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Exceptional sets and Antoine's necklace

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Abstract

We study Cantor sets which occur as minimal sets for homeomorphisms of $\mathbb{R}^n$. The minimality is modelled on an infinite product of finite cyclic groups and on a generalized adding machine. An interesting example is a homeomorphisms on $\mathbb{R}^3$ which has Antoine's Necklace as a minimal set. We also discuss some open problems concerning homeomorphism that have a Cantor set as a minimal set.

Key words: minimal set; solenoid, exceptional set, Antoine's necklace

1 Introduction

For a homeomorphism $f : X \to X$ of a topological space X, a nonempty compact subset $Y \subset X$ is a minimal set if for every $y \in Y$ the orbit of y is dense in Y. If X itself is a minimal set then f is said to be minimal. In [G], Gottschalk discussed the question of what sets can be minimal sets. A minimal set which is a Cantor set is called an exceptional set. Our main result is a collection of homeomorphisms that have interesting exceptional sets. The term exceptional set has a generalization to flows (direction 1-foliations) and higher dimensional foliations. For a k-dimensional foliated manifold M, a foliated subset of M is an exceptional set if it is the closure of every leaf in the subset and the intersection of any transversal with the subset is a Cantor set. Since minimal sets must be compact and perfect, they tend to be manifolds and an exceptional sets.

Gottschalk specifically questioned whether Antoine's Necklace, which is an interesting Cantor set (see Figure 3.2), could appear as a minimal set for a homeomorphism on $\mathbb{R}^3$. In [Z] Zang answers this in the affirmative with an explicit example. Each of the homeomorphisms $\mathfrak{g}$ and $\mathfrak{h}$ in Section 3.2 has Antoine's necklace as a minimal set, and Zang's example is included as the special case with $q_i = 4$ for all i.

In Section 2 we prove that certain homeomorphisms on a model symbol space G are minimal. In Section 3 we define homeomorphisms on D^2 and $\mathbb{R}^3$ that are conjugate to the model homeomorphisms when restricted to a Cantor set. Suspensions of these homeomorphisms could appropriately be called generalized solenoids. We discuss a generalization of Gottschalk's conjecture about Antoine's necklace and related open questions in Section 4.

2 The Model Space

Any topological group which is homeomorphic to the Cantor set is called a Cantor group. In this section we define a Cantor group G and two homeomorphisms $\alpha, \beta : G \to G$ which are minimal. These symbolic dynamical systems serve as our models for the dynamical systems in Section 3.

Let $q_1, q_2, \dots$ be an infinite sequence of positive integers. Let G be the group

$$G = \prod_{n=1}^{\infty} \mathbb{Z}_{q_n}$$

with elements denoted by $\mathbf{x} = (x_1, x_2, \dots)$. Define a basis of open sets for the topology to be cylinders

$$C_{y_1, y_2, \dots, y_k} = \{\mathbf{x} \,|\, x_i = y_i \text{ for all } i \leq k\}.$$

Our first homeomorphism only is minimal in the case where every pair q_i, q_j with $i \neq j$ are relatively prime. Define $\alpha : G \to G$ to be addition by the element $(1, 1, \dots)$,

$$\alpha(x_1, x_2, \dots) = (x_1 + 1 \bmod q_1, x_2 + 1 \bmod q_2, \dots).$$

It is easy to see that α is minimal as follows. Let $\mathbf{x} \in G$ and we will show that the orbit through $\mathbf{x}$ is dense. Given any point $\mathbf{y} \in G$, the first coordinate of $\alpha^n(\mathbf{x})$ must agree with the first coordinate of $\mathbf{y}$ for some $0 \leq n \leq q_1$. Next the first two coordinates of $\alpha^n(\mathbf{x})$ must agree with the first two coordinates of $\mathbf{y}$ for some $0 \leq n \leq q_1 q_2$. Continuing in this manner, for any k there is an n such that $\alpha^n(\mathbf{x})$ agrees with $\mathbf{y}$ in the first k coordinates. Hence the orbit through $\mathbf{x}$ intersects any neighborhood of any point $\mathbf{y} \in G$ and the orbit is dense.

The second homeomorphism is sometimes called a generalized adding machine or odometer. Define the homeomorphism $\beta : G \to G$ inductively as follows. Suppose $\beta(\mathbf{x}) = \mathbf{y}$. Then define y_1 by

$$y_1 = x_1 + 1 \bmod q_1.$$

For $i > 1$, define y_i inductively by

$$y_i = \begin{cases} x_i + 1 \bmod q_i & \text{if } y_j = 0 \text{ for all } j < i \\ x_i & \text{else.} \end{cases}$$

The argument that β is minimal is similar to that for α. Let $\mathbf{x} \in G$. Given any point $\mathbf{y} \in G$, the first coordinate of $\beta^n(\mathbf{x})$ must agree with the first coordinate of $\mathbf{y}$ for some $0 \leq n \leq q_1$. Since q_1 and q_2 are relatively prime, the first two coordinates of $\beta^n(\mathbf{x})$ must agree with the first two coordinates of $\mathbf{y}$ for some $0 \leq n \leq q_1 q_2$. Continuing in this manner, for any k there is an n such that $\beta^n(\mathbf{x})$ agrees with $\mathbf{y}$ in the first k coordinates. Hence the orbit through $\mathbf{x}$ intersects any open neighborhood of any point $\mathbf{y} \in G$ and the orbit is dense.

In Section 3 we define a homeomorphism between the G and an imbedded Cantor set and define homeomorphisms which are conjugate to α and β on the Cantor set.

3 The Homeomorphisms

3.1 Am Exceptional Set which is the Intersection of Nested Disks

Let $q_1, q_2, \ldots$ be an infinite sequence of positive integers such that each pair q_i and q_j with $i \neq j$ are relatively prime. Let D be a closed disk. Let $D_0, D_1, \ldots, D_{q_1-1}$ be disjoint closed disks contained in the interior of D such that a rotation of D by $2\pi/q_1$ takes D_i to $D_{i+1 \bmod q_1}$. Let $D_{(0,0)}, D_{(0,1)}, \ldots, D_{(0,q_2-1)}$ be disjoint closed disks which are contained in the interior of D_0 such that a rotation of D_0 by $2\pi/q_2$ takes $D_{(0,i)}$ to $D_{(0,i+1 \bmod q_2)}$. Let $D_{(i,j)}$ be the image of $D_{(0,j)}$ under a rotation of D by $i2\pi/q_1$. This is shown in Figure 1 with $q_1 = 3, q_2 = 5, q_3 = 7$. Continuing in this manner gives a nested sequence of compact sets $C_1 \supset C_2 \supset \cdots$ defined by

$$C_n = \bigcup_{\{(x_1,x_2,\ldots,x_n) | 0 < x_i \leq q_i - 1\}} D_{(x_1,x_2,\ldots,x_n)}.$$

Let C denote the Cantor set

$$C = \bigcap_{n=1}^{\infty} C_n.$$

Notice that if $q_i = 2$ for all i and every disk has its center on the $x-$axis then C is a standard "middle thirds" Cantor set (depending on the diameters of the disks.)

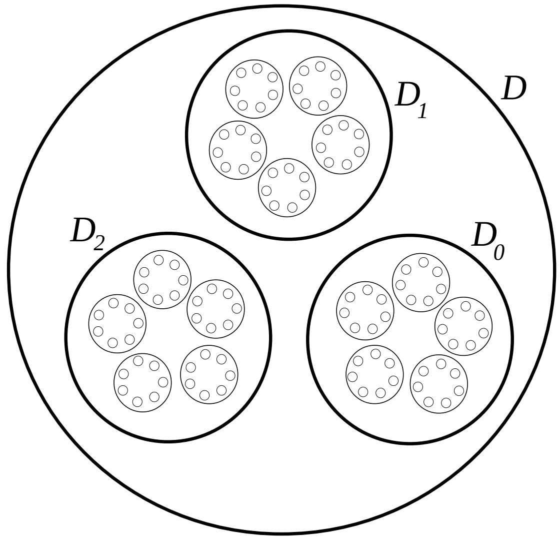

Figure 1. The first three steps in creating the Cantor set C.

The natural homeomorphism between C and G is

$$\Gamma(x_1, x_2, \dots) = D \cap D_{x_1} \cap D_{(x_1,x_2)} \cap \cdots .$$

We now define the homeomorphism on D which has C as a minimal set. Let $q_1, q_2, \dots$ be an infinite sequence of positive integers. Let g_1 be a rotation of D by $2\pi/q_1$. For each disk D_i define a larger disk $\overline{D}_i$ such that $\overline{D}_i$ contains D_i in its interior, all of the $\overline{D}_i$ are disjoint, and the center of $\overline{D}_i$ is the same as the center of D_i. Define a homeomorphism g_2 which is the identity on the compliment of the $\overline{D}_i$ and which is a rotation by $2\pi/q_2$ on each D_i. This can be done as follows. Foliate the annulus $\overline{D}_i - D_i$ by circles. Define a bump function ρ on these circles which is 1 on ∂D_i and 0 on $\partial\overline{D}_i$. Then define g_2 to be a rotation on each circle by $\rho 2\pi/q_2$. Define g_3 in an analogous manner so it rotates each $D_{(i,j)}$ by $2\pi/q_3$ and is the identity off of disks $\overline{D}_{(i,j)}$. Continuing in this manner we get infinitely many homeomorphisms g_n. Define $g : D \to D$ by

$$g(x) = \lim_{n\to\infty} g_n \circ \cdots \circ g_2 \circ g_1(x)$$

This is well defined and continuous on $D - C$ since any $x \in D - C$ is only moved by finitely many of the g_i. It is well defined on C because for any $x \in C$, the sequence $g_1(x), g_2 \circ g_1(x), g_3 \circ g_2 \circ g_1(x), \dots$ converges. It is continuous at points in C as follows. Let $x \in C$. For any $\epsilon > 0$, there exists an n such that the diameter of $D_{(x_1,x_2,\dots,x_n)}$ is less than ϵ. Let $\delta = d(x, \partial D_{x_1,x_2,\dots,x_n)})$. Then if y is a point such that $d(x, y) < \delta$ then $d(g(x), g(y)) < \epsilon$. Hence this is well defined and continuous on all of D. Now it is clear that for $x \in C$,

$$G(x) = \Gamma \circ \alpha \circ \Gamma^{-1}(x).$$

So g is a continuous homeomorphism on D which has C as a minimal set.

The homeomorphism which is conjugate to β on C is defined in a similar manner. Let $q_1, q_2, \ldots$ be an infinite sequence of positive integers, not necessarily pairwise relatively prime. Let h_1 be a rotation of D by $2\pi/q_1$. Define a disk $\overline{D}_0$ such that $\overline{D}_0$ contains D_0 in its interior, $\overline{D}_0 \cap D_j = \emptyset$ for $j \neq 0$, and the center of $\overline{D}_0$ is the same as the center of D_0. Define a homeomorphism h_2 which is the identity on the compliment of the $\overline{D}_0$ and which is a rotation by $2\pi/q_2$ on D_0. This can be done as before for g_2. Define h_3 in an analogous manner so it rotates $D_{(0,0)}$ by $2\pi/q_3$ and is the identity off of a disk $\overline{D}_{(0,0)}$. Continuing in this manner we get infinitely many homeomorphisms h_n. Define $h : D \to D$ by

$$h(x) = \lim_{n\to\infty} h_n \circ \cdots \circ h_2 \circ h_1(x)$$

This is well defined and continuous following the same arguments as for g. Now it is clear that for $x \in C$,

$$h(x) = \Gamma \circ \beta \circ \Gamma^{-1}(x).$$

So h is a continuous homeomorphism on D which has C as a minimal set.

3.2 A Homeomorphism on $\mathbb{R}^3$ with Antoine's Necklace as a Minimal Set

Antoine's Necklace can be defined in the following way. Let $q_1, q_2, \ldots$ be an infinite sequence of positive integers. Let $T = S^1 \times D^2$ be a solid torus coordinatized by (θ, ϕ, r) where $\theta \in [0, 2\pi]$ mod 2π is the coordinate on S^1 and (ϕ, r) are polar coordinates on D^2. Inside T define a chain of solid tori $T_0, \ldots, T_{q_1-1}$ as follows. (See figure 3.2.) For each $i \in \{0, 1, \ldots, q_1-1\}$ let $p_i = (i2\pi/q_1, 0, 0)$. For each i let γ_i be the circle of radius $3\pi/4q_1$, centered at p_i, and contained in $\{(\theta, \phi, r) \,|\, \phi = i2\pi/q_1\}$. Note that the linking number of γ_i with γ_j is ± 1 if $|i - j \bmod q_1| = 1$ and γ_i is the image of $\gamma_{i-1 \bmod q_1}$ under the rotation of T by $(\theta, \phi, r) \to (\theta + i2\pi/q_1, \phi + i2\pi/q_1, r)$. From now on we refer to this homeomorphism simply as rotation by $(i2\pi/q_1, i2\pi/q_1)$. For each i define T_i to be a torus neighborhood of γ_i such that all of the T_i are disjoint and a rotation of T by $(i2\pi/q_1, i2\pi/q_1)$ takes T_i to $T_{i+1 \bmod q_1}$. Denote the union of these tori by $C_1 = \bigcup_{i=0}^{q_1-1} T_i$.

In T_0 define a chain of q_2 pairwise disjoint solid tori, $T_{(0,0)}, \ldots, T_{(0,q_2-1)}$, such that a rotation of T_0 by $(2\pi/q_2, 2\pi/q_2)$ takes $T_{(0,i)}$ to $T_{(0,i+1 \bmod q_2)}$. Let $T_{(i,j)}$ be the image of $T_{(0,j)}$ under a rotation of T by $(2\pi/q_1, 2\pi/q_1)$ and let $C_2 = \bigcup_{j=0}^{q_2-1} \bigcup_{i=0}^{q_1-1} T_{(i,j)}$. Continuing in this manner results in a nested sequence of compact sets $\cdots \subset C_2 \subset C_1$ and Antoine's necklace is the set

$$A = \bigcap_{i=1}^{\infty} C_i.$$

The topologically interesting property of Antoine's necklace are that it is a Cantor set and it's complement is not simply connected when imbedded in R^3.

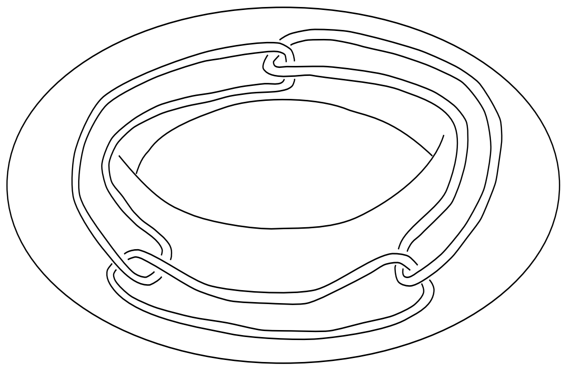

Figure 2. The first two steps in creating Antoine's Necklace.

The natural homeomorphism $\Pi : G \to A$ is

$$\Pi(x_1, x_2, \dots) = T \cap T_{x_1} \cap T_{(x_1,x_2)} \cap \cdots .$$

To define the homeomorphism on T which has A as a minimal set, let $q_1, q_2, \dots$ be an infinite sequence of positive integers such that each pair q_i and q_j with $i \neq j$ are relatively prime. Let $\mathfrak{g}_1$ be a rotation of T by $(2\pi/q_1, 2\pi/q_1)$. For each T_i define a solid torus $\overline{T}_i$ such that $\overline{T}_i$ contains T_i in its interior, $\overline{T}_i$ is also a torus neighborhood of γ_i, $\mathfrak{g}_1(\overline{T}_i) = \overline{T}_{i+1 \bmod q_1}$, and all of the $\overline{T}_i$ are disjoint. Define a homeomorphism $\mathfrak{g}_2$ which is the identity on the compliment of $\cup_i \overline{T}_i$ and which is a rotation by $(2\pi/q_2, 2\pi/q_2)$ on each T_i. This can be done as follows. Foliate the thickened torus $\overline{T}_i - T_i$ by tori. Define a bump function ρ on these tori which is 1 on ∂T_i and 0 on $\partial \overline{T}_i$. Then define $\mathfrak{g}_2$ to be a rotation on each torus in the foliation by $(\rho 2\pi/q_2, \rho 2\pi/q_2)$. Define $\mathfrak{g}_3$ in an analogous manner so it rotates each $T_{(i,j)}$ by $(2\pi/q_3, 2\pi/q_3)$ and is the identity off of the $\overline{T}_{(i,j)}$. Continuing in this manner we get infinitely many homeomorphisms $\mathfrak{g}_n$. Define $\mathfrak{g} : T \to T$ by

$$\mathfrak{g}(x) = \lim_{n\to\infty} \mathfrak{g}_n \circ \cdots \circ \mathfrak{g}_2 \circ \mathfrak{g}_1(x)$$

This is well defined and continuous on $T - A$ since any $x \in T - A$ is only moved by finitely many of the $\mathfrak{g}_i$. It is well defined on A because for any $x \in A$ the sequence $\mathfrak{g}_1(x), \mathfrak{g}_2 \circ \mathfrak{g}_1(x), \dots$ converges. The argument that it is continuous at points in A follows as the argument that g is continuous at points in C. Specifically, let $x \in A$. For any $\epsilon > 0$, there exists an n such that the diameter of $T_{(x_1,x_2,\dots,x_n)}$ is less than ϵ. Let $\delta = d(x, \partial T_{x_1,x_2,\dots,x_n)})$. Then if y is a point such that $d(x,y) < \delta$ then $d(g(x), g(y)) < \epsilon$. Hence this is well defined and continuous on all of A. It is clear from the definition that $\mathfrak{g}$ is conjugate to α,

$$\mathfrak{g}(x) = \Pi \circ \alpha \circ \Pi^{-1}(x)$$

So $\mathfrak{g}$ is a continuous homeomorphism on T which has A as a minimal set. It is possible to extend the definition of $\mathfrak{g}$ to all of $\mathbb{R}^3$ by defining as above on some imbedded solid torus T and making it the identity off of a neighborhood of T.

The homeomorphism on T which is conjugate to β on A is defined in a similar manner. Let $q_1, q_2, \ldots$ be an infinite sequence of positive integers, not necessarily pairwise relatively prime. Let $\mathfrak{h}_1$ be a rotation of T by $(2\pi/q_1, 2\pi/q_1)$. Define a solid torus $\overline{T}_0$ such that $\overline{T}_0$ contains T_0 in its interior, $\overline{T}_0$ is also a torus neighborhood of γ_0, and $\overline{T}_0 \cap T_i = \emptyset$ for all $i \neq 0$. Define a homeomorphism $\mathfrak{h}_2$ which is the identity on the compliment of the $\overline{T}_0$ and which is a rotation by $(2\pi/q_1, 2\pi/q_1)$ on T_0. This can be done as before for $\mathfrak{g}_2$. Define $\mathfrak{h}_3$ in an analogous manner so it rotates $T_{(0,0)}$ by $(2\pi/q_3, 2\pi/q_3)$ and is the identity off of a tours $\overline{T}_{(0,0)}$. Continuing in this manner we get infinitely many homeomorphisms $\mathfrak{h}_n$. Define $\mathfrak{h} : T \to T$ by

$$\mathfrak{h}(x) = \lim_{n\to\infty} \mathfrak{h}_n \circ \cdots \circ \mathfrak{h}_2 \circ \mathfrak{h}_1(x)$$

This is well defined and continuous following the same arguments as for $\mathfrak{g}$. Now it is clear that for $x \in A$,

$$\mathfrak{h}(x) = \Gamma \circ \beta \circ \Gamma^{-1}(x).$$

So $\mathfrak{h}$ is a continuous homeomorphism on T which has A as a minimal set. It is clear that $\mathfrak{h}$ could be extended to a homeomorphism on $\mathbb{R}^3$ which has A as a minimal set.

4 Remarks and Open Questions

Remark 1. An important topological invariant of a minimal set is the The D-function, developed by Ye in [Y]. Suppose that $f : X \to X$ is a continuous map of a compact Hausdorff space and that Y is a minimal set for f. The D-function for Y is the function $f_Y : \mathbb{N} \to \mathbb{N}$ which takes the natural number n to the number of distinct minimal sets of f^n which are contained in Y. The D-function for α is easy to compute (and hence so is the D-function for g and $\mathfrak{g}$.) If $q_1, q_2, ...$ is the sequence of pairwise relatively prime positive integers defining α then the D-function of α is

$$s(n) = \Pi_{\{q_i : q_i | n\}} q_i.$$

The D-function for β is more complicated. As remarked in [Y], for any function $s \in \mathcal{Y}$ one can choose the $\{q_i\}$ so that the D-function of β is s. (The space $\mathcal{Y}$ is the space of all possible D-functions as defined in [Y].)

In [Y], Ye defines a subshift which also has minimal sets with every possible D-function, and mentions the question of whether this subshift is conjugate to the adding machine β. The two systems are not conjugate because the β is not expansive but a subshift is, and expansiveness is preserved by conjugacy. (A dynamical system $f : X \to X$ is expansive if for every $x, y \in X$, $d(x,y) < d(f^n(x), f^n(y))$ for some positive integer n.) This observation was shown to the author by Brian Marcus during a personal conversation.

Remark 2. Our construction can be extended to a more general setting in several ways, one of which we describe loosely here. Let M ba a n-dimensional manifold imbedded in $M \times D^k$ with $k \geq M$, and let G be the group of isometries of M. Let G_1 be a discrete subgroup of G. Let $x \in M \times \{0\}$. For each $g_1 \in G_1$, let $x_{g_1} = g_1(x)$ (so $x = x_{id}$.) Also for each $g_1 \in G^1$, let M_{g_1} be a copy of M imbedded in $M \times d^n$ so that $x_{g_1} \in M_{g_1}$ and $g_1(M_{id}) = M_{g_1}$, where we are extending $g_1 : M \to M$ to $g_1 : M \times D^n \to M \times D^n$ via the identity on the D^n coordinate. One could repeat the process from Section 3 to obtain a Cantor set C with points $x_{(g_1, g_2, \ldots)} \in G$, where $G = \Pi_{i=1}^{\infty}$ and each G_i is a discrete subgroup of isometries of M. Such generalizations raise questions of knotting and linking, as well as the question of whether a $\mathbb{Z}$ action would be sufficient to make this a minimal set or if $\mathbb{Z}^m$, $m \leq \infty$ would be required. We note the special cases from Section 3 have $M = S^1$ and $k = 2$ or 3.

Open Questions One can generalize the question of Gottschalk as "Which imbeddings of Cantor sets can occur as an exceptional set for a homeomorphism on an $n-$dimensional manifold?"

A simple example of Cantor sets in $\mathbb{R}^3$ that cannot be a minimal set for any homeomorphism of $\mathbb{R}^3$ is as follows. It is clear that if a Cantor set has two points x, y such that some neighborhood of x in $\mathbb{R}^n$ is not homeomorphic (preserving C) to any neighborhood of y then there does not exist a homeomorphism with this Cantor set as a minimal set. Such an example can be defined similar to the definition of Antoine's Necklace by making some of the tori knotted, say all tori $T_{x_1, x_2, \ldots}$ knotted if $x_1 = 0$ but unknotted if $x_1 = 1$. An interesting question is: Is there an imbedding of a Cantor set that does not have this obstruction and which is not the minimal set for any homeomorphism?

Another interesting question is whether there exists a manifold which admits a periodic orbit free flow (map) but not a minimal flow (map). A solution to the Gottschalk conjecture of whether there exists a minimal flow on S^3 in the negative would provide such an example.

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