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Multidimensional Golden Means

Peter G. Anderson*

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Abstract

We investigate a geometric construction which yields periodic continued fractions and generalize it to higher dimensions. The simplest of these constructions yields a number which we call a two (or higher) dimensional golden mean, since it appears as a limit of ratios of a generalized Fibonacci sequence. Expressed as vectors, these golden points are eigenvectors of high dimensional analogues of $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, further justifying the appellation. Multiples of these golden points, considered “mod 1” (i.e., points on a torus), prove to be good probes for applications such as Monte Carlo integration and image processing. In [2] we exploit the two-dimensional example to derive pixel permutations in order to produce computer graphics images rapidly.

1 1D: Probing the Line

The golden mean, τ , can be represented as:

$$\frac{\sqrt{5}-1}{2} = \lim_{n \rightarrow \infty} \frac{F_{n-1}}{F_n} = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}$$

is the hardest irrational to approximate by rationals [4, 6, 7]. Hence, its modulo 1

multiples¹, $0, \tau, \{2\tau\}, \{3\tau\}, \dots$, provide an excellent list of points in the real interval to sample for estimates of integrals [5, 9], and to select points to search for the maximum of a unimodal function [10]. The discrete, or integer, analog is the sequence of multiples of F_{N-1} reduced modulo F_N . Applications for this permutation of the integers from 0 to F_{N-1} are table searching [8] and rapid computer graphics rendering [1, 2].

The sequence of points (either in the unit interval $I = [0, 1)$ or the integer interval $[0, F_N)$; i.e., where $x_n = \{n\tau\}$ or $x_n = nF_{N-1} \pmod{F_N}$) have the property that any sequence, such as $S_m = \{x_0, x_1, x_2, \dots, x_{m-1}\}$, of them is very evenly spread throughout their interval. No two are particularly close unless there are so many in the interval that crowding is necessary. In fact, the gaps between any two numerically successive elements of S_m can take on at most three different values, which are three successive powers of τ in the real case, and three successive Fibonacci numbers in the integer case. When x_m is added to S_m to form S_{m+1} , it bisects one of the largest gaps in a golden ratio or Fibonacci partitioning, respectively [8].

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¹We use the notation: $\{y\} = y - [y]$ is “ y (mod 1),” the fractional part of y .

2 Higher Dimensions

The object of our study is the generalization of the golden mean to points in higher dimensional spaces; that is, to discover a point worthy of the designation $\bar{\tau} \in R^N$, for $N \geq 2$, so that the points $\bar{x}_n = \{n\bar{\tau}\}$ fill up I^N in a smooth, uncrowded way. (We use the notation: if $\bar{v} = (v_1, v_2, \dots, v_N)$, then $\{\bar{v}\} = (\{v_1\}, \{v_2\}, \dots, \{v_N\})$.) The proof of the pudding, when $N = 1$, is a visual computer graphics display. The one-dimensional permutation of, say, integers $[0, 987)$, whose n^{th} element is $610n \pmod{987}$, gave us a shuffled order of display screen scan lines for artificial scene rendering. Slow-to-produce pictures, like fractals and ray tracings, would be shown by evenly spaced rendered scan lines, which, by being “all over the screen,” show us a recognizable image early. But 10,000 dots (1% of the computer screen) is only 10 scan lines, which is hardly enough to recognize an image. A two-dimensional shuffling of the individual pixels should give 10,000 dots evenly spread over a screen—this would give us a lot more information. See [2] for details of the computer graphics algorithms.

In the one-dimensional scan line shuffling case, we chose the above parameters since $F_{15} = 610$ and $F_{16} = 987$. Furthermore, $987 \approx 1,000$, which is approximately the number of scan lines on many computer screens. (Using $F_{16} \approx 1,000$ is like computer programmers’ use of $2^{10} \approx 1,000$. Programmers seldom complain about being restricted to powers of two for sizes of algorithm parameters, such as memory blocks. They have less reason to complain about Fibonacci numbers as parameters, since Fibonacci numbers are more plentiful than powers of 2; Fibonacci numbers are like powers of 1.618033988749.)

Our approach is to study the successful number τ via its continued fraction, and to develop higher dimensional analogies, $\bar{\tau}$.

3 Continued Fractions

$y \in I = [0, 1)$ has a continued fraction representation, $y = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \dots}}}}$ which

is usually written more compactly as $y = [0; a_1, a_2, a_3, \dots]$. The a_q are positive integers called the “partial quotients.” If one of these, say a_q , is rather large, then the continued fraction can be truncated after the appearance of a_{q-1} leaving a rational number, the $q - 1^{\text{st}}$ “convergent,” $[0; a_1, a_2, \dots, a_{q-1}] \approx y$, which is a very good approximation. The smaller the partial quotients, the harder it is to get a good rational approximation to y ; this is the principal underlying the statement: “ $\tau = [0; 1, 1, 1, \dots]$ is the hardest irrational to approximate rationally.” Our notion of an evenly spread sequence $\{x_0, \dots, x_{k-1}\}$ is closely related. Our sequence comes from an arithmetic progression, so if crowding occurs, we have x_r close to x_q ; but, if $|x_r - x_q|$ is small, then $x_{|r-q|}$ is close to 0 or 1. In that case, $|r - q|\tau$ is close to an integer; so we have a good rational approximation to τ .

We study continued fractions, like τ , whose partial quotients repeat with period 1. Suppose $z \in I$ has continued fraction $[0; p, p, p, \dots]$, where p is a positive integer. The modulo one multiples of z have a remarkable geometric relationship which we can generalize to higher dimensions. Since $\frac{1}{p+1} < z < \frac{1}{p}$, the numbers $\{nz\}$ for $0 \leq n \leq p$ evenly step through I , from left to right. The next element, however, is wrapped around: $\{(p+1)z\} = (p+1)z - 1$, and $\{(p+1)z\} < z$. The gap between $\{(p+1)z\}$ and z , as a fraction of the interval $[0, z]$ is the same as the fraction z is of the whole interval, 1. Algebraically, we are saying, $\frac{z - \{(p+1)z\}}{z} = \frac{z}{1}$, or $z - ((p+1)z - 1) = z^2$, or $z^2 + pz - 1 = 0$. (Let $w = z + p$ (note that $\{w\} = \{z\}$), so $w^2 - pw - 1 = 0$, or $w = p + 1/w$, and we get the continued

fraction $w = [p; p, p, p, \dots]$, immediately.) Express the geometric construction as an affine correspondence, T , satisfying:

$$\begin{aligned} T(0) &= z \\ T(1) &= 0 \\ T(z) &= \{(p+1)z\} = (p+1)z - 1 \end{aligned} \quad (1)$$

Since T is affine, $T(x) = mx + b$, and 1 allows us to solve for m , b , and therefore z easily. T can be generalized to higher dimensions. Vectors in N -space do not generally have “powers.” However, the “similarity” relationship between the two number triples, $(0, z, 1)$ and $(0, pz - 1, -z)$, can be generalized.

4 2D Golden Mean

In the previous section, $z \approx 1/p$, so the multiples $\{nz\}$ walked across I , re-entering I at the left to cut the interval from z to 0 in a mandated ratio. Suppose there is a 2-vector, $\bar{z} \approx \begin{pmatrix} p^{-2} \\ p^{-1} \end{pmatrix}$, so that the points $\{n\bar{z}\}$, for $n = 1, \dots, p^2$ fill I^2 with an even grid of p^2 points. The point, $p^2\bar{z}$ is near the lattice point $\begin{pmatrix} 1 \\ p \end{pmatrix}$. We set up a linear relationship to generalize (1), by requiring the point $p^2\bar{z} - \bar{i} - p\bar{j}$ (which is near $\bar{0}$; denote by $\bar{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\bar{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ the standard unit vectors in R^2) to fall within the parallelogram whose two sides are the vectors $-\bar{z}$ and $\overline{pj} - p\bar{z}$ “as $\bar{z}$ falls within I^2 .” This we specify using a linear correspondence, T , analogous to 1:

$$\begin{aligned} T(\bar{0}) &= \bar{0} \\ T(\bar{j}) &= -\bar{z} \\ T(\bar{i}) &= \bar{j} - p\bar{z} \\ T(\bar{z}) &= p^2\bar{z} - \bar{i} - p\bar{j} \end{aligned} \quad (2)$$

Let $\bar{z} = \begin{pmatrix} x \\ y \end{pmatrix}$. By inspection, 2 becomes:

$$\begin{pmatrix} -px & -x \\ 1 - py & -y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p^2x - 1 \\ p^2y - p \end{pmatrix} \quad (3)$$

We manipulate (3) and get:

$$\begin{aligned} x(x+p)^2 &= 1 \\ y &= x(x+p) \end{aligned} \quad (4)$$

The previous section’s one-dimensional discussion gave us $x(x+p) = 1$; the present (x, y) appears to be a worthy analog.

5 Higher Dimensional $\bar{\tau}$ ’s

The same construction can be carried out in R^N . The corresponding “golden point” is $\bar{z} = (x_1, x_2, \dots, x_N)^T$, with $x_k \approx p^{N+1-k}$, where $x_N(x_N + p)^N = 1$, and $x_k = x_N(x_N + p)^k$, for $k = 1, \dots, N-1$.

6 Higher Dimensional Continued Fractions

Brentjes [3] discusses a variety of approaches for generalizing the familiar continued fraction or Euclidean algorithm procedure to higher dimensions. One of the best of these generalizations is shown in Figure 1. Its purpose is to produce a set of non-negative integers corresponding to a given vector (or set) of non-negative numbers. For almost all sets S , the algorithm can run endlessly. On a set of measure zero (e.g., when S is a set of integers), a division by zero will eventually be called for; in that case the algorithm must terminate.

Figure 1 gives is the familiar algorithm used to obtain continued fractions: when S is the set $1, \alpha$, that algorithm produces the list of partial quotients for α .

input: a set, S , of positive real numbers
output: a sequence of positive integers

iterate the following:

let A be the largest element of S
let B be the second largest element of S
let $Q \leftarrow [A/B]$
output Q
replace A in S by the remainder $A - BQ$

Figure 1: A Generalized Euclidean algorithm.

Theorem. For any positive integer p , the algorithm of Figure 1, applied to $S = \{x, y, 1\}$ where x is the unique positive root of $x(x + p)^2 = 1$ and $y = x(x + p)$ produces output consisting of repeated p 's.

Proof. It is easy to establish that $0 < x < y < 1$, $1/y = x + p$, so $[1/y] = p$. The algorithm of Figure 1 replaces the largest element, $1 \in S$, with the remainder, $1 - py$, and produces an output value of p . S has become $\{1 - py, x, y\}$, which is a dilation of its original value by the factor y . We see this, since $1/y = x + p$ and $x/y = (x + y)^{-1} = y$. Finally, notice that the algorithm of Figure 1 will produce the same output for wS as for S , whenever w is positive; and we may replace S with its original value. Q.E.D.

So, $\{x, y, 1\}$ is a “fixed point” of the central operation of the algorithm of Figure 1. The N -dimensional analogies we described above are also fixed points of the algorithm of Figure 1. Consequently, Brentjes’s continued fractions generalization specifies hard-to-approximate irrational points that match those arising in our geometric construction.

7 Generalized $\bar{\tau}$'s and Fibonacci Sequences

We return to our original two-dimensional “golden point,” $\begin{pmatrix} x \\ y \end{pmatrix}$, where x is the real number satisfying $x(x + 1)^2 = 1$, and $y = x(x + 1)$. If $w = x + 1 = 1/y$, $w^3 = w^2 + 1$. So w (and inversely, y) is the limit of the ratio of two numbers in the sequence given by the following Fibonacci-type rule:

$$\begin{aligned} G_0 &= 0 \\ G_1 &= G_2 = 1 \\ G_n &= G_{n-1} + G_{n-3}, \text{ for } n > 2 \end{aligned} \quad (5)$$

(This sequence also counts rabbit populations in successive generations, but our rabbits take a month longer to mature than Fibonacci’s.)

Another way to arrive at this Fibonacci-like sequence is via a restatement of algorithm 1 and theorem 1 as an eigenvalue-eigenvector identification:

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = y \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \quad (6)$$

We deal with the inverse matrix:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - y \\ x \\ y \end{pmatrix} = y^{-1} \begin{pmatrix} 1 - y \\ x \\ y \end{pmatrix} \quad (7)$$

Which gives:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}^n \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} G_n \\ G_{n+1} \\ G_{n+2} \end{pmatrix} \quad (8)$$

8 Probing the Square

The points $\left\{ \begin{pmatrix} nx \\ ny \end{pmatrix} \right\}$ fill I^2 in a smooth, equidistributed way. Consequently, we expect

rational approximations to fill a square of discrete points smoothly. A rational approximation is $\begin{pmatrix} x \\ y \end{pmatrix} \approx \begin{pmatrix} G_n/G_{n+2} \\ G_{n+1}/G_{n+2} \end{pmatrix}$, which comes from (8) considered as a point in projective space—that is, divide through by the last coordinate and then drop it. There are only G_{n+2} multiples mod 1 of that approximation; they smoothly fill a square, but they do not provide a square grid (which we wanted for computer graphics and image processing—i.e., pixel coordinates). A close approximation which *sometimes* works very well is $Z = \begin{pmatrix} G_n/G_{n+2} \\ G_n/G_{n+1} \end{pmatrix}$. In case the three parameters, G_n , G_{n+1} , and G_{n+2} , are pairwise relatively prime, then the mod 1 multiples of Z give $G_{n+1}G_{n+2}$ distinct points. From another point of view, (G_n, G_n) is a (*sometimes* good) generator of the group $Z_{G_{n+1}} \times Z_{G_{n+2}}$, which is a cyclic group.

A phenomenon occurs in multiple dimensions that the one-dimensional study of continued fractions did not prepare us for. This was highlighted by our italic “*sometimes*” above. Although the ratios, G_n/G_{n+1} , tend to the limit y , they do not approach it monotonically. This is illustrated in table 1. However, the vector $(G_n/G_{n+2}, G_{n+1}/G_{n+2}, 1)$ *does* monotonically approach its limit, $(x, y, 1)$.

9 Using the High Dimensional $\overline{\tau}$

Here are indications of just a few of the many applications to which our “golden points” may be put.

9.1 Computer graphics

A typical graphics rendering program is:

for every pixel (x,y), plot f(x,y) at (x,y)

Use our permutations to determine the order of the pixels (2-D) or scan lines (1-D). This renders an *excellent* selection of pixels first and shows a low resolution image early. The first pixels or scan lines plotted can be artificially fat to illuminate the screen. This is important for program development and parameter space searching. More details are given in [2].

9.2 Image data-bank browsing

When images are transmission over limited bandwidths, the pixels or scan lines can be transmitted in the order suggested for graphics rendering. Users can decide quickly to wait for a full image or to skip to another.

9.3 Monte Carlo integration

An integral can be estimated by sampling:

$$\int_S f(x)dx \approx \frac{1}{N} \sum_{k=1}^N f(x_k) \quad (9)$$

For $S = I^n$, use the infinite probes. The error is generally less than that for sequences associated with “white noise.” (White noise indicates that successive samples are not correlated with one another. This is generally considered desirable in random number generators, such as *drand48()*, but not all what is needed for sampling a function to estimate its integral. See [9].).

9.4 Image morphology

Some non-linear image filtering operations based on set theory use all the translates, $E + (x, y)$, of a set E (the “structuring element”) that are contained in a set S (the set of black (or white) pixels of the given image). The translation pixels, (x, y) , can be ordered by our 2-D permutation; the process can be

terminated early, based on a stopping rule, to obtain an acceptable approximation of the desired operation.

Other image processing algorithms (filters) can similarly attack pixels in a shuffled order to allow rapid estimates of their effects.

9.5 Neural network training

Synaptic weights in an artificial neural network represent points in a space. These should be initially distributed smoothly. Similarly, exemplars forming a learning system's training set should be smoothly distributed in their space. Our x_k would be superior to white noise.

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n	G_n	$G_n/G_{n-1} - G_{98}/G_{97}$
0	0	
1	1	
2	1	-0.4
3	1	-0.4
4	2	0.5
5	3	0.03
6	4	-0.1
7	6	0.03
8	9	0.03
9	13	-0.02
10	19	-0.004
11	28	0.008
12	41	-0.001
13	60	-0.002
14	88	0.001
15	129	0.0003
16	189	-0.0004
17	277	0.00003
18	406	0.0001
19	595	-0.00005
20	872	-0.00002
21	1278	0.00002
22	1873	-0.00000002
23	2745	-0.000007
24	4023	0.000002
25	5896	0.000001
26	8641	-0.000001
27	12664	-0.0000001
28	18560	0.0000004
29	27201	-0.0000001
30	39865	-0.0000001
31	58425	0.00000007
32	85626	0.00000001
33	125491	-0.00000002
34	183916	0.000000004
35	269542	0.000000007
36	395033	-0.000000003
37	578949	-0.000000001
38	848491	0.000000001
39	1243524	-0.0000000001
40	1822473	-0.0000000004

Table 1: Notice the non-monotonic decrease of the error towards zero, and the particularly small error at $n = 22$.

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