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Frequency Doubling of RF-Over-Fiber Signal Based on Mach Zehnder Modulator

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Frequency Doubling of RF-Over-Fiber Signal Based on Mach Zehnder Modulator

by

Mohamed Maafa

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Science in Electrical Engineering

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________________________________________________________________________

Mohamed Maafa

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Date
Dedication

I dedicate this work to my loving family who have supported me all the time...
Abstract

This thesis proposes and explores a new technique to double the frequency of a RF-over-fiber signal in a linear fashion. The technique is based on combining the output of a well-known, commercial device (the Mach Zehnder Modulator) in a new fashion that has not been previously explored. This new combination technique uses polarization diversity of the electric field and a time shift. Analysis is performed using Taylor series and Jacobi-Anger expansions, as well as numerical analysis.
Acknowledgments

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Chapter 1

Analog Optical Modulation Techniques

1.1 Introduction

The main characteristic of optical transmitters is that they convert electrical signals into their corresponding optical domain. Thus, it is important to understand the concepts of optical modulation and the properties of modulated signals. This chapter provides an introduction to the concepts of modulation techniques of light wave sources. These techniques are known as direct and external modulation techniques [1].

1.2 Direct Modulation

The simplest method of Electrical-to Optical (EO) conversion where the RF signal will encode into light is called direct modulation [2]. Direct modulation can be done by changing the laser intensity or frequency by modulating its pump. This section will briefly introduce the main strategy used for direct modulation.

1.2.1 Direct Intensity Modulation

The basic idea of this method shown in Figure 1.1 [3], where the output power of laser diode based on the bias current. As it can be seen, there are three values of input current, $I_{th}$, $I_{bais}$, and $I_{sat}$. $I_{th}$ is the the minimum value of input current that is required to get output power. $I_{bais}$ refers to the optimum value of input current that accommodate the entire peak to
peak variation in input Rf signal. $I_{sat}$ is the threshold of required input current.

First, there will be no output power (light emitted from laser diode) until the input current get to the threshold value $I_{th}$, after that the output power will increase linearly until reaches the saturation.

![Output Optical Power](image)

Figure 1.1: Output Optical Power of Laser vs Input Current

### 1.3 External Modulation

As briefly introduced above, the physical processes of modulation consists in transferring the data from the electrical to the optical domain to be transmitted. Even though, the direct modulation seems like a low-complexity and cost-effective method of generating intensity modulated optical signals, there are many impairments associated with this method, a few examples of these impairments are relative intensity noise (RIN) that occurs with high
frequencies, and phase noise which is factor to limit the distance over long fibers.

In order, to avoid these kind of negative effects external modulators are used. The next section briefly introduces types of external modulators that are commonly used in optical communication and their main features.

1.3.1 Single-Drive Mach Zehnder Modulator SD-MZM

One of the most commonly commercially optical modulator is Mach Zehnder modulator [4]. MZM is knows as an electro-optic modulator, where the change in phase is based on the change of the refractive index. MZMs have been constructed using many electro-optic materials such as LiNbO3, GaAs, Si and polymers [1].

The structure of single-drive MZM is shown in Figure 1.2, the optical output from laser is coupled into an input waveguide that is split into path 1 and path 2 initially, both electrodes are biased with voltages $V_{b1}$ and $V_{b2}$ then, the initial phases are $\phi_1$ and $\phi_2$ [4]. The relative phase between two arms can be controlled by applied voltages across one or both arms where the change in refractive index will correspond to a change in phase. As it can be observed in Figure 1.2 each waveguide path is surrounded by electrodes, these electrodes are connected to the modulated signal voltage and the DC bias voltage which are used to vary the phase in both arms. Finally, at the recombination point the two paths will have different phase modulations, this phase modulation is turned into an intensity modulation. Thus this kind of MZM called a single-drive MZM, because both arms get the same modulated voltage.
The output field from MZM can be represented as

\[ E_{\text{out}} = \frac{1}{2} E_c e^{j\omega_c t} (e^{j\theta_1(t)} + e^{j\theta_2(t)}) \]  

(1.1)

where \( \omega_c \) is the carrier frequency, \( E_c \) is the magnitude of the carrier, \( \theta_1 \) and \( \theta_2 \) are the phase modulation of the both arm 1 and arm 2 respectively [4].
As it is known the optical output at the output of MZM equal to \( P_{MZM} = |E(t)_{out}|^2 \) [7].

Thus the output optical power from SD-MZM can be written as

\[
P_{MZM}(t) = \frac{P_{in}}{2} \left[ 1 + \cos \left( \frac{\pi (V_b + V_{RF})}{V_{\pi}} \right) \right], \tag{1.2}
\]

where \( P_{in} \) is the input optical power, \( V_b \) is the bias voltage, \( V_{\pi} \) is the half wave voltage of the modulator, and \( V_{RF} \) is the RF voltage [6].

For the best Mach-Zehnder modulator performance, it is really important to understand the effect of bias voltage on the output signal, so that the modulator phase must be biased to an optimal operating point for a given application. Figure 1.3 shows the typical transfer function between the output optical power and driving voltage applied to the electrode. This is the characteristic curve of a Mach-Zehnder intensity modulator [1].

As shown Figure 1.3, there are three important points on the curve:

- Quadrature Point (QP): which is located between the peak and null, this is the center of the linear zone where the modulator offers the maximum linearity whereas between branches there is a voltage difference equal to \( \frac{V_{\pi}}{2} \).

- Minimum Transmission Point (mTP): is located at the bottom and \( V_{\pi} \) voltage is required between the branches.

- Maximum Transmission Point (MTP): is located at the peak whereas between branches the relative phase is equal to zero.

Where \( V_{\pi} \) is the half wave voltage of the modulator, also as the voltage needed to go from a maximum to a minimum of amplitude in the modulated signal.
1.3.2 Dual-Drive Mach Zehnder Modulator DD-MZM

It is similar to SD-MZM, the only difference is that there is separate modulated signals voltage and DC bias voltage one to upper arms and the other two lower arm as seen in Figure 1.4.
This structure allows MZM to generate different output signals such as Single-Side-Band (SSB) that is used in different applications such as to overcome the effects of chromatic dispersion effect.

1.3.3 Electroabsorption Modulators

Electro-absorption modulator (EAM) relies on the absorption of material when an external electric field is applied. The intensity of output optical signal will be modulated by the light absorption of the material [6].
1.4 Frequency Doubling

There are different ways to achieve efficiency frequency doubling such as nonlinear optics or using optical modulator. In this section we will briefly talk about both methods.

1.4.1 Nonlinear Optics

Nonlinear optical phenomena occur when we apply an optical field and the response of the material would be nonlinear such as frequency doubling "Second-Harmonic Generation (SHG)". This is a nonlinear optical process that occurs when photons with the same frequency interact with a nonlinear material and generate new photons with twice the frequency, so that the atomic response that balances quadratically at frequency $2\omega$ with the amplitude of the applied optical field [8].

1.4.2 Optical Modulators

Mach-Zehnder modulator has been widely employed in the field of optical communication, due to the advantages of stability and low phase noise. In addition, the characteristics of bias-dependency of the MZM allows it to perform advanced electro-optic signal processing for instances of frequency doubling [6].

In order to obtain frequency doubling, MZ modulator should be biased at either peak or the null transmission point. Further study will be discussed in following chapters which will show the thesis contribution by take using two MZMs, $MZM_1$ is biased at peak and the second MZM operates at null and then sum the output power.
Chapter 2

Known Single-Drive MZM Output Optical Power Behavior

2.1 Introduction

This chapter highlights the operating principles of SD-MZM. Our contribution in this thesis is focussed on the performance of SD-MZM. It is therefore important to understand the operating principles as well as specific bias points of SD-MZM.

2.2 Principle of Operation

The output optical power from a SD-MZM can be written as

\[ P_{MZM}(t) = \frac{P_{in}}{2} \left[ 1 + \cos \left( \frac{\pi (V_b + V_{RF})}{V_\pi} \right) \right], \]  

(2.1)

where \( P_{in} \) is the input optical power, \( V_b \) is the bias voltage, and \( V_{RF} \) is the RF voltage.

We can simplify the above expression by making the following substitutions,

\[ a = \frac{\pi V_{RF}}{V_\pi}, \]  

(2.2)

\[ b = \frac{\pi V_b}{V_\pi}. \]  

(2.3)

The resulting equation for \( P_{MZM}(t) \) is

\[ P_{MZM}(t) = \frac{P_{in}}{2} \left[ 1 + \cos (a + b) \right]. \]  

(2.4)
To simplify even further, we can substitute the value of

\[ V_{RF} = A \sin(2\pi\Omega_{RF}t) \]  

(2.5)

as shown in Figure 2.1.

![Figure 2.1: Temporal RF signal vs time.](image)

Using the expression given in Equation 2.2 and substituting in our complete expression for \( V_{RF} \) resulting in,

\[ a = \pi A \sin(2\pi\Omega_{RF}t) \]

\( V_{\pi} \).

(2.6)

We can also define

\[ \beta = \frac{\pi A}{V_{\pi}}, \]

(2.7)

where \( \beta \) is the modulation index.

Combine Equations 2.5 and 2.6, yields

\[ a = \beta \sin(2\pi\Omega_{RF}t). \]

(2.8)
Furthermore, the argument of the sine wave can be simplified as

$$\delta = 2\pi \Omega_{RF} t, \quad (2.9)$$

then,

$$a = \beta \sin \delta. \quad (2.10)$$

Given the expressions defined in Equations 2.5 through 2.9, the general equation for $P_{MZM}(t)$ can be given as

$$P_{MZM}(t) = \frac{P_{in}}{2} \left[ 1 + \cos (\beta \sin \delta + b) \right]. \quad (2.11)$$

### 2.3 Operation at Quadrature Bias Point

To configure the SD-MZM at quadrature, we will set the bias voltage of SD-MZM at $V_b = \frac{3V_c}{2}$.

We can then substitute this value into equation 2.10 as follows

$$b = \frac{\pi \frac{3V_c}{2}}{V_c}, \quad (2.12)$$

$$b = \frac{3\pi}{2}. \quad (2.13)$$

Then the general equation of will be

$$P_{QMZM}(t) = \frac{P_{in}}{2} \left[ 1 + \cos \left( \beta \sin(\delta) + \frac{3\pi}{2} \right) \right]. \quad (2.14)$$

where $P_{QMZM}(t)$ is the output optical power of SD-MZM bias at quadrature point.

By using the trigonometric identity:

$$\cos(x + \frac{3\pi}{2}) = \sin(x), \quad (2.15)$$
the general equation of \( P_{QMZM}(t) \) can be written as

\[
P_{QMZM}(t) = \frac{P_{in}}{2} [1 + \sin(\beta \sin(\delta))].
\]  
(2.16)

This equation is plotted in Fig 2.2.

![Figure 2.2: Temporal Optical Power Out of MZM in case of Quadrature Bias Point vs Time.](image)

### 2.4 Operation at Peak Bias Point

To configure the SD-MZM at peak bias point, we will set the bias voltage, \( V_b \), equal to zero

\( V_b = 0: \)

\[
b = \frac{\pi V_b}{V_\pi},
\]  
(2.17)

\[
b = 0.
\]  
(2.18)
By plugging in the value of $b$ in $P_{MZM}(t)$, we can write the equation of $P_{MZM}(t)$ as

$$P_{MZM}(t) = \frac{P_{in}}{2} \left[ 1 + \cos (\beta \sin \delta + 0) \right]. \quad (2.19)$$

Thus, the general equation of output optical power from the MZM at peak bias voltage can be written as

$$P_{MZMpeak}(t) = \frac{P_{in}}{2} \left[ 1 + \cos (\beta \sin(2\pi\Omega_{RF}t)) \right]. \quad (2.20)$$

This equation is plotted in Figure 2.3. As can be seen the output signal has been doubled, however, the wave shape is not sinusoidal.

![Figure 2.3: Temporal Output Optical Power at Peak without Shift vs Time.](image)
2.5 Peak Bias Point vs Pure Sine Wave driven at $2\Omega$

Figure 2.4 shows the output power of MZM operates at peak and the pure sine wave vs time. It can be observed that the output of MZM is not close to the shape of pure sine wave.

![Figure 2.4: Temporal Output Optical Power at Peak without Shift & Pure Sine Wave vs Time.](image)

Figure 2.5, shows results of error where the output optical power of MZM subtract from pure sine wave. As it can seen the error is around 60% of normalized $P_{peak}$ input.

![Figure 2.5: Optical Signal Error between Peak without Shift & Pure Sine Wave vs Time.](image)
2.6 Operation at Null Bias Point

To activate the SD-MZM at null bias point, we set the bias voltage $V_b = V_\pi$:

$$b = \frac{\pi V_b}{V_\pi},$$

(2.21)

$$b = \pi.$$  \hspace{1cm} (2.22)

By plugging in the value of $b$ shown in Equation 2.21 in $P_{MZM}(t)$, we obtain

$$P_{MZM}(t) = \frac{P_{in}}{2} \left[ 1 + \cos (\beta \sin \delta + \pi) \right].$$

(2.23)

Using the trigonometric identity $\cos(x + \pi) = -\cos(x)$, we can now write the general equation of output optical power from MZM at null bias point without shift as

$$P_{MZMnull}(t) = \frac{P_{in}}{2} \left[ 1 - \cos (\beta \sin(2\pi \Omega_{RF} t)) \right].$$

(2.24)

This equation is plotted in Figure 2.6. As can be seen in this case as well the output signal has been doubled with different form compare to peak case.

![Figure 2.6: Temporal Output Optical Power at Null without Shift vs Time.](image-url)
2.7 Null Bias Point vs Pure Sine Wave driven at $2\Omega$

Figure 2.7 shows the output power of MZM operates at null and the pure sine wave vs time. It can be observed that the output of MZM is not close to the shape of pure sine wave. Figure 2.8, shows results of error where the output optical power of MZM subtract from pure sine wave. As it can seen it is similar to the error of MZMZ at peak where the error is around 60% of normalized $P_{\text{peak}}$ input.

Figure 2.7: Temporal Output Optical Power at Peak without Shift & Pure Sine Wave vs Time.

Figure 2.8: Optical Signal Error between Peak without Shift & Pure Sine Wave vs Time.
2.8 Total Output Power

As derived in section 2.4 the general equation for output optical power from the MZM at peak bias point without time shift is:

\[ P_{MZM\text{peak}}(t) = \frac{P_{in}}{2} \left[ 1 + \cos(\beta \sin(\delta)) \right]. \tag{2.25} \]

In addition, the general equation for output optical power from the MZM at null bias point without time shift is:

\[ P_{MZM\text{null}}(t) = \frac{P_{in}}{2} \left[ 1 - \cos(\beta \sin(\delta)) \right]. \tag{2.26} \]

Now by combine both of them by using polarization beam combiner (PBC), the output optical power can be defined as

\[ P_{PBC}(t) = P_{MZM\text{peak}}(t) + P_{MZM\text{null}}(t). \tag{2.27} \]

Substituting in expressions given in Equations 2.24 and 2.25 results in:

\[ P_{PBC}(t) = \frac{P_{in}}{2} \left[ 1 + \cos(\beta \sin(\delta)) \right] + \frac{P_{in}}{2} \left[ 1 - \cos(\beta \sin(\delta)) \right], \tag{2.28} \]

\[ P_{PBC}(t) = \frac{P_{in}}{2} + \frac{P_{in}}{2} + \cos(\beta \sin(\delta)) - \cos(\beta \sin(\delta)), \tag{2.29} \]

\[ P_{PBC}(t) = P_{in}. \tag{2.30} \]
The output result is shown in Figure 2.9.

![Figure 2.9: Temporal RF Signal Output of Photodiode vs Time.](image)

In conclusion, the output signal in both cases peak and null the frequency has been doubled, however the wave shape is not sinusoidal.
Chapter 3

Frequency Doubling Technique

3.1 Introduction

This Chapter illustrates the contribution of this thesis which is frequency doubling technique using time shift principle. To achieve frequency doubling, the output power from SD-MZM operating at peak bias voltage will have a shift in time. We will then combine it with the output of a SD-MZM operating at null bias voltage without time shifting by polarization beam combiner (PBC), then pass the output signal through photodiode and finally get the output signal.

3.2 Proposed Frequency Doubling Technique Scheme

The frequency doubling technique diagram is illustrated in Figure 3.1. A continuous wave (CW) beam is divided into two equal power for each sub-MZM by using coupler 50/50. RF resource, at $\Omega$ is splitted, then applied to drive the two sub-MZMs. A phase shift (PS) is employed to impose the phase shift to the on RF signals into MZM biased at peak. The PCs are used to change polarization of optical signals signals to match the TE mode of the planer waveguide of the MZM, as a result we obtain a maximum output power. The polarization beam combiner (PBC) was used to combine two orthogonal polarizations from MZM1 and MZM2 into a single fiber, so that the two input signal polarizations orthogonal
to each other, as a result we can remove the coherence and the interference.

At the end the output optical power from The polarization beam combiner send through photodiode (PD) to convert into the desired RF electrical signal.

Figure 3.1: Schematic of Frequency Doubling Technique
3.3 Operation at Peak Bias Point with Time Shift

To derive the output optical power of an SD-MZM at peak bias point with time shift, we will shift time as \( t = t - t_s \), where \( t_s \) is the shifting value:

\[
P_{MZM_{peak}}(t) = \frac{P_{in}}{2} \left[ 1 + \cos(\beta \sin(2\pi \Omega_{RF}(t - t_s))) \right],
\]

(3.1)

\[
P_{MZM_{peak}}(t) = \frac{P_{in}}{2} \left[ 1 + \cos(\beta \sin(2\pi \Omega_{RF}t - 2\pi \Omega_{RF}t_s)) \right].
\]

(3.2)

Using the phase shift

\[
\Delta = 2\pi \Omega_{RF}t_s,
\]

(3.3)

\[
P_{MZM_{peak}}(t) = \frac{P_{in}}{2} \left[ 1 + \cos(\beta \sin(\delta - \Delta)) \right].
\]

(3.4)

We now consider the trigonometric identity:

\[
\sin(x - y) = \sin(x) \cos(y) - \cos(x) \sin(y).
\]

(3.5)

Substituting in Equation 3.5 the output optical power of an SD-MZM at peak bias point with time shift will then be

\[
P_{MZM_{peak}}(t) = \frac{P_{in}}{2} \{ 1 + \cos[\beta \sin(\delta) \cos(\Delta) - \beta \cos(\delta) \sin(\Delta)] \}.
\]

(3.6)

We can determine the impact of phase shift on the output optical power by adjusting the value \( \Delta \).

**Taking** \( \Delta = 0 \)

\[
P_{MZM_{peak}}(t) = \frac{P_{in}}{2} \{ 1 + \cos[\beta \sin(\delta) \cos(0) - \beta \cos(\delta) \sin(0)] \}.
\]

(3.7)
We know that \( \cos(0) = 1, \sin(0) = 0 \), so the equation can be simplified to:

\[
P_{\text{MZM peak}}(t) = \frac{P_{\text{in}}}{2} \left[ 1 + \cos[\beta \sin(2\pi \Omega_{RF} t)] \right]. \tag{3.8}
\]

Figure 3.2 shows the optical output of an SD-MZM with zero phase shift plotted against time. As it can be observed, it looks exactly like the general case of activation at peak bias point without shift as expected.

![Figure 3.2: Temporal Output Optical Power at Peak without Shift \( \Delta = 0 \) vs Time](image)

**Taking \( \Delta = \pi \)**

Now we can analyze the impact on output optical power when \( \Delta = \pi \)

\[
P_{\text{MZM peak}}(t) = \frac{P_{\text{in}}}{2} \left[ 1 + \cos[\beta \sin(\delta) \cos(\pi) - \beta \cos(\delta) \sin(\pi)] \right]. \tag{3.9}
\]

We know that \( \cos(\pi) = -1, \sin(\pi) = 0 \), so the equation can be simplified to

\[
P_{\text{MZM peak}}(t) = \frac{P_{\text{in}}}{2} \left[ 1 + \cos[-\beta \sin(\delta)] \right]. \tag{3.10}
\]
By using the trigonometric identity \( \cos(-x) = \cos(x) \), the output optical power can now be represented as
\[
P_{MZMpeak}(t) = \frac{P_{in}}{2} \{1 + \cos[\beta \sin(2\pi \Omega_{RF} t)]\}. \tag{3.11}
\]

By comparing equations 3.8 and 3.11, we see that the output optical power is identical for phase shift values of zero and \( \pi \) as shown in Figure 3.2.

**Taking** \( \Delta = \frac{\pi}{2} \)

Now we will illustrate the thesis contribution by analyze the impact on output optical power when \( \Delta = \frac{\pi}{2} \)
\[
P_{MZMpeak}(t) = \frac{P_{in}}{2} \{1 + \cos[\beta \cos(\delta)] \cos(\frac{\pi}{2}) - \beta \cos(\delta) \sin(\frac{\pi}{2})]\}. \tag{3.12}
\]

By the definition of the \( \cos(\frac{\pi}{2}) = 0 \), similarly \( \sin(\frac{\pi}{2}) = 1 \), then the equation can be simplified to:
\[
P_{MZMpeak}(t) = \frac{P_{in}}{2} \{1 + \cos[-\beta \cos(\delta)]\}. \tag{3.13}
\]

By using trigonometric identity \( \cos(-x) = \cos(x) \), the output optical power can now be represented as
\[
P_{MZMpeak}(t) = \frac{P_{in}}{2} \{1 + \cos[\beta \cos(2\pi \Omega_{RF} t)]\}. \tag{3.14}
\]

This power profile is plotted in Figure 3.3.
3.4 Total Output Optical Power

To derive the output optical power of a PBC, we can sum the output optical power from the MZM at peak bias point with $\Delta = \frac{\pi}{2}$

$$P_{MZMpeak}(t) = \frac{P_{in}}{2} [1 + \cos (\beta \cos (2\pi \Omega_{RF} t))]$$

(3.15)

with the output optical power from the MZM at null bias point with $\Delta = 0$

$$P_{MZMnull}(t) = \frac{P_{in}}{2} [1 - \cos (\beta \sin (2\pi \Omega_{RF} t))]$$

(3.16)

This output optical power of PBC can thus be represented as:

$$P_{PBC}(t) = P_{MZMpeak}(t) + P_{MZMnull}(t)$$

(3.17)
Substituting Equation 3.15 and 3.16 into the general equation shown in 3.17 results in

\[
P_{PB}(t) = \frac{P_{in}}{2} + \frac{P_{in}}{2} \cos (\beta \cos \delta) - \frac{P_{in}}{2} \cos (\beta \sin \delta), \quad (3.18)
\]

\[
P_{PB}(t) = P_{in} + \frac{P_{in}}{2} \cos (\beta \cos \delta) - \frac{P_{in}}{2} \cos (\beta \sin \delta), \quad (3.19)
\]

\[
P_{PB}(t) = P_{in} + \frac{P_{in}}{2} [\cos (\beta \cos \delta) - \cos (\beta \sin \delta)]. \quad (3.20)
\]

Consider the \( \cos(x) \) Taylor series

\[
\cos(x) = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \frac{1}{8!} x^8. \quad (3.21)
\]

Using

\[
x = \beta \cos(\delta),
\]

we obtain

\[
\cos(\beta \cos(\delta)) = 1 - \frac{1}{2!} \left[\beta \cos(\delta)\right]^2 + \frac{1}{4!} \left[\beta \cos(\delta)\right]^4 - \frac{1}{6!} \left[\beta \cos(\delta)\right]^6 + \frac{1}{8!} \left[\beta \cos(\delta)\right]^8. \quad (3.22)
\]

Using

\[
x = \beta \sin(\delta),
\]

we obtain

\[
\cos(\beta \sin(\delta)) = 1 - \frac{1}{2!} \left[\beta \sin(\delta)\right]^2 + \frac{1}{4!} \left[\beta \sin(\delta)\right]^4 - \frac{1}{6!} \left[\beta \sin(\delta)\right]^6 + \frac{1}{8!} \left[\beta \sin(\delta)\right]^8. \quad (3.23)
\]

Now, we will derive the difference

\[
\cos (\beta \cos(\delta)) - \cos (\beta \sin(\delta)) = 1 - \frac{1}{2!} \left[\beta \cos(\delta)\right]^2 + \frac{1}{4!} \left[\beta \cos(\delta)\right]^4 - \frac{1}{6!} \left[\beta \cos(\delta)\right]^6 + \frac{1}{8!} \left[\beta \cos(\delta)\right]^8 \\
- \left[ 1 - \frac{1}{2!} \left[\beta \sin(\delta)\right]^2 + \frac{1}{4!} \left[\beta \sin(\delta)\right]^4 - \frac{1}{6!} \left[\beta \sin(\delta)\right]^6 + \frac{1}{8!} \left[\beta \sin(\delta)\right]^8 \right]. \quad (3.24)
\]
\[
\cos (\beta \cos(\delta)) - \cos (\beta \sin(\delta)) = 1 - 1 - \frac{1}{2!} \left[ \beta \cos(\delta) \right]^2 + \frac{1}{2!} \left[ \beta \sin(\delta) \right]^2 \\
+ \frac{1}{4!} \left[ \beta \cos(\delta) \right]^4 - \frac{1}{4!} \left[ \beta \sin(\delta) \right]^4 - \frac{1}{6!} \left[ \beta \cos(\delta) \right]^6 + \frac{1}{6!} \left[ \beta \sin(\delta) \right]^6 + \frac{1}{8!} \left[ \beta \cos(\delta) \right]^8 - \frac{1}{8!} \left[ \beta \sin(\delta) \right]^8.
\]
(3.27)

We simplify the first term as follows

\[
- \frac{1}{2!} \left[ \beta \cos(\delta) \right]^2 + \frac{1}{2!} \left[ \beta \sin(\delta) \right]^2 = - \frac{\beta^2}{2!} \left[ \cos^2(\delta) - \sin^2(\delta) \right],
\]
(3.28)

by using the identity

\[
\cos^2(x) - \sin^2(x) = \cos(2x).
\]
(3.29)

yields

\[
- \frac{1}{2!} \left[ \beta \cos(\delta) \right]^2 + \frac{1}{2!} \left[ \beta \sin(\delta) \right]^2 = - \frac{\beta^2}{2!} \left[ \cos(2\delta) \right].
\]
(3.30)

Then, we simplify the second term as follows

\[
\frac{1}{4!} \left[ \beta \cos(\delta) \right]^4 - \frac{1}{4!} \left[ \beta \sin(\delta) \right]^4 = \frac{\beta^4}{4!} \left[ \cos^4(\delta) - \sin^4(\delta) \right],
\]
(3.31)

\[
\frac{1}{4!} \left[ \beta \cos(\delta) \right]^4 - \frac{1}{4!} \left[ \beta \sin(\delta) \right]^4 = \frac{\beta^4}{4!} \left[ \cos^2(\delta) - \sin^2(\delta) \right] \left[ \cos^2(\delta) + \sin^2(\delta) \right],
\]
(3.32)

\[
\frac{1}{4!} \left[ \beta \cos(\delta) \right]^4 - \frac{1}{4!} \left[ \beta \sin(\delta) \right]^4 = \frac{\beta^4}{4!} \left[ \cos(2\delta) \right] \left[ 1 \right],
\]
(3.33)

\[
\frac{1}{4!} \left[ \beta \cos(\delta) \right]^4 - \frac{1}{4!} \left[ \beta \sin(\delta) \right]^4 = \frac{\beta^4}{4!} \cos(2\delta).
\]
(3.34)

Next, we simplify the third term as follows

\[
- \frac{1}{6!} \left[ \beta \cos(\delta) \right]^6 + \frac{1}{6!} \left[ \beta \sin(\delta) \right]^6 = - \frac{\beta^6}{6!} \left[ \cos^6(\delta) - \sin^6(\delta) \right],
\]
(3.35)

\[
- \frac{1}{6!} \left[ \beta \cos(\delta) \right]^6 + \frac{1}{6!} \left[ \beta \sin(\delta) \right]^6 = - \frac{\beta^6}{6!} \left[ \cos^2(\delta) \right]^3 - \left[ \sin^2(\delta) \right]^3.
\]
(3.36)
We use the identity
\[ x^3 - y^3 = (x - y)(x^2 + xy + y^2). \] (3.37)

\[ -\frac{1}{6!}[\beta \cos(\delta)]^6 + \frac{1}{6!}[\beta \sin(\delta)]^6 = -\beta^6 \left[ -\cos^2(\delta) - \sin^2(\delta) \right] \left[ \cos(\delta) + \cos^2(\delta) \sin^2(\delta) + \sin^4(\delta) \right], \] (3.38)

\[ -\frac{1}{6!}[\beta \cos(\delta)]^6 + \frac{1}{6!}[\beta \sin(\delta)]^6 = -\beta^6 \left[ \cos^2(\delta) + \sin^2(\delta) + \cos^2(\delta) \sin^2(\delta) \right]. \] (3.39)

By using the trigonometric identity,
\[ x^4 + y^4 = [x^2 + y^2]^2 - 2x^2y^2, \] (3.40)

we obtain
\[ -\frac{1}{6!}[\beta \cos(\delta)]^6 + \frac{1}{6!}[\beta \sin(\delta)]^6 = -\beta^6 \left[ -\cos(2\delta) \left[ \cos^2(\delta) + \sin^2(\delta) \right] - 2 \cos^2(\delta) \sin^2(\delta) \right. \]
\[ + \cos^2(\delta) \sin^2(\delta) \left], \] (3.41)

\[ -\frac{1}{6!}[\beta \cos(\delta)]^6 + \frac{1}{6!}[\beta \sin(\delta)]^6 = -\beta^6 \left[ \cos(2\delta) \left[ 1 \right]^2 - \cos^2(\delta) \sin^2(\delta) \right], \] (3.42)

\[ -\frac{1}{6!}[\beta \cos(\delta)]^6 + \frac{1}{6!}[\beta \sin(\delta)]^6 = -\beta^6 \left[ \cos(2\delta) \left[ 1 - \cos^2(\delta) \sin^2(\delta) \right] \right], \] (3.43)

by using the identity
\[ \cos^2(\alpha) \sin^2(\alpha) = \frac{1 - \cos(4\alpha)}{8}, \] (3.44)

\[ -\frac{1}{6!}[\beta \cos(\delta)]^6 + \frac{1}{6!}[\beta \sin(\delta)]^6 = -\beta^6 \cos(2\delta) \left[ 1 - \frac{1 - \cos(4\delta)}{8} \right]. \] (3.45)
Finally, simplify forth term as follows

$$\frac{1}{8!} [\beta \cos(\delta)]^8 - \frac{1}{8!} [\beta \sin(\delta)]^8 = \frac{\beta^8}{8!} \left[ \cos^8(\delta) - \sin^8(\delta) \right].$$  (3.46)

$$\frac{1}{8!} [\beta \cos(\delta)]^8 - \frac{1}{8!} [\beta \sin(\delta)]^8 = \frac{\beta^8}{8!} \left[ \cos^4(\delta) - \sin^4(\delta) \right] \left[ \cos^4(\delta) + \sin^4(\delta) \right].$$  (3.47)

From simplify of the second term, we know that

$$\cos^4(\delta) - \sin^4(\delta) = \cos(2\delta),$$  (3.48)

$$\frac{1}{8!} [\beta \cos(\delta)]^8 - \frac{1}{8!} [\beta \sin(\delta)]^8 = \frac{\beta^8}{8!} \cos(2\delta) \left[ \cos^4(\delta) + \sin^4(\delta) \right].$$  (3.49)

By using identity

$$\cos^4(\alpha) + \sin^4(\alpha) = \frac{3}{4} + \frac{1}{4} \cos(4\alpha),$$  (3.50)

$$\frac{1}{8!} [\beta \cos(\delta)]^8 - \frac{1}{8!} [\beta \sin(\delta)]^8 = \frac{\beta^8}{8!} \cos(2\delta) \left[ \frac{3}{4} + \frac{1}{4} \cos(4\delta) \right].$$  (3.51)

Now, by substituting all simplifications in Eq.(66), we will get,

$$\left[ \cos (\beta \cos(\delta)) \right] - \left[ \cos (\beta \sin(\delta)) \right] = 0 - \frac{\beta^2}{2!} \cos(2\delta) + \frac{\beta^4}{4!} \cos(2\delta)$$

$$- \frac{\beta^6}{6!} \cos(2\delta) \left[ 1 - \frac{1 - \cos(4\delta)}{8} \right] + \frac{\beta^8}{8!} \cos(2\delta) \left[ \frac{3}{4} + \frac{1}{4} \cos(4\delta) \right].$$  (3.52)

Now, by plugging in general equation of output optical power of PBC with shift,

$$P_{PBC}(t) = P_{in} + \frac{P_{in}}{2} \left[ \left[ \cos (\beta \cos(\delta)) \right] - \left[ \cos (\beta \sin(\delta)) \right] \right],$$  (3.53)

$$P_{PBC}(t) = P_{in} + \frac{P_{in}}{2} \left[ - \frac{\beta^2}{2!} \cos(2\delta) + \frac{\beta^4}{4!} \cos(2\delta) - \frac{\beta^6}{6!} \cos(2\delta) \left[ 1 - \frac{1 - \cos(4\delta)}{8} \right] \right.$$

$$+ \frac{\beta^8}{8!} \cos(2\delta) \left[ \frac{3}{4} + \frac{1}{4} \cos(4\delta) \right].$$  (3.54)
By taking \( \cos(2\delta) \) as common factor, the general output optical power of PBC with shift can be written as

\[
P_{PBC}(t) = P_{in} + \frac{P_{in} \cos(2\delta)}{2} \left[ -\frac{\beta^2}{2!} + \frac{\beta^4}{4!} - \frac{\beta^6}{6!} \left[ 1 - \frac{1 - \cos(4\delta)}{8} \right] + \frac{\beta^8}{8!} \left[ \frac{3}{4} + \frac{1}{4} \cos(4\delta) \right] \right].
\]

(3.55)

More simply so that each spectral component is clearly displayed as follows

\[
P_{PBC}(t) = P_{in} + \frac{P_{in} \cos(2\delta)}{2} \left[ -\frac{\beta^2}{2!} + \frac{\beta^4}{4!} - \frac{\beta^6}{8 \times 6!} + \frac{\beta^6 \cos(4\delta)}{8 \times 6!} + \frac{3\beta^8}{4 \times 8!} + \frac{\beta^8 \cos(4\delta)}{4 \times 8!} \right].
\]

(3.56)

\[
P_{PBC}(t) = P_{in} + \frac{\beta^2 P_{in} \cos(2\delta)}{2 \times 2!} + \frac{\beta^4 P_{in} \cos(2\delta)}{2 \times 4!} - \frac{\beta^6 P_{in} \cos(2\delta)}{2 \times 6!} + \frac{\beta^6 P_{in} \cos(2\delta)}{2 \times 8 \times 6!} + \frac{\beta^8 P_{in} \cos(2\delta) \cos(4\delta)}{2 \times 8 \times 6!}
+ \frac{3\beta^8 P_{in} \cos(2\delta)}{2 \times 4 \times 8!} + \frac{\beta^8 P_{in} \cos(2\delta) \cos(4\delta)}{2 \times 4 \times 8!}.
\]

(3.57)

By using the following formulas:

\[
\cos(2\theta) \cos(4\theta) = \frac{1}{2} [\cos(2\theta - 4\theta) + \cos(2\theta + 4\theta)],
\]

(3.58)

\[
\cos(2\theta) \cos(4\theta) = \frac{1}{2} [\cos(-2\theta) + \cos(6\theta)].
\]

(3.59)

Because the function \( \cos(x) \) is even

\[
\cos(-\theta) = \cos(\theta),
\]

(3.60)

and therefore

\[
\cos(2\theta) \cos(4\theta) = \frac{1}{2} [\cos(2\theta) + \cos(6\theta)].
\]

(3.61)
More simply,

\[
P_{PBC}(t) = P_{\text{in}} + \frac{\beta^2 P_{\text{in}} \cos(2\delta)}{2 \times 2!} + \frac{\beta^4 P_{\text{in}} \cos(2\delta)}{2 \times 4!} - \frac{\beta^6 P_{\text{in}} \cos(2\delta)}{2 \times 6!} + \frac{\beta^8 P_{\text{in}} \cos(2\delta)}{2 \times 8 \times 6!} + \frac{\beta^6 P_{\text{in}} \cos(6\delta)}{2 \times 2 \times 8 \times 6!} + \frac{3\beta^8 P_{\text{in}} \cos(2\delta)}{2 \times 4 \times 8!} + \frac{\beta^8 P_{\text{in}} \cos(6\delta)}{2 \times 2 \times 4 \times 8!}.
\] (3.62)

The final expression can be written as

\[
P_{PBC}(t) = P_{\text{in}} + \cos(2\delta) \left[ \frac{\beta^2 P_{\text{in}}}{2 \times 2!} + \frac{\beta^4 P_{\text{in}}}{2 \times 4!} - \frac{\beta^6 P_{\text{in}}}{2 \times 6!} + \frac{\beta^8 P_{\text{in}}}{2 \times 2 \times 6 \times 6!} + \frac{3\beta^8 P_{\text{in}}}{2 \times 4 \times 8!} + \frac{\beta^8 P_{\text{in}}}{2 \times 2 \times 4 \times 8!} \right] + \cos(6\delta) \left[ \frac{\beta^6 P_{\text{in}}}{2 \times 2 \times 8 \times 6!} + \frac{\beta^8 P_{\text{in}}}{2 \times 2 \times 4 \times 8!} \right].
\] (3.63)

Figure 3.4 shows the input and output RF signals. We can see the output frequency is double the input frequency where the harmonic show at 2F and 6F, however the harmonic at 4F has been disappear, which is the intended object. This shows that we have doubled double the input frequency by changing the characteristic of SD-MZM.
3.5 Operation at Quadrature Bias Point with Doubling Input Frequency

There is another way to double RF frequency by doubling the input RF frequency to SD-MZM operates at quadrature bias point. As explained before the output optical power from SD-MZM can be written as

\[ P_{MZM}(t) = \frac{P_{in}}{2} \left[ 1 + \cos \left( \frac{\pi(V_b + V_{RF})}{V_\pi} \right) \right], \quad (3.64) \]
where $P_{in}$ is the input optical power, $V_b$ is the bias voltage, and $V_{RF}$ is the RF voltage.

We can simplify it as

$$a = \frac{\pi V_{RF}}{V_{\pi}}$$  \hspace{1cm} (3.65)

when $V_{RF} = A \sin(4\pi \Omega_{RF} t)$, where we double the frequency as $2\Omega$, as shown in Fig 3.5. By plugging the value of $V_{RF}$ in Equation 3.65, we obtain

$$a = \frac{\pi A \sin(4\pi \Omega_{RF} t)}{V_{\pi}}$$  \hspace{1cm} (3.66)

when

$$\beta = \frac{\pi A}{V_{\pi}}$$  \hspace{1cm} (3.67)

$$a = \beta \sin(4\pi \Omega_{RF} t),$$  \hspace{1cm} (3.68)

when

$$\delta = 2\pi \Omega_{RF} t$$  \hspace{1cm} (3.69)
yeilds

\[ a = \beta \sin 2(\delta). \tag{3.70} \]

\[ b = \frac{\pi V_b}{V_\pi}. \tag{3.71} \]

Then the general equation of \( P_{MZM}(t) \) will be

\[ P_{MZM}(t) = \frac{P_{in}}{2} [1 + \cos (a + b)], \tag{3.72} \]

\[ P_{MZM}(t) = \frac{P_{in}}{2} \left[ 1 + \cos \left( \frac{\pi V_{RF}}{V_\pi} + \frac{\pi V_b}{V_\pi} \right) \right]. \tag{3.73} \]

Thus, the general equation of MZM with doubling RF input frequency can be written as

\[ P_{MZM}(t) = \frac{P_{in}}{2} \left[ 1 + \cos \left( \beta \sin 2\delta + \frac{3\pi}{2} \right) \right]. \tag{3.74} \]

When we configure the SD-MZM at quadrature point the value of \( V_b = \frac{3V_s}{2} \)

\[ b = \frac{3\pi}{2}, \tag{3.75} \]

\[ b = \frac{3\pi}{2}. \tag{3.76} \]

By plugging in the values of \( b \) shown in equation 3.76 into \( P_{QMZM}(t) \), then the general equation of will be

\[ P_{QMZM}(t) = \frac{P_{in}}{2} \left[ 1 + \cos \left( \beta \sin(2\delta) + \frac{3\pi}{2} \right) \right]. \tag{3.77} \]

By using trigonometric identity:

\[ \cos(x + \frac{3\pi}{2}) = \sin(x). \tag{3.78} \]
The general equation of \( P_{Q\text{M}Z\text{M}}(t) \) can be simplified to

\[
P_{Q\text{M}Z\text{M}}(t) = \frac{P_{\text{in}}}{2} \left[ 1 + \sin(\beta \sin(2\delta)) \right].
\] (3.79)

Equation 3.79 is plotted in Fig 3.6. As can be seen the frequency has been doubled.

For simplicity, we will use \( \sin(x) \) series:

\[
\sin(x) = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7.
\] (3.80)

Where

\[
x = \beta \sin(2\delta),
\] (3.81)

yields

\[
\sin(x) = \beta \sin(2\delta) - \frac{1}{3!} [\beta \sin(2\delta)]^3 + \frac{1}{5!} [\beta \sin(2\delta)]^5 - \frac{1}{7!} [\beta \sin(2\delta)]^7.
\] (3.82)
Thus, the final equation of SD-MZM of doubling Input RF frequency can be written as

$$P_{Q\text{MZM}}(t) = \frac{P_{in}}{2} \left[ 1 + \beta \sin(2\delta) - \frac{1}{3!}[\beta \sin(2\delta)]^3 + \frac{1}{5!}[\beta \sin(2\delta)]^5 - \frac{1}{7!}[\beta \sin(2\delta)]^7 \right]. \quad (3.83)$$

In conclusion, we double the input frequency in case of SD-MZM operates at quadrature bias point by doubling the input RF frequency.

3.6 Comparison between Analytical and Numerical Approaches vs Pure Sine Wave vs Operation at Quadrature with Doubling Input Frequency

According to the previous analysis, we derive the output power versus time in both as analytic approach, using Taylor expansion, and a numerical approach. The results are plotted in Figure 3.4. Now Let us determine the accuracy of both methods by comparing to a pure sine wave and Quadrature bias driven at $2 \Omega$.

3.6.1 Numerical Approach vs Pure Sine Wave

Figure 3.7 shows the output power in case of numerical method and the pure sine wave vs time. It can be observed that they are close to each other with minimal difference.
Figure 3.7: Temporal optical Power out PBC of Numerical approach & Pure sine wave vs Time.

From Figure 3.14, shows results of error where the output optical power subtract from pure sine wave. As it can seen the error is around 5% of normalized $P_{peak}$ input.

Figure 3.8: Optical Signal Error [Numerical approach - Pure sine Wave] vs Time
3.6.2 Numerical Approach vs Quadrature Bias Driven at $2\Omega$

Figure 3.9 shows the output power in case of numerical method and the output of SD-MZM operates at Quadrature bias driven at $2\Omega$ vs time. As it can be seen the difference is more pronounced than in the case of a pure sine wave. Figure 3.10 illustrates the error as a result of the output optical power subtract numerical output power from the output of SD-MZM operates at quadrature bias driven at $2\Omega$. As can be seen the error is more than we had in Figure 3.14 wherr the error is around 25% of normalized $P_{peak}$ input.
3.6.3 Analytical Approach vs Pure Sine Wave

Similarly, Figure 3.11 shows the output power in case of analytical method and the pure sine wave vs time. As can be seen the analytical output power generally is similar to numerical output with more error as we can see in 3.12. This is a result of the usage of Taylor series expansion for approximation of the output power where the error is around 5% of normalized $P_{\text{peak}}$ input.
3.6.4 Analytical Approach vs Quadrature bias driven at $2\Omega$

Figure 3.13 shows the output power in the case of analytical method and the output of SD-MZM operating at Quadrature bias driven at $2\Omega$ vs time. As it can be seen, it is similar to the numerical approach with more error value once again that is because of use of the Taylor Expansion where the error is close to around 25% of normalized $P_{\text{peak}}$ input.
Figure 3.13: Temporal Optical Power out PBC of Analytical Approach & Quadrature Bias driven at $2\Omega$ vs Time.

Figure 3.14: Optical Signal Error [Analytical Approach - Quadrature Bias driven at $2\Omega$] vs Time
Chapter 4

Spectral Domain Analysis — Numerical and Taylor Series Expansion

4.1 Introduction

In the previous chapter, we demonstrated the output power in the temporal domain. We can also represent the output power in the spectral domain for each scenario. The performance is characterized using fast fourier transform (FFT), and then the output spectral powers are graphically represented so they can be compared.

4.2 Input RF Signal Spectral Power

First we will start with the input RF signal. Figure 4.1 shows the spectral power of our RF input signal versus normalized frequency. The output spectrum has been done by taking fast fourier transform (FFT) to input RF signal Equation 2.2. It can be observed that the output frequency just appears at F, which represents the fourier transform to pure sine wave.
4.3 Taylor Expansion

Figure 4.2 illustrates the spectral output power of the output RF signal shown in Figure 3.4. The output spectrum has been done by taking fast fourier transform to Equation 3.55 which the temporal output in case of Taylor Expansion as follows:

\[ P(f) = |P(t)|^2 \]  \hspace{1cm} (4.1)

then get the output spectrum power in unit of dBm as follows

\[ P(f)_{dBm} = 10 \log[1000p(f)]. \]  \hspace{1cm} (4.2)

As we can see the output frequency has been doubled at 2F a result of our technique where the input frequency was at F as shown in 4.1.
Figure 4.2: Normalized Output RF Spectral Power vs Normalized Frequency

4.4 Numerical Method

Figure 4.3 shows the numerical spectral output power versus normalized frequency. The output spectrum has been done by taking fast fourier transform to Equation 3.20 which the temporal output in case of Numerical Method as follows:

\[ P(f) = |P(t)|^2 \]  

(4.3)

then get the output spectrum power in unit of \(dBm\) as follows

\[ P(f)_{dBm} = 10 \log[1000p(f)]. \]  

(4.4)

As we can see the frequency has been doubled which has similar Taylor expansion output.
In conclusion we have successfully doubled the input frequency by using our frequency doubling technique and this has been proved through numerical and analytical approaches.

4.5 Quadrature Bias Driven at $2\omega$

In Section 3.4 we learned that another technique to double output frequency is by operating SD-MZM at quadrature bias point with double the input RF signal frequency. Figure 4.4 shows the spectral domain of Figure 3.6. As we have illustrated in Section 1.1.2 when operating SD-MZM at its quadrature bias point we will have odd harmonics. So in the case when the SD-MZM is driven at $2\omega$, the frequency harmonics appear at $2F$ and $3(2F)=6F$. The output spectrum has been done by taking fast fourier transform to Equation 3.75 which
the temporal output in case of Numerical Method as follows:

\[ P(f) = |P_{Q\text{MZM}}(t)|^2 \]  \hspace{1cm} (4.5)

then get the output spectrum power in unit of dBm as follows

\[ P(f)_{\text{dBm}} = 10 \log[1000p(f)]. \]  \hspace{1cm} (4.6)

4.6 Numerical vs Taylor Expansion vs Quadrature Bias Driven at 2Ω

The results in Figure 4.5, Figure 4.6, and Figure 4.7 show that the behavior of the second and sixth harmonics of numerical, Tylor Expansion and SD-MZM operating at quadrature
Bias and driven at 2Ω respectively. When we vary the input voltage from 0.1V to full swing the output in each case as we can see generally they have the same response whereas the difference between second and sixth harmonic get smaller while we move toward full swing.

Figure 4.5: Amplitude Scaling of Voltage Swing vs Normalized RF Tone Peaks of Numerical

Figure 4.8 shows another way to compare the three methods, which is by comparing the output power of second tones in all methods while we going to full swing. As a result, it can be observed that the output power of second tone of our frequency doubling technique has more power than Quadrature Bias Driven at 2Ω. In addition, by comparing the difference between second and sixth harmonic while we going to full swing as shown in 4.9 the swing increases the difference value gets smaller in the three methods.

However, it is worth mentioning the the case of SD-MZM operates at quadrature Bias and driven at 2Ω has slightly different pattern because as we mentioned before, the outputs are
Figure 4.6: Amplitude Scaling of Voltage Swing vs Normalized RF Tones Peak of Taylor Expansion

Figure 4.7: Amplitude Scaling of Voltage Swing vs Normalized RF Tone Peaks of Quadrature Bias Driven at $2\Omega$

the odd harmonics.
4.7 Peak and Null Bias Cases

As we have discussed in Section 2.4, one way to double the input frequency, is to operate the SD-MZM at peak. the output frequency will be doubled, but the output signal will not
have a traditional or desired sine wave shape as shown in Figure 2.3.

Figure 4.10 shows the RF spectral power of Figure 2.3. As we can see the output harmonics are at 2F, 4F and 6F and so on.

![Figure 4.10: Normalized RF Spectral Power of Peak vs Normalized Frequency](image)

Similarly, when we operate SD-MZM at null, the output frequency will be doubled and we will get the even harmonics as shown in Figure 4.11. Figure 4.12 explores the spectrum harmonics in both operation cases varying the voltage from 0.1 to full swing. As we can see the harmonics of both peak and null have the same response.

Figure 4.13 shows another way to look at the response of peak and null output by comparing the difference between second and fourth, also second and sixth harmonics while going to full swing. As expected, it looks similar to our frequency technique and quadraure bias driven at 2Ω where as the swinging increases the difference value gets smaller.
As we discussed the frequency doubling technique based on operating one SD-MZM at peak and the other SD-MZM at null and then combining both output signals by polarization beam combiner (PBC). In the previous section we discussed the output in each case.
Now by comparing them we will understand the differences and advantages of the frequency technique.

As can be seen in Figure 4.14 the output second tones of of Frequency Doubling Technique& peak& null. The second tone of frequency doubling technique has more output power than peak and null outputs. However, the difference between the second and sixth harmonics in all cases has the same value as shown in 4.15.

In conclusion, the output signal shape in temporal domain using our frequency doubling technique is closer to desired pure sine wave shape.
Figure 4.14: Amplitude Scaling of Voltage Swing vs Normalized Output of Second Tones of Frequency Doubling Technique & Peak & Null vs The Difference between Second and Sixth Tones

Figure 4.15: Amplitude Scaling of Voltage Swing vs The Difference between Second and Sixth Tones of Frequency Doubling Technique & Peak & Null
Chapter 5

Spectral Domain Analysis — Jacobi Anger Expansion

5.1 Introduction

This chapter illustrates the output of frequency doubling technique by using the Jacobi Anger expansion. This method is an expansion of exponentials of trigonometric functions as exponential terms in the basis of their harmonics. In addition, by using Jacobi Anger expansion we can analyze each terms optical field which allows us to simplify our derivations instead of dealing with the whole output expression.

The general expression for the modulated output of the general MZM as in section 1.3.1 is given as

\[ E_{\text{out}}(t) = \frac{1}{2} E_{in}(e^{i\theta_1} + e^{i\theta_2}). \] (5.1)

Because it is a general equation, we will substitute the value of each variable as follows.

Firstly, we will start with RF inputs \( \theta_1(t) \) and \( \theta_2(t) \)

\[ \theta_1(t) = \frac{\pi}{V_{\pi}} V_{1RF}(t), \] (5.2)

\[ \theta_2(t) = \frac{\pi}{V_{\pi}} V_{2RF}(t) + \phi_b, \] (5.3)

\[ \phi_b = \frac{\pi}{V_{\pi}} V_b, \] (5.4)
where $V_{1RF}$ and $V_{2RF}$ are the RF voltage inputs, $\phi_b$ is a phase bias applied to one arm of the MZM, $V_b$ is bias voltage, and $V_\pi$ is the half wave voltage of the MZM.

Now we can expand $V_{RF}(t)$ into further detail as follows,

$$V_{1RF}(t) = A \cos(2\pi \Omega_{RF} t + \phi), \quad (5.5)$$

$$V_{2RF}(t) = A \cos(2\pi \Omega_{RF} t), \quad (5.6)$$

where $A$ is the magnitudes of the RF signals, $\Omega_{RF}$ is the RF modulation frequency, and $\phi$ is the phase difference between the two RF signals.

By substituting all variables, the expression for $\theta_1(t)$ and $\theta_2(t)$ can be written as

$$\theta_1(t) = \frac{\pi A}{V_\pi} \cos(2\pi \Omega_{RF} t + \phi), \quad (5.7)$$

$$\theta_2(t) = \frac{\pi A}{V_\pi} \cos(2\pi \Omega_{RF} t) + \phi_b. \quad (5.8)$$

We define the term called the modulation index, $\beta$, as

$$\beta = \frac{\pi A}{V_\pi}. \quad (5.9)$$

The expressions for $\theta_1(t)$ and $\theta_2(t)$ can be written as,

$$\theta_1(t) = \beta \cos(2\pi \Omega_{RF} t + \phi), \quad (5.10)$$

$$\theta_2(t) = \beta \cos(2\pi \Omega_{RF} t) + \phi_b. \quad (5.11)$$

Now we can write the general expression for the modulated output of the MZM as follows,

$$E_{out1}(t) = E_c e^{j\omega_c t} (e^{j\beta \cos(2\pi \Omega_{RF} t + \phi)} + e^{j\beta \cos(2\pi \Omega_{RF} t) + \phi_b}), \quad (5.12)$$
where $E_c = j\frac{1}{2}E_{in}$.

For a single-drive, push-pull MZM, the phase difference is

$$\varphi = \pi,$$  \hspace{1cm} (5.13)

Using this expressions for $\theta_1(t)$ and $\theta_2(t)$ can be written as,

$$\theta_1(t) = \beta \cos(2\pi \Omega_{RF}t + \pi),$$  \hspace{1cm} (5.14)

which can be further simplified to,

$$\theta_1(t) = -\beta \cos(2\pi \Omega_{RF}t),$$  \hspace{1cm} (5.15)

$$\theta_2(t) = \beta \cos(2\pi \Omega_{RF}t) + \phi_b.$$  \hspace{1cm} (5.16)

As a result we can write the general expression for the modulated output of the MZM as follows,

$$E_{out1}(t) = E_c e^{j\omega_c t} (e^{-j\beta \cos(2\pi \Omega_{RF}t)} + e^{j[\beta \cos(2\pi \Omega_{RF}t) + \phi_b]}).$$  \hspace{1cm} (5.17)

To understand the optical field, we should analyze our signal in both temporal domain and spectral domain. One of the methods we can use in frequency domain is Jacobi-Anger expansion methods, where we can analyze and derive the harmonics of a signal.

We can define the Jacobi-Anger expansion as

$$e^{iz \cos(\theta)} = \sum_{m=-\infty}^{\infty} i^m J_m(z)e^{im\theta} = J_0(z) + 2 \sum_{m=1}^{\infty} i^m J_m(z) \cos(m\theta),$$  \hspace{1cm} (5.18)

where $J_m(z)$ is defined as the mth order of the Bessel function of the first kind as shown in Figure 5.1.
The general expression for the modulated output of the MZM does not match the expression of Jacobi-Anger. So we will substitute the following values for the MZM as follows,

\[ z = \beta, \quad (5.19) \]

\[ \theta_1 = 2\pi\Omega_{RF} t + \varphi. \quad (5.20) \]

For the SD-MZM, \( \varphi = \pi \), which yields

\[ \theta_1 = 2\pi\Omega_{RF} t + \pi. \quad (5.21) \]

The second arm of the MZM yields

\[ \theta_2 = 2\pi\Omega_{RF} t. \quad (5.22) \]
Thus, $\theta_2 = \theta$, and $\theta_1 = \theta + \pi$.

The resulting equation is

$$E_{out1}(t) = E_c e^{j\omega_c t} (e^{j\beta \cos(2\pi \Omega_{RF} t + \pi)} + e^{j\beta \cos(2\pi \Omega_{RF} t + \phi_b)}),$$  \hspace{1cm} (5.23)$$

$$E_{out1}(t) = E_c e^{j\omega_c t} (e^{j\beta \cos(\theta + \pi)} + e^{j\beta \cos(\theta + \phi_b)}).$$  \hspace{1cm} (5.24)$$

Now, this format is proper to apply the Jacobi-Anger expansion. Then, the general form can be written as,

$$E(t) = E_c e^{j\omega_c t} \sum_{m=-\infty}^{\infty} \left[ i^m J_0(\beta)e^{im\pi} e^{im\theta} + i^m J_1(\beta)e^{im\theta} e^{ip\phi_b} \right].$$  \hspace{1cm} (5.25)$$

By taking the $i^m J_m(\beta)e^{im\theta}$ term as a common factor, we obtain

$$E(t) = E_c e^{j\omega_c t} \sum_{m=-\infty}^{\infty} \left[ i^m J_m(\beta)e^{im\theta} [e^{im\pi} + e^{ip\phi_b}] \right].$$  \hspace{1cm} (5.26)$$

Where $E_c = j^{1/2} E_{in2}$.

\section*{5.2 Optical Electrical Field in Jacobi-Anger Format at Null}

\subsection*{5.2.1 Spectral Optical Field}

The general expression for the modulated output of the MZM in the Jacobi Anger format is given as

$$E(t) = E_c e^{j\omega_c t} \sum_{m=-\infty}^{\infty} i^m J_m(\beta)e^{im\theta} [e^{im\pi} + e^{ip\phi_b}].$$  \hspace{1cm} (5.27)$$

To activate at null $\phi_b = \pi$, and the output can be written as

$$E(t)_N = E_c e^{j\omega_c t} \sum_{m=-\infty}^{\infty} i^m J_m(\beta)e^{im\theta} [e^{im\pi} + e^{in}],$$  \hspace{1cm} (5.28)$$
\[ E(t)_N = E_c e^{i\omega_c t} \sum_{m=-\infty}^{\infty} i^m J_m(\beta)e^{im\beta}[e^{im\pi} - 1]. \] (5.29)

Now we take \( m \) from -6 to 6 as follows.

Taking \( m = 0 \) yields

\[ E(t)_{N_0} = E_c e^{i\omega_c t} i^0 J_0(\beta)e^{i0\beta}[e^{i0\pi} - 1]. \] (5.30)

By using the algebraic properties

\[ i^0 = 1, \] (5.31)
\[ e^{i0\pi} = 1, \] (5.32)

we obtain

\[ E(t)_{N_0} = E_c e^{i\omega_c t} \times 1 \times J_0(\beta)1[1 - 1], \] (5.33)
\[ E(t)_{N_0} = 0. \] (5.34)

This means that the carrier has been suppress. This is one of the characteristic of SD-MZM when we active at null.

Taking \( m = 1 \) yields

\[ E(t)_{N_1} = E_c e^{i\omega_c t} i^1 J_1(\beta)e^{i\beta}[e^{i\pi} - 1]. \] (5.35)

By using the algebraic properties

\[ (i)^1 = i, \] (5.36)
\[ e^{i1\pi} = -1, \] (5.37)

we obtain

\[ E(t)_{N_1} = E_c e^{i\omega_c t}[iJ_1(\beta)e^{i\theta}[-1 - 1]], \] (5.38)
\[ E(t)_{N_1} = -2iE_1 e^{i\omega t} J_1(\beta)e^{i\theta}. \] (5.39)

Taking \( m = -1 \) yields

\[ E(t)_{N_{-1}} = E_1 e^{i\omega t} i^{-1} J_{-1}(\beta)e^{-i\theta} [e^{-i\pi} - 1]. \] (5.40)

By using the algebraic properties

\[ (i)^{-1} = -i, \] (5.41)

\[ e^{-i\pi} = -1, \] (5.42)

and the following lemma

\[ J_{-1}(x) = (-1)^1 J_1(x), \] (5.43)

\[ J_{-1}(x) = -J_1(x), \] (5.44)

we obtain

\[ E(t)_{N_{-1}} = E_1 e^{i\omega t} \left[ -i \times -J_1(\beta)e^{-i\theta} [-1 - 1] \right]. \] (5.45)

\[ E(t)_{N_{-1}} = -2iE_1 e^{i\omega t} J_1(\beta)e^{-i\theta}. \] (5.46)

Taking \( m = 2 \) yields

\[ E(t)_{N_2} = E_1 e^{i\omega t} i^{2} J_2(\beta)e^{i2\theta} [e^{i2\pi} - 1]. \] (5.47)

By using the algebraic properties

\[ (i)^2 = -1, \] (5.48)

\[ e^{i2\pi} = 1, \] (5.49)

we obtain

\[ E(t)_{N_2} = E_1 e^{i\omega t} \left[ -1 \times J_2(\beta)e^{i2\theta} [1 - 1] \right]. \] (5.50)
\[ E(t)_{N_2} = 0. \] (5.51)

Fifth by taking \( m = -2 \)

\[ E(t)_{N_{-2}} = E_c e^{i\omega_c t} i^{-2} J_{-2}(\beta)e^{i2\theta} [e^{-i2\pi t} - 1]. \] (5.52)

By using the algebraic properties

\( (i)^{-2} = -1, \) (5.53)

\( e^{-is2\pi} = 1, \) (5.54)

and the following lemma

\[ J_{-2}(x) = (-1)^2 J_2(X), \] (5.55)

\[ J_{-2}(x) = J_2(X), \] (5.56)

we obtain

\[ E(t)_{N_{-2}} = E_c e^{i\omega_c t} - 1J_2(\beta)e^{i2\theta}[1 - 1], \] (5.57)

\[ E(t)_{N_{-2}} = 0. \] (5.58)

Taking \( m = 3 \) yields

\[ E(t)_{N_3} = E_c e^{i\omega_c t} i^3 J_3(\beta)e^{i3\theta} [e^{i3\pi} - 1]. \] (5.59)

By using the algebraic properties

\( (i)^3 = -i, \) (5.60)

\( e^{is1\pi} = -1, \) (5.61)

we obtain

\[ E(t)_{N_3} = E_c e^{i\omega_c t} - iJ_3(\beta)e^{i3\theta}[-1 - 1], \] (5.62)
\[ E(t)_{N_3} = 2iE_c e^{i\omega t} J_3(\beta)e^{i3\theta}. \] (5.63)

Taking \( m = -3 \) yields

\[ E(t)_{N_3} = E_c e^{i\omega t} i^{-3} J_3(\beta)e^{-i3\theta} [e^{-i3\pi} - 1]. \] (5.64)

By using the algebraic properties

\[(i)^{-3} = i, \] (5.65)
\[e^{-i3*\pi} = -1, \] (5.66)

and the following lemma

\[ J_{-3}(x) = (-1)^3 J_3(x), \] (5.67)
\[ J_{-3}(x) = -J_3(x), \] (5.68)

we obtain

\[ E(t)_{N_3} = E_c e^{i\omega t} i \times -J_3(\beta)e^{-i3\theta} [-1 - 1], \] (5.69)
\[ E(t)_{N_3} = 2iE_c e^{i\omega t} J_3(\beta)e^{-i3\theta}. \] (5.70)

Taking \( m = 4 \) yields

\[ E(t)_{N_4} = E_c e^{i\omega t} i^4 J_4(\beta)e^{i4\theta} [e^{i4\pi} - 1]. \] (5.71)

By using the algebraic properties

\[(i)^4 = 1, \] (5.72)
\[e^{i4*\pi} = 1, \] (5.73)

we obtain

\[ E(t)_{N_4} = E_c e^{i\omega t} \times 1 \times J_4(\beta)e^{i4\theta} [1 - 1], \] (5.74)
\[ E(t)_{N_4} = 0. \] (5.75)

Taking \( m = -4 \) yields
\[
E(t)_{N_{-4}} = E_c e^{i\omega_c t} i^{-4} J_{-4}(\beta) e^{i\theta} e^{-i4\pi} - 1. \] (5.76)

By using the algebraic properties
\[ (i)^{-4} = 1, \] (5.77)
\[ e^{-i4\pi} = 1, \] (5.78)

and the following lemma
\[ J_{-4}(x) = (-1)^4 J_4(X), \] (5.79)
\[ J_{-4}(x) = J_4(X), \] (5.80)

we obtain
\[
E(t)_{N_{-4}} = E_c e^{i\omega_c t} 1 \times J_4(\beta) e^{i\theta} [ e^{-i4\pi} - 1, \] (5.81)
\[ E(t)_{N_{-4}} = 0. \] (5.82)

Taking \( m = 5 \) yields
\[
E(t)_{N_5} = E_c e^{i\omega_c t} i^5 J_5(\beta) e^{i5\theta} [ e^{i5\pi} - 1. \] (5.83)

By using the algebraic properties
\[ (i)^5 = i, \] (5.84)
\[ e^{i5\pi} = -1, \] (5.85)

We obtain
\[
E(t)_{N_5} = E_c e^{i\omega_c t} i \times J_5(\beta) e^{i5\theta} [-1 - 1. \] (5.86)
\[ E(t)_{N_5} = -2iE_c e^{j\omega_t} J_5(\beta)e^{j\delta_0}. \]  
(5.87)

Taking \( m = -5 \) yields

\[ E(t)_{N_{-5}} = E_c e^{j\omega_t} i^{-5} J_5(\beta)e^{-j\delta_0} [e^{-i\delta_\pi} - 1]. \]  
(5.88)

By using the algebraic properties

\[ (i)^{-5} = -i, \]  
(5.89)

\[ e^{-i\delta_{5\pi}} = -1, \]  
(5.90)

and the following lemma

\[ J_{-5}(x) = (-1)^5 J_5(x), \]  
(5.91)

\[ J_{-5}(x) = -J_5(x), \]  
(5.92)

we obtain

\[ E(t)_{N_{-5}} = E_c e^{j\omega_t} i \times -i \times J_5(\beta)e^{-j\delta_0} [-1 - 1], \]  
(5.93)

\[ E(t)_{N_{-5}} = -2iE_c e^{j\omega_t} J_5(\beta)e^{-j\delta_0}. \]  
(5.94)

Taking \( m = 6 \) yields

\[ E(t)_{N_6} = E_c e^{j\omega_t} i^6 J_6(\beta)e^{j\delta_0} [e^{j\delta_\pi} - 1]. \]  
(5.95)

By using the algebraic properties

\[ (i)^6 = -1, \]  
(5.96)

\[ e^{j\delta_{6\pi}} = 1, \]  
(5.97)

we obtain

\[ E(t)_{N_6} = E_c e^{j\omega_t} \times -1 \times J_6(\beta)e^{j\delta_0} [1 - 1], \]  
(5.98)
\[ E(t)_{N_6} = 0. \]  
(5.99)

Taking \( m = -6 \) yields

\[ E(t)_{N,-6} = E_c e^{i\omega_t \times -6} J_{-6}(\beta) e^{i\theta} [e^{-i6\pi} - 1]. \]  
(5.100)

By using the algebraic properties

\[ (i)^{-6} = -1, \]  
(5.101)

\[ e^{-i\times 6\pi} = 1, \]  
(5.102)

and the following lemma

\[ J_{-6}(x) = (-1)^6 J_6(x), \]  
(5.103)

\[ J_{-6}(x) = J_6(x), \]  
(5.104)

we obtain

\[ E(t)_{N,-6} = E_c e^{i\omega_t \times -6} \times -1 \times J_6(\beta) e^{i\theta} [1 - 1], \]  
(5.105)

\[ E(t)_{N,-6} = 0. \]  
(5.106)

### 5.2.2 Total Spectrum Of Optical Field

We can get the output optical field by sum all output of all cases follows

\[ E(t)_{N_{T}} = E(t)_{N_0} + E(t)_{N_1} + E(t)_{N,-1} + E(t)_{N_2} + E(t)_{N,-2} + E(t)_{N_3} + E(t)_{N,-3} + E(t)_{N_4} + E(t)_{N,-4} + E(t)_{N_5} + E(t)_{N,-5} \]

\[ + E(t)_{N_6} + E(t)_{N,-6} \]  
(5.107)

\[ E(t)_{N_T} = 0 + E(t)_{N_1} + E(t)_{N,-1} + 0 + 0 + E(t)_{N_3} + E(t)_{N,-3} + 0 + 0 + E(t)_{N_5} + E(t)_{N,-5} + 0 + 0, \]  
(5.108)

\[ E(t)_{N_T} = E(t)_{N_1} + E(t)_{N,-1} + E(t)_{N_3} + E(t)_{N,-3} + E(t)_{N_5} + E(t)_{N,-5}. \]  
(5.109)
By substituting the value of each term, we obtain

\[
E(t)_{N_T} = -2iE_e e^{i\omega t} J_1(\beta)e^{i\theta} - 2iE_e e^{i\omega t} J_3(\beta)e^{i\theta} + 2iE_e e^{i\omega t} J_3(\beta)e^{i\omega} + 2iE_e e^{i\omega t} J_3(\beta)e^{-i\omega} \\
- 2iE_e e^{i\omega t} J_5(\beta)e^{i5\omega} - 2iE_e e^{i\omega t} J_5(\beta)e^{-i5\omega}.
\] (5.110)

More simply

\[
E(t)_{N_T} = -2iE_e e^{i\omega t} J_1(\beta)[e^{i\theta} + e^{-i\theta}] + 2iE_e e^{i\omega t} J_3(\beta)[e^{i3\theta} + e^{-i3\theta}] - 2iE_e e^{i\omega t} J_5(\beta)[e^{i5\theta} + e^{-i5\theta}],
\] (5.111)

\[
E(t)_{N_T} = 2iE_e e^{i\omega t} \left\{- J_1(\beta)[e^{i\theta} + e^{-i\theta}] + J_3(\beta)[e^{i3\theta} + e^{-i3\theta}] - J_5(\beta)[e^{i5\theta} + e^{-i5\theta}] \right\}.
\] (5.112)

### 5.2.3 Total Spectrum Of Optical Power

We can get the output power as follows

\[
P_{N_T} = |E(t)_{N_T}|^2 = E(t)_{N_T} E(t)_{N_T}^*. \] (5.113)

By substituting the value of \(E(t)_{N_T}\) we obtain

\[
P_{N_T} = \left[ 2iE_e e^{i\omega t} \left\{- J_1(\beta)[e^{i\theta} + e^{-i\theta}] + J_3(\beta)[e^{i3\theta} + e^{-i3\theta}] - J_5(\beta)[e^{i5\theta} + e^{-i5\theta}] \right\} \right] \times \left[ -2iE_e e^{-i\omega t} \left\{- J_1(\beta)[e^{i\theta} + e^{-i\theta}] + J_3(\beta)[e^{i3\theta} + e^{-i3\theta}] - J_5(\beta)[e^{i5\theta} + e^{-i5\theta}] \right\} \right],
\] (5.114)

\[
P_{N_T} = 4P_e \left[ J_1^2(\beta)[e^{i\theta} + e^{-i\theta}]^2 + J_3^2(\beta)[e^{i3\theta} + e^{-i3\theta}]^2 + J_5^2(\beta)[e^{i5\theta} + e^{-i5\theta}]^2 - J_1(\beta)J_3(\beta)[e^{i\theta} + e^{-i\theta}][e^{i3\theta} + e^{-i3\theta}] \\
+ J_1(\beta)J_5(\beta)[e^{i\theta} + e^{-i\theta}][e^{i5\theta} + e^{-i5\theta}] - J_1(\beta)J_3(\beta)[e^{i\theta} + e^{-i\theta}][e^{i3\theta} + e^{-i3\theta}] - J_3(\beta)J_5(\beta)[e^{i3\theta} + e^{-i3\theta}][e^{i5\theta} + e^{-i5\theta}] \right] \\
+ J_1(\beta)J_5(\beta)[e^{i\theta} + e^{-i\theta}][e^{i5\theta} + e^{-i5\theta}] - J_3(\beta)J_5(\beta)[e^{i3\theta} + e^{-i3\theta}][e^{i5\theta} + e^{-i5\theta}].
\] (5.115)
More simply

\[ P_{NT} = 4P_c \left[ J_1^2(\beta)\left[ e^{i\theta} + e^{-i\theta}\right]^2 + J_3^2(\beta)\left[ e^{i3\theta} + e^{-i3\theta}\right]^2 + J_5^2(\beta)\left[ e^{i5\theta} + e^{-i5\theta}\right]^2 - 2J_1(\beta)J_3(\beta)\left[ e^{i\theta} + e^{-i\theta}\right]\left[ e^{i3\theta} + e^{-i3\theta}\right] \right. \]

\[ + 2J_1(\beta)J_5(\beta)\left[ e^{i\theta} + e^{-i\theta}\right]\left[ e^{i5\theta} + e^{-i5\theta}\right] - 2J_3(\beta)J_5(\beta)\left[ e^{i3\theta} + e^{-i3\theta}\right]\left[ e^{i5\theta} + e^{-i5\theta}\right] \right]. \] (5.116)

By using the following formulas:

\[ \cos(x) = \frac{1}{2}\left[ e^{ix} + e^{-ix}\right], \] (5.117)

\[ e^{ix} + e^{-ix} = 2\cos(x). \] (5.118)

\[ \cos(3x) = \frac{1}{2}\left[ e^{i3x} + e^{-i3x}\right], \] (5.119)

\[ e^{i3x} + e^{-i3x} = 2\cos(3x). \] (5.120)

\[ \cos(5x) = \frac{1}{2}\left[ e^{i5x} + e^{-i5x}\right], \] (5.121)

\[ e^{i5x} + e^{-i5x} = 2\cos(5x). \] (5.122)

The output power can therefore be written as

\[ P_{NT} = 4P_c \left\{ 4J_1^2(\beta)\cos^2(\theta) + 4J_3^2(\beta)\cos^2(3\theta) + 4J_5^2(\beta)\cos^2(5\theta) - 2 \times 2 \times J_1(\beta)J_3(\beta)\cos(\theta)\cos(3\theta) \right. \]

\[ + 2 \times 2 \times J_1(\beta)J_5(\beta)\cos(\theta)\cos(5\theta) - 2 \times 2 \times J_3(\beta)J_5(\beta)\cos(3\theta)\cos(5\theta) \right\}. \] (5.123)

By using the following formulas:

\[ \cos^2(\theta) = \frac{1}{2}[1 + \cos(2\theta)], \] (5.124)

\[ \cos(\theta)\cos(3\theta) = \frac{1}{2}[\cos(\theta - 3\theta) + \cos(\theta + 3\theta)], \] (5.125)

\[ \cos(\theta)\cos(3\theta) = \frac{1}{2}[\cos(-2\theta) + \cos(4\theta)]. \] (5.126)
Because the function $\cos(x)$ is even

$$
\cos(-\theta) = \cos(\theta),
$$

(5.127)

and therefore

$$
\cos(\theta) \cos(3\theta) = \frac{1}{2} [\cos(2\theta) + \cos(4\theta)].
$$

(5.128)

By using the same reasoning in case of $\theta$ and $5\theta$, we obtain

$$
\cos(\theta) \cos(5\theta) = \frac{1}{2} [\cos(4\theta) + \cos(6\theta)].
$$

(5.129)

And in case of $3\theta$ and $5\theta$, we obtain

$$
\cos(3\theta) \cos(5\theta) = \frac{1}{2} [\cos(2\theta) + \cos(8\theta)].
$$

(5.130)

Now the output power can be written as

$$
P_{NT} = 4P_c \left\{ 4J_1^2(\beta) \frac{1}{2} [1 + \cos(2\theta)] + 4J_3^2(\beta) \frac{1}{2} [1 + \cos(2\times3\theta)] + 4J_5^2(\beta) \frac{1}{2} [1 + \cos(2\times5\theta)] \\
- 2 \times 2 \times 2J_1(\beta) J_3(\beta) \frac{1}{2} [\cos(2\theta) + \cos(4\theta)] + 2 \times 2 \times 2J_1(\beta) J_5(\beta) \frac{1}{2} [\cos(4\theta) + \cos(6\theta)] \\
- 2 \times 2 \times 2J_3(\beta) J_5(\beta) \frac{1}{2} [\cos(2\theta) + \cos(8\theta)] \right\}.
$$

(5.131)

More simply

$$
P_{NT} = 4P_c \left\{ 2J_1^2(\beta) + 2J_1^2(\beta) \cos(2\theta) + 2J_3^2(\beta) \cos(6\theta) + 2J_5^2(\beta) \cos(10\theta) \\
- 2 \times 2J_1(\beta) J_3(\beta) \cos(2\theta) - 2 \times 2J_1(\beta) J_5(\beta) \cos(6\theta) \right\}.
$$

(5.132)
The final expression can be written as

\[
P_{N_T} = 4P_c \left\{ 2J_1^2(\beta) + 2J_3^2(\beta) + 2J_5^2(\beta) + 2J_1^2(\beta) \cos(2\theta) - 4J_1(\beta)J_3(\beta) \cos(2\theta) - 4J_3(\beta)J_5(\beta) \cos(2\theta) \\
- 4J_1(\beta)J_3(\beta) \cos(4\theta) + 4J_1(\beta)J_5(\beta) \cos(4\theta) + 2J_3^2(\beta) \cos(6\theta) + 4J_1(\beta)J_5(\beta) \cos(6\theta) \\
- 4J_3(\beta)J_5(\beta) \cos(8\theta) + 2J_5^2(\beta) \cos(10\theta) \right\}.
\] (5.133)

Figure 5.2 shows the output power of each term. Where PNcw represents terms without \(\cos(\theta)\), PN2 represents terms with \(\cos(2\theta)\), PN4 represents terms with \(\cos(4\theta)\), PN6 represents terms with \(\cos(6\theta)\), PN8 represents terms with \(\cos(8\theta)\), and PN10 represents terms with \(\cos(10\theta)\) as it has been written in Equation 5.133.
5.3 Optical Electrical Field in Jacobi-Anger Format at Peak

5.3.1 Spectral Optical Field

The general expression for the modulated output of the MZM is given as

\[ E(t) = E_c e^{j \omega_c t} \sum_{m=-\infty}^{\infty} i^m J_m(\beta) e^{im\theta} e^{im\pi} + e^{i\phi_b} \].

(5.134)

To activate at peak we can set \( \phi_b = 0 \), and the output can be written as

\[ E(t)_p = E_c e^{j \omega_c t} \sum_{m=-\infty}^{\infty} i^m J_m(\beta) e^{im\theta} e^{im\pi} + e^{0} \],

(5.135)

\[ E(t)_p = E_c e^{j \omega_c t} \sum_{m=-\infty}^{\infty} i^m J_m(\beta) e^{im\theta} e^{im\pi} + 1 \].

(5.136)

By considering the shift in phase between two RF inputs happen with this arm by \( \frac{\pi}{2} \), we obtain

\[ E(t) = E_c e^{j \omega_c t} \sum_{m=-\infty}^{\infty} i^m J_m(\beta) e^{im[\theta - \frac{\pi}{2}]} e^{im\pi + 1} \].

(5.137)

By using the identity

\[ e^{x-y} = e^x e^{-y}, \]

(5.138)

the general expression for the modulated output of the MZM when activates at peak can be written as

\[ E(t) = E_c e^{j \omega_c t} \sum_{m=-\infty}^{\infty} [i^m J_m(\beta) e^{im\theta} e^{-im\frac{\pi}{2}} e^{im\pi} + 1]. \]

(5.139)

Now we take \( m \) from -6 to 6 as follows.

Taking \( m = 0 \) yields

\[ E(t)_{p0} = E_c e^{j \omega_c t} J_0(\beta) e^{i\theta} e^{-i\frac{\pi}{2}} e^{i0\pi} + 1. \]

(5.140)
By using the algebraic properties

\[ i^0 = 1, \]  
\[ e^{i0\pi} = 1, \]  
\[ e^{i0\pi} = 1, \]

we obtain

\[ E(t)p_0 = E_c e^{i\omega c t} J_0(\beta) [J_0(\beta)], \]  
\[ E(t)p_0 = 2E_c e^{i\omega c t} J_0(\beta). \]

Taking \( m = 1 \) yields

\[ E(t)p_1 = E_c e^{i\omega c t} J_1(\beta) e^{i\theta} e^{-i\pi/2} [e^{i\pi} + 1]. \]

By using the algebraic properties

\[ (i)^1 = i, \]  
\[ e^{-i\pi/2} = -i, \]  
\[ e^{i1\pi} = -1, \]

we obtain

\[ E(t)p_1 = E_c e^{i\omega c t} J_1(\beta) e^{i\theta} e^{-i\pi/2} [e^{i\pi} + 1], \]  
\[ E(t)p_1 = 0. \]

Taking \( m = -1 \) yields

\[ E(t)p_{-1} = E_c e^{i\omega c t} J_1(\beta) e^{-i\theta} e^{-i\pi/2} [e^{-i\pi} + 1]. \]
By using the algebraic properties

\[(i)^{-1} = -i, \quad (5.153)\]

\[e^{\frac{i\pi}{2}} = i, \quad (5.154)\]

\[e^{-i\times1\pi} = -1, \quad (5.155)\]

we obtain

\[E(t)_{p-1} = E_c e^{ij_0, t} - i \times J_1(\beta)e^{-i\theta}(i) \times 1[-1 + 1], \quad (5.156)\]

\[E(t)_{p-1} = 0. \quad (5.157)\]

Taking \(m = 2\) yields

\[E(t)_{p_2} = E_c e^{ij_0, t} i^2 J_2(\beta)e^{i2\theta} e^{-i2\pi} [e^{i2\pi} + 1]. \quad (5.158)\]

By using the algebraic properties

\[(i)^2 = -1, \quad (5.159)\]

\[e^{-i2\frac{\pi}{2}} = -1, \quad (5.160)\]

\[e^{i2\times\pi} = 1, \quad (5.161)\]

we obtain

\[E(t)_{p_2} = E_c e^{ij_0, t} - 1 \times J_2(\beta)e^{i2\theta}(-1)[1 + 1], \quad (5.162)\]

\[E(t)_{p_2} = 2E_c e^{ij_0, t} e^{i2\theta} J_2(\beta). \quad (5.163)\]

Taking \(m = -2\) yields

\[E(t)_{p_{-2}} = E_c e^{ij_0, t} i^{-2} J_{-2}(\beta)e^{-i2\theta} e^{i2\pi} [e^{-i2\pi} + 1]. \quad (5.164)\]
By using the algebraic properties

\[(i)^{-2} = -1,\]  
\[e^{i2\pi} = -1,\]  
\[e^{-i2\pi} = 1,\]

and the following lemma

\[J_{-2}(x) = (-1)^2 J_2(x),\]  
\[J_{-2}(x) = J_2(x),\]

we obtain

\[E(t)_{p_{-2}} = E_c e^{i\omega t} - 1 \times J_2(\beta)e^{-i2\theta}(i)[1 + 1],\]  
\[E(t)_{p_{-2}} = 2E_c e^{i\omega t} e^{-i2\theta} J_2(\beta).\]

Taking \(m = 3\) yields

\[E(t)_{p_3} = E_c e^{i\omega t} i^3 J_3(\beta)e^{i3\theta} e^{-i3\pi}[e^{i3\pi} + 1].\]

By using the algebraic properties

\[(i)^3 = -i,\]  
\[e^{-i3\pi} = -i,\]  
\[e^{i3\pi} = -1,\]

we obtain

\[E(t)_{p_3} = E_c e^{i\omega t} - i \times J_3(\beta)e^{i3\theta}(-i) \times 1[-1 + 1],\]  
\[E(t)_{p_3} = 0.\]
Taking $m = -3$ yields

$$E(t)_{p_{-3}} = E_c e^{i\omega c t} - J_3(\beta) e^{-3i\theta} e^{i\frac{3\pi}{2}} [e^{-i3\pi} + 1]. \quad (5.178)$$

By using the algebraic properties

$$(i)^{-3} = i, \quad (5.179)$$
$$e^{i\frac{3\pi}{2}} = i, \quad (5.180)$$
$$e^{-i3\pi} = -1, \quad (5.181)$$

we obtain

$$E(t)_{p_{-3}} = E_c e^{i\omega c t} - i \times J_3(\beta)(i) \times 1[-1 + 1], \quad (5.182)$$

$$E(t)_{p_{-3}} = 0. \quad (5.183)$$

Taking $m = 4$ yields

$$E(t)_{p_{4}} = E_c e^{i\omega c t} i^4 J_4(\beta) e^{-4i\theta} e^{i\frac{4\pi}{2}} [e^{i4\pi} + 1]. \quad (5.184)$$

By using the algebraic properties

$$(i)^4 = 1, \quad (5.185)$$
$$e^{-4i\frac{\pi}{2}} = 1, \quad (5.186)$$
$$e^{i4\pi} = 1, \quad (5.187)$$

we obtain

$$E(t)_{p_{4}} = E_c e^{i\omega c t} i \times J_4(\beta) e^{-4i\theta}(1)[1 + 1], \quad (5.188)$$

$$E(t)_{p_{4}} = 2E_c e^{i\omega c t} e^{4i\theta} J_4(\beta). \quad (5.189)$$
Taking \( m = -4 \) yields

\[
E(t)_{p_{-4}} = E_c e^{i\omega c t} i^{-4} J_{-4}(\beta) e^{-i4\theta} e^{i\frac{4\pi}{2}} [e^{-i4\pi} + 1].
\] (5.190)

By using the algebraic properties

\[
(i)^{-4} = 1, \quad (5.191)
\]
\[
e^{i4\frac{\pi}{2}} = 1, \quad (5.192)
\]
\[
e^{-i4\pi} = 1, \quad (5.193)
\]

and the following lemma

\[
J_{-4}(x) = (-1)^4 J_4(x),
\] (5.194)
\[
J_{-4}(x) = J_4(x),
\] (5.195)

we obtain

\[
E(t)_{p_{-4}} = E_c e^{i\omega c t} 1 \times J_4(\beta) e^{-i4\theta} (1)[1 + 1],
\] (5.196)
\[
E(t)_{p_{-4}} = 2E_c e^{i\omega c t} e^{-i4\theta} J_4(\beta).
\] (5.197)

Taking \( m = 5 \) yields

\[
E(t)_{p_5} = E_c e^{i\omega c t} i^5 J_5(\beta) e^{i5\theta} e^{-i5\pi} [e^{i5\pi} + 1].
\] (5.198)

By using the algebraic properties

\[
(i)^5 = i, \quad (5.199)
\]
\[
e^{-i5\frac{\pi}{2}} = -i, \quad (5.200)
\]
\[
e^{i5\pi} = -1, \quad (5.201)
\]
we obtain

\[ E(t)_{p_5} = E_c e^{i \omega t} i \times J_5(\beta) e^{i \beta \theta} (-i) \times 1[-1 + 1], \quad (5.202) \]

\[ E(t)_{p_5} = 0. \quad (5.203) \]

Taking \( m = -5 \) yields

\[ E(t)_{p_{-5}} = E_c e^{i \omega t} i^{-5} J_{-5}(\beta) e^{-5i \theta} e^{i \frac{5}{2} \pi} [e^{-i \frac{5}{2} \pi} + 1]. \quad (5.204) \]

By using the algebraic properties

\[ (i)^{-5} = -i, \quad (5.205) \]

\[ e^{i \frac{5}{2} \pi} = i, \quad (5.206) \]

\[ e^{-i \frac{5}{2} \pi} = -1, \quad (5.207) \]

we obtain

\[ E(t)_{p_{-5}} = E_c e^{i \omega t} i \times (i) \times 1[-1 + 1], \quad (5.208) \]

\[ E(t)_{p_{-5}} = 0. \quad (5.209) \]

Taking \( m = 6 \) yields

\[ E(t)_{p_6} = E_c e^{i \omega t} i^6 J_6(\beta) e^{i \beta \theta} e^{-i \frac{6}{2} \pi} [e^{-i \frac{6}{2} \pi} + 1]. \quad (5.210) \]

By using the algebraic properties

\[ (i)^6 = -1, \quad (5.211) \]

\[ e^{-i \frac{6}{2} \pi} = -1, \quad (5.212) \]

\[ e^{i \frac{6}{2} \pi} = 1, \quad (5.213) \]
we obtain

\[ E(t)p_6 = E_c e^{i\omega_c t} - 1 \times J_6(\beta)e^{i6\theta}(-1)[1 + 1]. \]  
\[ (5.214) \]

\[ E(t)p_6 = 2E_c e^{i\omega_c t} e^{i6\theta} J_6(\beta). \]  
\[ (5.215) \]

Taking \( m = -6 \) yields

\[ E(t)p_{-6} = E_c e^{i\omega_c t} e^{-i6\theta} J_{-6}(\beta) e^{i6\xi} [e^{-i6\pi} + 1]. \]  
\[ (5.216) \]

By using the algebraic properties

\[ (i)^{-6} = -1, \]  
\[ (5.217) \]

\[ e^{i6\frac{\xi}{2}} = -1, \]  
\[ (5.218) \]

\[ e^{-i\pi 6\xi} = 1, \]  
\[ (5.219) \]

and the following lemma

\[ J_{-6}(x) = (-1)^6 J_6(X), \]  
\[ (5.220) \]

\[ J_{-6}(x) = J_6(X), \]  
\[ (5.221) \]

we obtain

\[ E(t)p_{-6} = E_c e^{i\omega_c t} - 1 \times J_6(\beta)e^{-i6\theta}(-1)[1 + 1]. \]  
\[ (5.222) \]

\[ E(t)p_{-6} = 2E_c e^{i\omega_c t} e^{-i6\theta} J_6(\beta). \]  
\[ (5.223) \]
5.3.2 Total Spectrum of Optical Field

We can get the output optical field by sum all output of all cases follows

\[ E(t)_{P_r} = E(t)_{p_0} + E(t)_{p_1} + E(t)_{p_2} + E(t)_{p_3} + E(t)_{p_4} + E(t)_{p_5} + E(t)_{p_6} + E(t)_{p_7}, \]

\[ E(t)_{P_r} = E(t)_{p_0} + 0 + 0 + E(t)_{p_2} + E(t)_{p_3} + 0 + 0 + E(t)_{p_5} + E(t)_{p_6} + 0 + 0, \]

\[ E(t)_{P_r} = E(t)_{p_0} + E(t)_{p_2} + E(t)_{p_3} + E(t)_{p_5} + E(t)_{p_6}. \]

By substituting the value of each term, we obtain

\[ E(t)_{P_r} = 2E_e e^{i\omega t} J_0(\beta) + 2E_e e^{i\omega t} e^{i2\theta} J_2(\beta) + 2E_e e^{i\omega t} e^{-i2\theta} J_2(\beta) + 2E_e e^{i\omega t} e^{i4\theta} J_4(\beta) + 2E_e e^{i\omega t} e^{-i4\theta} J_4(\beta) + 2E_e e^{i\omega t} e^{i6\theta} J_6(\beta) + 2E_e e^{i\omega t} e^{-i6\theta} J_6(\beta). \]

More simply

\[ E(t)_{P_r} = 2E_e e^{i\omega t} J_0(\beta) + 2E_e e^{i\omega t} J_2(\beta) \left[ e^{i2\theta} + e^{-i2\theta} \right] + 2E_e e^{i\omega t} J_4(\beta) \left[ e^{i4\theta} + e^{-i4\theta} \right] + 2E_e e^{i\omega t} J_6(\beta) \left[ e^{i6\theta} + e^{-i6\theta} \right]. \]

\[ (5.228) \]

\[ E(t)_{P_r} = 2E_e e^{i\omega t} \left[ J_0(\beta) + J_2(\beta) \left[ e^{i2\theta} + e^{-i2\theta} \right] + J_4(\beta) \left[ e^{i4\theta} + e^{-i4\theta} \right] + J_6(\beta) \left[ e^{i6\theta} + e^{-i6\theta} \right] \right]. \]

\[ (5.229) \]

5.3.3 Total Spectrum Of Optical Power

We can get the output power as follows

\[ P_{P_r} = |E(t)_{P_r}|^2 = E(t)_{P_r} E(t)_{P_r}^*. \]

\[ (5.230) \]
\[ P_{Pr} = 2E_c e^{i\omega t}\left[ J_0(\beta) + J_2(\beta)\left( e^{i2\theta} + e^{-i2\theta} \right) + J_4(\beta)\left( e^{i4\theta} + e^{-i4\theta} \right) + J_6(\beta)\left( e^{i6\theta} + e^{-i6\theta} \right) \right] \]
\[ \times 2E_c^* e^{-i\omega t}\left[ J_0(\beta) + J_2(\beta)\left( e^{i2\theta} + e^{-i2\theta} \right) + J_4(\beta)\left( e^{i4\theta} + e^{-i4\theta} \right) + J_6(\beta)\left( e^{i6\theta} + e^{-i6\theta} \right) \right], \] (5.231)

which can be further simplified to,

\[ P_{Pr} = 4P_c \left\{ J_0^2(\beta) + J_0(\beta)J_2(\beta)\left( e^{i2\theta} + e^{-i2\theta} \right) + J_0(\beta)J_4(\beta)\left( e^{i4\theta} + e^{-i4\theta} \right) + J_0(\beta)J_6(\beta)\left( e^{i6\theta} + e^{-i6\theta} \right) \right\} + J_2(\beta)J_4(\beta)\left( e^{i4\theta} + e^{-i4\theta} \right) + J_4(\beta)J_6(\beta)\left( e^{i6\theta} + e^{-i6\theta} \right) + J_0(\beta)J_6(\beta)\left( e^{i6\theta} + e^{-i6\theta} \right) + J_2(\beta)J_6(\beta)\left( e^{i6\theta} + e^{-i6\theta} \right) \]
\[ + J_4(\beta)J_6(\beta)\left( e^{i6\theta} + e^{-i6\theta} \right) \} \] (5.232)

More simply

\[ P_{Pr} = 4P_c \left\{ J_0^2(\beta) + J_2^2(\beta)\left( e^{i2\theta} + e^{-i2\theta} \right)^2 + J_4^2(\beta)\left( e^{i4\theta} + e^{-i4\theta} \right)^2 + J_6^2(\beta)\left( e^{i6\theta} + e^{-i6\theta} \right)^2 \right\} \]
\[ + 2J_0(\beta)J_2(\beta)\left( e^{i2\theta} + e^{-i2\theta} \right) + 2J_0(\beta)J_4(\beta)\left( e^{i4\theta} + e^{-i4\theta} \right) + 2J_0(\beta)J_6(\beta)\left( e^{i6\theta} + e^{-i6\theta} \right) \]
\[ + 2J_2(\beta)J_4(\beta)\left( e^{i4\theta} + e^{-i4\theta} \right) + 2J_2(\beta)J_6(\beta)\left( e^{i6\theta} + e^{-i6\theta} \right) \]
\[ + 2J_4(\beta)J_6(\beta)\left( e^{i6\theta} + e^{-i6\theta} \right) \} \] (5.233)

By using the following formulas:

\[ \cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix} \right), \] (5.234)
\[ e^{ix} + e^{-ix} = 2\cos(x). \] (5.235)
\[
\cos(2x) = \frac{1}{2} [e^{i2x} + e^{-i2x}], \quad (5.236)
\]
\[
e^{i2x} + e^{-i2x} = 2 \cos(2x). \quad (5.237)
\]
\[
\cos(4x) = \frac{1}{2} [e^{i4x} + e^{-i4x}], \quad (5.238)
\]
\[
e^{i4x} + e^{-i4x} = 2 \cos(4x). \quad (5.239)
\]
\[
\cos(6x) = \frac{1}{2} [e^{i6x} + e^{-i6x}], \quad (5.240)
\]
\[
e^{i26x} + e^{-i6x} = 2 \cos(6x). \quad (5.241)
\]

The output power can therefore be written as

\[
P_{PT} = 4P_c \left\{ J_0^2(\beta) + 4J_2^2(\beta) \cos^2(2\theta) + 4J_4^2(\beta) \cos^2(4\theta) + 4J_6^2(\beta) \cos^2(6\theta) + 4J_0(\beta)J_2(\beta) \cos(2\theta) + 4J_0(\beta)J_4(\beta) \cos(4\theta) + 4J_0(\beta)J_6(\beta) \cos(6\theta) + 2 \times 2 \times 2J_2(\beta)J_4(\beta) \cos(2\theta) \cos(4\theta) + 2 \times 2 \times 2J_4(\beta)J_6(\beta) \cos(2\theta) \cos(6\theta) + 2 \times 2 \times 2J_2(\beta)J_6(\beta) \cos(4\theta) \cos(6\theta) \right\}. \quad (5.242)
\]

By using the following formulas:

\[
\cos^2(\theta) = \frac{1}{2} [1 + \cos(2\theta)]. \quad (5.243)
\]
\[
\cos(2\theta) \cos(4\theta) = \frac{1}{2} [\cos(2\theta - 4\theta) + \cos(2\theta + 4\theta)], \quad (5.244)
\]
\[
\cos(2\theta) \cos(4\theta) = \frac{1}{2} [\cos(-2\theta) + \cos(6\theta)]. \quad (5.245)
\]

Because the function \(\cos(x)\) is even

\[
\cos(-\theta) = \cos(\theta), \quad (5.246)
\]
and therefore
\[ \cos(2\theta) \cos(4\theta) = \frac{1}{2} \left[ \cos(2\theta) + \cos(6\theta) \right]. \] (5.247)

By using the same reasoning in case of 2\( \theta \) and 6\( \theta \), we obtain
\[ \cos(2\theta) \cos(6\theta) = \frac{1}{2} \left[ \cos(4\theta) + \cos(8\theta) \right]. \] (5.248)

And in case of 4\( \theta \) and 6\( \theta \), we obtain
\[ \cos(4\theta) \cos(6\theta) = \frac{1}{2} \left[ \cos(4\theta) + \cos(10\theta) \right]. \] (5.249)

Now the output power can be written as
\[
P_{PT} = 4 P_c \left\{ J_0^2(\beta) + 4 J_2^2(\beta) \frac{1}{2} [1 + \cos(4\theta)] + 4 J_4^2(\beta) \frac{1}{2} [1 + \cos(8\theta)] + 4 J_6^2(\beta) \frac{1}{2} [1 + \cos(12\theta)] + 4 J_0(\beta) J_2(\beta) \cos(2\theta) + 4 J_0(\beta) J_4(\beta) \cos(4\theta) + 4 J_0(\beta) J_6(\beta) \cos(6\theta) + 2 \times 2 \times 2 J_2(\beta) J_4(\beta) \frac{1}{2} [\cos(2\theta) + \cos(6\theta)] + 2 \times 2 \times 2 J_2(\beta) J_6(\beta) \frac{1}{2} [\cos(4\theta) + \cos(8\theta)] + 2 \times 2 \times 2 J_4(\beta) J_6(\beta) \frac{1}{2} [\cos(2\theta) + \cos(10\theta)] \right\}. \] (5.250)

More simply
\[
P_{PT} = 4 P_c \left\{ J_0^2(\beta) + 2 J_2^2(\beta) + 2 J_2^2(\beta) \cos(4\theta) + 2 J_4^2(\beta) + 2 J_4^2(\beta) \cos(8\theta) + 2 J_6^2(\beta) + 2 J_6^2(\beta) \cos(12\theta) + 4 J_0(\beta) J_2(\beta) \cos(2\theta) + 4 J_0(\beta) J_4(\beta) \cos(4\theta) + 4 J_0(\beta) J_6(\beta) \cos(6\theta) + 2 \times 2 J_2(\beta) J_4(\beta) \cos(2\theta) + 2 \times 2 J_2(\beta) J_6(\beta) \cos(6\theta) + 2 \times 2 J_4(\beta) J_6(\beta) \cos(2\theta) \right\}. \] (5.251)
The final expression of total output power at peak bias point can be written as

\[
P_{P_r} = 4P_c \left\{ J_0^2(\beta) + 2J_2^2(\beta) + 2J_4^2(\beta) + 2J_6^2(\beta) + 4J_0(\beta)J_2(\beta) \cos(2\theta) + 4J_2(\beta)J_4(\beta) \cos(2\theta) \\
+ 4J_4(\beta)J_6(\beta) \cos(2\theta) + 4J_2^2(\beta) \cos(4\theta) + 4J_0(\beta)J_4(\beta) \cos(4\theta) + 4J_2(\beta)J_6(\beta) \cos(4\theta) + 4J_0(\beta)J_6(\beta) \cos(6\theta) \\
+ 4J_2(\beta)J_4(\beta) \cos(6\theta) + 4J_4^2(\beta) \cos(8\theta) + 4J_2(\beta)J_6(\beta) \cos(8\theta) + 4J_4(\beta)J_6(\beta) \cos(10\theta) + 4J_2^2(\beta) \cos(12\theta) \right\}. 
\]

(5.252)

Similarly, Figure 5.3 shows the output power of each term in case of operation at peak with time shifting. Where \(P_{p_{cw}}\) represents terms without \(\cos(\theta)\) and it has more value than in case of null because it has \(J_0\) terms, \(P_{p2}\) represents terms with \(\cos(2\theta)\), \(P_{p4}\) represents terms with \(\cos(4\theta)\), \(P_{p6}\) represents terms with \(\cos(6\theta)\), \(P_{p8}\) represents terms with \(\cos(8\theta)\), \(P_{p10}\) represents terms with \(\cos(10\theta)\), and \(P_{n12}\) represents terms with \(\cos(12\theta)\) as it has been written in Equation 5.252. It can be observed that they are close to each other with only a slight difference.

### 5.4 Total Output Optical Power in Jacobi-Anger Format

We can get the total output power of frequency doubled technique as follows

\[
P_T = P_{N_T} + P_{P_T},
\]

(5.253)
Figure 5.3: Modulation Index $\beta$ vs The Power Tones at Peak Bias Point

\[ P_T = 4P \left\{ 2J_1^2(\beta) + 2J_3^2(\beta) + 2J_5^2(\beta) + 2J_1^2(\beta) \cos(2\theta) - 4J_1(\beta)J_3(\beta) \cos(2\theta) - 4J_3(\beta)J_5(\beta) \cos(2\theta) \right. \]

\[ - 4J_1(\beta)J_3(\beta) \cos(4\theta) + 4J_1(\beta)J_5(\beta) \cos(4\theta) + 2J_3^2(\beta) \cos(6\theta) + 4J_1(\beta)J_3(\beta) \cos(6\theta) \]

\[ - 4J_3(\beta)J_5(\beta) \cos(8\theta) + 2J_5^2(\beta) \cos(10\theta) \}\]

\[ + 4P_c \left\{ J_0^2(\beta) + 2J_2^2(\beta) + 2J_4^2(\beta) + 2J_6^2(\beta) + 4J_0(\beta)J_2(\beta) \cos(2\theta) + 4J_2(\beta)J_4(\beta) \cos(2\theta) \right. \]

\[ + 4J_4(\beta)J_6(\beta) \cos(2\theta) + 2J_6^2(\beta) \cos(4\theta) + 4J_0(\beta)J_4(\beta) \cos(4\theta) + 4J_2(\beta)J_6(\beta) \cos(4\theta) + 4J_0(\beta)J_6(\beta) \cos(6\theta) \]

\[ + 4J_2(\beta)J_4(\beta) \cos(6\theta) + 2J_4^2(\beta) \cos(8\theta) + 4J_2(\beta)J_6(\beta) \cos(8\theta) + 4J_4(\beta)J_6(\beta) \cos(10\theta) + 4J_6^2(\beta) \cos(12\theta) \}\].

\[ (5.254) \]
More Simply

\[ P_T = 4P_c \left( J_0^2(\beta) + 2J_1^2(\beta) + 2J_2^2(\beta) + 2J_3^2(\beta) + 2J_4^2(\beta) + 2J_5^2(\beta) + 2J_6^2(\beta) \right. \]

\[ + 2J_1^2(\beta) \cos(2\theta) + 4J_0(\beta)J_2(\beta) \cos(2\theta) - 4J_1(\beta)J_3(\beta) \cos(2\theta) + 4J_2(\beta)J_4(\beta) \cos(2\theta) - 4J_3(\beta)J_5(\beta) \cos(2\theta) \]

\[ + 4J_2(\beta)J_4(\beta) \cos(2\theta) + 4J_4(\beta)J_6(\beta) \cos(2\theta) \]

\[ + 2J_2^2(\beta) \cos(4\theta) + 4J_0(\beta)J_4(\beta) \cos(4\theta) - 4J_1(\beta)J_5(\beta) \cos(4\theta) + 4J_2(\beta)J_6(\beta) \cos(4\theta) + 2J_3^2(\beta) \cos(6\theta) + 4J_0(\beta)J_6(\beta) \cos(6\theta) + 4J_1(\beta)J_5(\beta) \cos(6\theta) + 4J_2(\beta)J_6(\beta) \cos(6\theta) \]

\[ + 2J_4^2(\beta) \cos(8\theta) + 4J_2(\beta)J_6(\beta) \cos(8\theta) - 4J_3(\beta)J_5(\beta) \cos(8\theta) + 2J_5^2(\beta) \cos(10\theta) + 4J_4(\beta)J_6(\beta) \cos(10\theta) + 4J_6^2(\beta) \cos(12\theta) \]. (5.255)

The Final expression can be written as

\[ P_T = 4P_c \left( J_0^2(\beta) + 2J_1^2(\beta) + 2J_2^2(\beta) + 2J_3^2(\beta) + 2J_4^2(\beta) + 2J_5^2(\beta) + 2J_6^2(\beta) \right. \]

\[ + \left[ 2J_1^2(\beta) + 4J_0(\beta)J_2(\beta) - 4J_1(\beta)J_3(\beta) + 4J_2(\beta)J_4(\beta) - 4J_3(\beta)J_5(\beta) + 4J_4(\beta)J_6(\beta) \right] \cos(2\theta) \]

\[ + \left[ 2J_2^2(\beta) + 4J_0(\beta)J_4(\beta) - 4J_1(\beta)J_5(\beta) + 4J_2(\beta)J_6(\beta) \right] \cos(4\theta) \]

\[ + \left[ 2J_3^2(\beta) + 4J_0(\beta)J_6(\beta) + 4J_1(\beta)J_5(\beta) + 4J_2(\beta)J_4(\beta) \right] \cos(6\theta) \]

\[ + \left[ 2J_4^2(\beta) + 4J_2(\beta)J_6(\beta) - 4J_3(\beta)J_5(\beta) \right] \cos(8\theta) \]

\[ + \left[ 2J_5^2(\beta) + 4J_4(\beta)J_6(\beta) \right] \cos(10\theta) + \left[ 4J_6^2(\beta) \cos(12\theta) \right] \}. (5.256) \]

Figure 5.4 illustrates the total output power of each term when using the frequency doubled technique. Where Pcw represents the sum of PNcw and Ppcw, P2 represents the sum of PN2 and Pp2, P4 represents the sum of PN4 and Pp4, P6 represents the sum of PN6 and Pp6, P8 represents the sum of PN8 and Pp8, and P10 represents the sum of PN10 and Pp10.
As it has been written in Equation 5.252. It can be observed that P2 has the most value with 
represents all terms that include cos(2θ) . Recall Equations 5.5 & 5.6 which illustrates that 
the input frequency where Ω, and compare that to the output equation we can conclude that 
the frequency has been doubled to 2Ω, 4Ω, 6Ω, 8Ω, and so on. This confirms the technique 
proposed in this paper. In addition it can be observe that the output power of second tone 
has the highest power comparing to the rest, and the following power is the the power of 
sixth tone, and that is exactly match the output of Taylor series Expansion.

Figure 5.4: Modulation Index β vs The Power Tones
As we have seen in Section 4.3 and 4.4, the spectral output of frequency doubling frequency have tones at 2F and 6F. Similarly Figure 5.5 shows that the power of $\cos(6\theta)$ is bigger than the power of $\cos(4\theta)$ where the subtract of the difference power between the power of $\cos(2\theta)$ and the power of $\cos(6\theta)$ is less than the difference power between the power of $\cos(2\theta)$ from the power of $\cos(4\theta)$.
Chapter 6

Conclusion

This thesis performed a detailed analysis of a frequency doubling technique using Mach Zehnder modulator. An analysis was first performed on the outputs of MZMs without time shift in chapter 2 in order to emphasize important principle of MZM operates at maximum or minimum bias points. In the later chapters, the derivations of frequency doubling technique has been demonstrated using Taylor Series Expansion and Jacobi-Anger Expansion. The results obtained show the error between the pure sine wave by using analytic and numerical methods, whereas the output signal from our technique was doubled in frequency with nice sinusoid shape.
Bibliography


