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TRANVERSE DISKS, SYMBOLIC DYNAMICS, HOMOLOGY DIRECTION VECTORS, AND THURSTON-NIELSON THEORY

WILLIAM BASENER

Abstract. We review some properties of transverse disks and use symbolic dynamics to determine rotation vectors from the return map to a transverse disk. We also prove connections between the symbolic dynamics and Nielsen equivalence of orbits.

homology direction; Nielsen equivalence; transverse disk; global cross section;

1. Introduction

Let $M$ be a $n$-dimensional manifold and $\varphi : \mathbb{R} \times M \to M$ be a $C^1$ nonsingular flow on $M$. A transverse disk $\Sigma$ for $\varphi$ is a compact $(n - 1)$-dimensional disk imbedded in $M$ and transverse to the flow. Since $\Sigma$ is compact, by transverse to the flow we mean that there exists an open $(n - 1)$-dimensional disk $E$ containing $\Sigma$ that is transverse to the flow. We call $\Sigma$ a global transverse disk if it is a transverse disk and the positive and negative orbits through any $x \in M$ both intersect $\Sigma$. It is proven in [2] that every $C^1$ nonsingular flow on a manifold of dimension greater than 2 has a global transverse disk.

Transverse disks have been used as a tool in various settings. In [8], Carlos Gutierrez uses transverse disks in dimension 3 to prove a result concerning prime knotting of orbits. Based on Gutierrez’ work, it is proven in [4] that if a flow has a dense orbit and $H_2(M) = 0$ then there exists a dense open set $N$ such that any periodic orbit intersecting $N$ is a prime knot. Marcy Barge and Bob Williams use transverse disks in torus flows to classify Denjoy Continua and prove results concerning continued fractions. They are a common tool in the study of flows on surfaces, especially regarding Cherry flows and billiards.

We take the point of view that the return map to a transverse disk captures all of the topology of a flow. Specifically, in ?? it is proven that, for flows $\varphi, \varphi'$ with transverse disks $\Sigma, \Sigma'$ and return maps $h, h'$, the return maps $h$ and $h'$ are conjugate if and only if the flows are topologically equivalent. In this paper we prove that important topological invariants can be “read off” from the symbolic dynamics of the return map to a transverse disk. Specifically, from the symbolic dynamics of the return map one can determine the space of homology directions for the flow, the Abelian Nielsen classes of periodic orbits. One can also determine information about the periodic Nielsen classes and strong Nielsen classes of orbits as described in Section 4.

2. Basic Definitions

For the rest of this paper, assume that $M$ is 3-dimensional and that $\Sigma$ is a global transverse disk for $\varphi$. Associated with $\Sigma$ is a first return map $h : \Sigma \to \Sigma$. In [1] it is proven that if $\Sigma$ is chosen correctly then there exists a partition of $\Sigma$ into points,
1-manifolds, and 2-manifolds such that \( h \) is continuous on each submanifold. The images of these manifolds also forms a partition of \( \Sigma \). We make this rigorous in the following definitions.

A natural structure that we need is an M complex, which is a generalization of a CW complex and is defined as follows.

**DEFINITION 1.** An M complex is a topological space defined as follows.

For each \( n = 0, 1, \ldots, N \), let \( \{ e_n^\alpha \} \) be a set of compact \( n \)-dimensional manifolds with boundary where \( \alpha \) runs over some finite indexing set. For each \( e_n^\alpha \), we denote the interior of \( e_n^\alpha \) by \( e_n^\alpha \). The \( e_n^\alpha \) are called M-cells, being manifolds which play the role of cells in the definition of a CW complex.

1. Let \( X^0 = \{ e_0^\alpha \} \) be a discrete set of points.
2. Inductively define \( X^n \), called the \( n \)-skeleton, from \( X^{n-1} \) by attaching each \( e_n^\alpha \) by maps \( \psi_\alpha : \partial e_n^\alpha \to X^{n-1} \). That is, \( X^n \) is the identification space of \( X^{n-1} \! \! \times_\psi \! \! \prod_\alpha e_n^\alpha \) under \( x \sim \psi_\alpha(x) \) for \( x \in \partial e_n^\alpha \).

Following the notational conventions in [10] for CW complexes, if \( C \) denotes the set of cells and attaching maps then \( |C| = X^N \) denotes the resulting topological space.

The following is our definition of M-cellwise continuous.

**DEFINITION 2.** Suppose that \( C_d \) and \( C_r \) are M complexes and \( h : |C_d| \to |C_r| \) is a (not necessarily continuous) map. (The notation is chosen because \( C_d \) is the cell complex on the domain of \( h \) and \( C_r \) is the cell complex on the range of \( h \).) If \( h \) restricted to any M-cell of \( C_d \) is continuous and the image of any M-cell of \( C_d \) under \( h \) is an M-cell of \( C_r \) then we say \( h \) is M-cellwise continuous. For us \( h \) will be a bijection.

So an M-cellwise continuous map is essentially a piecewise continuous map where the regions of continuity are M-cells. For the rest of this paper we will use the term cell for short, instead of M-cell.

Consider again the case of our 3-dimensional \( M \). Define \( N : \Sigma \to \mathbb{N} \) by

\[
N(x) = \min\{ n > 0 : h^n(x) \in \text{int} \Sigma \}.
\]

As is proven in [1] that for a generic global transverse disk \( \Sigma \), we can choose \( C_d \) satisfying the following properties:

1. The union of the two cells of \( C_d \) is the set \( \{ x \in \Sigma : N(x) = 1 \} \).
2. The union of the one cells in \( \text{int} \Sigma \) is the set \( \{ x \in \Sigma : N(x) = 2 \} \).
3. The union of the zero cells in \( \text{int} \Sigma \) is the set \( \{ x \in \Sigma : N(x) = 3 \} \).

In this case, the one cells of \( C_d \) on \( \partial \Sigma \) are the images under \( h \) of the one cells in \( \text{int} \Sigma \) and the zero cells in \( \partial \Sigma \) are the images of the zero cells in \( \text{int} \Sigma \). It is clear then that \( C_d \) and \( C_r \) agree on \( \partial \Sigma \).

It is sometimes required for \( \Sigma \) to obey further restrictions. It is possible to perturb any global cross section \( \Sigma \) by an arbitrarily small amount so that every intersection between \( h(\partial \Sigma) \) and \( h^{-1}(\partial \Sigma) \) occurs at the intersection of 1-cells and is transverse. Such a perturbation exists by standard arguments regarding transversality. Assuming \( \Sigma \) is so perturbed, it is possible to reduce the M-complexes \( C_d \) and \( C_r \) (by subdividing some of the cells) to get new complexes \( C'_d \) and \( C'_r \) such that
every cell of $C'_d$ and $C'_r$ is simply connected and so that $h(A) \cap B$ is path connected for every pair of cells $A, B$ in $C'_d$. For such a $\Sigma$, we say $\Sigma$ and the complexes $C'_d$ and $C'_r$ are in topologically reduced form. Observe that if $\Sigma$ and the complexes $C'_d$ and $C'_r$ are in topologically reduced form then $C'_d$ no longer satisfies properties (1) through (3) above.

3. Symbolic Dynamics in Homology and the Fundamental Group

For our 2-dimensional global transverse disk $\Sigma$, the set of top dimensional cells of $C_d$ is $\{e_1^2, \ldots, e^K_2\}$. For every cell $e^n_\alpha$ with $n \in \{1, 2\}$, there is a unique $e^2_k$ such that $e^n_\alpha \subset e^2_k$ and $h$ restricted to $e^n_\alpha \cup e^2_k$ is continuous. This is proven in [1], although the reason is simple. For a point $a \in e^n_\alpha$, $h(a) \in \partial \Sigma$ and for all $x$ in a 2-cell near $a$, $h(x)$ is near $h(a)$. See Figure 1. For each 2-dimensional cell $e^2_k$, let $A_k$ denote the union of $e^2_k$ with all lower dimensional cells having this property. Equivalently, $A_k$ is the largest set containing $e^2_k$ such that $h$ restricted to $A_k$ is continuous. So the discontinuity set of $h$ is the boundaries of the regions $A_k$. Clearly, $\Sigma$ is the disjoint union

$$\Sigma = \bigcup_{k=1}^{K} A_k.$$
If the cross section $\Sigma$ and complexes $C'_0$ and $C'r'$ are in reduced form it is still possible to write $\Sigma$ as the union of nonintersecting regions $A_1, \ldots, A_K$ where each $A_k$ is the union of a 2-cell with some lower dimensional cells that are contained in its closure. Moreover, the return map $h$ will be continuous on each $A_k$. The difference is that the discontinuity set of $h$ will be contained in the boundaries of the $A_k's$, but may not be equal to the union of the boundaries of the $A_k$.

Define $A : \Sigma \to \{A_1, \ldots, A_k\}$ by letting $A(X)$ be the unique region that contains $x$. Let

$$a_n(x) = A(h^n(x)).$$

and let

$$\pi(x) = \{\ldots, a_{-1}(x), a_0(x), a_1(x), \ldots\}$$

Let $\Sigma_K$ be the space of all bi-infinite sequences of the symbols $\{A_1, \ldots, A_k\}$, endowed with the usual cylinder topology making $\Sigma_K$ a Cantor set. Then $\pi$ is a function from $\Sigma$ to $\Sigma_K$ that takes a point $x$ in $\Sigma$ to the itinerary of $x$. Define the usual shift map $\sigma : \Sigma_K \to \Sigma_K$ by the rule that the $i$th coordinate of $\sigma(\pi)$ is equal to the $(i+1)$th coordinate of $\pi$. Then $\pi(\Sigma)$ is a shift invariant subset of $\Sigma_K$.

The sequence $\pi(x)$ keeps account of the way $O(x)$ winds around the manifold $M$. Let $b$ be any point in $\Sigma$, and for every $x \in \Sigma$ let $b(x)$ be a path from $b$ to $x$. For each $A_k$, choose a point $x_k \in A_k$. For each $k = 1, \ldots, K$, define the loop $\gamma_k$ by

$$\gamma(A_k) = b(x_k) \cup \overrightarrow{h(x_k) h(b(x_k))},$$

where $\overrightarrow{xy}$ denotes the orbit segment from $x$ to $y$. Observe that the homotopy equivalence class of $\gamma(A_k)$ does not depend on the choice of the paths $b(x)$ or on the choice of $x_k \in A_k$. Let $[\gamma(A_k)]$ denote the equivalence class of $\gamma(A_k)$ in $\pi_1(b, M)$ and let $[\gamma(A_k)]$ denote the equivalence class of $\gamma(A_k)$ in $H_1(M; \mathbb{R})$. For a point $x \in M$, define

$$\alpha(x) = \gamma(a(x))$$

$$\alpha_n(x) = \gamma(a_n(x))$$

$$\pi(x) = \{\ldots, a_{-1}(x), a_0(x), a_1(x), \ldots\}$$

We consider $\pi(x)$ as a point in $\Sigma_K$, where $\Sigma_K$ is now the space of bi-infinite sequences of the $K$ symbols $\{\alpha(A_1), \ldots, \alpha(A_K)\}$.

The winding of an orbit the the homology of a manifold can be measured in a number of ways. We use ideas from [6] and track the homology of the orbit in the space of homology directions, $D_M = H_1(M; \mathbb{R})/(x \sim rx, r > 0)$, with the topology of a sphere together with a point representing the zero class. Let $p$ be the natural projection from $H_1(M; \mathbb{R})$ to $D_M$ which is continuous everywhere except at zero. Each point in $D_M$ represents a direction vector in the homology space for $M$. The flow gives rise to a subset $D_\varphi \subset D_M$ as follows. Let $R$ denote the set of all $x$ in $M$ for which there exists a sequence of points $x_n$ and real numbers $t_n$ such that $x_n \to x$, $\varphi(t_n, x_n) \to x_n$ and $t_n \to 0$ (all as $n \to \infty$), called the recurrent set of $\varphi$. Such a sequence $(x_n, t_n)$ is called a closing sequence at $x$. The set of nonwandering points of a flow $\varphi$, denoted by $NW(\varphi)$, is the set of all points $p \in M$ such that every neighborhood $U$ of $p$, there is a time $t > 1$ such that $\varphi(t, U) \cap U \neq \emptyset$. It is obvious that a point has a closing sequence if and only if it is a nonwandering point. For each pair $(x_n, t_n)$ in a given closing sequence, let $\gamma_n$ be a short path from $x_n$ to $x$, joined to the flowline from $x_n$ to $\varphi(t_n, x_n)$, and then joined to a short path from $\varphi(t_n, x_n)$ to $x$. For each closing sequence, $p[\gamma_n]$ is a sequence in $D_M$ and
hence has accumulation points. Each such accumulation point is called a homology direction, and, as in [6], we denote the space of all homology directions by \( \mathcal{D}_\varphi \).

Returning to our global cross section \( \Sigma \), for each point \( x \in \Sigma \) and \( n \in \mathbb{N} \), let

\[
d_n(x) = p([\alpha_0(x)] + \cdots + [\alpha_n(x)]).
\]

For a fixed \( x \in \Sigma \), we define the \( \omega \)-limit set of the sequence \( \{d_n(x)\}_{n \in \mathbb{N}} \) to be

\[
\omega_d(x) = \bigcap_{N \in \mathbb{N}} \bigcup_{n>N} d_n(x).
\]

set of points in this omega limit set are the homology directions for \( x \). We call the set \( D_\Sigma = \{\omega_d(x) : x \in \Sigma\} \) the collection of homology directions for \( \Sigma \). Clearly \( D_\Sigma \) is a compact nonempty subset of \( D_M \). Let \( NW(\varphi) \) denote the nonwandering set of \( \varphi \) and let \( D_{\Sigma,NW} = \{\omega_d(x) : x \in \Sigma \cap NW(\varphi)\} \) which we call the collection of nonwandering homology directions for \( \Sigma \). Observe that for each \( d \in D_\Sigma \) there is an obit segment beginning and ending in \( \Sigma \) that, when closed by a segment in \( \Sigma \), represents a class close to \( d \).

**THEOREM 1.** For a smooth flow on a manifold \( M \), \( D_\varphi = D_{\Sigma,NW} \).

*Proof.* Suppose \( d \in D_\varphi \). Then for some closing sequence \( (t_n, x_n) \), \( d \) can be approximated by a long flowline from \( x_n \) to \( \varphi(t_n, x_n) \). By pushing \( x \), \( x_n \) and \( \varphi(t_n, x_n) \) each a bounded amount along the flow until they are all in \( \Sigma \), one obtains a loop \( d_n(x) \). Since the amount that we push \( x \), \( x_n \) and \( \varphi(t_n, x_n) \) along the flow to get them in \( \Sigma \) is bounded above by the maximum first return time for \( \Sigma \), the homology class of \( d_n(x) \) can be made arbitrarily close to the class of \( \gamma_n \). Hence, \( d \in D_{\Sigma,NW} \).

Suppose \( d \in D_{\Sigma,NW} \). Then \( d \) can be approximated by loops generated from a closing sequence consisting of points in \( \Sigma \). Thus, \( d \in D_\varphi \). \( \square \)

Observe that \( D_\Sigma \) determines how orbits wind around the homology of \( M \) but, in contrast to the homological rotation vectors of Franks [7], it does not indicate how quickly the orbits go around these directions. This is the most that can be determined from the return map \( h : \Sigma \to \Sigma \) because \( h \) determines the flow up to topological equivalence and hence does not determine the velocity of orbits.

4. **Nielsen Equivalence of Orbits**

Let \( \gamma \) and \( \gamma' \) be periodic orbits in \( M \) and \( x \) and \( x' \) be points in \( \gamma \cap \Sigma \) and \( \gamma' \cap \Sigma \) respectively. For a periodic point \( y \in \Sigma \), let \( \text{per}(y) \) denote the period of \( y \) under the return map \( h : \Sigma \to \Sigma \). For a thorough treatment of Thurston-Neilson Theory, see [5] and [11]. The following are trivial.

(1) \( \gamma \) and \( \gamma' \) are Abelian Nielsen equivalent (they are homologous) if and only if

\[
\Sigma_{i=0}^{\text{per}(x)} [\alpha_i(x)] = \Sigma_{j=0}^{\text{per}(x')} [\alpha_j(x')].
\]

(2) \( \gamma \) and \( \gamma' \) are periodic Nielsen equivalent (they are freely homotopic) if

\[
\Sigma_{i=0}^{\text{per}(x)} [[\alpha_i(x)]] = \Sigma_{j=0}^{\text{per}(x')} [[\alpha_j(x')]].
\]
(3) Suppose that $\Sigma$ and the complexes $C_d'$ and $C_r'$ are in reduced form. Then $\gamma$ and $\gamma'$ are strongly Nielsen equivalent (they are isotopic) if

$$\overline{\gamma}(x) = \sigma^n(\overline{\gamma}(x'))$$

for some $n \in \mathbb{N}$.

It is an interesting question under what conditions is item (2) an if and only if. That is, under what conditions is it true that $\gamma$ and $\gamma'$ are periodic Nielsen equivalent (they are freely homotopic) if and only if

$$\sum_{i=0}^{\text{per}(x)}[[\alpha_i(x)]] = \sum_{j=0}^{\text{per}(x')}[[\alpha_j(x')]].$$

**References**


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