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A short constructive proof that nonsingular flows only exist on manifolds of zero Euler characteristic

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A SHORT CONSTRUCTIVE PROOF THAT NONSINGULAR FLOWS ONLY EXIST ON MANIFOLDS OF ZERO EULER CHARACTERISTIC

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Abstract. It is a well-known result that a closed manifold that admits a nonsingular flow (ie a flow without fixed points) only if its Euler Characteristic is zero. We provide a short proof of this by constructing a cell complex on the ambient manifold that is induced by the flow and showing that this complex has zero Euler Characteristic. We use the method of a global transverse disk with the hope that this method will be useful in studying global topological properties of nonsingular flows.

1. Introduction

It is well-known that a closed (compact, no boundary) manifold admits a nonsingular flow (equivalently, fixed point free flow or oriented 1-foliation) only if the Euler characteristic of the manifold is zero. This can be proven using the Morse index. The sum of the Morse index of each of the fixed points of a flow is equal to the Euler Characteristic of the ambient manifold. Since there are no fixed points, this sum is zero. We provide a constructive proof that the Euler Characteristic is zero by constructing a natural cell complex on the manifold. A very simple calculation shows that the Euler characteristic of the cell complex is zero. Moreover, the flow is trivial on the cells of the complex, being either transverse or tangent to the cells, and hence the cell complex describes global topological properties of the flow.

Before proving our main theorem we recall some definitions and theorems. Our main tool will be a *global transverse disk*, defined as follows.

DEFINITION 1. *Let M be a manifold with flow φ. A transverse disk is an imbedded closed $(n-1)$-dimensional disk Σ which is topologically transverse to φ. By topologically transverse to Σ, we mean that there is an imbedded open $(n-1)$-dimensional disk Γ containing Σ such that the flow is topologically transverse at each point in Γ.*

DEFINITION 2. *A global transverse disk is a transverse disk Σ such that for all $x \in M$, the orbit through x intersects Σ in both positive and negative time.*

An M complex, defined in detail below, is a generalization of a CW complex where each cell is homeomorphic to a k-dimensional manifold without boundary, not necessarily D_k.

DEFINITION 3. *An **M complex** is a topological space defined as follows.*

For each $n = 0, 1, \dots, N$, let $\{\overline{e^n_\alpha}\}$ be a set of compact n-dimensional manifolds with interiors $\{e^n_\alpha\}$, where α runs over some finite indexing set. The e^n_α are called M-cells, being manifolds which play the role of cells in the definition of a CW complex.

1. *Let $X^0 = \{e^0_\alpha\}$ be a discrete set of points.*
2. *Inductively define X^n, called the $n-$skeleton, from X^{n-1} by attaching each e^n_α by maps $\psi_\alpha : \partial\overline{e^n_\alpha} \to X^{n-1}$. That is, X^n is the identification space of $X^{n-1}\coprod_\alpha \overline{e^n_\alpha}$ under $x \sim \psi_\alpha(x)$ for $x \in \partial\overline{e^n_\alpha}$.*

Following the notational conventions in [7] for CW complexes, if C denotes the set of cells and attaching maps then $|C| = \cup_n X^n$ denotes the resulting topological space.

We will need the following two theorems. The first theorem was proven in [4] in 1996 for the case $n = 3$, and in higher dimensions in [1].

THEOREM 1. *Let Σ' be a global transverse disk for a flow φ on an n-dimensional manifold M. There exists a perturbation of Σ so that the new Σ is a global transverse disk and the first return map $h : \Sigma \to \Sigma$ satisfies the following property: There exists M complexes C_D and C_R such that $|C_d| = |C_r| = \Sigma$ and for each cell $e \in C_d$, $h|_e$ is a homeomorphism onto a cell of C_r.*

We will refer to C_d as the domain cell complex on Σ and refer to C_r as the range cell complex on Σ. The M complexes C_d and C_r can be chosen so that $h(x) \in \partial\Sigma$ if and only of $x \in \overline{(X^{n-1} \cap \mathrm{int}\Sigma)}$, where X^{n-1} is the $(n-1)$-skeleton of C_d. In this case $\overline{(X^{n-1} \cap \mathrm{int}\Sigma)}$ is the discontinuity set of h. It is also shown in [1] that $C_d|_{\partial\Sigma} = C_d|_{\partial\Sigma}$. That is, a subset $e \subseteq \partial\Sigma$ ia a cell of C_d if and only if it is a cell of C_r.

The following theorem is proven in [2]

THEOREM 2. *If M is a closed n-dimensional manifold with $n > 2$ and φ is a nonsingular flow on M then there exists a global transverse disk for φ.*

We are now ready to state and prove our main theorem.

THEOREM 3. *If M is a closed n-dimensional manifold with $n > 2$ and φ is a nonsingular flow on M then the Euler characteristic of M is zero.*

Proof. By theorems 1 and 2 there exists a transverse disk Σ with M complexes C_r and C_d. Define a CW complex C'_d on Σ as such that C'_d is a subcomplex of C_d. Let C'_r be the CW complex whose cells consist of $h(e)$ for each cell $e \in C'_d$. For each $x \in \Sigma$, let $\gamma(x)$ be the orbit segment beginning at x and ending at the first return of x to Σ. For each cell $e \in C'_d$, let $e^v = \cup_{x \in e} \gamma(x)$ be the "vertical cell over e." Let C be the cell complex on M which consists of the following cells:

- The cells of C'_r
- The cells e^v for each cell $e \in C'_r$ with $e \subseteq \partial\Sigma$
- the single n-dimensional cell $\cup_{x \in \mathrm{int}\Sigma} \gamma(x)$.

Examples of C are shown in figures 1 and 2

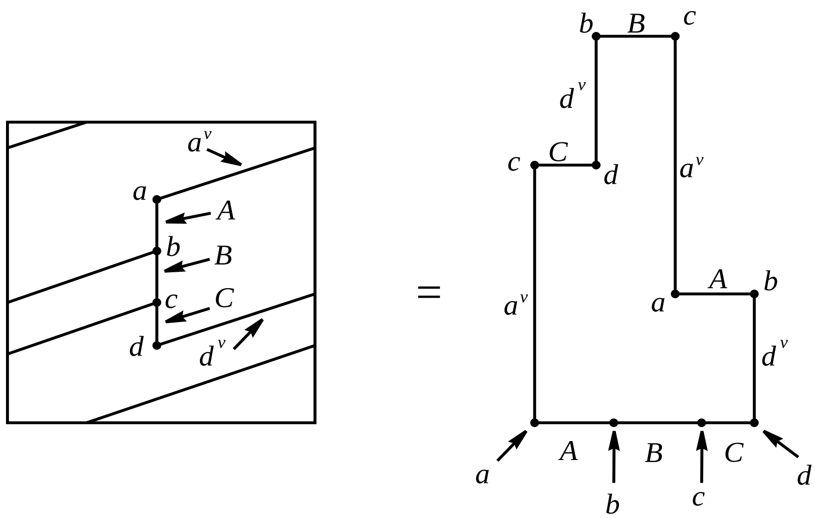

FIGURE 1. An example of the induced cell complex C for a nonsingular flow on the 2-dimensional torus.

By Definition of C above, the Euler Characteristic of M is the sum

$$\sum_{e \in C} (-1)^{\dim(e)} = \sum_{e \in C'_r} (-1)^{\dim(e)} + \sum_{e \in C'_r | e \subseteq \partial\Sigma} (-1)^{\dim(e^v)} + (-1)^n$$

$$= \sum_{e \in C'_r | e \subseteq \mathrm{int}\Sigma} (-1)^{\dim(e)} + \sum_{e \in C'_r | e \subseteq \partial\Sigma} (-1)^{\dim(e)} + \sum_{e \in C'_r | e \subseteq \partial\Sigma} (-1)^{\dim(e^v)} + (-1)^n.$$

Since $\mathrm{int}\Sigma$ is an open $(n-1)$-dimensional disk the sum becomes

$$= (-1)^{n-1} \sum_{e \in C'_r | e \subseteq \partial\Sigma} (-1)^{\dim(e)} + \sum_{e \in C'_r | e \subseteq \partial\Sigma} (-1)^{\dim(e^v)} + (-1)^n$$

$$= \sum_{e \in C'_r | e \subseteq \partial\Sigma} (-1)^{\dim(e)} + \sum_{e \in \partial C'_r} (-1)^{\dim(e^v)}$$

$$= \sum_{e \in \partial C'_r} (-1)^{\dim(e)} + (-1)^{\dim(e^v)}$$

$$= 0$$

□

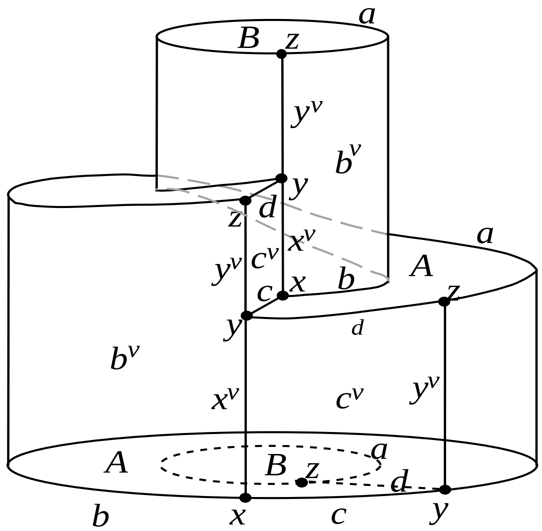

FIGURE 2. An example of the induced cell complex C for a nonsingular flow on a 3-manifold which is homotopic to S^3.

REFERENCES

[1] W. Basener, Minimal Flows and Global Cross Sections which are Disks, to appear in *Topology and its Applications*

[2] W. Basener, Every Nonsingular FLow in Dimension Greater than 2 has a Global Transverse Disk, preprint

[3] A. Candel and L. Conlon, Foliations I (Americain Mathematical Society, 2000)

[4] C. Gutierrez, Knots and Minimal Flows on S^3, *Topology* **241** (1995) 679-698

[5] C. Hog-Angloni and W. Metxler, Geometric Aspects of Two-Dimensional Complexes, Two-Dimensional Homotopy (1993) 1-50

[6] A. Katok and B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems (Cambridge University Press, 1995)

[7] C. P. Rourke and B. J. Sanderson, Introduction to Piecewise-Linear Topology (Springer Verlag, 1971)

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