence of the length of the straight segment \( l \) over stresses.

**Table 6.2: Parameters of the initial geometry of the QCA tooth profile.**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dedendum circular arc radius, ( r_d )</td>
<td>2.0 mm</td>
</tr>
<tr>
<td>Addendum circular arc radius, ( r_a )</td>
<td>1.0 mm</td>
</tr>
<tr>
<td>Root circular arc radius, ( r_r )</td>
<td>0.3 mm</td>
</tr>
<tr>
<td>Tip circular arc radius, ( r_t )</td>
<td>0.3 mm</td>
</tr>
</tbody>
</table>

**Figure 6.12: Comparison of geometries of the QCA tooth profile with different lengths of the straight segment \( l \).**
Several analyses have been performed varying the length of the straight segment \( l \). Figure 6.12 shows geometries of a tooth of the flexible spline with QCA tooth profile for values of the length of the straight segment \( l \) equal to 0.0, 0.1, 0.3, and 0.5 mm. The length of the straight segment \( l \) considerably influences the obtained flank geometry of the QCA profile teeth. When the length of the straight segment \( l \) is equal to 0.0 mm, meaning there is no straight segment, the addendum \( r_a \) and the dedendum \( r_d \) circular arcs connect tangentially at the pitch point of the gear with QCA tooth profile. However, when the length of the straight segment \( l \) is equal to 0.1 mm, there is a short straight segment with the pitch circle \( r_P \) of the gear passing through its center (considering the straight segment length reduction coefficient \( c_l \) equal to 0.5). The straight segment \( l \) tangentially connects the addendum \( r_a \) and the dedendum \( r_d \) circular arcs. With a length of the straight segment \( l \) different than 0.0 mm, the top land region is increased while the thickness of the QCA tooth at the base, or between roots, is slightly reduced.

When the length of the straight segment \( l \) is equal to 0.3 mm, the straight segment around the pitch circle \( r_P \) region of the tooth is considerably longer, further increasing the top land region and reducing the tooth thickness at the base of the QCA profile teeth. Besides, longer lengths of the straight segment \( l \) considerably reduce the lengths of arc of the addendum \( r_a \) and the dedendum \( r_d \) circular arcs, as shown in Figure 6.12. This is specially evident with a length of the straight segment \( l \) equal to 0.5 mm, where the straight segment constitutes a considerable part of the flank geometry of the teeth. In this case, the tooth flank geometry has almost no appreciable addendum \( r_a \) and dedendum \( r_d \) circular arcs due to the space on the tooth flank geometry taken by the straight segment \( l \), while the top land region is larger despite the tooth thickness at the base of the tooth being slightly smaller.

This section shows the obtained results for lengths of the straight segment \( l \) equal to 0.0, 0.1, and 0.2 mm due to the significant increase that stresses show with larger lengths of the straight segment \( l \). Figure 6.13 shows the variation of maximum tensile stress on the flexible spline for the different values of the length of the straight segment \( l \). When the length of the straight segment \( l \) is larger than 0.0 mm, the maximum tensile stress experienced by the flexible spline rises remarkably. Besides, the oscillatory behavior of the maximum tensile stress also rises with the length of the straight segment \( l \).
Chapter 6. Two-dimensional stress analysis of different tooth profile geometries

The lowest maximum tensile stress on the flexible spline is obtained when the length of the straight segment $l$ is equal to 0.0 mm, which means that, in order to improve the mechanical performance of SWG drives in terms of maximum tensile stress experienced by the flexible spline, there should not be a straight segment on the pitch region of the QCA profile teeth.

Figure 6.14 shows the variation of absolute maximum compressive stress on the flexible spline for lengths of the straight segment $l$ between 0.0 and 0.2 mm. When the length of the straight segment $l$ increases, the absolute maximum compressive stress experienced by the flexible spline slightly reduces. However, the stress reduction provided by increments of the length of the straight segment $l$ of 0.1 mm over the absolute maximum compressive stress is considerably smaller than the increase in stresses shown in the maximum tensile stress in Figure 6.13.

Figure 6.15 shows the variation of maximum von Mises stress on the flexible spline for lengths of the straight segment $l$ equal to 0.0, 0.1, and 0.2 mm. Considering the length of the straight segment $l$ equal to 0.0 mm, the maximum von Mises stress experienced by the flexible spline reaches its lowest value. On the other hand, the maximum von Mises stress significantly increases with larger values of the length of the straight segment $l$, similarly to the maximum tensile stress shown in Figure 6.13.
6.4. Quadruple circular arc tooth profile geometry

**Figure 6.14:** Variation of absolute maximum compressive stress on the flexible spline with different lengths of the straight segment $l$.

**Figure 6.15:** Variation of maximum von Mises stress on the flexible spline with different lengths of the straight segment $l$. 
Further increasing the length of the straight segment $l$ beyond 0.2 mm leads to significantly high stress due to contact between the topmost part of the straight segment $l$ of the flexible spline and the bottommost part of the straight segment $l$ of the ring gear. As an example, this is illustrated in Figure 6.16, which shows the major axis region of the SWG drive after the deflection has been applied on a SWG drive case design with the length of the straight segment $l$ equal to 0.8 mm for both the QCA profile teeth of the flexible spline and the ring gear. The stresses resulting from the tooth-to-tooth contact between the flexible spline and the ring gear are concentrated near the tip of the teeth of the flexible spline due to contact with the dedendum circular arc $r_d$ of the ring gear or the bottom of its straight segment $l$.

This tooth contact is similar to the contact observed in Figure 6.11, where the almost straight flanks of the involute profile teeth lead to contact between the tip of the teeth of the flexible spline and the dedendum region of the teeth of the ring gear when the pressure angle $\alpha$ is the same for both members of the SWG drive. When employing the QCA tooth profile here, the
pressure angle \( \alpha \) is also considered the same for both elements of the drive, which leads to the non-suitable tooth contact illustrated in Figure 6.16 with longer lengths of the straight segment \( l \). Consequently, it can be assumed that the deflection \( d \) imposed by the wave generator on the flexible spline produces a slight reduction on the pressure angle of the flexible spline \( \alpha_{fs} \), which causes contact between the tip of the teeth of the flexible spline and the dedendum region of the teeth of the ring gear.

These results indicate that stresses considerably increase with the length of the straight segment \( l \), meaning that, in order to improve the mechanical performance of SWG drives, the lowest possible length of the straight segment \( l \) should be selected.

### 6.4.2 Results obtained with different radii of the dedendum arc \( r_d \)

This section shows the results obtained by modifying the radius of the dedendum arc \( r_d \) with an improved length of the straight segment \( l \) equal to 0.0 mm in the reference SWG drive case design of parameters shown in Table 4.1. Table 6.3 shows the updated parameters of the QCA tooth profile after studying the influence of the length of the straight segment \( l \).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>[units]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Straight segment length, ( l )</td>
<td>0.0 mm</td>
</tr>
<tr>
<td>Addendum circular arc radius, ( r_a )</td>
<td>1.0 mm</td>
</tr>
<tr>
<td>Root circular arc radius, ( r_r )</td>
<td>0.3 mm</td>
</tr>
<tr>
<td>Tip circular arc radius, ( r_t )</td>
<td>0.3 mm</td>
</tr>
</tbody>
</table>

Figure 6.17 shows several geometries of a tooth with the QCA profile with radii of the dedendum arc \( r_d \) equal to 2, 5, 10, and 20 mm. When the radius of the dedendum arc \( r_d \) is equal to 2 mm, the curvature of the dedendum circular arc can be identified on the flank geometry of the QCA tooth profile. A low radius of the dedendum arc \( r_d \) leads to considerably wide thickness at the base of the tooth, between roots. However, when the radius of the dedendum arc \( r_d \) increases to 5 mm, the dedendum circular arc gets straightened which reduces the thickness at the base of the tooth. The dedendum circular arc does not influence the geometry of the top land of the teeth with QCA profile, as opposed to the straight segment \( l \).
Increasing the radius of the dedendum arc $r_d$ to 10 mm straightens it further to resemble a straight segment between the pitch point of the tooth and the root circular arc $r_r$. When the radius of the dedendum arc $r_d$ is equal to 20 mm, there is almost no observable curvature or difference between this geometry and the geometry when the radius of the dedendum arc $r_d$ is equal to 10 mm.

Results are obtained for radii of the dedendum arc $r_d$ equal to 2, 5, 10, 20, and 40 mm. These values have been selected due to the fact that low values of the radius of the dedendum arc $r_d$ lead to remarkably high stresses. On the other hand, increasing the radius of the dedendum arc $r_d$ makes the resulting stresses to approximate asymptotically to a particular value due to the lack of influence of larger radii $r_d$ on the QCA tooth profile geometry.

Figure 6.18 shows the variation of maximum tensile stress on the flexible spline for the selected radii of the dedendum arc $r_d$. The highest maximum tensile stress is obtained when the radius of the dedendum arc $r_d$ is equal to 2 mm. The maximum tensile stress experienced by the flexible spline remarkably reduces when increasing the radius of the dedendum arc $r_d$ to 5 mm.
Further increasing the radius of the dedendum arc $r_d$ to 10 mm continues to reduce the maximum tensile stress, although to a lesser extent than by changing the radius of the dedendum arc $r_d$ from 2 to 5 mm.

The larger the radius of the dedendum arc $r_d$, the lower the maximum tensile stress on the flexible spline, as shown in Figure 6.18 when the radius of the dedendum arc $r_d$ is equal to 40 mm. Consequently, the maximum tensile stress approximates a minimum value the larger the radius of the employed dedendum arc $r_d$ is.

Figure 6.19 shows the variation of absolute maximum compressive stress on the flexible spline for radii of the dedendum arc $r_d$ between 2 to 40 mm. Similarly to the maximum tensile stress shown in Figure 6.18, the maximum absolute compressive stress experienced by the flexible spline remarkably reduces when the radius of the dedendum arc $r_d$ increases from 2 to 40 mm.

Figure 6.20 shows the variation of maximum von Mises stress on the flexible spline for values of the radius of the dedendum arc $r_d$ equal to 2, 5, 10, 20, and 40 mm. The highest variation of maximum von Mises stress is obtained when the radius of the dedendum arc $r_d$ is equal to 2 mm, similarly to the maximum tensile and absolute compressive stresses in Figures 6.18 and
Chapter 6. Two-dimensional stress analysis of different tooth profile geometries

Rotation angle of the wave generator [deg]

Absolute maximum compressive stress on the flexible spline (\(\sigma_3\)) [MPa]

\(r_d = 2\) mm
\(r_d = 5\) mm
\(r_d = 10\) mm
\(r_d = 20\) mm
\(r_d = 40\) mm

0 30 60 90 120 150 180

175 185 195 205

FIGURE 6.19: Variation of absolute maximum compressive stress on the flexible spline with different radii of the dedendum arc \(r_d\).

Maximum von Mises stress on the flexible spline (\(\sigma\)) [MPa]

\(r_d = 2\) mm
\(r_d = 5\) mm
\(r_d = 10\) mm
\(r_d = 20\) mm
\(r_d = 40\) mm

0 30 60 90 120 150 180

250 255 260 265 270

FIGURE 6.20: Variation of maximum von Mises stress on the flexible spline with different radii of the dedendum arc \(r_d\).
6.4. Quadruple circular arc tooth profile geometry

Increasing the radius of the dedendum arc \( r_d \) allows the maximum von Mises stress experienced by the flexible spline to be reduced. For this reason, enhanced mechanical performance of SWG drives can be obtained with large radii of the dedendum arc \( r_d \). The resulting geometry of the QCA tooth profile resembles a straight segment below the pitch point due to the radius of the dedendum arc \( r_d \) being so large, as shown in Figure 6.17. Consequently, the dedendum circular arc \( r_d \) could be substituted by a straight segment so as long as it begins at the pitch point of the teeth and progresses towards the roots.

6.4.3 Results obtained with different radii of the addendum arc \( r_a \)

With a selected improved value of the radius of the dedendum circular arc \( r_d \) equal to 20 mm, the following parameter influencing the flank geometry of the QCA tooth profile is the radius of the addendum circular arc \( r_a \). This radius defines the geometry of the circular arc in the addendum region of the teeth which connects tangentially to both the tip circular arc \( r_t \) and the top of the straight segment \( l \). Table 6.4 shows the remaining parameters for the study of the influence of the radius of the addendum arc \( r_a \) over stresses in SWG drives.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>units</th>
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</thead>
<tbody>
<tr>
<td>Straight segment length, ( l )</td>
<td>0.0 mm</td>
</tr>
<tr>
<td>Dedendum circular arc radius, ( r_d )</td>
<td>20.0 mm</td>
</tr>
<tr>
<td>Root circular arc radius, ( r_r )</td>
<td>0.3 mm</td>
</tr>
<tr>
<td>Tip circular arc radius, ( r_t )</td>
<td>0.3 mm</td>
</tr>
</tbody>
</table>

Figure 6.21 shows several geometries of the QCA tooth profile with radii of the addendum arc \( r_a \) equal to 1.0, 1.5, 2.0, and 3.0 mm. When the radius of the addendum arc \( r_a \) is equal to 1.0 mm, the top land region of the tooth is slightly reduced. This radius of the addendum arc \( r_a \) provides the largest curvature near the tip of the tooth. On the other hand, when the radius of the addendum arc \( r_a \) increases to 1.5 mm, the top land region gets extended due to the straightening of the addendum circular arc. Further increasing the radius of the addendum arc \( r_a \) to 2.0 mm turns the tooth flank geometry of the QCA profile into an almost straight line progressing from the straight segment \( l \) to the tip circular arc \( r_t \). This is especially evident when the radius of the addendum arc \( r_a \) is equal to 3.0 mm.
Larger radii of the addendum \( r_a \) and dedendum \( r_d \) arcs convert the tooth flank geometry of the QCA profile into a considerably straight geometry extending between the tip \( r_t \) and root \( r_r \) circular arcs. These arcs simply connect the flank to the top land and the root region of the teeth with a smooth tangent arc that reduces the concentration of stresses, especially near the root of the teeth.

Several analyses have been performed varying the radius of the addendum arc \( r_a \) with the improved values of the length of the straight segment \( l \) and the radius of the dedendum arc \( r_d \). Here, results are shown for radii of the addendum arc \( r_a \) equal to 1, 2, and 3 mm, due to the fact that a radius of the addendum arc \( r_a \) below or equal to 1 mm causes the addendum circular arc to become too small, leading to truncation of the tip of the tooth geometry where the tip does not reach the addendum circle \( r_A \) of the gear. On the other hand, when the radius of the addendum arc \( r_a \) is increased beyond 3 mm, the stresses considerably increase.

Figure 6.22 shows the variation of maximum tensile stress on the flexible spline for radii of
the addendum arc $r_a$ equal to 1, 2, and 3 mm. The lowest maximum tensile stress is obtained with the radius of the addendum arc $r_a$ equal to 1 mm. However, the maximum tensile stress experienced by the flexible spline considerably increases with larger radii of the addendum circular arc $r_a$, as shown when the radius is equal to 2 mm and, especially, 3 mm.

Figure 6.23 shows the variation of absolute maximum compressive stress on the flexible spline for the same radii of the addendum arc $r_a$ shown in Figure 6.22. The different variations of absolute maximum compressive stress almost exactly coincide. This means that the radius of the addendum circular arc $r_a$ of the QCA tooth profile does not influence the compressive stress in SWG drives.

Figure 6.24 shows the variation of maximum von Mises stress on the flexible spline for radii of the addendum arc $r_a$ equal to 1, 2, and 3 mm. This figure shows considerably similar results to those provided by the maximum tensile stress in Figure 6.22. It can be assumed that the best mechanical performance of SWG drives as a function of the radius of the addendum circular arc $r_a$ can be obtained with lower radii. Besides, the addendum circular arc $r_a$ does not influence the compressive stress.
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Figure 6.23: Variation of absolute maximum compressive stress on the flexible spline with different radii of the addendum arc $r_a$.

Figure 6.24: Variation of maximum von Mises stress on the flexible spline with different radii of the addendum arc $r_a$. 
6.4. Quadruple circular arc tooth profile geometry

6.4.4 Results obtained with different radii of the root arc \( r_r \)

The root circular arc \( r_r \) serves as the fillet of the teeth with the QCA tooth profile due to tangentially connecting the dedendum circular arc \( r_d \) with the dedendum or root circle \( r_D \) of the gear. Results are shown for the SWG drive case design with the previously found improved values of the parameters of the QCA tooth profile shown in Table 6.5. These parameters are the length of the straight segment \( l \), the radius of the dedendum arc \( r_d \), and the radius of the addendum arc \( r_a \) selected as 1 mm. The radius of the root arc \( r_r \) is varied here to study its influence over stresses in SWG drives.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>[units]</th>
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<tbody>
<tr>
<td>Straight segment length, ( l )</td>
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<tr>
<td>Dedendum circular arc radius, ( r_d )</td>
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</tr>
<tr>
<td>Addendum circular arc radius, ( r_a )</td>
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</tr>
<tr>
<td>Tip circular arc radius, ( r_t )</td>
<td>0.3 mm</td>
</tr>
</tbody>
</table>

Table 6.5: Partially-improved (3/4) parameters of the geometry of the QCA tooth profile.

![Comparison of geometries of the QCA tooth profile with different radii of the root arc \( r_r \).](image)

**Figure 6.25:** Comparison of geometries of the QCA tooth profile with different radii of the root arc \( r_r \).
Figure 6.25 shows several geometries of the QCA tooth profile with radii of the root arc $r_r$ equal to 0.1, 0.2, 0.4, and 0.6 mm. When the radius of the root arc $r_r$ is equal to 0.1 mm, the fillet region is considerably small and of a high curvature. Increasing the radius of the root arc $r_r$ to 0.2 mm increases the fillet and, consequently, the thickness at the base of the tooth. This is further illustrated when the radius of the root arc $r_r$ is equal to 0.4 mm and, especially, when it is equal to 0.6 mm.

When the radius of the root arc $r_r$ is equal to 0.6 mm, the root circular arc extends from the dedendum circular arc $r_d$ on the tooth flank to almost the center of the root region between consecutive teeth. This radius of the root arc $r_r$ provides the widest tooth thickness at the base of the tooth with QCA profile.

The results here are shown for radii of the root arc $r_r$ between 0.2 and 0.6 mm in increments of 0.1 mm. When the radius of the root arc $r_r$ is below 0.2 mm, the stresses rise considerably due to the resulting small fillet. This concentrates the stresses near the root region, which is subjected to large bending stresses alternating with compressive stresses twice per revolution of the wave generator as input member of the SWG drive. On the other hand, radii of the root arc $r_r$ above 0.6 mm are too large, making the root circular arc extend beyond the center of the root region between consecutive teeth.

Figure 6.26 shows the variation of maximum tensile stress on the flexible spline for the selected radii of the root arc $r_r$. The highest maximum tensile stress is provided by the radius of the root arc $r_r$ equal to 0.2 mm. The maximum tensile stress considerably reduces when the radius of the root arc $r_r$ is increased. Consequently, the lowest variation of maximum tensile stress on the flexible spline is obtained for a radius of the root arc $r_r$ equal to 0.6 mm. This is also the maximum suitable value of the radius of the root arc $r_r$ with the selected parameters of the SWG drive.

Figure 6.27 shows the variation of absolute maximum compressive stress on the flexible spline for the same radii of the root arc $r_r$ shown in Figure 6.26. Similarly to the maximum tensile stress, the absolute maximum compressive stress considerably reduces with larger radii of the root arc $r_r$. However, the absolute maximum compressive stress slightly increases when varying the radius of the root arc $r_r$ from 0.5 to 0.6 mm. The lowest variation of absolute maximum
6.4. Quadruple circular arc tooth profile geometry

Rotation angle of the wave generator [deg]

Maximum tensile stress on the flexible spline (σ₁) [MPa]

<table>
<thead>
<tr>
<th>r (mm)</th>
<th>0.2</th>
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<th>0.4</th>
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</tbody>
</table>

Figure 6.26: Variation of maximum tensile stress on the flexible spline with different radii of the root arc r.

Absolute maximum compressive stress on the flexible spline (σ₁) [MPa]

<table>
<thead>
<tr>
<th>r (mm)</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
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<td>70</td>
<td>80</td>
<td>90</td>
</tr>
</tbody>
</table>

Figure 6.27: Variation of absolute maximum compressive stress on the flexible spline with different radii of the root arc r.

Compressive stress is obtained when the radius of the root arc r is equal to 0.5 mm.

Figure 6.28 shows the variation of maximum von Mises stress on the flexible spline for radii of the root arc r between 0.2 and 0.6 mm. The maximum von Mises stress considerably reduces with larger values of the radius of the root arc r, similarly to the maximum tensile and absolute.

Rotation angle of the wave generator [deg]
compressive stresses. The highest maximum von Mises stress is obtained with the radius of the root arc \( r_r \) equal to 0.2 mm, whereas the lowest stress is obtained when the radius of the root arc \( r_r \) is equal to 0.6 mm.

For these reasons, it is evident that improved mechanical performance of SWG drives is obtained when employing larger radii of the root circular arc \( r_r \). This reinforces the root region of the teeth with the QCA profile due to the resulting larger fillet and thickness at the base of the tooth. The maximum stresses in SWG drives are experienced in this region of the drive twice per revolution of the wave generator.

**6.4.5 Results obtained with different radii of the tip arc \( r_t \)**

The radius of the tip circular arc, \( r_t \), tangentially connects the top land region to the addendum circular arc \( r_a \). The tip circular arc \( r_t \) provides the QCA profile teeth with a smooth transition from the flank to the tip of the teeth.

The influence of the radius of the tip arc \( r_t \) is evaluated here considering the previously obtained improved values of the length of the straight segment \( l \), radii of dedendum \( r_d \) and
6.4. Quadruple circular arc tooth profile geometry

addendum $r_a$ circular arcs, and the radius of the root circular arc $r_r$ selected as 0.6 mm as shown in Table 6.6.

**Table 6.6:** Partially-improved (4/4) parameters of the geometry of the QCA tooth profile.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>[units]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Straight segment length, $l$</td>
<td>0.0 mm</td>
</tr>
<tr>
<td>Dedendum circular arc radius, $r_d$</td>
<td>20.0 mm</td>
</tr>
<tr>
<td>Addendum circular arc radius, $r_a$</td>
<td>1.0 mm</td>
</tr>
<tr>
<td>Root circular arc radius, $r_r$</td>
<td>0.6 mm</td>
</tr>
</tbody>
</table>

Figure 6.29 shows several geometries of the QCA tooth profile with radii of the tip arc $r_t$ equal to 0.1, 0.2, 0.4, and 0.6 mm. When the radius of the tip arc $r_t$ is equal to 0.1 mm, there is a small radius or corner between the top land and the addendum circular arc $r_a$. However, a larger radius of the tip circular arc $r_t$ equal to 0.2 mm provides a smoother connection between
the top land and the addendum arc $r_a$. Larger radii $r_t$ such as 0.4 and 0.6 mm turn the tip of the QCA profile teeth almost into a curve that connects both flanks of each tooth.

This section shows results for values of the radius of the tip arc $r_t$ equal to 0.1, 0.3, 0.5, and 0.7 mm. This is because for radii of the tip arc $r_t$ below 0.1 mm, there is no appreciable curve or arc tangentially connecting the top land to the addendum circular arc $r_a$. On the other hand, radii of the tip arc $r_t$ larger than 0.7 mm are similar to the radius of the addendum arc $r_a$ which almost makes them both coincide as a single extended addendum circular arc connecting the top land with the straight segment $l$ (or the dedendum circular arc $r_d$ as in this case).

Figure 6.30: Variation of maximum von Mises stress on the flexible spline with different radii of the tip arc $r_t$.

Figure 6.30 shows the variation of maximum von Mises stress on the flexible spline for the selected radii of the tip arc $r_t$. The maximum von Mises stress experienced by the flexible spline is approximately the same regardless of the value of the employed radius of the tip arc $r_t$. There is no significant influence over stresses by the radius of the tip arc $r_t$, which renders the study of the maximum tensile and absolute compressive stresses unnecessary. For these reasons, an improved geometry of the QCA profile teeth as a function of the radius of the tip arc $r_t$ cannot be determined. The radius of the tip arc $r_t$ equal to 0.5 mm is selected as the reference value for
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further analyses with the QCA tooth profile.

6.4.6 Mechanical performance of the improved QCA tooth profile geometry

After evaluating and improving all the parameters defining the flank geometry of the QCA tooth profile in the previous sections, conclusions and insight can be drawn as to which parameter is the most influential and how to enhance the mechanical performance of SWG drives with this tooth profile. Figure 6.31 left shows the initial geometry of the QCA tooth profile based on the parameters provided in Table 4.1, whereas the resulting improved geometry of the QCA tooth profile is shown on the right side of the figure. The parameters for both the initial and the improved geometries of the QCA tooth profile are shown in Table 6.7 together with the obtained average maximum tensile, absolute compressive, and von Mises stresses on the flexible spline.

![Initial and Improved Geometries of the QCA Tooth Profile](image)

The initial geometry of the flanks of the QCA tooth profile includes a larger top land region as compared to the improved geometry with a rounder top land. The smaller radius of the root arc $r_r$ in the initial geometry results on a smaller thickness at the base of the tooth. On the other hand, the considerably larger radius of the dedendum arc $r_d$ employed in the improved geometry almost constitutes a straight segment similar to segment $l$. However, the dedendum arc $r_d$ is located between the pitch point of the gear and the beginning of the root arc $r_r$. The straight segment $l$ in the initial geometry is on the center of each flank of the teeth where the pitch circle $r_p$ of the gear goes through.
TABLE 6.7: Parameters of the QCA tooth profile and resulting stresses before and after improvement.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Initial</th>
<th>Improved</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radius of tip circular arc, (r_t)</td>
<td>0.3</td>
<td>0.5</td>
<td>mm</td>
</tr>
<tr>
<td>Radius of addendum circular arc, (r_a)</td>
<td>1.0</td>
<td>1.0</td>
<td>mm</td>
</tr>
<tr>
<td>Length of straight segment, (l)</td>
<td>0.2</td>
<td>0.0</td>
<td>mm</td>
</tr>
<tr>
<td>Radius of dedendum circular arc, (r_d)</td>
<td>2.0</td>
<td>20.0</td>
<td>mm</td>
</tr>
<tr>
<td>Radius of root circular arc, (r_r)</td>
<td>0.3</td>
<td>0.6</td>
<td>mm</td>
</tr>
<tr>
<td>Resulting average maximum stress</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tensile, (\sigma_1)</td>
<td>297.314</td>
<td>235.864</td>
<td>MPa</td>
</tr>
<tr>
<td>Absolute compressive, (</td>
<td>\sigma_3</td>
<td>)</td>
<td>197.017</td>
</tr>
<tr>
<td>Von Mises, (\sigma)</td>
<td>288.101</td>
<td>233.517</td>
<td>MPa</td>
</tr>
</tbody>
</table>

The presence of the straight segment \(l\) in the initial geometry leads the addendum circular arc \(r_a\) to begin at a higher point than in the improved geometry, although the radius of the addendum arc \(r_a\) is the same for both geometries, as shown in Table 6.7. With the initial QCA profile geometry, the tooth-to-tooth contact between the flexible spline and the ring gear is located on a higher spot along the flank of the tooth. This leads to higher tensile stresses on the root region. The smaller radius of the root arc \(r_r\) in the initial geometry further increases the tensile stresses on the root region.

Table 6.7 shows the previously obtained average maximum stresses, which are considerably higher than those obtained with the improved geometry. The average maximum tensile stress is reduced approximately 62 MPa when utilizing the improved geometry while the average maximum von Mises stress gets reduced almost 55 MPa. The average absolute maximum compressive stress also reduces, although only approximately 22 MPa. This is because it is the lowest stress, as well as principally influenced by the geometry of the wave generator imposing the deflection \(d\) on the flexible spline at all times. In these studies to improve the QCA tooth profile, the rim of the flexible spline and the geometry of the wave generator remained the same, which justifies the lower reduction of the average maximum compressive stress as compared to the maximum tensile and von Mises stresses.

Figure 6.32 shows the distribution of tensile stresses near the major and minor axes regions of the SWG drive with the improved geometry of the QCA tooth profile when the wave generator has rotated 45 degrees counterclockwise. Near the minor axis of the drive, the inner diameter of the flexible spline is subjected to tensile stress due to its inward deflection in this
6.4. Quadruple circular arc tooth profile geometry

**Figure 6.32:** Distribution of tensile stresses around the major and minor axes regions of the SWG drive with improved QCA tooth profile.
region. Near the major axis region, however, the flexible spline is subjected to high tensile stress on the root regions. The maximum tensile stress at this stage is located on the next root that the lobe of the wave generator is moving towards to, as shown in Figure 6.32.

Figure 6.33 shows the distribution of compressive stresses near the major and minor axes regions of the SWG drive for the same case shown in Figure 6.32. In this case, however, the flexible spline is subjected to high compressive stress on the root regions near the minor axis of the drive due to its inward deflection that makes the teeth bend towards each other, using the rim of the flexible spline as a hinge or pivoting point. In fact, the maximum compressive stress is located on a root of the flexible spline close to the minor axis region. Near the major axis region, where the lobes of the wave generator are, the flexible spline is subjected to compressive stress on its inner diameter due to the continuously imposed deflection $d$. This compressive stress reaches a magnitude sufficient to deem the tooth contact stresses between the flexible spline and the ring gear too small to be shown in the distribution of compressive stresses near the major axis region.

Finally, Figure 6.34 shows the distribution of von Mises stresses near the major and minor axes regions of the SWG drive with the improved geometry of the QCA tooth profile when the wave generator has rotated 45 degrees counterclockwise. Near the minor axis region, both the inner diameter and the roots of the flexible spline are subjected to von Mises stress due to the tension and compression that these regions experience, as shown in Figures 6.32 and 6.33. The flexible spline is also subjected to von Mises stress in both its roots and inner diameter near the major axis region. Here, the flexible spline experiences the highest von Mises stress, similarly to the distribution of tensile stresses. Compression is experienced on the inner diameter of the flexible spline near the major axis region.

Considerable improvement and reduction of stresses has been obtained on the SWG drive by comparing these results to those obtained with the initial values of the parameters defining the flank geometry of the QCA tooth profile shown in Figure 5.13. However, the maximum stresses are still located on the roots of the flexible spline, which alternates experiencing maximum tensile and maximum compressive stresses twice per revolution of the wave generator as input member of the drive. These results further emphasize the need to improve and enhance
6.4. Quadruple circular arc tooth profile geometry

**Figure 6.33:** Distribution of compressive stresses around the major and minor axes regions of the SWG drive with improved QCA tooth profile.
Figure 6.34: Distribution of von Mises stresses around the major and minor axes regions of the SWG drive with improved QCA tooth profile.
the mechanical performance of SWG drives.

**Backlash influence with the improved QCA tooth profile**

Section 6.2 shows the influence of backlash on stresses in SWG drives considering the initial parameters shown in Table 4.1. Here, this section shows the influence of backlash over stresses considering the improved parameters of the QCA tooth profile shown in Table 6.7.

Backlash $B$ is provided by reducing the angular tooth thickness reduction coefficient $c_P$ of the teeth of the ring gear. The tooth thickness of the ring gear is reduced due to being the least critical element for mechanical performance of SWG drives. The flexible spline is the critical element and reducing the thickness of its teeth would further weaken it.

![Figure 6.35: Variation of maximum von Mises stress on the flexible spline with different angular tooth thickness reduction coefficients $c_P$ for the ring gear.](image)

Figure 6.35 shows the variation of maximum von Mises stress on the flexible spline for angular tooth thickness reduction coefficients $c_P$ of the ring gear equal to 0.96, 0.98, and 1.00. The maximum von Mises stress on the flexible spline is similar for any value of the angular tooth thickness reduction coefficient $c_P$. The presence of backlash $B$ by reducing the angular tooth
thickness reduction coefficient $c_P$ of the ring gear teeth does not influence the magnitude of the resulting stresses employing the improved parameters of the QCA tooth profile.

## 6.5 Double circular arc tooth profile geometry

Similarly to the study of the parameters defining the QCA tooth profile, this section shows results from analyses performed using the DCA tooth profile and modifying its input parameters with the aim of improving the mechanical performance of SWG drives. The input parameters of the DCA tooth profile are shown in Table 6.8 with the considered initial values for the analyses in this section, before any improvement. These parameters have been selected based on the improved parameters of the QCA tooth profile. The radius of the addendum circular arc $r_a$ is equal to 1 mm and the radius of the dedendum circular arc $r_d$ is equal to 0.5 mm. This is because the dedendum circular arc $r_d$ in the DCA tooth profile is equivalent to the root circular arc $r_r$ in the QCA tooth profile, serving as the fillet of the teeth and the connection between the straight segment $l$ and the root of the teeth with the DCA profile.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radius of addendum circular arc, $r_a$</td>
<td>1.0 mm</td>
</tr>
<tr>
<td>Radius of dedendum circular arc, $r_d$</td>
<td>0.5 mm</td>
</tr>
<tr>
<td>Angular tooth thickness reduction coef., $c_P$</td>
<td>1.0 -</td>
</tr>
<tr>
<td>Straight segment length reduction coef., $c_l$</td>
<td>0.0 -</td>
</tr>
</tbody>
</table>

The angular tooth thickness reduction coefficient $c_P$ is 1.0 to ensure that the tooth thickness is maximized and equal for both the teeth of the flexible spline and the ring gear. Similarly to the analyses performed with the QCA tooth profile, there is no backlash between the teeth of the flexible spline and the ring gear. On the other hand, the straight segment length reduction coefficient $c_l$ is selected as 0.0, locating the top end of the straight segment $l$ at the pitch point of the teeth. The straight segment $l$ then progresses towards the root of the teeth.

As opposed to the QCA tooth profile, where the length of the straight segment $l$ is an input parameter, in the DCA tooth profile it is an output parameter which depends on the other parameters of the DCA tooth profile, especially the straight segment length reduction coefficient $c_l$ and the radius of the dedendum circular arc $r_d$. In this case with the input parameters of Table
6.5.1 Results obtained with different radii of the dedendum arc $r_d$

The study of the influence of the parameters defining the DCA tooth profile begins with the variation of the radius of the dedendum circular arc $r_d$, as the most influential parameter over stresses. Figure 6.36 shows several geometries of a tooth with the DCA profile of parameters shown in Table 6.8 and radii of the dedendum circular arc $r_d$ equal to 0.1, 0.2, 0.4, and 0.6 mm. Increasing the radius of the dedendum arc $r_d$ enlarges the thickness at the base of the teeth.

![Comparison of geometries of the DCA tooth profile with different radii of the dedendum arc $r_d$.](image)
Table 6.9 shows the resulting straight segment lengths $l_{fs}$ and $l_{rg}$ of the teeth of the flexible spline and the ring gear with DCA tooth profile, respectively. The straight segment lengths $l_{fs}$ and $l_{rg}$ are shown as a function of the different radii of the dedendum arc $r_d$ shown in Figure 6.36. When the radius of the dedendum arc $r_d$ is equal to 0.1 mm, the straight segment length $l$ is the longest due to the small fillet on the DCA profile. Both lengths of the straight segment $l_{fs}$ and $l_{rg}$ reduce with increasing values of the radius of the dedendum arc $r_d$, which also reduces the available active profile of the teeth of the flexible spline and the ring gear. Regardless of the radius of the dedendum circular arc $r_d$, the straight segment $l$ always begins at the pitch point of the teeth due to the straight segment length reduction coefficient $c_l$ being equal to 0.0.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$r_d$ [mm]</th>
<th>$l_{fs}$ [mm]</th>
<th>$l_{rg}$ [mm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.7825</td>
<td>0.7802</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.7130</td>
<td>0.7097</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.5742</td>
<td>0.6391</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>0.4357</td>
<td>0.4268</td>
<td></td>
</tr>
</tbody>
</table>

Results are shown for radii of the dedendum circular arc $r_d$ between 0.3 and 0.7 mm in increments of 0.1 mm. This is because resulting stresses reduce considerably with radii of the dedendum arc $r_d$ between 0.3 and 0.6 mm. However, for radii of the dedendum arc $r_d$ below 0.3 mm, stresses are remarkably large. On the other hand, radii of the dedendum arc $r_d$ larger than 0.7 mm lead to extremely large fillets on the DCA tooth profile, which extend beyond the center of the roots between teeth.

Figure 6.37 shows the variation of maximum tensile stress on the flexible spline for radii of the dedendum circular arc $r_d$ between 0.3 and 0.7 mm. The maximum tensile stress on the flexible spline reduces considerably with the radius of the dedendum arc $r_d$, reaching the lowest value when the radius of the dedendum arc $r_d$ is equal to 0.6 mm. However, when the radius of the dedendum arc $r_d$ increases further to 0.7 mm, the maximum tensile stress on the flexible spline considerably rises due to concentration of stresses on the center of the roots of the flexible spline.

Regarding the oscillatory behavior of the maximum tensile stress shown in Figure 6.37, it
is considerable although similar for different radii of the dedendum arc $r_d$. This is because modifying the radius of the dedendum arc $r_d$ does not influence the number of pairs of teeth in contact in SWG drives, which is the main factor influencing the oscillatory behavior of the maximum tensile stress. Besides, the location of the pitch point of the teeth of both the flexible spline and the ring gear is independent of the different geometries of the DCA tooth profile as a function of the radius of the dedendum circular arc $r_d$. The pitch point of both the flexible spline and the ring gear constitutes the contact point between them after the deflection has been applied.

Figure 6.38 shows the variation of absolute maximum compressive stress on the flexible spline for the same radii of the dedendum circular arc $r_d$ shown in Figure 6.37. Similarly to the maximum tensile stress, the absolute maximum compressive stress considerably reduces with the radius of the dedendum arc $r_d$, reaching the lowest value when the radius of the dedendum arc $r_d$ is equal to 0.6 mm. The amount of reduction of stresses provided by modifying the radius of the dedendum arc $r_d$ on the maximum tensile and absolute compressive stresses is reduced with larger radii of the dedendum arc $r_d$. Except for the case when the radius of the dedendum arc $r_d$ is equal to 0.7 mm, further increasing the radius of the dedendum arc $r_d$ provides a slightly
lower reduction of stresses on the flexible spline.

The oscillatory behavior of the absolute maximum compressive stress shown in Figure 6.38 is also similar regardless of the radius of the dedendum arc \( r_d \). The tooth-to-tooth contact is not influenced by the radius of the dedendum arc \( r_d \) due to the pitch point being the same at all times.

Figure 6.39 shows the variation of maximum von Mises stress on the flexible spline for radii of the dedendum arc \( r_d \) between 0.3 and 0.7 mm. The lowest maximum von Mises stress is achieved when the radius of the dedendum arc \( r_d \) is equal to 0.6 mm, although it is considerably similar to the maximum von Mises stress when the radius of the dedendum arc \( r_d \) is equal to 0.5 mm. This is due to their stress similarity, also shown in the maximum tensile and absolute compressive stresses in Figures 6.37 and 6.38, respectively. In the maximum von Mises stress, the stresses also reduce with larger radii of the dedendum arc \( r_d \).

### 6.5.2 Results obtained with different radii of the addendum arc \( r_a \)

The following parameter defining the flank geometry of a tooth with the DCA profile is the radius of the addendum circular arc \( r_a \). Figure 6.40 shows several geometries of a tooth with the
DCA profile and different values for the radius of the addendum arc $r_a$ to illustrate its effect on the flank geometry. When the radius of the addendum arc $r_a$ is equal to 1.0 mm, the straight segment $l$ connects with the top land region of the tooth with an arc of large curvature. Increasing the radius of the addendum arc $r_a$ makes differentiating between the straight segment $l$ region and the addendum circular arc progressively more difficult. The curvature of the addendum arc reduces as the radius of the addendum arc $r_a$ increases through 1.5 mm and 2.0 mm. When the radius of the addendum arc $r_a$ is equal to 3.0 mm, the flank geometry of the DCA profile tooth seems to consist of a straight flank from the dedendum circular arc $r_d$ or fillet to the top land region.

The straight segment $l$ on the DCA tooth profile does not change with different values of the radius of the addendum arc $r_a$. Although the radius of the addendum circular arc $r_a$ changes, the top-most point of the addendum arc $r_a$ is not restricted to comply with certain conditions on the addendum radius $r_A$ as opposed to the bottommost point of the dedendum arc $r_d$. The latter has to be the point tangentially connecting the dedendum circle of radius $r_D$ to the dedendum circular arc $r_d$. On the other hand, at the top of the tooth, the addendum circular arc $r_a$ is exempted to transition to the top land region of the tooth tangentially. Consequently, the topmost
point of the addendum circular arc \( r_a \) changes location with different values of its radius \( r_a \), as shown in Figure 6.40.

In this section, the obtained results with the DCA tooth profile are shown. These analyses are performed by modifying the value of the radius of the addendum arc \( r_a \) and keeping constant the previously mentioned parameters. The radius of the dedendum arc \( r_d \) is equal to a selected improved value of 0.6 mm. Both the radius of the addendum arc \( r_a \) of the flexible spline and the ring gear are varied here.

After several analyses modifying the value of the radius of the addendum arc \( r_a \), it is observed that modification does not significantly influence the obtained stresses on the flexible spline. For this reason, Figure 6.41 shows the variation of maximum von Mises stress on the flexible spline for radii of the addendum arc \( r_a \) equal to 1.0, 1.5, and 2.0 mm. The lowest maximum von Mises stress is obtained when the radius of the addendum arc \( r_a \) is equal to 1.0 mm.
6.5. Double circular arc tooth profile geometry

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Rotation angle of the wave generator [deg]

Maximum von Mises stress on the flexible spline (\(\sigma\)) [MPa]

\(r_a = 2.0\) mm

\(r_a = 1.5\) mm

\(r_a = 1.0\) mm

Figure 6.41: Variation of maximum von Mises stress on the flexible spline with different radii of the addendum arc \(r_a\).

6.5.3 Results obtained with different straight segment length reduction coefficients \(c_l\)

In SWG drives with the DCA tooth profile, straight segment \(l\) and its location are mainly governed by the radius of the dedendum circular arc \(r_d\) and the straight segment length reduction coefficient \(c_l\). The radius of the dedendum arc \(r_d\) conditions where the bottommost point of the straight segment \(l\) of the DCA profile tooth is located. On the other hand, the straight segment length reduction coefficient \(c_l\) determines the location of the topmost point of the straight segment \(l\) with respect to the pitch point of the tooth and the resulting length \(l\) of the straight segment.

Although the influence of the straight segment length reduction coefficient \(c_l\) was not studied in the section pertaining to the QCA tooth profile, its influence is shown in this section with the DCA tooth profile to verify the unsuitability of the straight segment \(l\) when it progresses beyond the pitch circle \(r_P\) towards the tip of the teeth.

Figure 6.42 shows several tooth geometries with the DCA tooth profile and different values of the straight segment length reduction coefficient \(c_l\) equal to 0.0, 0.1, 0.3, and 0.5. The dashed
line indicates the pitch circle of the gear for reference of the location of the pitch point on the straight segment $l$ as a function of the straight segment length reduction coefficient $c_l$.

When the straight segment length reduction coefficient $c_l$ is equal to 0.0, the pitch point coincides with the topmost point of the straight segment $l$. Increasing the straight segment length reduction coefficient $c_l$ raises the location of the topmost point of the straight segment $l$ along the tooth flank of DCA profile. This also increases the length $l$ of the straight segment, as shown in Table 6.10. This table shows the resulting lengths $l_{fs}$ and $l_{rg}$ of the straight segments of the DCA profile teeth of the flexible spline and the ring gear, respectively.

Further increasing the straight segment length reduction coefficient $c_l$ to 0.5 leads to considerably straightened flanks on the teeth with DCA profile, as shown in Figure 6.42. In this case, the pitch point of teeth is located on the center of the straight segment $l$. With larger values of the straight segment length reduction coefficient $c_l$, the DCA profile teeth become similar to the involute profile teeth.
Several analyses have been performed modifying the straight segment length reduction coefficient $c_l$ and studying the resulting stresses in SWG drives with the DCA tooth profile. These analyses are performed while keeping the remaining parameters constant and equal to their previous values. The straight segment length reduction coefficient $c_l$ is modified for both the teeth of the flexible spline and the ring gear.

![Figure 6.43: Variation of maximum von Mises stress on the flexible spline with different straight segment length reduction coefficients $c_l$.](image)

Figure 6.43 shows the variation of maximum von Mises stress on the flexible spline for values of the straight segment length reduction coefficient $c_l$ equal to 0.0, 0.1, and 0.2. When the straight segment length reduction coefficient $c_l$ is equal to 0.0, the maximum von Mises stress
experienced by the flexible spline is the lowest. The oscillatory behavior of the maximum von Mises stress is also reduced in this case.

Similarly to the results obtained with the QCA tooth profile, the smallest possible straight segment $l$ on the DCA tooth profile provides the lowest maximum von Mises stresses on the flexible spline. This is obtained with the straight segment length reduction coefficient $c_l$ equal to 0.0, which limits the straight segment $l$ on the flank of the DCA profile teeth between the pitch point and the topmost point of the dedendum circular arc $r_d$.

### 6.5.4 Mechanical performance of the improved DCA tooth profile geometry

The previous sections focused on studying the influence of each parameter defining the geometry of the DCA tooth profile over stresses in order to reduce them in SWG drives. Conclusions can be drawn here regarding which parameter is the most influential, as well as insight about how to enhance the mechanical performance of SWG drives.

![Figure 6.44: Geometries of the DCA tooth profile before and after improvement study.](image)

Figure 6.44 left shows the initial geometry of the DCA tooth profile of parameters shown in Table 6.8. The right side of the figure shows the resulting improved geometry after studying each parameter of the DCA tooth profile. Unfortunately, the geometries are almost identical due to the use of parameters based on the improved parameters of the QCA tooth profile at the beginning of the analyses with the DCA tooth profile.

Table 6.11 shows both the initial and the improved values of the parameters of the DCA tooth profile, as well as the resulting average maximum tensile, absolute compressive, and von
Mises stresses on the flexible spline. As shown in Figure 6.44, the difference between the initial and the improved geometries of the DCA tooth profile is almost inappreciable. The geometries differ on the radius of the dedendum circular arc \( r_d \) employed. The improved geometry uses a slightly larger radius of the dedendum arc \( r_d \) equal to 0.6 mm for the fillet region of each tooth, which provides a smoother transition from the flank of the tooth to the root region.

### Table 6.11: Parameters of the DCA tooth profile and resulting stresses before and after improvement.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Initial</th>
<th>Improved</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radius of addendum circular arc ( r_a )</td>
<td>1.0</td>
<td>1.0</td>
<td>mm</td>
</tr>
<tr>
<td>Radius of dedendum circular arc ( r_d )</td>
<td>0.5</td>
<td>0.6</td>
<td>mm</td>
</tr>
<tr>
<td>Resulting average maximum stress</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tensile, ( \sigma_1 )</td>
<td>241.670</td>
<td>239.151</td>
<td>MPa</td>
</tr>
<tr>
<td>Absolute compressive, (</td>
<td>\sigma_3</td>
<td>)</td>
<td>169.079</td>
</tr>
<tr>
<td>Von Mises, ( \sigma )</td>
<td>236.526</td>
<td>236.429</td>
<td>MPa</td>
</tr>
</tbody>
</table>

Regarding stresses in SWG drives, the slightly larger radius of the dedendum arc \( r_d \) of the improved geometry of the DCA tooth profile leads to a slight reduction of the average maximum stresses on the flexible spline. This small reduction of stresses is also caused by the similarity of the initial DCA tooth profile and the improved DCA tooth profile. The initial profile already results in remarkably low stresses. In this case, it is difficult to further and/or significantly lower the stresses obtained with the DCA tooth profile.

### 6.6 Comparison of improved tooth profile geometries

The two-dimensional model allows to identify the influence of the geometry of their teeth over the mechanical performance of SWG drives. This provides insight about which parameter defining the geometry is more influential over stresses in SWG drives with the involute, the QCA, and the DCA tooth profiles. The two-dimensional model implemented these tooth profiles and this chapter focused on the evaluation of their influence over stresses with the aim to improve the mechanical performance of SWG drives as a function of their tooth profile geometry.

The involute tooth profile has proven unsuitable in SWG drives. This is due to the appearance of edge contact between the teeth of the flexible spline and the ring gear. Their flank
geometry is almost completely straight because of the large number of teeth in SWG drives. Although proposed by Musser [1] and considerably advantageous in other types of gear drives, the involute tooth profile is not recommended in SWG drives, for which other tooth profiles have been developed.

The improved geometries of the QCA and DCA tooth profiles have been obtained after studying each parameter independently; the parameters resulting in the lowest average maximum stresses will be employed in further analyses of SWG drives. The improved parameters of the QCA and the DCA tooth profiles are shown in Tables 6.7 and 6.11, respectively.

Table 6.12 shows the average maximum tensile, absolute compressive, and von Mises stresses on the flexible spline for the improved combination of parameters of each tooth profile geometry. The use of the involute tooth profile geometry in SWG drives results in the highest average maximum stresses, with averages approximately 370 MPa for tensile, absolute compressive, and von Mises stresses. These are considerably larger than the average maximum stresses obtained with the QCA and the DCA tooth profiles, deeming the involute tooth profile unsuitable in SWG drives.

| Tooth profile            | Stresses: Tensile, $\sigma_1$ [MPa] | Compressive, $|\sigma_3|$ [MPa] | Von Mises, $\sigma$ [MPa] |
|--------------------------|------------------------------------|---------------------------------|---------------------------|
| Involute                 | 373.672                            | 370.947                         | 369.366                   |
| Quadruple circular arc   | 235.864                            | 174.912                         | 233.517                   |
| Double circular arc      | 239.151                            | 168.022                         | 236.429                   |

On the other hand, the QCA tooth profile results in an average maximum tensile stress of 235.864 MPa, barely higher than that obtained with the DCA tooth profile. The QCA also leads to slightly lower average maximum von Mises stress on the flexible spline than the DCA tooth profile, with an average equal to 233.517 MPa. These results deem the QCA tooth profile the best performer evaluated with the two-dimensional model with the aim of improving the mechanical performance of SWG drives.

The DCA tooth profile also proves competitive and suitable for SWG drives together with the QCA tooth profile. The DCA tooth profile results in the lowest average maximum absolute
compressive stress with a value equal to 168.022 MPa, slightly lower than that obtained with the QCA tooth profile.

Comparing the performance of the geometries themselves, the straight flanks of the involute tooth profile provided more similar results than using large straight segments on the flanks of the QCA and DCA tooth profiles. This also occurred when the top portion of the straight segment protruded from the pitch circle towards the tip of the teeth. For these reasons, the use of straight sections on the geometry of the tooth flanks of the flexible spline and the ring gear should be limited. A straight segment should be incidental and located between the pitch circle and the root of the teeth for enhanced mechanical performance of SWG drives.

To sum up, the best mechanical performance in terms of lower stresses from the considered geometries of the teeth is achieved with the smoothest possible transition between the pitch region of the teeth to the roots. By having larger radii for the fillet regions, the base of the teeth are further strengthened, which results in lower stresses and improved mechanical performance of SWG drives.

The analyses performed with the two-dimensional model provide useful insight towards improving the mechanical performance of SWG drives and understanding their behavior. However, a three-dimensional model of SWG drives is necessary to take into account and evaluate the effect of the deformation of the cup-shaped spring of the flexible spline over stresses, as well as over the tooth-to-tooth contact.
Chapter 7

Three-dimensional simulation of meshing and stress analysis

7.1 Chapter overview

This chapter shows the initial results and improvements of the mechanical performance of SWG drives evaluated with the three-dimensional finite element model for simulation of meshing and stress analysis.

Section 7.2 establishes the context of the results in terms of the input parameters of the reference SWG drive design, as well as the geometry of the cup-shaped spring of the flexible spline. Then, Section 7.3 explains the initial state and steps of the three-dimensional model, which leads into the application of deflection to localize the tooth contact between the flexible spline and the ring gear in Section 7.4.

The simulation of meshing with the three-dimensional model of SWG drives is illustrated in Section 7.5, while Section 7.6 explains the complex state of deflection of the flexible spline, including the geometry of its cup-spring throughout the meshing. Section 7.7 shows the differences between the two- and three-dimensional models of SWG drives in terms of tensile, compressive, and von Mises stresses.

Sections 7.8 and 7.9 compare the stresses between the flexible spline and the ring gear and the effect of the transmission of load with the three-dimensional model, respectively.

Similarly to the two-dimensional model of SWG drives, Section 7.10 shows the influence of the rim thickness of the flexible spline over stresses with the three-dimensional model. Then,
Section 7.11 provides a summary of the preliminary results with this model.

Finally, Sections 7.12 and 7.13 illustrate the most important contribution of this thesis dissertation in terms of the implemented micro-geometry modifications of SWG drives and their effectiveness towards reducing the stresses obtained with the three-dimensional model and, consequently, improving the mechanical performance of this type of gear drive.

## 7.2 Context of the results

The results from the three-dimensional finite element model of SWG drives are obtained for the reference SWG drive design of parameters shown in Table 7.1, unless otherwise specified. These parameters are based on the improved values of the parameters previously found with the two-dimensional model. The traditional layout of SWG drives, where the wave generator is the input member and the flexible spline is the output member, is selected for these analyses with the ring gear fixed [1].

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Flexible spline</th>
<th>Ring gear</th>
<th>[units]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of teeth, $N$</td>
<td>120</td>
<td>122</td>
<td>-</td>
</tr>
<tr>
<td>Module, $m$</td>
<td>1</td>
<td></td>
<td>mm</td>
</tr>
<tr>
<td>Pressure angle, $\alpha$</td>
<td>20 deg</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Addendum coef., $h_a$</td>
<td>0.6</td>
<td></td>
<td>-</td>
</tr>
<tr>
<td>Dedendum coef., $h_d$</td>
<td>0.8</td>
<td></td>
<td>-</td>
</tr>
<tr>
<td>Face width, $F_w$</td>
<td>10 mm</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rim thickness, $t_r$</td>
<td>1 mm</td>
<td>2 mm</td>
<td></td>
</tr>
<tr>
<td>Major axis, $a$</td>
<td>59.2 mm</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Type of tooth profile</td>
<td>QCA</td>
<td></td>
<td>-</td>
</tr>
<tr>
<td>Radius of tip circular arc, $r_t$</td>
<td>0.5 mm</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Radius of addendum circular arc, $r_a$</td>
<td>1.0 mm</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Length of straight segment, $l$</td>
<td>0.0 mm</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Radius of dedendum circular arc, $r_d$</td>
<td>20.0 mm</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Radius of root circular arc, $r_r$</td>
<td>0.6 mm</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Angular tooth thickness reduction coef., $c_P$</td>
<td>1.0</td>
<td>0.95</td>
<td>-</td>
</tr>
<tr>
<td>Straight segment length reduction coef., $c_L$</td>
<td>0.5</td>
<td></td>
<td>-</td>
</tr>
<tr>
<td>Modulus of elasticity, $E$</td>
<td>210 GPa</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Poisson’s ratio, $\nu$</td>
<td>0.29</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Output torque, $T$</td>
<td>20 Nm</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The flexible spline and the ring gear have 120 and 122 teeth, respectively, which provides a gear ratio $m_G$ equal to -60. This gear ratio requires the wave generator to rotate 60 times in a direction for the flexible spline to rotate one revolution in the opposite direction.

![Flexible spline tooth](image1)

![Ring gear tooth](image2)

**Figure 7.1**: Geometry of the QCA profile geometry of a tooth of the flexible spline and the ring gear.

The selected tooth profile for the teeth of the flexible spline and the ring gear is the QCA profile with the improved parameters obtained previously with the two-dimensional model. Figure 7.1 shows a tooth of the flexible spline and a tooth of the ring gear with the parameters shown in Table 7.1. The face width $F_w$ of both the flexible spline and the ring gear is equal to 10 mm.

The pitch point of the teeth is located at the center of the straight segment of the QCA tooth profile with the straight segment length reduction coefficient $c_l$ equal to 0.5. However, using an angular tooth thickness reduction coefficient $c_P$ (1.0) for both the flexible spline and the ring gear led initial analyses with the three-dimensional model to fail due to extreme wedging of the teeth of the flexible spline into the tooth slots of the ring gear. This is because with the same angular tooth thickness reduction coefficient $c_P$ equal to 1.0 for both flexible spline and ring gear, the tooth thickness and the tooth slot width of the flexible spline and the ring gear are identical. Considering the cup-spring of the flexible spline, the open-end of the flexible spline deflects largely and its teeth are pushed into the tooth slots of the ring gear, which makes the three-dimensional finite element model fail. For this reason, a slightly reduced angular tooth thickness reduction coefficient $c_P$ equal to 0.95 has been selected for the teeth of the ring gear, the
stronger element for mechanical performance of SWG drives. This parameter change enlarges the width of the tooth slots of the ring gear, allowing the teeth of the flexible spline to contact without crashing the model.

Similarly to the results obtained with the two-dimensional model, the selected material for the flexible spline and the ring gear is steel of mechanical properties shown in Table 7.1. The mechanical properties are the modulus of elasticity $E$ and the Poisson’s ratio $\nu$. The output torque $T$ used to simulate the transmission of load during the meshing of SWG drives is equal to 20 Nm and applied to the flexible spline as output member of the drive. On the other hand, the wave generator and the pushing pins are rigid surface elements which do not require any mechanical properties in the model [64].

Traditional SWG drives employ the elliptical wave generator geometry as proposed by Musser in 1959 [1]. For this reason, the analyses performed with the three-dimensional model include the previously found improved elliptical wave generator of parameters shown in Table 7.2. During the analyses with the two-dimensional model, it was determined that the improved elliptical wave generator geometry required the minor axis reduction coefficient $C$ to be equal to 1.0. This allows the wave generator to be the largest and contact the inner diameter of the flexible spline along its entire geometry. Finally, the face width $F_w$ of the wave generator is 10 mm, similarly to that of the flexible spline and the ring gear.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>[units]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type</td>
<td>Elliptical -</td>
</tr>
<tr>
<td>Minor axis coef., $C$</td>
<td>1.0 -</td>
</tr>
<tr>
<td>Face width, $F_w$</td>
<td>10 mm</td>
</tr>
</tbody>
</table>

Similarly to the results from the two-dimensional model the unit system employed here is the SI. Distances, forces, and, consequently, stresses are measured in millimeters, Newtons, and Mega Pascals, respectively. The resulting stresses are observed in the form of both tensile and compressive stresses. These are obtained from the maximum principal stress $\sigma_1$ and the absolute value of the minimum principal stress $\sigma_3$ [64]. The mechanical strength of SWG drives analyzed with the three-dimensional model is evaluated in terms of von Mises stress.
A module has been developed in the custom-made software IGD for the automatic generation of parameterized analyses of the three-dimensional model of SWG drives. IGD generates the finite element models and each model includes all the necessary defining parameters for a SWG drive design. Then, ABAQUS performs the finite element analysis of each model [64]. Initial models considered in this chapter includes 36 steps to simulate the meshing of the drive by rotating the wave generator half a revolution. Further analyses simulate the meshing through 45 degrees of rotation of the wave generator with 9 steps only, due to the computational cost of each three-dimensional finite element model.

A three-dimensional finite element model corresponding to the SWG drive design of parameters shown in Tables 7.1 and 7.2 includes 575,613 nodes and 451,964 finite elements for the five parts in the model. This number of nodes and elements is remarkably larger than that of the two-dimensional finite element models due to the expansion of the parts into three dimensions, as well as the inclusion of the cup-shaped spring of the flexible spline. For this reason, the analysis of this model requires approximately 22 hours plus 3 hours of post-processing using four cores of a processor Intel(R) Xeon(R) CPU E3-1240 v6 of a desktop computer.

![Flexible spline tooth](image1)

**Figure 7.2:** Finite element meshes of a tooth of the flexible spline and a tooth of the ring gear.

Figure 7.2 shows the finite element meshes of a tooth of the flexible spline and a tooth of the ring gear for the SWG drive reference design of parameters in Tables 7.1 and 7.2. These are the tooth meshes employed for each tooth of the flexible spline and the ring gear throughout the analyses. The number of elements on each section of the active tooth profile is the same as
those selected for the analyses with the two-dimensional model. For the expansion into three-
dimensions along the longitudinal direction or z-axis of the drive, the number of elements along
the face width is specified in IGD. However, the shape of the elements on the lower rim section
of the flexible spline is modified to connect with the elements in the cup-spring of the flexible
spline, as shown in Figure 7.2.

7.2.1 Geometry of the cup-shaped spring of the flexible spline

The main difference between the two- and three-dimensional models of SWG drives consists
of the inclusion of the cup-shaped spring of the flexible spline. The analyses performed with
the three-dimensional model incorporate a cup-spring design of parameters shown in Table
7.3. Figure 7.3 shows the actual geometry of the cup of the flexible spline in the results with the
three-dimensional model. The figure also illustrates the different sections of the entire geometry
of the flexible spline, which will be referenced throughout the results.
7.3 Initial state and steps of the three-dimensional model

In order to simulate the meshing of SWG drives in the three-dimensional model, several steps organize the different boundary conditions and interactions from the initial state of the model with interference between the elements to the deformed state and meshing condition, with the wave generator rotating as input member of the drive.

Figure 7.4 left shows the initial state of the different elements of SWG drives included in the three-dimensional model where the wave generator interferes with the flexible spline near the major axis regions of the drive while the teeth of the latter interfere with those of the ring.

### Table 7.3: Parameters of the cup-spring of the reference flexible spline design.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>[units]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cup length, $l_c$</td>
<td>40 mm</td>
</tr>
<tr>
<td>Cup thickness coef., $c_c$</td>
<td>0.8</td>
</tr>
<tr>
<td>Cup fillet radius, $r_{cf}$</td>
<td>2 mm</td>
</tr>
<tr>
<td>Cup inner radius, $r_{ci}$</td>
<td>20 mm</td>
</tr>
</tbody>
</table>

The main regions of the flexible spline are the teeth (near the open-end of the cup), the cylinder section, and the closed-end of the cup-spring. The open-end is separated from the cylinder section by the teeth of parameters shown in Table 7.1. The cylinder section is connected to the lower rim of the teeth of the flexible spline. Then, the fillet of the cup-spring provides the transition from the cylinder section to the closed-end of the cup, where the latter includes a bore to connect the flexible spline to the output shaft.

With a cup length $l_c$ of 40 mm, the axial length of the complete flexible spline is equal to 50 mm, which is slightly smaller than the radius of the entire SWG drive. The cup thickness coefficient $c_c$ is equal to 0.8, which provides a 20 percent reduction of the radial dimension of the rim of the flexible spline, as shown in Figure 7.2. This means that, while the rim thickness $t_r$ of the flexible spline is equal to 1 mm, as shown in Table 7.1, the thickness of the cup-spring is equal to 0.8 mm. The geometry of the fillet of the cup of the flexible spline is defined by the cup fillet radius $r_{cf}$ equal to 2 mm. Finally, the cup inner radius $r_{ci}$ defines the bore on the closed-end of the flexible spline. This radius $r_{ci}$ is equal to 20 mm and includes a coupling to simulate the transmission of load by the flexible spline.
Figure 7.4: Initial state of the three-dimensional finite element model and detail of the mapped meshing of the flexible spline and the ring gear.

gear near the minor axis. The pushing pins fit inside the flexible spline to impose the deflection $d$ during Step 1. The right side of the figure shows the mapped meshing of the flexible spline and the ring gear near the major axis region of the SWG drive reference design. Similarly to the two-dimensional model of SWG drives, the mesh divides each tooth in two halves and refines the elements near the tooth flanks. Coarser elements are used for the rim regions, and these are specially modified on the flexible spline to connect with the elements of its cup-shaped spring.

Similarly to Figure 4.3 from the two-dimensional model results, Figure 7.5 shows the variation of maximum tensile stress on the flexible spline throughout a full analysis, with the three-dimensional finite element model of SWG drives as an example. The initial steps set up the model for the simulation of meshing from Step 4 and onwards. The steps are shown in the horizontal axis from Step 1, when the pushing pins deflect the flexible spline into its meshing state with the ring gear, to Step 39, which finalizes half a rotation of the wave generator as input member. During Step 2, and after the deflection $d$ is applied, the wave generator engages with the inside of the flexible spline to relieve the pushing pins from contacting the flexible spline. Then, the interaction of the pushing pins and the flexible spline is deactivated for the rest of the analysis. The pushing pins are not employed from Step 2 and onwards. Step 2 also activates the contact interaction between the teeth of the flexible spline and the ring gear.
During Step 3, the output torque $T$ is applied to the output member of the drive, in this case, the flexible spline. Then, the three-dimensional finite element model is ready for the simulation of meshing beginning in Step 4 when the wave generator begins to rotate a small increment of its total rotation angle for each step until the end of the analysis.

As shown in Figure 7.5, the low maximum tensile stress experienced by the flexible spline in Step 1 is caused by the application of the deflection $d$ by the pushing pins. The maximum tensile stress considerably rises in Step 2 due to the activation of the contact between the wave generator and the flexible spline and the teeth of the flexible spline and the ring gear. The final increase of tensile stresses occurs during Step 3 by the application of the output torque $T$. From Step 4 and onward, the maximum tensile stress experienced by the flexible spline oscillates slightly due to the engagement and disengagement of different pairs of teeth in contact between the flexible spline and the ring gear.

Finally, the steps from Step 3 until the end of the analysis are used for stress analysis of SWG
drives with the three-dimensional model in order to show the influence of load transmission, as well as the influence of the meshing over stresses. For initial results shown in this chapter, the simulation of meshing is performed up to half a rotation of the wave generator. As noted, further results only include 45 degrees of rotation of the wave generator due to the computational cost of each three-dimensional model.

7.4 Application of deflection for tooth contact localization

In order to set up the three-dimensional model for stress analysis of SWG drives after being generated in IGD, the flexible spline has to be deflected into its meshing position with the ring gear. To do so, the pushing pins contact with the inner diameter of the flexible spline below its teeth and deflect it during Step 1.

Figure 7.6 shows the distribution of compressive stresses on the flexible spline near the region where the contact between a pushing pin and the flexible spline occurs. This region is located on the top area of the major axis of the drive and a similar contact region is located on the bottom of the major axis. Both of these regions will transfer their contact from the pushing pins to the wave generator once the deflection has been fully applied. Figure 7.6 a) shows the distribution of compressive stress as soon as the pushing pins have contacted the flexible spline. At this point the ring gear is not experiencing any stresses due to its lack of contact with the flexible spline. The flexible spline, however, experiences compressive stress, especially on its inner diameter, due to the contact with the pushing pins applying the deflection $d$.

The compressive stress experienced by the flexible spline progressively increases while the pushing pins apply the deflection $d$ as shown in Figure 7.6 b). At this stage, the pushing pins have reached their final position and fully deflected the flexible spline into meshing. However, the contact between the teeth of the flexible spline and the ring gear is not yet active.

Similarly to the two-dimensional model of SWG drives, when the deflection $d$ is fully applied by the pushing pins their geometry coincides with that of the wave generator as shown in Figure 4.5. This allows the contact interaction between the flexible spline and the pushing pins to be released. At the same time, the contact interaction between the flexible spline and the
7.4. Application of deflection for tooth contact localization

a) Min. Principal Stress $\sigma_3$ (MPa)

b) Min. Principal Stress $\sigma_3$ (MPa)

c) Min. Principal Stress $\sigma_3$ (MPa)

Figure 7.6: Distribution of compressive stresses on the flexible spline: (a) beginning of deflection, (b) maximum deflection, and (c) tooth contact activation (Step 2).
wave generator is activated. From this moment of the model onwards, the pushing pins do not participate in the analysis.

The change of contact of the flexible spline from the pushing pins to the wave generator is performed during Step 2 together with the activation of the contact interaction between the teeth. This serves to localize the tooth contacts between the flexible spline and the ring gear, as shown in Figure 7.6 (c), where the high concentration of compressive stresses on the flanks and edges of the teeth of the flexible spline indicate the contact with the teeth of the ring gear. At this stage, the compressive stresses on the flexible spline reach a remarkably large value due to significant edge contact. This is caused by the deflection of the cup-spring of the flexible spline, which highly affects the performance of SWG drives. In the two-dimensional model, the lack of the cup-shaped spring of the flexible spline allowed the flexible spline to homogeneously deflect along its entire circumference, which led the contact between the wave generator and the flexible spline to result in considerably larger stresses than between the flexible spline and the ring gear. Here, however, the inclusion of the cup-spring leads to edge contact which significantly raises the compressive stress on the flexible spline, deeming the stresses between the wave generator and the flexible spline less influential, as shown in Figure 7.6 (c).

Beyond the presence of edge contact on the flexible spline, several teeth of the flexible spline are experiencing double flank tooth contact with the teeth of the ring gear. Due to the deflection of the cup-spring of the flexible spline, its teeth experience considerably forced wedging inside the tooth slots of the ring gear. Both edge contact and double flank contact constitute serious issues during operation of traditional gear drives, leading to significant reduction of their mechanical performance and risk of failure. In this case, these issues will be targeted for improvement in following analyses.

Figure 7.7 shows the distribution of tensile stresses on the flexible spline for the same initial stages of the three-dimensional model as shown in Figure 7.6. When the application of deflection by the pushing pins begins, the flexible spline experiences tensile stresses on the root region of its teeth near the major axis where the deflection $d$ is applied. These stresses are especially large on the root and fillet regions of the flexible spline where it connects to its cup-spring, as shown in Figure 7.7 (a). This is caused by the outward deflection of the flexible spline teeth
7.4. Application of deflection for tooth contact localization

a) Max. Principal Stress $\sigma_1$ (MPa)

b) Max. Principal Stress $\sigma_1$ (MPa)

c) Max. Principal Stress $\sigma_1$ (MPa)

**Figure 7.7:** Distribution of tensile stresses on the flexible spline: (a) beginning of deflection, (b) maximum deflection, and (c) tooth contact activation (Step 2).
Chapter 7. Three-dimensional simulation of meshing and stress analysis

along the major axis.

During the application of deflection, the tensile stress on the flexible spline considerably rises due to the need to deflect the entire cup-spring of the flexible spline outwards along the major axis and inwards along the minor axis. When the deflection $d$ is fully applied to the flexible spline as shown in Figure 7.7(b), the tensile stress on the flexible spline is considerably larger than that obtained with the two-dimensional model of SWG drives. Here, the contact between the teeth of the flexible spline and the ring gear is not active yet, which leads the tensile stresses to still be concentrated on the root and fillet regions of the flexible spline, especially close to the cup-spring.

Finally, when the contact between the teeth of the flexible spline and the ring gear is activated, the tensile stresses on the flexible spline rise considerably, as shown in Figure 7.7(c). A large concentration of tensile stresses is experienced around the tooth contacts between the flexible spline and the ring gear. This is caused by the wedging of the teeth of the flexible spline into the tooth slots of the ring gear, which produces significant deflection of the tooth flanks and edges of the flexible spline meshing with the ring gear.

![Figure 7.8: Distribution of compressive stresses on the ring gear after contact is found.](image)
7.5 Simulation of meshing by rotating the wave generator

Figure 7.8 shows the distribution of compressive stresses on the ring gear around the minor axis region where its teeth contact with those of the flexible spline. As opposed to the edge contact experienced by the flexible spline, the contact on the teeth of the ring gear is slightly displaced from the edge of the tooth surfaces. However, due to the large wedging of the teeth of the flexible spline, the ring gear experiences large compressive stresses on its tooth flanks which extend far beyond the flank surfaces towards the rim section. The teeth of the ring gear are also contacting on both flanks of several teeth, which would considerably lower the mechanical performance of SWG drives.

These results illustrate the importance of including the cup-shaped spring of the flexible spline due to its large and significant influence over stresses. The level of stresses shown in these preliminary results is considerably larger than those shown in the preliminary results with the two-dimensional model in Chapter 4.

7.5 Simulation of meshing by rotating the wave generator

The simulation of meshing of SWG drives with the three-dimensional model is performed similarly to the two-dimensional model, by rotating the wave generator as input member of the drive in small increments during each step of the model. For the SWG drive design of parameters shown in Section 7.2, this is performed until half a revolution of the wave generator is completed counterclockwise, resulting in the full motion of a tooth of the flexible spline from a tooth slot of the ring gear to the following tooth slot. A full rotation of the wave generator is not necessary due to its two lobes producing the same motion of the flexible spline twice per revolution.

Figure 7.9 shows the distribution of compressive stresses near the top meshing region of the SWG drive throughout a small rotation of the wave generator. The figure shows a planar view of both the flexible spline and the ring gear teeth to illustrate the evolution of the tooth-to-tooth contact between different pairs of teeth as the wave generator rotates. The tooth contact occurs between five teeth of the flexible spline and four teeth of the ring gear, meaning that the ring gear experiences double flank contact with the flexible spline, as shown in Figure 7.9 a). The
rightmost two teeth in contact of the flexible spline are experiencing slightly larger compressive stresses than the leftmost teeth.

When the wave generator rotates further, the compressive stress on the rightmost tooth of the flexible spline considerably lowers while another tooth to the left begins to contact the ring gear, as shown in Figure 7.9 b). This leads to six teeth of the flexible spline being in contact with five teeth of the ring gear per contact region of the SWG drive. In this case, both the flexible spline and the ring gear experience double flank contact on the center teeth of the meshing
As the wave generator continues its rotation, the right tooth of the flexible spline fully loses contact with the ring gear and will continue moving away from the tooth slot of the ring gear. On the other hand, the compressive stresses on the leftmost tooth of the flexible spline in contact rise, as shown in Figure 7.9 c). These transfers of contact and changes on compressive stress as the wave generator rotates illustrate the transfer of load from each tooth to the following, which constitutes the simulation of meshing with the three-dimensional model of SWG drives. The tooth contacts in Figure 7.9 a) and c) are very similar, although displaced one tooth of the flexible spline as it occurs during the meshing of SWG drives.

**Figure 7.10:** Distribution of compressive stresses on the tooth flanks of the flexible spline throughout a small rotation of the wave generator: (a) five-teeth contact, (b) beginning of six-teeth contact, and (c) end of six-teeth contact.
Figure 7.10 shows the distribution of compressive stresses on the tooth flanks and the open-end of the flexible spline to further illustrate the simulation of meshing. The contact between five teeth of the flexible spline and four teeth of the ring gear is shown in Figure 7.10 (a). The compressive stresses on the tooth flanks of the flexible spline progress slightly along their face width, which approximately indicates the contact pattern that would appear in this SWG drive design during operation. The wedging of the teeth of the flexible spline inside the tooth slots of the ring gear is such that the compressive stress also progresses toward the rim and inner diameter regions of the flexible spline.

When the wave generator continues its rotation, the contact between the teeth of the flexible spline and the ring gear switches from one tooth to the next. This is shown in Figure 7.10 (b) by the reduction of stresses on the rightmost tooth of the flexible spline while the leftmost tooth begins to contact the ring gear. At this stage, the tooth contact is found between six teeth of the flexible spline and five teeth of the ring gear per contact region of the SWG drive. Finally, Figure 7.10 (c) shows the meshing of the drive as the wave generator rotates further and the contact between the flexible spline and the ring gear switches back to include five and four teeth of the flexible spline and the ring gear, respectively. However, this contact arrangement occurs one tooth of the flexible spline further along the direction of rotation of the wave generator.

With the parameters of the SWG drive considered in this case as shown in Section 7.2, the resulting contact between the teeth of the flexible spline and the ring gear is considerably undesirable. This is because the presence of edge contact between the teeth significantly lowers the mechanical performance and expected life of any gear drive.

7.6 State of deflection of the cup-shaped spring of the flexible spline

The main advantage of the three-over the two-dimensional finite element model of SWG drives consists of the inclusion of the cup-shaped spring of the flexible spline. The inclusion of the full tooth geometries of the flexible spline and the ring gear by expanding them along their face width is also a considerable improvement from the two-dimensional model. Consequently, the three-dimensional model allows for the study and evaluation of the actual deflected state of the full geometry of the flexible spline, something that was impossible with the
two-dimensional model because the flexible spline was deflected outwards on a constant plane of analysis.

For these reasons, this section focuses on studying the state of deflection of the flexible spline during the meshing of SWG drives, as well as further understanding the distributions of tensile, compressive, and von Mises stresses as obtained with the three-dimensional model. This helps to determine the need for modifications and improvement of the drive considering the entire deflected geometry of the flexible spline.

Figure 7.11 shows the distribution of compressive stresses inside the cup-spring of the flexible spline along the major and minor axes of the drive during the meshing. Along the major axis of the drive, the flexible spline experiences large compressive stresses on its inside diameter close to the lobes of the wave generator, as shown in Figure 7.11 left. The deflection $d$ imposed by the lobes of the wave generator at all times twists the cup-spring around its bore in the closed end. The open-end of the cup is deflected outwards along the major axis, which leads the teeth of the flexible spline to not be parallel to the z-axis on their deflected state. This
means that the tip edge of the teeth on the open-end of the cup-spring is radially farther away from their original position without deflection. The tip edge of the teeth near the cup-spring is slightly closer radially. The section of the cup-spring of the flexible spline along the major axis resembles an open truncated cone.

On the other hand, the teeth of the flexible spline are deflected inwards along the minor axis due to deflection $d$ imposed along the major axis, as shown in Figure 7.11 right. This is similar to the results from the two-dimensional model of SWG drives. However, it is evident here that beyond simply deflecting inwards, the teeth of the flexible spline are also slightly rotated due to the inclusion of the cup-spring of the flexible spline, which leads its teeth to pivot around the bore on the closed-end of the cup. Figure 7.11 right also shows the large concentration of compressive stresses near the lobes of the wave generator where they contact with the inner diameter of the flexible spline on the side closer to the closed-end of the cup. This concentration of stresses is caused by edge contact between the wave generator and the flexible spline, which will be targeted for modification and improvement in further analyses.

![Figure 7.12: Distribution of tensile stresses inside the deflected cup of the flexible spline along major and minor axes.](image-url)
7.6. State of deflection of the cup-shaped spring of the flexible spline

Figure 7.12 shows the distribution of tensile stresses inside the deflected cup-spring of the flexible spline during the meshing of the drive. This figure shows where the flexible spline experiences the largest tensile stresses, as they considerably influence the mechanical performance of this element of the drive. Along the major axis, where the flexible spline is deflected outwards, the largest tensile stresses occur near the root of the teeth of the flexible spline where they attach to the cup-spring. The outward deflection strains the outer fibers of the cup-spring, as shown in Figure 7.12 left.

The opposite phenomenon occurs along the minor axis of the drive where the cup-spring of the flexible is stretched on the inner diameter, as shown in Figure 7.12 left (center) and right (top and bottom). The large tensile stresses inside the flexible spline are caused by its contact with the wave generator as the flexible spline teeth are deflected inwards along the minor axis. The corner of the inner diameter of the flexible spline with the open-end of the cup contacts the wave generator near the minor axis, which consequently deflects the teeth of the flexible spline slightly outwards along the minor axis. This produces further twisting of the flexible spline, whose cup rotates around the bore on the closed-end while the teeth are both twisted inwards by the cup deflection and outwards by the contact with the wave generator.

Finally, Figure 7.13 shows the distribution of von Mises stresses inside the deflected flexible spline along the major and minor axes. The flexible spline is deflected outwards along the major axis and inwards along the minor axis. However, the distribution of von Mises stresses is slightly more difficult to analyze here as compared to the two-dimensional model due to the complex state of tensile and compressive stresses as shown in Figures 7.11 and 7.12. Here, large von Mises stresses are shown on the inner diameter of the flexible spline near the lobes of the wave generator below the face of the teeth closer to the closed-end of the cup, as well as below the face of the teeth on the open-end of the cup along the minor axis of the drive.

For this reason, it is important to positively identify which stressed regions are loaded in tension and which in compression when studying von Mises stresses on the flexible spline. This is different from the two-dimensional model where, depending on the planar region of the drive, the stresses are tensile or compressive. The stresses obtained with the three-dimensional model are also dependent on the longitudinal region of the drive under study.
Figure 7.13 left shows two regions with large stresses on the inner diameter of the flexible spline below its teeth:

- Major axis region, closer to closed-end
- Minor axis region, open-end

The first region corresponds to the compressive stresses due to edge contact between the flexible spline and the wave generator, as shown in Figure 7.11, whereas the second region of von Mises stresses corresponds to the tensile stresses along the minor axis, as shown in Figure 7.12. These same regions are shown in Figure 7.13 right, although rotated 90 degrees around the z-axis or longitudinal direction of the drive.
7.7 Understanding the state of stress on the three-dimensional model

The complicated state of stress on the deformed flexible spline, including the cup-spring, is remarkably influenced by the geometry of the different elements of the drive. The influence of the geometries causing edge contact and large concentrations of stresses is such that the location of maximum stresses must be studied carefully, in this case with parameters shown in Section 7.2.

![Perspective](image)

Figure 7.14: Location of the maximum compressive stress on the flexible spline.

Figure 7.14 shows the distribution of compressive stresses on the flexible spline around the bottommost contact region of the drive where the maximum compressive stress is located on the main tooth contact between the flexible spline and the ring gear. Here, the absolute maximum compressive stress reaches a value of 1,925.584 MPa, which is considerably larger than the previously obtained stresses with the two-dimensional model without considering the full geometry of the teeth and the cup-spring of the flexible spline. In the results from the two-dimensional model, the location of the maximum compressive stress is typically located below the rim of the flexible spline where contact with the wave generator occurs, as shown in Section...
4.6.

Perspective

Figure 7.15 shows the distribution of tensile stresses on the flexible spline around the bottommost contact region where the maximum tensile stress is located on the root region of the teeth of the flexible spline closer to the closed-end of the flexible spline. This is near the connection between the teeth and the cup-spring of the flexible spline and is caused by the large outwards deflection of the flexible spline teeth shown in Figure 7.12. The maximum tensile stress reaches a value of 790.452 MPa, which is large, although significantly smaller than the absolute maximum compressive stress.

For this reason, Figure 7.16 shows the location of maximum von Mises stress on the flexible spline, which coincides with the location of the maximum compressive stress shown in Figure 7.14. As opposed to the two-dimensional model of SWG drives, where the results typically show the maximum von Mises stress to coincide with the maximum tensile stress, here, the current geometry of the SWG drive under study results in such large edge contact between the teeth of the flexible spline and the tooth flanks of the ring gear that the von Mises stress
7.8. Stress comparison of deformable elements

Due to the use of a flexible gear as compared to traditional gear drives, stress analysis of SWG drives has to be carefully performed. Similarly to the two-dimensional finite element model, the three-dimensional model provides the stresses that the flexible spline and the ring gear are subjected to. The maximum von Mises stress is located at the bottommost region of contact between the flexible spline and the ring gear, and it reaches a value of 1,829.518 MPa.

These results further illustrate the large influence of the cup-shaped spring of the flexible spline over the results with the three-dimensional model as opposed to the results obtained with the two-dimensional model. For this reason, careful evaluation of the resulting stresses and their maximum values has to be performed here.
Chapter 7. Three-dimensional simulation of meshing and stress analysis

Gear experience as the deformable elements of the drive. This section focuses on the evaluation of the resulting stresses on the deformable elements of the drive to determine the critical element for mechanical performance of SWG drives.

<table>
<thead>
<tr>
<th>Von Mises Stress $\sigma$ (MPa)</th>
<th>Flexible spline</th>
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<tbody>
<tr>
<td>1829.316</td>
<td></td>
</tr>
<tr>
<td>1676.991</td>
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<table>
<thead>
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<th>Ring gear</th>
</tr>
</thead>
<tbody>
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<td></td>
</tr>
<tr>
<td>330.660</td>
<td></td>
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<tr>
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<td></td>
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</tbody>
</table>

**Figure 7.17:** Comparison of distributions of von Mises stresses between the flexible spline and the ring gear near a lobe of the wave generator.

Figure 7.17 shows the distribution of von Mises stresses on the face of the teeth of the flexible spline and the ring gear in contact, near a lobe of the wave generator. The flexible spline and the ring gear have five and four teeth in contact, respectively. Consequently, the ring gear is experiencing double flank contact on one of its teeth. However, the von Mises stresses experienced by the flexible spline are considerably larger than those experienced by the ring gear. The maximum von Mises stress on the flexible spline reaches 1,829.316 MPa, as opposed to the maximum von Mises stress on the ring gear which is only 360.720 MPa.

As opposed to the two-dimensional model, the resulting stresses on the inner diameter of the flexible spline due to contact with the wave generator are considerably lower than the stresses on the tooth flanks in contact, as shown in Figure 7.17. This results from the current
7.8. Stress comparison of deformable elements

(unimproved) combination of input parameters as well as the large influence of the cup-shaped spring of the flexible spline over stresses.

![Graph](image)

**Figure 7.18**: Variation of maximum von Mises stress on the flexible spline and the ring gear.

The maximum von Mises stress on the flexible spline and the ring gear are shown in Figure 7.18. The maximum von Mises stress on the flexible spline reaches an average of 1,160.326 MPa, which is remarkably larger than that of the ring gear, which only reaches 331.311 MPa. Similarly to the two-dimensional model results, the large difference between the stresses experienced by the flexible spline and the ring gear in the three-dimensional model indicates that the flexible spline is the critical element to evaluate the mechanical performance of SWG drives.

For these reasons, when evaluating the resulting mechanical performance of SWG drives, the study of the resulting stresses on the flexible spline suffices, as it is subjected to considerably larger stresses than the ring gear. This is caused by the deflection imposed by the wave generator at all times, as well as the tooth-to-tooth contact between the flexible spline and the ring gear.
Chapter 7. Three-dimensional simulation of meshing and stress analysis

7.9 Effect of transmitted torque on stresses

Similarly to the two-dimensional finite element model of SWG drives, the three-dimensional model considers the transmission of load by the drive, which makes the model more realistic in order to evaluate the mechanical performance of this type of gear drive. The transmission of load is implemented in the form of output torque $T$ on the output element of the drive.

In these analyses, the traditional layout of SWG drives where the wave generator is the input and the flexible spline the output member of the drive is considered. For this reason, the output torque $T$ is applied on the bore of the closed-end of the cup-shaped spring of the flexible spline, as it is typically attached to the output shaft from the drive [1].

![Graph showing change of distribution of von Mises stresses on the contact region of the flexible spline when torque is applied.](image)

**Figure 7.19:** Change of the distribution of von Mises stresses on the contact region of the flexible spline when torque is applied.
Figure 7.19 shows the distribution of von Mises stress on the flexible spline near the major axis of the drive when the torque is not applied and when the output torque $T$ equal to 20 Nm is applied. When the torque is not applied, the maximum von Mises stress on the flexible spline is 1,502.917 MPa, as opposed to reaching a value of 1,829.316 MPa when the torque is applied. This change in the maximum von Mises stress indicates the influence of the transmitted load over stresses obtained with the three-dimensional model.

The results of the application of the output torque $T$ are the same as those obtained with the two-dimensional model and shown in the literature [30]. This is due to the reduced stresses region created behind the movement of the wave generator as shown in Figure 7.19 when torque is applied, similar to the results obtained with the two-dimensional model as shown in Figure 4.11. However, with the three-dimensional model, the stresses seem to reduce ahead of the lobe of the wave generator. This is caused by the large tooth-to-tooth contact stresses experienced by the flexible spline as well as the disappearance of double flank contact on the center tooth of the flexible spline in contact. Without torque applied, the center tooth of the flexible spline is experiencing double flank contact. The scale of the distribution of von Mises stresses is also different when applying or not the output torque $T$.

Figure 7.20 shows the variation of maximum von Mises stress on the flexible spline with and without torque applied. The maximum von Mises stress reaches an average value of 1,160.326 MPa when torque is applied and reaches an average value of 964.739 MPa when torque is not applied. This difference is not as large as the difference of average maximum von Mises stress from the two-dimensional model.

The figure also illustrates the difference in oscillatory behavior between the case with and without torque applied. When torque is applied, the oscillatory behavior of the maximum von Mises stress is slightly reduced as compared to the case when torque is not applied. This is caused by the change of the tooth-to-tooth contact between both cases, as shown in Figure 7.19. The elimination of double flank contact on the flexible spline reduces the influence of the tooth-to-tooth contact over the alternating component of the oscillatory behavior of the maximum von Mises stress. When torque is not applied, the flexible spline experiences double flank contact and the alternating component of the oscillatory behavior is slightly larger due to the influence
of the tooth-to-tooth contact.

7.10 Influence of the rim thickness of the flexible spline over stresses

This section shows the influence of the rim thickness $t_r$ of the flexible spline over stresses obtained with the three-dimensional model. The analyses consider different rim thicknesses $t_r$ of the flexible spline in order to evaluate and compare the resulting stresses. The remaining parameters of the SWG drive design under study as specified in Section 7.2 are kept constant and half a revolution of the wave generator is performed to simulate the meshing.

7.10.1 Flexible spline rim thickness geometries

The rim thickness $t_r$ of the flexible spline affects both the geometry of the flexible spline below the teeth and that of its cup-shaped spring in the three-dimensional model. In the two-dimensional model, however, the rim thickness $t_r$ of the flexible spline only influences a planar
geometry of the teeth by modifying the amount of material below the teeth or rim region of the flexible spline.

![Diagram showing comparison of geometries of the flexible spline as a function of its rim thickness](image)

**Figure 7.21**: Comparison of geometries of the flexible spline as a function of its rim thickness $t_r$.

Figure 7.21 shows a section of the cup-spring of the flexible spline with different rim thicknesses $t_r$ equal to 0.6, 1, and 2 mm. The rim of the flexible spline consists of the region below the teeth and connects with the cup-spring towards its closed-end. When the rim thickness $t_r$ is equal to 0.6 mm, both the rim of the flexible spline and the thickness of the cup-spring are significantly reduced along the cylinder and closed-end regions. On the other hand, increasing the rim thickness $t_r$ of the flexible spline to 1 mm enlarges the amount of material below the teeth, as well as increases the thickness of the cup-spring of the flexible. The thickness of the cup-spring is slightly smaller than that of the rim due to being determined by the product of the rim thickness $t_r$ and the cup thickness coefficient $c_c$ shown in Table 7.3.

When the rim thickness $t_r$ of the flexible spline is 2 mm, the rim region and the cup-spring are considerably thick. Regardless, for any value of the rim thickness $t_r$ of the flexible spline, the geometry of its teeth remains the same. A larger amount of material below the teeth and in the cup-spring of the flexible spline may significantly increase the torsional stiffness and strength of the flexible spline.
7.10.2 Results obtained as a function of the rim thickness of the flexible spline

The results shown in this section consider rim thicknesses of the flexible spline between 0.6 and 2.0 mm in intervals of 0.2 mm. When the rim thickness $t_r$ of the flexible spline is below 0.6 mm, the analyses with the three-dimensional model tend to fail due to the considerably thin rim and cup-spring of the flexible spline. On the other hand, a rim thickness $t_r$ of the flexible spline above 2.0 mm results in proportionally higher stresses.

![Variation of maximum von Mises stress on the flexible spline as a function of its rim thickness $t_r$.](image)

Figure 7.22 shows the variation of maximum von Mises stress on the flexible spline for rim thicknesses $t_r$ between 0.6 and 2.0 mm. The maximum von Mises stress on the flexible spline rises linearly with the value of its rim thickness $t_r$. When the rim thickness $t_r$ of the flexible spline is equal to 0.6 mm, the maximum von Mises stress reaches its lowest average with a value equal to 701.241 MPa. However, when the rim thickness $t_r$ of the flexible spline is equal to 2.0 mm, the average maximum von Mises stress is equal to 2,508.026 MPa. This average maximum von Mises stress is considerably larger than when employing lower values of the rim thickness $t_r$ of the flexible spline.
Regarding the oscillatory behavior of the maximum von Mises stress shown in Figure 7.22, the alternating component of the oscillation slightly increases with larger values of the rim thickness $t_r$ of the flexible spline. When the deflection $d$ has been applied, the initial maximum von Mises stress also increases with larger values of the rim thickness $t_r$ of the flexible spline.

These results coincide with those obtained with the two-dimensional model shown in Chapter 4. Increasing the rim thickness $t_r$ of the flexible spline considerably increases the strength and torsional stiffness of the flexible spline. This increases the stresses on the flexible spline with larger rim thicknesses due to the fact that the deflection $d$ imposed by the wave generator is the same regardless of the employed rim thickness $t_r$ of the flexible spline. Consequently, deflecting a flexible spline with a large amount of material on its rim and cup-spring results in large stresses.

For these reasons, the selection of the rim thickness $t_r$ of the flexible spline for a particular SWG drive design requires a trade-off between the strength of the material and the load to be transmitted. Larger loads require larger rim thicknesses of the flexible spline and, consequently, torsional stiffness in order to withstand the transmission of torque. On the other hand, lower loads to-be-transmitted can take advantage of lower rim thicknesses of the flexible spline so as to reduce the weight and material of the drive. For a given load, smaller rim thicknesses may also improve the fatigue life of the flexible spline due to reduced oscillatory behavior and overall stresses.

### 7.11 Summary of initial three-dimensional results

The three-dimensional model of SWG drives has shown considerably worse initial results in terms of mechanical performance as compared to those from the two-dimensional model. This is because of the inclusion of the cup-shaped spring of the flexible spline and the consideration of the entire face width of the teeth of both the flexible spline and the ring gear in the analyses. This finite element model provides information about the complex state of deflection of the entire geometry of the flexible spline throughout the operation of SWG drives.

This section summarizes the results from the three-dimensional model and justifies the need for micro-geometry modifications in order to improve the mechanical performance of the SWG
drive case under study with parameters shown in Section 7.2. Figure 7.23 shows the variations of maximum stresses on the flexible spline with the three-dimensional model. Compared to the results from the two-dimensional model, the main difference lies with the inversion of the maximum tensile and absolute compressive stresses. Here, the absolute maximum compressive stress is considerably larger than the maximum tensile stress. On the other hand, the magnitude of maximum stresses experienced by the flexible spline is significantly larger than that obtained with the two-dimensional model of SWG drives.

The maximum tensile stress is the lowest variation with an average value equal to 784.547 MPa, whereas the absolute maximum compressive stress reaches an average value of 1,318.391 MPa. The maximum von Mises stress lies between the maximum tensile and absolute compressive stresses with an average value equal to 1,160.326 MPa, as shown in Figure 7.23. An additional particularity of these results lies with the considerably larger range of stresses between the maximum von Mises and tensile stresses. In the results from the two-dimensional model, the average maximum tensile and von Mises stresses were similar. With the three-dimensional model, however, the maximum absolute compressive and von Mises stresses are similar in their average values as well as oscillatory behavior, whereas the oscillatory behavior of the maximum tensile stress is negligible. This indicates that the stresses experienced by the flexible spline are
dominated by the compressive stress caused by the different contact interactions of the model.

### 7.11.1 Contact between the wave generator and the flexible spline

Figure 7.24 shows the distribution of compressive stresses on the inner diameter of the flexible spline near the major axis of the SWG drive where a lobe of the wave generator contacts the flexible spline. There is a large concentration of stresses on the inner diameter of the flexible spline due to contact with the edge of the wave generator closer to the closed-end of the flexible spline. This concentration of stresses occurs near both lobes of the wave generator.

![Figure 7.24: Distribution of compressive stresses on the inner diameter of the flexible spline.](image)

On the other hand, and near the minor axis of the drive, the flexible spline open-end edge contacts the wave generator due to the inward deflection of the teeth of the flexible spline. Alternatively, the teeth near the major axis are deflected outwards by the wave generator at all times.

In this case, the geometry of the wave generator is a rigid surface without curvature along the z-axis. To avoid edge contact between the wave generator and the inner diameter of the
flexible spline, it is necessary to add curvature along the z-axis or longitudinal direction of the drive; this translates into the removal of material from the edges of the wave generator. The removal of material may be largely beneficial by simply removing microns from the surface. This is a common micro-geometry modification called longitudinal crowning and is applied to the tooth flanks of traditional gear drives in order to avoid edge contact when large loads are transmitted and misalignments are present [13].

7.11.2 Contact between the teeth of the drive

Figure 7.25 shows the distribution of compressive stresses on the teeth of the flexible spline and the ring gear near the major axis region of the drive where the meshing occurs. The points of view in the figure serve to show the contact patterns on both the teeth of the flexible spline and the ring gear, as well as where the teeth are in contact. With the current parameters of the SWG drive under study, the ring gear experiences double flank tooth contact and the compressive stresses created by its contact with the teeth of the flexible spline spread to the face of the teeth. This is caused by the proximity of the contact pattern to the edge of the teeth of the ring gear.

A tooth of the flexible spline also experiences double flank contact. However, the main concern on the contact patterns present on the teeth of the flexible spline lies in the location of the contact patterns along the tooth flanks. All of the contact patterns on the flexible spline are located on the edge of its teeth, as shown in Figure 7.25. This is a significantly disadvantageous situation in traditional gear drives, as edge contacts lead to large stresses and failure of the gears. Here, the significant edge contacts on the teeth of the flexible spline dominate the stresses experienced by the entire SWG drive and lead to considerably poor mechanical performance.

In the non-deflected state of the flexible spline, its teeth are parallel to the z-axis, meaning there is no curvature or inclination of the teeth with respect to the longitudinal direction. Consequently, when the flexible spline is deflected along the major axis, the teeth of the flexible spline are inclined with respect to the z-axis, while the teeth of the ring gear keep their original position. This situation leads to remarkable edge contacts on the teeth of the flexible spline with the current parameters of the SWG drive under study.
Similarly to when trying to avoid edge contact between the flexible spline and the wave generator, the addition of curvature on the teeth of the flexible spline along the longitudinal direction when is not being deflected may prove effective towards reducing the resulting stresses with the three-dimensional model. In traditional gear drives, this is performed by removing microns of material from the tooth flanks of the gear in a parabolic or circular manner. This modification, called longitudinal crowning, removes more material closer to the edges of the tooth flanks while leaving the center section of the teeth intact. Due to the edge contacts between the teeth of the flexible spline and the ring gear, the mechanical performance of SWG drives may significantly improve by the application of this micro-geometry modification. Although it is the critical element for mechanical performance of the drive, this modification is applied to the flexible spline due to the complexity of applying crowning to internal teeth such as those of the ring gear [13].
7.12 Implemented micro-geometry modifications

The three-dimensional finite element model of SWG drives allows to evaluate the effect of the cup-spring of the flexible spline over stresses experienced by the flexible spline and the ring gear. The wave generator deflects the teeth on the open-end of the flexible spline into meshing with the teeth of the ring gear. In the two-dimensional model, this deflection \( d \) expands the geometry of the flexible spline along the major axis of the drive and contacts it along the minor axis. However, when the cup-shaped spring of the flexible spline is considered, the teeth of the flexible spline deflect outwards along the major axis while their sector of the cup-spring bends around the bore of the closed-end of the cup. On the other hand, the teeth of the flexible spline deflect inwards along the minor axis and their cup sector also has to bend around the bore of the closed-end, but in the opposite direction. This deflection of the flexible spline leads to a twisting condition of the cylinder region of the cup-spring, which remarkably increases its stresses due to significant edge contacts and concentration of stresses and, consequently, reduce the effective life of the flexible spline.

Edge contact is a common weakness of traditional gear drives that most frequently appears when large loads are transmitted by the drive, leading to misalignments. In order to eliminate the possibility of edge contacts resulting in large stresses, micro-geometry modifications are applied to the flank surfaces of the teeth in traditional gear drives. These modifications can be included in the main cutting processes of the gears, as well as in the later grinding and finishing processes. The micro-geometry modifications are applied by slightly reducing the amount of material on the areas where edge contact and concentration of stresses are expected to appear once load is transmitted by the drive and misalignments occur. In traditional gear drives, typical micro-geometry modifications include crowning of flank surfaces along the longitudinal direction and the profile direction, tip and root reliefs, and front- and back-end reliefs, among others [13].

Due to the twisting of the cup-spring of the flexible spline and the deflected state of its teeth once the deflection \( d \) has been applied, three micro-geometry modifications have been implemented in the three-dimensional model for the geometries of the wave generator and the tooth flanks of the flexible spline. These micro-geometry modifications are:
7.12. Implemented micro-geometry modifications

- Crowning: on the outer surface of the wave generator
- Slope: on the outer surface of the wave generator
- Crowning: on the tooth flanks of the flexible spline

The crowning applied to the teeth of the flexible spline and the wave generator reduces the amount of material on these surfaces by a specified amount in IGD. This reduction of material leaves the center points of the surface on the actual unmodified location, while the outer edges are lowered. The points between the edges and the center of the surface are connected by a parabolic or circular curve, as specified in IGD.

The crowning on the wave generator intends to avoid edge contact between the inner diameter of the flexible spline and the edges of the wave generator. If the wave generator is not modified, the regions of the flexible spline near the major axis contact with the edge of the wave generator closer to the closed-end of the cup of the flexible spline. On the other hand, the open-end edges of the flexible spline near the minor axis contact the wave generator in the deflected state of the flexible spline. For these reasons, removing material from the edges of the wave generator may avoid edge contact between the wave generator and the flexible spline.

![Diagram](image)

**Figure 7.26:** Parabolic crowning micro-geometry modification applied to the outer surface of the wave generator.

Figure 7.26 shows the crowning micro-geometry modification applied to the outer surface of the wave generator in the coordinate system of the wave generator $S_{wg}$ where the y-axis goes along the major axis of the drive. The y-axis is truncated for the purpose of readability. This
micro-geometry modification is shown on a section of the wave generator and is identical along the entire geometry of the wave generator. The crowning is applied along the longitudinal direction, thus lowering the front and back edges of the wave generator and leaving the center section unmodified.

The initial geometry of the wave generator is straight and parallel to the z-axis, while the applied parabolic crowning connects the front and back edges of the wave generator with its center section with a parabolic curve. The amount of parabolic crowning applied in this example is 100 microns, which is the magnitude that the points on the edges of the wave generator are moved towards the center of the drive along its entire circumference, as shown in Figure 7.26. However, this modification leaves every point of the wave generator on its original plane x-y, which means that the axial dimension or face width $F_w$ of the wave generator along the z-axis remains unchanged.

The second micro-geometry modification implemented on the wave generator is the slope modification. This modification intends to compensate for the twisting state of deflection of the cylinder section of the cup-spring of the flexible spline. Figure 7.27 shows the slope modification applied to the outer geometry of the wave generator in its coordinate system $S_{wg}$. The y-axis on the top and the x-axis on the bottom of the figure are truncated for the purpose of readability. The top of the figure shows a section of the wave generator with the slope modification on the major axis region (y-axis), while the bottom of the picture shows the same modification on the minor axis region (x-axis). The slope micro-geometry modification changes the outer geometry of the wave generator to be inclined and not parallel to the z-axis or longitudinal direction of the drive, which is parallel if unmodified. Similarly to the crowning on the wave generator, the amount of slope micro-geometry modification applied to the edges of the wave generator is specified in IGD in microns.

On the top of Figure 7.27, the major axis region is shown where the edge of the wave generator below the open-end of the flexible spline is expanded by the input amount. In this case, it is expanded by 50 microns. Oppositely, the edge of the wave generator facing the closed-end of the flexible spline near the major axis is reduced by the input amount. These edges of the wave generator connect from one to the other with a straight line which includes the unmodified
7.12. Implemented micro-geometry modifications

The slope modification affects the minor axis regions of the wave generator in the opposite way that it modifies the major axis regions. On the bottom of Figure 7.27, the section of the wave generator near the minor axis region is shown where the points on the edge of the wave generator underneath the open-end of the flexible spline are lowered by the input amount. Alternatively, the edge of the wave generator facing the closed-end of the cup is raised by the input amount.

The slope micro-geometry modification changes progressively along each quarter of the wave generator from a major axis region to the following minor axis region, in which the modification is inverted. Similarly to the crowning of the wave generator, the slope modification...
does not modify the axial location of the points of the wave generator, which remain in the same x-y plane before and after the modification is applied.

![Diagram of modified flank geometry](image)

**Figure 7.28:** Parabolic crowning micro-geometry modification applied to the flank surfaces of a flexible spline tooth.

Finally, the third micro-geometry modification implemented in the model aims to avoid edge contact between the teeth of the flexible spline and the ring gear. The crowning of the flank surfaces of the teeth of the flexible spline is applied along the longitudinal direction to a magnitude of microns specified in IGD. Similarly to the crowning of the wave generator, the crowning of the flank surfaces of the flexible spline reduces the amount of material on both sides of the teeth by the input amount while the center section remains unchanged, as shown in Figure 7.28. A tooth of the flexible spline is shown with 100 microns of parabolic crowning applied as compared to the initial flank geometry.

As the open-end of the cup-spring is deflected outwards by the wave generator along the major axis, the bending of the cup-spring of the flexible spline around the bore of its closed-end causes the edge of its tooth surfaces to contact the flank surfaces of the teeth of the ring gear. This causes a remarkably large concentration of stresses due to edge contact, as well as unsuitable contact patterns for transmission of load between the teeth of the flexible spline and the ring gear. For this reason, the crowning of the flank surfaces of the teeth of the flexible spline serves to eliminate the edge contacts by reducing the amount of material on both edges of the tooth surfaces of the flexible spline. The crowning is applied on the teeth of the flexible spline...
because they are external teeth. Grinding processes to apply micro-geometry modifications are challenging on internal teeth compared to external teeth [13].

These three micro-geometry modifications, crowning and slope of the wave generator and crowning of the flexible spline, are evaluated in further analyses with the three-dimensional model of SWG drives with the aim of reducing stresses caused by edge contacts between the different elements of the drive. This justifies the use of this model, as these modifications cannot be implemented or evaluated in a two-dimensional model.

### 7.13 Stress reduction with micro-geometry modifications

This section focuses on the application of micro-geometry modifications on the outer geometry of the wave generator and the flank surfaces of the teeth of the flexible spline with the aim of eliminating edge contacts and concentrations of stresses. This, in turn, reduces the resulting stresses, which indicates improvement of the mechanical performance of SWG drives.

The results shown in this section are evaluated in terms of their variation of maximum and average value of von Mises stresses as a function of the magnitude of micro-geometry modifications applied. The amount of modification resulting in the lowest stresses is utilized for further analyses. Finally, the combination of micro-geometry modifications resulting in the lowest stresses is explained.

Due to the remarkably large computational resources required to run and store each three-dimensional model simulating half an input rotation of the wave generator, the analyses in this section perform an input rotation of the wave generator of only 45 degrees as opposed to 180 degrees as shown in previous results. The resulting stresses on the flexible spline when simulating 45 degrees of rotation of the wave generator are considerably similar to those obtained when simulating larger input rotations. A quarter rotation of the wave generator illustrates the influence over stresses of changes on the input parameters, as well as the micro-geometry modifications applied.

The results shown here include the parameters of the SWG drive case under study shown in Section 7.2 with the addition of micro-geometry modifications to the surfaces of the wave generator and the tooth flanks of the flexible spline.
Chapter 7. Three-dimensional simulation of meshing and stress analysis

7.13.1 Results obtained with crowning on the wave generator

Since the deflection applied by the wave generator to the flexible spline dominates the operation of SWG drives, the application of micro-geometry modifications on the wave generator to eliminate edge contact with the flexible spline constitutes the first priority. The parabolic crowning micro-geometry modification is applied to the wave generator along its longitudinal direction, lowering its edges to avoid edge contact.

These results are shown for values of the parabolic crowning applied to the wave generator between 0 and 120 µm in increments of 20 µm. The geometry of the wave generator with 0 µm of parabolic crowning is the geometry shown in previous results. On the other hand, beyond 120 µm of crowning, the resulting stresses rise due to concentration of stresses between the center sections of the wave generator and the inner diameter of the flexible spline. The illustration of the geometries is omitted here due to the small difference between themselves and their similarity to Figure 7.26.

![Graph](image)

**Figure 7.29:** Variation of maximum von Mises stress on the flexible spline as a function of the crowning applied to the wave generator.

Figure 7.29 shows the variation of maximum von Mises stress on the flexible spline as a function of the parabolic crowning applied to the wave generator. When the wave generator is not
modified, the maximum von Mises stress is the highest, which significantly reduces as a small amount of parabolic crowning is applied, as shown when applying 20 and 40 µm of crowning. The lowest variation of maximum von Mises stress is obtained with 60 µm of parabolic crowning on the wave generator. Further increasing the crowning slightly increases the stresses experienced by the flexible spline, as shown when the micro-geometry modification applied is equal to 80, 100, and 120 µm.

On the other hand, as the parabolic crowning applied to the wave generator increases beyond 60 µm, the oscillatory behavior of the maximum von Mises stress reduces while the average stress is similar, as shown in Figure 7.29. This is caused by the reduction of edge contact between the wave generator and the inner diameter of the flexible spline, as well as the freedom provided to the teeth of the flexible spline to deflect away from the ring gear when wedged into the tooth slots of the ring gear. However, the resulting maximum von Mises stress on the flexible spline is still considerably larger than previous values obtained with the two-dimensional model of SWG drives. This indicates the need for further study and improvement of the drive with the three-dimensional model.

**Improved wave generator geometry with crowning**

Due to its reduced oscillatory behavior, as well as low maximum von Mises stress on the flexible spline, 80 µm of parabolic crowning on the wave generator is considered the improved value of this micro-geometry modification. This value of parabolic crowning is used in further analyses in combination with other micro-geometry modifications.

Figure 7.30 shows the variations of maximum stresses on the flexible spline with 80 µm of parabolic crowning applied to the wave generator. The maximum absolute compressive and tensile stresses remain the largest and lowest level of stress, respectively. The maximum von Mises stress is similar to the absolute compressive stress, but is slightly reduced due to the influence of the tensile stress. This indicates the remaining presence of edge contact between the teeth of the flexible spline and the ring gear, which still dominates the compressive stress due to the large concentration of stresses created on the tooth flanks.
Chapter 7. Three-dimensional simulation of meshing and stress analysis

The oscillatory behavior of the maximum stresses show very little oscillation for the case of the maximum tensile stress. However, the maximum absolute compressive and von Mises stresses oscillate in a similar manner, as shown in Figure 7.30. Their similarities in oscillatory behavior indicates the dominance of compressive stresses on the flexible spline due to large concentrations of stress which require further improvement.

7.13.2 Results obtained with crowning on the tooth flanks of the flexible spline

Once the contact between the wave generator and the inner diameter of the flexible spline is improved by the application of crowning on the wave generator, the next contact targeted for improvement is between the teeth of the flexible spline and the ring gear. This is performed by applying crowning on the flank surfaces of the teeth of the flexible spline while maintaining the improved value of crowning on the wave generator.

The results shown here are obtained with 80 µm of parabolic crowning on the wave generator, while the amount of parabolic crowning applied to the tooth flanks of the flexible spline is varied between 0 and 40 µm in increments of 10 µm. The use of 80 µm of parabolic crowning on
the wave generator as opposed to a larger value allows the modification of the flanks of the flexible spline to more evidently show their influence over stresses. Alternatively, beyond 40 µm of parabolic crowning on the flexible spline, the resulting stresses increase with the parameters of the SWG drive under study and no other micro-geometry modifications.

The different geometries of the tooth flanks of the flexible spline as a function of the parabolic crowning applied are similar to Figure 7.28 and on the order of µm. Therefore, their illustration is not necessary here.

![Graph showing variation of maximum von Mises stress on the flexible spline](image)

**Figure 7.31:** Variation of maximum von Mises stress on the flexible spline as a function of the crowning applied to the tooth flanks.

Figure 7.31 shows the variation of maximum von Mises stress on the flexible spline as a function of the parabolic crowning applied to its tooth flanks. 80 µm of parabolic crowning is constantly applied to the wave generator and 45 degrees of input rotation are employed. The highest variation of maximum von Mises stress on the flexible spline is obtained when 10 µm of parabolic crowning is applied to its tooth flanks. However, when the parabolic crowning increases to 20 µm, the maximum von Mises stress on the flexible spline considerably reduces. When further increasing the crowning to 30 and 40 µm, the maximum von Mises reduces and reaches its lowest values.
The oscillatory behavior of the maximum von Mises stress significantly reduces when the parabolic crowning applied to the tooth flanks of the flexible spline is larger than 20 \( \mu \text{m} \). This is caused by the elimination of edge contact between the teeth of the flexible spline and the ring gear. The elimination of edge contact remarkably reduces the concentration of stresses on the flexible spline and, therefore, the resulting maximum von Mises stress, as shown in Figure 7.31.

**Improved flexible spline geometry with crowning on the tooth flanks**

The amount of parabolic crowning on the flanks surfaces of the teeth of the flexible spline equal to 40 \( \mu \text{m} \) is selected as the improved value of this micro-geometry modification due to the significant reduction of maximum von Mises stress on the flexible spline.

*Figure 7.32: Variations of maximum stresses on the flexible spline with 40 microns of crowning applied to the tooth flanks.*

Figure 7.32 shows the variations of maximum stresses on the flexible spline with 40 \( \mu \text{m} \) of parabolic crowning on the tooth flanks of the flexible spline and 80 \( \mu \text{m} \) of parabolic crowning on the wave generator. The initial value of the maximum absolute compressive stress on the flexible spline is equal to 202.410 MPa. The oscillatory behavior of the maximum absolute compressive stress is remarkable here due to the appearance and disappearance of edge contact...
between the teeth of the flexible spline and the ring gear as a function of the input position of the wave generator. The oscillatory behavior is such that certain values of the maximum absolute compressive stress are lower than the maximum tensile and von Mises stresses experienced by the flexible spline while other values are higher.

On the other hand, the maximum tensile and von Mises stresses are inverted, meaning that the maximum tensile stress is higher than the maximum von Mises stress experienced by the flexible spline as shown in Figure 7.32. Besides, the oscillatory behavior of these variations of maximum stresses is considerably reduced due to the significantly reduced influence of the tooth-to-tooth contact over stresses on the flexible spline. Large concentrations of stresses due to unsuitable contacts have been almost eliminated and, consequently, the mechanical performance of the flexible spline has remarkably improved.

The average values of the maximum stresses shown in Figure 7.32 are still large as compared to previous results obtained with the two-dimensional finite element model of SWG drives. For this reason, further improvement of the drive is still possible.

### 7.13.3 Improved SWG drive with all implemented micro-geometry modifications

After several analyses have been performed combining different magnitudes of parabolic crowning and slope on the wave generator and parabolic crowning on the tooth flanks of the flexible spline, the combination of micro-geometry modifications resulting on the lowest stresses as evaluated with the three-dimensional finite element model of SWG drives consists of the following modifications:

- 100 $\mu$m of parabolic crowning on the wave generator
- 90 $\mu$m of slope on the wave generator
- 45 $\mu$m of parabolic crowning on the flexible spline

This combination of micro-geometry modifications on the SWG drive case design of parameters shown in Section 7.2 is capable of eliminating edge contact and large concentrations of stresses between the wave generator and the flexible spline and between the flexible spline and the ring gear.
Figure 7.33 shows the geometry of the sections of the wave generator near the major and minor axes of the drive. The micro-geometry modifications are applied on the longitudinal direction, which runs below the teeth and along their face width $F_w$. The initial geometry of the wave generator was parallel to the $z$-axis around the entire geometry of the elliptical wave generator. However, the improved geometry of the wave generator does not possess any section of its geometry parallel to the $z$-axis.

With 100 $\mu$m of parabolic crowning applied to the wave generator, its outer geometry poses certain curvature which avoids edge contact and concentration of stresses between the wave generator and the flexible spline. On the other hand, 90 $\mu$m of slope twists the outer geometry of the wave generator so that its edges are closer or further to the center of the SWG drive depending on the region of the wave generator. Near the major axis, the edge of the outer
geometry of the wave generator closer to the open-end of the drive (negative z-axis) is closer to the center, allowing the teeth of the flexible spline to deflect towards the center of the drive when in contact with the ring gear. Alternatively, but still near the major axis, the edge of the outer geometry of the wave generator facing the closed-end of the flexible spline is closer to the flexible spline in order to produce its deflection, as shown in Figure 7.33. Near the minor axis of the drive, the modification of the outer geometry of the flexible spline is opposite. This is due to the influence of the slope micro-geometry modification, designed to compensate for the twisted and complex state of deflection of the flexible spline.

Regardless of the region of the modified wave generator being observed, the center section of the wave generator maintains its initial position on the center of the face width $F_w$, as shown in Figure 7.33. Due to the combination of parabolic crowning and slope on the wave generator, the total magnitudes of modification applied are $+20$ and $-190 \mu m$ depending on the side and region of the wave generator. This is the combination of 100 and 90 $\mu m$ of crowning and slope modifications, respectively, which raise or lower the outer geometry of the wave generator in different manners.

![Geometry of the tooth flanks of the flexible spline with parabolic crowning micro-geometry modification.](image)

**Figure 7.34:** Geometry of the tooth flanks of the flexible spline with parabolic crowning micro-geometry modification.

Figure 7.34 shows the geometry of a tooth of the flexible spline with the improved micro-geometry modification equal to 45 $\mu m$ of parabolic crowning on the tooth flanks. The parabolic crowning reduces the material on the teeth of the flexible spline closer to each side of the teeth
along the face width $F_w$. Besides, this modification provides curvature to the tooth flank geometry designed to avoid edge contacts between the teeth of the flexible spline and the ring gear.

Similarly to the micro-geometry modifications applied to the wave generator, the center section of the flank surfaces of the teeth of the flexible spline is not modified, as shown in Figure 7.34. This allows the contact pattern on the teeth of the flexible spline to move away from its edge and towards the center of the face width $F_w$ along the longitudinal direction. Consequently, the mechanical performance of the flexible spline transmitting load increases by the reduction of large concentrations of stresses between the teeth of the drive.

Due to the achievement of the lowest stresses on the flexible spline, the results here are shown for half a rotation of the wave generator as input member of the drive for comparison to the results shown in Section 7.11. Figure 7.35 shows the variations of maximum stresses on the flexible spline with the combination of micro-geometry modifications resulting on the lowest stresses. The maximum absolute compressive stresses is the largest with an average value equal to 388.952 MPa, whereas the maximum tensile stress is the lowest with an average equal to 269.033 MPa. The range between these two variations of maximum stresses is considerably

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.35.png}
\caption{Variations of maximum stresses on the flexible spline with the combination of micro-geometry modifications resulting in the lowest stresses.}
\end{figure}
reduced as compared to the SWG drive of stresses shown in Figure 7.23 and no micro-geometry modifications applied. Here, the maximum von Mises stress is close to the maximum tensile stress with an average value equal to 293.161 MPa, which is considerably more reduced than the previous results without micro-geometry modifications applied.

The oscillatory behavior of the maximum stresses with micro-geometry modifications is similar for the absolute compressive and von Mises stresses, as shown in Figure 7.35. This indicates the dominance of the compressive stress as the largest stress to which the flexible spline is subjected, as well as the significant influence of the tooth-to-tooth contact between the flexible spline and the ring gear over stresses. On the other hand, the maximum tensile stress experienced by the flexible spline remains almost constant.

These results in terms of maximum stresses demonstrate the effectiveness of the inclusion of micro-geometry modifications on the surfaces of the wave generator and the flexible spline to improve the mechanical performance of SWG drives. This is performed by reducing the resulting stresses on the flexible spline as a function of the input parameters of the drive.

**Contact between the wave generator and the flexible spline with micro-geometry modifications**

One of the main disadvantages of SWG drives observed on the obtained results without the application of micro-geometry modifications is the appearance of edge contacts and concentration of stresses between the interaction of the wave generator and the flexible spline. Figure 7.36 shows the distribution of compressive stresses on the inner diameter of the flexible spline near the major axis of the drive with the improved combination of micro-geometry modifications on the wave generator and the flexible spline. The compressive stress resulting from the contact between the wave generator and the flexible spline is considerably lower than that shown in Figure 7.24 without micro-geometry modifications applied.

On the other hand, with micro-geometry modifications, the contact between the wave generator and the flexible spline near the major axis region (where the deflection $d$ is applied) is approximately centered along the face width $F_w$ of the teeth of the flexible spline as shown in
Figure 7.36: Distribution of compressive stresses on the inner diameter of the flexible spline with modifications.

Figure 7.36. There is no edge contact between the wave generator and the inner diameter of the flexible spline, which significantly reduces the stresses experienced by the flexible spline.

The addition of the micro-geometry modifications parabolic crowning and slope on the outer geometry of the wave generator proves useful towards reducing stresses by eliminating the edge contact and concentration of stresses on the interface between the wave generator and the flexible spline.

**Contact between the teeth of the SWG drive with micro-geometry modifications**

The second interface where the preliminary results showed large concentration of stresses due to edge contact is the tooth-to-tooth contact between the flexible spline and the ring gear. Figure 7.37 shows the distribution of compressive stresses on the tooth flanks of the flexible spline and the ring gear near the major axis of the drive where the teeth mesh. As opposed to the results shown in Figure 7.25, where the flexible spline and the ring gear were in contact between several pairs of teeth with the contact patterns on or near the edge of the tooth flanks,
with micro-geometry modifications the number of pairs of teeth in contact per meshing region of the SWG drive is only two. This may initially seem disadvantageous when transmitting large loads with SWG drives. However, when comparing results with the same output torque $T$ transmitted, the compressive stress experienced by the drive with micro-geometry modifications is considerably lower.

![Diagram showing the distribution of compressive stresses on the teeth of the flexible spline and the ring gear with modifications.]

**Figure 7.37:** Distribution of compressive stresses on the teeth of the flexible spline and the ring gear with modifications.

The contact patterns on the flexible spline have moved inwards along the face width $F_w$ of the teeth, consequently reducing the previously large concentrations of stresses due to edge contact between the teeth of the flexible spline and the ring gear. With this combination of micro-geometry modifications, there is neither edge contact nor double flank contact between the flexible spline and the ring gear, as shown in Figure 7.37, which considerably improves the mechanical performance of SWG drives.
7.13.4 Comparison of overall mechanical performance with and without micro-geometry modifications

Table 7.4 shows the average maximum tensile, absolute compressive, and von Mises stresses on the flexible spline for both cases with and without micro-geometry modifications. These results are obtained for the simulation of meshing consisting of half a rotation of the wave generator as input member of the SWG drive. In the initial case without micro-geometry modifications, the average maximum stresses are remarkably large, with values over 1,000 MPa. These are considerably larger than the strength of the considered steel. This means that the SWG drive without micro-geometry modifications would fail due to being subjected to remarkably large stresses.

<table>
<thead>
<tr>
<th>Micro-geometry modification</th>
<th>Initial</th>
<th>Improved</th>
<th>[units]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wave generator parabolic crowning</td>
<td>-</td>
<td>100</td>
<td>µm</td>
</tr>
<tr>
<td>Wave generator slope</td>
<td>-</td>
<td>90</td>
<td>µm</td>
</tr>
<tr>
<td>Flexible spline parabolic crowning</td>
<td>-</td>
<td>45</td>
<td>µm</td>
</tr>
</tbody>
</table>

Table 7.4: Parameters of the DCA tooth profile and resulting stresses before and after improvement.

<table>
<thead>
<tr>
<th>Resulting average maximum stress</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Tensile, (σ_1)</td>
<td>784.547</td>
<td>269.033</td>
<td>MPa</td>
</tr>
<tr>
<td>Absolute compressive, (</td>
<td>σ_3</td>
<td>)</td>
<td>1,318.391</td>
</tr>
<tr>
<td>Von Mises, (σ)</td>
<td>1,160.326</td>
<td>293.161</td>
<td>MPa</td>
</tr>
</tbody>
</table>

On the other hand, with micro-geometry modifications applied, the average maximum stresses experienced by the flexible spline are considerably reduced. The improvement of mechanical performance on the SWG drive with modifications provides a reduction of the average maximum tensile stress of more than 500 MPa, while the average maximum absolute compressive stress reduces approximately 930 MPa, as shown in Table 7.4. This is a remarkable reduction of stresses where the three-dimensional model has proven the effectiveness of micro-geometry modifications towards avoiding unsuitable edge contacts and concentrations of stresses between the different elements of SWG drives. In terms of the average maximum von Mises stress, the difference between applying micro-geometry modifications or not is over 850 MPa, which constitutes a large improvement on the mechanical performance of the drive.
The three-dimensional model of SWG drives has evidenced the large influence of the cup-spring of the flexible spline over stresses in this type of gear drive as opposed to the two-dimensional model, where the modification of the input parameters of the geometries of the different elements of the drive was sufficient to remarkably reduce the resulting stresses. The three-dimensional model shows that applying micro-geometry modifications on the geometries of the wave generator and flexible spline allows edge contacts and concentration of stresses to be avoided, and the mechanical performance of SWG drives to be improved significantly.
Chapter 8

Conclusions

8.1 Conclusions and recommendations

From the performed research, the following conclusions can be drawn:

- The computational code to generate fully parameterized finite element models of SWG drives has been developed in a custom-made software. Two- and three-dimensional finite element models have been developed to evaluate the influence of the geometries of the different elements of this type of gear drive on a planar slice of their toothed region with the two-dimensional model, or on their entire geometry including the cup-shaped spring of the flexible spline with the three-dimensional model. These models satisfactorily serve to perform stress analysis by the finite element method throughout the complex process of meshing of SWG drives with two regions of tooth contact.

- The simulation of meshing of SWG drives is performed by rotating the wave generator in small increments of the total angle of rotation to be simulated. This is because the wave generator is typically the input member in the most common layout of this type of gear drive. However, depending on the selected layout to simulate, the flexible spline or the ring gear are fixed while the other member constitutes the output. The output element is loaded with a constant torque to simulate the transmission of load by the drive throughout the meshing.

- The two- and three-dimensional finite element models are used to evaluate the resulting stresses on the flexible spline and the ring gear. Comparing the resulting maximum
stresses of different designs serves to evaluate the mechanical performance of SWG drives and determine the combination of input parameters or the particular geometries that result in the lowest stresses. The stresses obtained with the models are remarkably influenced by the geometry of the different members of the drive and modifying their parameters with the aim of lowering stresses significantly improves mechanical performance.

- Due to the deflection, the flexible spline experiences considerably larger stresses than the ring gear. For this reason, the flexible spline is determined to be the critical element for mechanical performance of SWG drives. Its stresses must be targeted for reduction to improve the behavior of this type of gear drive.

- Evaluating the distributions of tensile and compressive stresses on the flexible spline indicates that it experiences large tensile stresses on the root region of the teeth near the major axis of the drive, as well as on the inner diameter near the minor axis. Concurrently, the flexible spline experiences large compressive stresses on the inner diameter near the major axis of the drive and on the roots of its teeth near the minor axis. Compressive stresses are also present on the tooth flanks due to the contact between the teeth of the flexible spline and the ring gear.

- The root regions of the teeth of the flexible spline experience compressive stress near the minor axis of the drive and tensile stress near the major axis. Consequently, the root regions of the teeth of the flexible spline constitute the critical focus area to improve the mechanical performance of SWG drives due to being subjected to large tensile and compressive stresses alternatively and repeating twice per revolution of the wave generator.

- The literature about SWG drives indicates that the flexible spline experiences larger stresses on the region ahead of the movement of the lobes of the wave generator when transmitting load. The results obtained with the two-dimensional model of SWG drives are in agreement with the literature and show both the increase of overall resulting stresses on the flexible spline when torque is transmitted as well as the reduction of stresses behind
the movement of the lobes of the wave generator. The resulting stresses on the tooth contacts between the flexible spline and the ring gear also increase due to the transmission of load.

• The two-dimensional model serves to evaluate the influence of the rim thickness of the flexible spline over stresses. With larger rim thicknesses, the resulting stresses significantly increase due to the larger amount of material to be deflected by the wave generator. This deflection is constant, which means that thicker rims of the flexible spline incorporate more material to be deflected the same amount as thinner rims. The analyses with the two-dimensional model emphasize the required trade-off on the selection of the rim thickness of the flexible spline due to the influence of the transmitted torque as a function of the rim thickness. Smaller rim thicknesses result in considerably larger stresses when larger loads are transmitted, whereas thicker rims are less influenced by the transmitted torque. For this reason, a larger rim thickness may be beneficial given the particular load to be transmitted by the drive.

• Four different geometries of the wave generator have been implemented and they are called simplified, elliptical, parabolic, and four roller geometries. The simplified geometry is based on a wave generator consisting of two rollers as lobes of the wave generator while the elliptical geometry was proposed by Musser in 1959. The parabolic wave generator geometry is newly proposed in this thesis and consists of a combination of parabolic regions as lobes of the wave generator with circular arcs connecting them. Finally, the four roller wave generator geometry was proposed in 1970 with four rollers providing the deflection to the flexible spline with the aim of increasing the number of teeth in contact between the flexible spline and the ring gear.

• The four roller wave generator is proven unsuitable due to the remarkably large resulting stresses, which justify its lack of use after it was patented. The improved geometries of the simplified and parabolic wave generator result in the lowest overall stresses with the two-dimensional model, while the improved elliptical wave generator geometry results
in the lowest compressive stress on the flexible spline due to its smooth contact with this element.

- The improved wave generator geometries have also been evaluated using the alternative layout of SWG drives, where the flexible spline is fixed and the ring gear serves as output member. This leads to approximately the same results as when using the traditional layout.

- Beyond the initial definition of the parabolic wave generator, two additional geometries have been evaluated with the two-dimensional model. These geometries aim to reduce stresses by employing asymmetric parabolic regions for the wave generator to rotate in a single direction. One of the geometries consists of slightly rotating the parabolic regions of the parabolic wave generator around their center, while another geometry uses different combinations of parameters for each quadrant of the parabolic wave generator. Both geometries lead to either significantly large stresses on the flexible spline due to imposing an excessively large deflection or no improvement of the mechanical performance of the drive.

- Three different geometries for the teeth of the flexible spline and the ring gear have been implemented. The first geometry, the involute tooth profile, is the most commonly used tooth profile in traditional gear drives. The additional tooth profiles are two directly-defined geometries based on profiles currently used in SWG drives, which are called double circular arc (DCA) and quadruple circular arc (QCA). The DCA tooth profile combines two circular arcs and a straight segment tangentially connected from the root to the tip of the teeth. The QCA tooth profile combines two circular arcs, a straight segment, and two additional circular arcs which tangentially connect the dedendum circle to the addendum circle of the gear.

- The involute tooth profile is unsuitable in SWG drives due to resulting in large stresses. The geometry of the QCA profile has been improved by modifying the input parameters with the aim of reducing the resulting stresses with the two-dimensional model. The initially evaluated geometry of the DCA tooth profile is based on the improved geometry of
the QCA tooth profile, which leads to minimal improvement by modifying the parameters of the DCA geometry. Both directly-defined tooth profiles result in similarly low stresses and prove advantageous towards improving the mechanical performance of SWG drives.

- The addition of small amounts of backlash between the teeth of the flexible spline and the ring gear reduces the stresses obtained with the two-dimensional model, whereas larger amounts of backlash provide no further improvement.

- The deflection applied to the flexible spline produces a slight reduction on the pressure angle of its teeth without deflection applied, which influences the location of the contact pattern. Consequently, directly-defined tooth profiles experience large stresses when the straight segment progresses above the pitch circle of the gear. The tip geometry of the directly-defined profiles does not significantly influence compressive stresses.

- The three-dimensional finite element model is based on the two-dimensional model by expanding its parts along the face width so that the full tooth flank geometry of the teeth of the flexible spline and the ring gear, as well as the outer geometry of the wave generator, can be evaluated. The additional difference between the models consists of the inclusion of the cup-shaped spring of the flexible spline in the three-dimensional model. The cup-spring of the flexible is attached to its teeth at the rim region. The boundary conditions and load applied to the inner diameter of the flexible spline in the two-dimensional model are applied to the bore on the closed-end of the cup-spring in the three-dimensional model to represent the attachment of the flexible spline to the output shaft of SWG drives.

- The three-dimensional model proves the flexible spline to be the critical element for mechanical performance of SWG drives due to experiencing significantly larger stresses than the ring gear because of the constantly applied deflection by the wave generator. The results also satisfy the literature in terms of the transmission of load by showing an overall increase of stresses when load is transmitted, as well as a region of reduced stresses behind the movement of the lobes of the wave generator and changing the tooth contact between the flexible spline and the ring gear from being on two sides to a single side.
• The influence of the rim thickness of the flexible spline has also been evaluated with the three-dimensional model and provides similar results as those obtained with the two-dimensional model, where stresses increase with larger rim thicknesses. Here, the rim thickness of the flexible spline also influences the thickness of its cup-spring which, in turn, increases the stiffness of the entire flexible spline if thicker rims are selected. The selection of the rim thickness of the flexible spline for a particular application requires careful consideration, taking into account the material of the flexible spline, the load to be transmitted, and the overall weight of the drive.

• The three-dimensional model represents more accurately the influence of the different geometries of SWG drives over stresses. The expansion of the toothed region of the drive and the inclusion of the cup-spring of the flexible spline significantly affects stresses and shows unsuitable contacts which must be targeted for improvement by modifying the input parameters of the drive or applying micro-geometry modifications with the aim of reducing stresses.

• Micro-geometry modifications have been implemented in the three-dimensional model for the geometries of the wave generator and the tooth flanks of the flexible spline. These micro-geometry modifications include crowning and slope applied to the wave generator outer geometry with the aim of eliminating the areas of concentration of stresses on the inner diameter of the flexible spline due to the continuously imposed deflection. Crowning provides curvature to the outer geometry of the wave generator along the longitudinal direction while the slope modification aims to accommodate the complex state of deflection of the flexible spline with the cup-spring. This is obtained by inclining the outer geometry of the wave generator in one direction near the major axis, which progressively changes to the opposite direction near the minor axis. Crowning on the tooth flanks of the flexible spline is implemented with the aim of eliminating the areas of concentration of stresses due to edge contact with the ring gear, as well as the wedging and double-side contact of the teeth of the flexible spline inside the tooth slots of the ring gear.

• The influence of each micro-geometry modification over stresses has been evaluated. Only
applying crowning to the wave generator considerably reduces the stresses experienced by the flexible spline, whereas the slope modification requires the use of additional micro-geometry modifications. Crowning of the teeth of the flexible spline also proves advantageous towards reducing the stresses of the drive.

- Combining the implemented micro-geometry modifications satisfactorily eliminates the areas of concentration of stresses on the inner diameter of the flexible spline caused by edge contact with the wave generator. The edge contact between the teeth of the flexible spline and the ring gear is also eliminated and the contact pattern on the tooth flanks of the flexible spline is moved inwards along the face width. Micro-geometry modifications significantly reduce the resulting stresses, which considerably improves the mechanical performance of SWG drives.

8.2 Future work opportunities

The use of the two- and three-dimensional models goes beyond this thesis dissertation and can be applied on several studies of SWG drives. An in-depth evaluation of the influence of the different geometries of the wave generator over stresses can be performed with the three-dimensional model. This can also be done with the tooth profile geometries. Combined with the use of micro-geometry modifications, these studies can prove advantageous to further improve the mechanical performance of SWG drives with the application of the three-dimensional model.

The developed finite element models can be used to evaluate the mechanical performance of commercially available SWG drives, as well as to improve their behavior by modifying their input parameters and applying micro-geometry modifications.

Similarly to the results with the two-dimensional model, the three-dimensional model can be used to evaluate the performance of SWG drives when operating with the alternative layout where the flexible spline is fixed and the ring gear is the output member. The different layout possibilities of the two- and three-dimensional model can be expanded to consider the wave
generator as the fixed element of the drive while the flexible spline serves as the input and the ring gear as the output and vice versa.

The two-dimensional model can be used to determine the motions of a given geometry of the tooth profile of the flexible spline and from it generate the conjugated tooth profile of the ring gear. Similarly, from a given geometry of the ring gear, the conjugated tooth profile geometry of the flexible spline can be generated. This process can also be applied along the entire face width of the teeth with the three-dimensional model. This would allow to extend the contact area between the flexible spline and the ring gear.

Although both models satisfactorily represent the behavior and transmission of motion and load of SWG drives, simplifications have been made to generate the finite element models and perform the work of this thesis. For this reason, a future work opportunity consists of further developing the two- and three-dimensional models into even more realistic approaches and reduce their computational cost. This can be done in terms of modifying the wave generator to include the typical flexible race ball bearing around its entire geometry, as well as the balls themselves to more realistically represent the behavior of SWG drives. On the other hand, the geometry of the cup-spring of the flexible spline can be further developed to include different geometries of the attachment to the output shaft of the drive. This can be done with the ring gear as well by implementing the geometry of its attachment and fixing mechanism to the housing of the drive.

Traditional gear drives employ micro-geometry modifications to compensate for misalignments and concentration of stresses when transmitting load. Further developing different micro-geometry modifications for the geometry of the elements of SWG drives together with the study of misalignments throughout their operation constitutes an additional future work opportunity for which the developed finite element models may serve as foundation.

To sum up, the future work opportunities consist of further developing the finite element models of SWG drives, their possibilities, and use in actual applications. Doing so will further enhance the potential of the models to evaluate and improve the mechanical performance of SWG drives without the need for costly prototyping and experiments. Continuing with the development and use of these models will provide additional insight for the design process of this
type of gear drive which provides outstanding performance characteristics with a significantly
different meshing process than traditional gear drives.
Appendix A

Experimental testing of efficiency losses due to deflection

A.1 Appendix overview

This appendix illustrates the experiments performed at the department of Research and Development of The Gleason Works. The authors of these experiments and the report from which the content of this appendix is based of are Dr. Hermann J. Stadtfeld and Dr. Haris Ligata. Their experiments are based on the work performed in this thesis dissertation and constitute an initial approach to determine the efficiency losses stemming from the rotation of a constantly deflected body, as is the case of the flexible spline in SWG drives.

A.2 Introduction and motivation

In the middle of 2019, the department of Research and Development at The Gleason Works engaged in the research efforts of this thesis dissertation towards further understanding the operation of SWG drives and developing tools for their design. Gleason’s team decided to determine the efficiency of the deflection mechanism of SWG drives with experiments.

The to-be-performed tests consisted of two parts:

- Measurement of the static deflection of the flexible spline.
- Determination of the torque needed for rotating the deflected flexible spline.
From these tests, the efficiency would be measured as the torque required to rotate the deflected geometry of the flexible spline. This appendix summarizes the experimental approach, the calculation of the deflection required in SWG drives, and the measurement of the static deflection.

A.3 Manufacturing of the deflection test setup of the flexible spline

Gleason’s team designed a deflection stand to measure the static deflection of the flexible spline. This stand consists of two radial ball bearing supports of 65 mm of outside diameter as shown in Figure A.1. One of the bearing supports is held stationary while the other can be displaced axially with a screw operated by hand. The displacement of one bearing support produces the deflection of a ring emulating the geometry of the flexible spline and taking the shape of an ellipse once the deflection is applied. The inner and outer diameters of the ring are ground to a thickness equal to 4.5 mm. The material of the ring is steel 6150 which has been hardened, press quenched, and deep freeze tempered.

Figure A.1: Test setup for measurement of the deformation of the flexible spline.
A.4 Scaling of the flexible spline

The flexible spline is modeled as a solid ring with an outer diameter equal to 136 mm, slightly larger than the diameter of the reference design used throughout this thesis dissertation (equal to 121.2 mm). This enlargement is aimed to more easily measure the magnitude of the deflection experienced by the flexible spline around its geometry. However, the number of teeth and the addendum and dedendum coefficients are the same as the design studied in this thesis. Table A.1 shows a comparison between the dimensions utilized throughout this thesis dissertation and the experiments performed by Gleason’s team.

<table>
<thead>
<tr>
<th></th>
<th>RIT</th>
<th>Gleason</th>
<th>units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of teeth, ( N )</td>
<td>120</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>Module, ( m )</td>
<td>1.0</td>
<td>1.122112</td>
<td>mm</td>
</tr>
<tr>
<td>Addendum coef., ( h_a )</td>
<td>0.6</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>Dedendum coef., ( h_d )</td>
<td>0.8</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>Rim thickness, ( t_r )</td>
<td>1.0</td>
<td>1.122112</td>
<td>mm</td>
</tr>
<tr>
<td>Nominal thickness</td>
<td>2.4</td>
<td>2.693069</td>
<td>mm</td>
</tr>
<tr>
<td>Effective thickness</td>
<td>1.8</td>
<td>2.019802</td>
<td>mm</td>
</tr>
<tr>
<td>Pitch diameter, ( 2r_p )</td>
<td>120.0</td>
<td>134.6535</td>
<td>mm</td>
</tr>
<tr>
<td>Outside diameter</td>
<td>121.2</td>
<td>136.0</td>
<td>mm</td>
</tr>
<tr>
<td>Whole depth, ( m(h_a+h_d) )</td>
<td>1.4</td>
<td>1.570957</td>
<td>mm</td>
</tr>
<tr>
<td>Diametral deflection</td>
<td>2.8</td>
<td>3.141914</td>
<td>mm</td>
</tr>
<tr>
<td>Effective deflected diam.</td>
<td>118.8</td>
<td>133.3069</td>
<td>mm</td>
</tr>
</tbody>
</table>

A.5 Deflections needed on the flexible spline

In order to measure the static deflection of the flexible spline, a major assumption is made throughout its calculation procedure. This assumption is considering that the difference between the radial deflection near the major axis of the ellipse (above or below the bearing supports) and the region 45 degrees away from the major axis must be equal or larger than the whole depth of the teeth of the flexible spline. This assumption means the flexible spline and the ring gear are in contact on two regions extending 45 degrees around the major axis of the ellipse. The whole depth of the ring considered throughout these experiments is 1.57 mm, as shown in Table A.1.
Appendix A. Experimental testing of efficiency losses due to deflection

Figure A.2: Deflection of the flexible spline.

Figure A.2 shows the necessary deflections to be measured in this experiment. These deflections are:

- The radial distance between the tip of the undeformed flexible spline and the root of the ring gear equal to $a_G$.

- The undeformed flexible spline deflecting beyond the bearing displacement equal to $a_G$ to engage with the ring gear.

- The ring in the experiment has to deflect inward by an amount equal to $b_G$ at 45 degrees from the major axis during the application of deflection.

The difference between the deflection near the major axis of the ellipse and the inwards deflection at 45 degrees has to be equal to the whole depth of the teeth for the teeth of the flexible spline to engage with those of the ring gear at this position.
A.6 Determination of the values of deflections \( a_G \) and \( b_G \)

The experimental test is performed to determine the necessary deflections on the flexible spline of SWG drives with the initial setup of dial gages shown in Figure A.3. The dial indicators are:

- Top dial indicator (aligned with the major axis of the ellipse) to measure \( a_G \).
- Dial indicator at 45 degrees to measure \( b_G \).

However, the current arrangement does not allow for the direct measurement of \( a_G \) and \( b_G \). Under the deflection force applied by the displaced bearing with the experimental setup, the measurement of \( a_G \) and \( b_G \) would be different than the desired radial deflections.

Figure A.4 shows the arrangement of dial indicators in the experimental setup. These indicators would measure twice the amount of \( a_G \) and an unknown quantity for \( b_G \), as the dial
indicator at 45 degrees would be displaced together with the movable bearing support. In order to make accurate measurements of the deflection $b_G$, the dial indicator should move with the bearing support as shown in Figure A.4.

### A.7 Measurement results

The experiments are performed with the arrangement of dial indicators shown in Figure A.4. Table A.2 and Figure A.5 summarize the obtained results of measured deflections of the test ring.

<table>
<thead>
<tr>
<th>$2a_G$</th>
<th>$a_G$</th>
<th>$b_G$</th>
<th>$a_G + b_G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.254</td>
<td>0.127</td>
<td>0.064</td>
<td>0.191</td>
</tr>
<tr>
<td>0.508</td>
<td>0.254</td>
<td>0.127</td>
<td>0.381</td>
</tr>
<tr>
<td>0.762</td>
<td>0.381</td>
<td>0.254</td>
<td>0.635</td>
</tr>
<tr>
<td>1.016</td>
<td>0.508</td>
<td>0.381</td>
<td>0.889</td>
</tr>
<tr>
<td>1.270</td>
<td>0.635</td>
<td>0.533</td>
<td>1.168</td>
</tr>
<tr>
<td>1.524</td>
<td>0.762</td>
<td>0.737</td>
<td>1.499</td>
</tr>
<tr>
<td>1.626</td>
<td>0.813</td>
<td>0.762</td>
<td>1.575</td>
</tr>
</tbody>
</table>
Figure A.5: Measured deflection of the flexible spline.

The total deflection is equal to 1.575 mm and the whole depth of the teeth of the flexible spline as computed in Table A.1. The achievement of this total deflection constitutes the last point of measurement of these experiments.

A.8 Update of existing design

The experimental test setup is used for the proof of concept towards measuring the necessary deflections on the flexible spline of the SWG drive, along with proving its feasibility. However, during the test, it appears that the torque required to displace the movable bearing support changes. This indicates that the ring being tested is experiencing misalignments throughout the application of deflection, leading to edge contact and varying clearance with the bearing supports.
Appendix A. Experimental testing of efficiency losses due to deflection

A relatively simple design update is proposed here, as shown in Figure A.6, where rollers should be used to hold the test ring in place along the longitudinal direction of the bearing supports. These rollers supporting the ring would achieve the following:

- Prevent misalignments and large variations of torque developed throughout the application of deflection between the ring and the bearing supports.

- Provide lower efficiency losses as those obtained with the containment plate shown in Figure A.4 with very low friction if the rollers are made of teflon.

Updating the experimental test setup in this manner would more realistically represent the behavior and operation of the deflected flexible spline.

A.9 Measurement of the rotation torque

The final goal of this initial experimental test setup is to determine the torque required to rotate the deformed flexible spline of SWG drives. Measuring this torque would be used to quantify the mechanical losses throughout the operation of this type of gear drive stemming alone from the deflection applied to the flexible spline at all times.

Figure A.7 shows the proposed experimental test setup for measurement of the deflection-induced torque. The expected mechanical losses consist of the bearing losses due to the roller
elements and their races, the friction between the bearing and the ring under study, and the
bending of the ring itself, which constitutes the internal friction of the material of the ring.

Updating the experimental setup is required in order to measure the torque losses. The main
conversion of the setup is the addition of a frame to support a steel disk with a rubber ring as
shown in Figure A.7. This disk would be used to rotate the deflected ring under study while
the disk itself would be rotated using a torquemeter to determine the torque required to rotate
the deflected ring. Different torques may be measured as a function of the deflection applied to
the test ring.

If the mechanical losses created by the pure bending of the test ring are required, it is nec-
essary to determine the losses stemming from the bearing supports applying the deflection,
although these losses are assumed to be small. One approach to measure such losses due to
the bearings would be to apply a radial load at the outer race of one bearing and determine the
torque needed for its rotation.
A.10 Conclusions

The first objective of the experimental tests of the flexible spline is completed. This is in terms of the design of the test setup, the calculation of the required deflections on the test ring, and the measurement of those deflections.

Completing these tasks requires the design, development, and manufacturing of the test setup, as well as scaling the deflections of the flexible spline to the test ring. The scaling of the deflections in order to obtain realistic measurements is based on the work performed throughout this thesis dissertation.

The performed experiments prove the measurement of the required deflections of the flexible spline to be possible with the designed test setup.

Rotating the deflected ring under study is possible. However, variations of the torque required to rotate the deflected ring are experienced. These variations indicate the presence of misalignments between the bearing supports of the test setup and the deflected ring. It is necessary to improve the experimental test setup to avoid misalignments and excessive friction in order to properly measure the required torque to rotate the deflected ring. This torque would provide the mechanical losses originated by the rotation of a deflected body.

A possible update of the test setup is proposed to include teflon rollers to support the deflected ring along the longitudinal dimension of the bearing supports. These rollers would reduce the variation of torque and the misalignment between the bearings and the deflected ring, as well as their frictional losses. An additional update is proposed to measure the losses from the operation of the deflected ring alone.
Bibliography


