Construction of Multi-Mode Fiber Modes using Phase Masks

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Dedicated to my family
Abstract

Transverse fiber modes are the distribution of electromagnetic fields in the cross-section of an optical fiber through which light propagates in the fiber. Research is being conducted to explore the usage of fiber modes with the purpose of increasing the capacity and reach of optical fibers. Mode Division Multiplexing, an area in which research is being conducted with the same purpose, is a multiplexing scheme used in optical networks that maps data channels onto different modes, and multiplexes these modes into one fiber for transmission. This thesis focuses on the study of modes in multi-mode fibers, and the simulation of mode creation in gradient-index multi-mode fibers.

Starting from published related works, equations for electric fields in step-index and gradient-index multi-mode fibers are derived, and examples of intensity profiles in several modes are shown. For the first time, we seek to study modal cross-talk between modes. Given the creation process, not all power is coupled into the desired mode, resulting in some power being coupled into undesired modes. We also seek to increase the power coupling between the created mode and the desired mode. We report a 23% power coupling increase up from previous works. Finally, we seek to study the effect of noise introduced by the components used in the process of creating modes. We report an approximate 30dB signal-to-noise ratio is sufficient to create a mode with maximum power coupling.
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Chapter 1

Introduction

The ever growing demand for higher speeds of information transmission is leading to the exploration of new and more efficient ways of transmitting information. As the existing high speed links are approaching their full capacity [1], it is imperative to explore new ways to accommodate the demand for high speeds, and to provision for future higher speeds demand. This demand is more accentuated in data centers and other local-area networks (LAN), as the amount of information to transmit in these networks keeps increasing. Multimode fibers (MMF) are the type of fibers widely employed in these networks [2]. It is desired to transmit data at 10 Gb/s and higher bit rates over long distances of MMF, but the modal dispersion in these fibers limits the bit rate-transmission length product in existing infrastructures [2].
1.1 Telecommunications Optical Fibers

There are mainly three types of optical fibers utilized in the Telecommunications industry: single-mode fiber, step-index multi-mode fiber, and gradient-index multi-mode fiber. Single-mode fibers have a smaller core of a high refractive index $n_1$ surrounded by a cladding of a low refractive index $n_2$. Because of the small size of the core, single-mode fibers allow only one distribution of electromagnetic field, namely mode. They are generally used in long haul communications. Step-index and gradient-index multi-mode fibers are types of multi-mode fibers. Multi-mode fibers have a larger core compared with single-mode fibers. In step-index multi-mode fibers, the refractive index of the core is constant across the core and is higher than that of the cladding. Gradient-index multi-mode fibers have a varying refractive index in the core, following a decreasing power law with the maximum value at the center of the core.
1.2 Modal Dispersion

As light, encoded with information to transmit, is launched in a MMF, several different transverse modes are excited in which light propagates [3]. The existence of these modes depends on the launch condition at the transmitter, among other dependencies. The choice of MMF in LAN is based on the ease of light launch into the fiber, and therefore a less expensive infrastructure over single mode fibers (SMF). Since these modes have distinct group velocities, the information carried by the modes reach the receiver at different times. This physical phenomenon is known as modal dispersion. The direct effect of modal dispersion is Inter-symbol Interference (ISI). A pulse leaving the transmitter reaches the receiver after having overlapped with its neighboring pulse [3]. Figure 1.1 shows two distinct pulses at the transmitter and the subsequent four pulses at the receiver. ISI due to modal dispersion in MMF is the dominant limitation to the bit rate-transmission length in MMF.
There have been numerous approaches to compensate for modal dispersion effects using digital signal processing [14] and other inline techniques using adaptive optics. The compensation for modal dispersion using a spatial light modulator (SLM) proves to be more efficient since it does not require a prior knowledge of the refractive index profile of the transmission fiber [2]. By adaptively changing the phase mask applied on the input light beam, a 10Gbit/s transmission over 11.1 km has been demonstrated [2].

### 1.3 Mode Division Multiplexing

The plurality of modes in MMF has been long viewed as a negative effect on a communication link due to the modal dispersion explained above. However, the existence of multiple modes is now been seen as a way of increasing the bandwidth of a MMF through Mode Division Multiplexing (MDM) [15]. In a MDM, each mode is considered as an independent channel transmitting separate signals, as shown in Figure 1.2. Each signal is mapped to a specific mode, and a multiplexer at the transmitter couples all the signals in a single transmission fiber.
1.4 Thesis Overview

The focus of the thesis is to conduct a study of modes in gradient-index fibers and to simulate their creation using MATLAB. In chapter 2, a theory on transverse modes in step-index and gradient-index multi-mode fibers is given. The chapter explains the geometry of these two types of fibers and gives the equation for the electric field in each type. In chapter 3, a detailed study of mode creation is conducted and the results are shown. Chapter 4 is an extension of chapter 3 where phase noise is added in the creation of modes and the results are shown.

1.4.1 Previous Related Work

In 2005, a technique to compensate for multi-mode fiber dispersion by adaptive optics was reported by Shen et al. [14]. In this technique a Spatial Light Modulator is used to control the launched mode into an output fiber. This
is accomplished by estimating ISI at the receiver side, and this information is fed back to the transmitter, which adjusts the SLM in order to minimize ISI. This loop continues until the minimum ISI is reached.

In 2011, yet another technique that uses an SLM for mode creation was reported by Stepniak et al. [8]. In this technique, however, the SLM takes on two values, 0 or $\pi$. Instead of utilizing ISI to adjust the SLM as in the previous work did, the pattern applied to the SLM corresponds to the target mode desired at the receiver. This method only works with a type of gradient-index multi-mode fiber modes, namely Laguerre polynomial-based modes, since the phase of these modes exhibits a $\pi$ phase shift.

\subsection*{1.4.2 Advancement of the State of the Art by the Current Thesis}

In this thesis, we have extended the work by Stepniak et al. [8] by studying the modal cross-talk in gradient-index multi-mode fibers. This is an important study because modes are not created with 100%. Some of the power is coupled into other modes which causes the problems of modal dispersion. Knowing the power coupling between modes can help when designing a network that can only tolerated a given modal dispersion.

Also in this thesis, a new technique that increases the power coupling
by up to 23\% is proposed. Finally, the noise of the SLM is considered in the process of creating gradient-index multi-mode fibers. This is a valid study because opto-electronics components of the SLM will inherently introduce noise to the pattern applied to the SLM.
Chapter 2

Mode Theory in Multi-mode Fibers

2.1 Geometry of Fibers

An optical fiber is a cylindrical dielectric waveguide that consists of an inner region, namely core, of high refractive index surrounded by an outer region, namely cladding, of lower refractive index as shown in Figure 2.1. Two types of multi-mode fibers will be studied in this section: step-index fibers and gradient-index fibers.
Figure 2.1: Refractive index profile in: (a) multi-mode step-index fiber, (b) a single-mode step-index fiber, and (c) a multi-mode gradient-index fiber.


2.2 Multi-mode Step-index Fibers

In multi-mode step-index fibers, the core fills the whole region $r \leq a$ with a refractive index $n_1$, and the cladding region $r > a$ is of index $n_2$. The normalized index difference of a multi-mode step-index fiber is expressed as

\[
\Delta = \frac{n_1^2 - n_2^2}{2n_1^2} \approx \frac{n_1 - n_2}{n_1}. \tag{2.1}
\]

In most commercial fibers, $n_1 \approx n_2$, and thus the approximation in Eq. (2.1). This condition is called the weak-guidance condition. The numerical aperture (NA) defines the acceptance cone in which the incident ray will experience total internal reflection. The NA is given by

\[
NA = \sqrt{n_1^2 - n_2^2} = n_1\sqrt{2\Delta}. \tag{2.2}
\]

2.2.1 Electric Field in Multi-mode Step-index Fiber

Given the weak-guidance approximation, linearly polarized (LP) modes arise in optical fibers [7]. These modes consist of an x-polarized electric component and a y-polarized magnetic component [5]. The field components are

\[
E = E_x(r, \theta, z)\hat{a}_x = E_{x0}(r, \theta)exp(-i\beta z)\hat{a}_x \tag{2.3}
\]
and
\[ H = H_y(r, \theta, z) \hat{a}_y = H_{y0}(r, \theta) e^{i \beta z} \hat{a}_y \tag{2.4} \]

where \( r \) is the radius from the center of the core and \( \theta \) the azimuthal angle.

Using Maxwell’s equations and Helmholtz equations

\[ \nabla^2 E + k^2 E = 0 \tag{2.5} \]

and

\[ \nabla^2 H + k^2 H = 0 \tag{2.6} \]

where the transverse Laplacian

\[ \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{2.7} \]

and assuming tranverse variation in both \( r \) and \( \theta \) [5], the wave equation in either region is given by

\[ \frac{\partial^2 E_x}{\partial r^2} + \frac{1}{r} \frac{\partial E_x}{\partial r} + \frac{1}{r^2} \frac{\partial^2 E_x}{\partial \theta^2} + \beta_t^2 E_x = 0 \tag{2.8} \]

where

- \( \beta_t^2 = (n_1^2 k_0^2 - \beta^2) \) is the transverse propagation constant
- \( k_0 = \frac{2\pi}{\lambda_0} \) is the free space wave number
- \( \beta \) is the propagation constant in the \( z \) direction
Assuming the solution for \( E_x \) to be a series of modes separable in \( r \), \( \theta \), and \( z \), the solution \( E_x \) takes on a form of

\[
E_x = \sum_i R_i(r)\Phi_i(\theta) \exp(-i\beta_i z).
\]  

(2.9)

Each term \( E_x = R\Phi \exp(-i\beta_i z) \) in Eq. (2.9) is itself a solution to Eq. (2.8), and can be substituted in Eq. (2.8) to obtain

\[
\frac{r^2}{R} \frac{d^2 R}{dr^2} + \frac{r}{R} \frac{dR}{dr} + r^2 \beta_i^2 = -\frac{1}{\Phi} \frac{d^2 \Phi}{d\theta^2}.
\]  

(2.10)

Since \( r \) and \( \theta \) are independent, it follows that both sides of Eq. (2.10) are equal to a constant, namely \( l^2 \), and are separable into

\[
\frac{d^2 \Phi}{d\theta^2} + l^2 \Phi = 0
\]  

(2.11)

and

\[
\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + [\beta_i^2 - \frac{l^2}{r^2}] R = 0.
\]  

(2.12)

The solutions to Eq. (2.11) and Eq. (2.12) are

\[
\Phi(\theta) = \begin{cases} 
\cos(l\theta + \alpha) \\
\sin(l\theta + \alpha)
\end{cases}
\]  

(2.13)
and

\[ E_x = \begin{cases} 
J_l(w_{a}^{l}) & r \leq a \\
K_l(w_{a}^{l}) & r > a 
\end{cases} \]  \hspace{1cm} (2.14)

respectively, where:

- \( J_l \) and \( K_l \) are ordinary Bessel functions of first kind of order \( l \) and modified Bessel function of first kind of order \( l \), respectively,

- \( u = a\sqrt{n_1^2k_0^2 - \beta^2} \) and \( w = a\sqrt{\beta^2 - n_2^2k_0^2} \) are normalized transverse phase and attenuation constants, respectively.

The complete electric field \( E_{lm} \) and magnetic field \( H_{lm} \) are

\[ E_{lm} = \begin{cases} 
J_l(u_{a}^{l,m})\cos(l\theta)\exp(-i\beta z), & r \leq a \\
\frac{J_l(u_{a}^{l,m})}{K_l(u_{a}^{l,m})}K_l(w_{a}^{l,m})\cos(l\theta)\exp(-i\beta z), & r > a 
\end{cases} \]  \hspace{1cm} (2.15)

\[ H_{lm} = \begin{cases} 
J_l(u_{a}^{l,m})\cos(l\theta)\exp(-i\beta z), & r \leq a \\
\frac{J_l(u_{a}^{l,m})}{K_l(w_{a}^{l,m})}K_l(w_{a}^{l,m})\cos(l\theta)\exp(-i\beta z), & r > a 
\end{cases} \]  \hspace{1cm} (2.16)

where \( l \) and \( m \) designate the azimuthal mode number and the radial mode number, respectively.
2.2.2 Eigenvalues of Multi-mode Step-index Fibers

Some conditions, such as the core radius, the operating wavelength, and the index difference between core and cladding, have to be met for modes to be excited in the fiber, and these parameters have to be set so that the boundary condition is met: the electric and magnetic fields must be continuous at the core-cladding boundary \( r = a \). This condition is called the characteristic equation or dispersion relation. In weakly guiding fibers, this condition is approximated by the condition that Eq. (2.14) be continuous at \( r = a \), and is satisfied when

\[
\frac{J_{l-1}(u)}{J_l(u)} = -\frac{w}{u} \frac{K_{l-1}(w)}{K_l(w)}
\]  

(2.17)

where

\[ V^2 = u^2 + w^2, \]  

(2.18)

and \( V \) is the V-parameter given by \( V = 2\pi \frac{a}{\lambda_0} NA \). The solution to Eq. (2.17) yields values for \( u \) and \( w \) for which the given mode is excited.

However, this equation is a transcendental function and cannot be solved using the algebraic methods. The equation can be rewritten as

\[
\frac{J_{l-1}(u)}{J_l(u)} + \frac{w}{u} \frac{K_{l-1}(w)}{K_l(w)} = 0
\]  

(2.19)

and use a computer tool such as MATLAB to find the roots of the
equation. A graphical method is another way of solving this equation. The right-hand side and the left-hand side of Eq. (2.17) are plotted against $u$ and their intersections yield the values of $u$ and $w$ given $V$. Figure 2.2 shows a graphical solution to Eq. (2.17).
Figure 2.2: Graphical solution for multi-mode step-index $E_{0,m}$ and $E_{1,m}$. modes
Figure 2.3: Graphical solution for multi-mode step-index $E_{3,m}$ and $E_{4,m}$. Modes
2.2.3 Intensity Profiles of some Step-index Modes

The intensity profile of a step-index multi-mode fiber modes is given by

\[ I_{l,m} = |E_{l,m}|^2. \]  \hspace{1cm} (2.20)

Replacing the values of \( u_{l,m} \) and \( w_{l,m} \) obtained from the graphical solution of Eq. (2.17) into Eq. (2.15), and setting \( z = 0 \), the intensity plots of \( E_{01} \), \( E_{04} \), \( E_{32} \), and \( E_{34} \) are shown in Figure 2.4.
Figure 2.4: Intensity plots for four E modes in multi-mode step-index fiber.
2.3 Multi-mode Gradient-index Fibers

In gradient-index fibers, the core does not have a uniform refractive index as it is the case in step-index fibers. The refractive index in the core decreases with increasing radial distance from the center of the core. The *slowly varying index* criterion, used in geometrical optics, is used in gradient-index fibers to overcome the diffraction and scattering effects that occur when the refractive index abruptly changes \[9\]. This criterion is defined as

\[
|\lambda \frac{dn}{dx}/n| << 1
\]  \hspace{1cm} (2.21)

and is interpreted as follows: the variation in refractive index that occurs over a distance comparable to a wavelength is much less than the average
refractive index in core region.

The most used refractive index profile in gradient-index fibers is parabolic, i.e., the light takes on a sinusoidal path as it propagates down the fiber as shown in Figure 2.1. The advantage of this profile over a step-index profile is that the modal dispersion is minimized [9]. The parabolic refractive index profile is given by

\[ n^2(r) = n_1^2 \left[ 1 - 2\Delta \left( \frac{r}{a} \right)^p \right] \quad (2.22) \]

where

- \( \Delta = \frac{n_1^2 - n_2^2}{2n_1^2} \approx \frac{n_1 - n_2}{n_1} \)
- \( p \), the grade profile parameter, determines the steepness of the profile
- \( n_1 \) highest refractive index at the center of the core
- \( n_2 \) refractive index in the cladding

The grade profile parameter of \( p = 2 \) is used for best reduction of modal dispersion [9] and will be used for the remainder of this section.
2.3.1 Electric Field in Multi-mode Gradient-index Fibers

In gradient-index fibers, Maxwell’s equations and Helmholtz equations are given by

\[ \nabla^2 E + (n^2(r)k_0^2 - \beta^2)E = 0 \quad (2.23) \]

and

\[ \nabla^2 H + (n^2(r)k_0^2 - \beta^2)H = 0. \quad (2.24) \]

To solve these equations, a similar method as in step-index fibers is used. By substituting one term of Eq. (2.9) into Eq. (2.23), the wave equation for a gradient-index fiber becomes

\[ \frac{d^2 R(r)}{dr^2} + \frac{1}{r} \frac{dR(r)}{dr} + \left[ n^2(r)k_0^2 - \beta^2 - \frac{l^2}{r^2} \right] R(r) = 0 \quad (2.25) \]

and

\[ \frac{d^2 \Phi}{d\theta^2} + l^2 \Phi = 0. \quad (2.26) \]

Eq. (2.26) is the same as Eq. (2.11) and yields the same solution. Eq. (2.25), however, is different from Eq. (2.12) in that \( n^2(r) \) is not a constant. However, a qualitatively similar function to a Bessel function results as a solution.
Substituting Eq. (2.22) with $p = 2$ in Eq. (2.25) yields a Laguerre- or Hermite-Gaussian function. In other cases where $p \neq 2$, the WKB method, invented by Kramers and L. Brillouin, is used to approximate the solution [9]. The complete electric field in a gradient-index fiber using the Laguerre form solution is

$$E_{lm} = \begin{cases} L_{l-1}^m(\rho^2)\rho^l e^{\frac{-\rho^2}{2}} \cos(l\theta) e^{-i\beta z} & r \leq a \\ \frac{J_l(u_{lm})}{K_l(w_{lm})} K_{l}(w_{ra}) e^{i\beta z} e^{-\frac{\rho^2}{2}} \cos(l\theta) & r > a \end{cases} \quad (2.27)$$

where

$$\rho = \sqrt{\frac{k_0 n_1}{a} r} \quad (2.28)$$

and the Laguerre function is

$$L_n^l = \sum_{s=0}^{n} \frac{(n+l)!(\rho^2)^s}{(l+s)!(n-s)!s!} \quad (2.29).$$

### 2.3.2 Orthogonality of Gradient-index Fiber Modes

The coefficient that defines how good two modes of electric fields $E_{l_1,m_1}$ and $E_{l_2,m_2}$ overlap is defined as an overlap integral given by

$$\eta = \frac{\left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_{l_1,m_1}(x,y) E_{l_2,m_2}^*(x,y) dx dy \right|^2}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |E_{l_1,m_1}(x,y)|^2 dx dy \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |E_{l_2,m_2}(x,y)|^2 dx dy}. \quad (2.30)$$
In polar coordinates, Eq. (2.30) is

\[ \eta = \frac{\left[ \int \int E_{l_1, m_1} (r, \theta) E_{l_2, m_2}^* (r, \theta) r dr d\theta \right]^2}{\int \int |E_{l_1, m_1} (r, \theta)|^2 r dr d\theta \int \int |E_{l_2, m_2} (r, \theta)|^2 r dr d\theta}. \]  \hspace{1cm} (2.31)

Substituting the electric fields in Eq. (2.31) by their respective expressions in Eq. (2.15), the numerator becomes

\[ \left| \int \int L_{l_1}^{m_1'} (\rho^2 r^2) L_{l_2}^{m_2'} (\rho^2 r^2) (\rho r)^{l_1 + l_2} e^{-\rho^2 r^2} \cos(l_1 \theta) \cos(l_2 \theta) r dr d\theta \right|^2 \]  \hspace{1cm} (2.32)

where \( m_n' = m_n - 1 \)

By substituting \( \rho^2 r^2 \) by \( \varphi, r dr = \frac{1}{2\rho^4} d\varphi \), Eq. (2.32) becomes

\[ \frac{1}{4\rho^4} \left| \int \int L_{l_1}^{m_1'} (\varphi) L_{l_2}^{m_2'} (\varphi) \varphi^{l_1 + l_2} e^{-\varphi} \cos(l_1 \theta) \cos(l_2 \theta) d\varphi d\theta \right|^2 \]  \hspace{1cm} (2.33)

Since \( \varphi \) and \( \theta \) vary independently, Eq. (2.33) can be written as

\[ \frac{1}{4\rho^4} \left| \int \cos(l_1 \theta) \cos(l_2 \theta) d\theta \int \int L_{l_1}^{m_1'} (\varphi) L_{l_2}^{m_2'} (\varphi) \varphi^{l_1 + l_2} e^{-\varphi} d\varphi \right|^2 \]  \hspace{1cm} (2.34)
The Laguerre functions are orthogonal over \([0, \infty)\) with respect to the weighting function \(x^le^{-x}\), i.e,

\[
\int_0^\infty L_{m_1}^{l_1}(x) L_{m_2}^{l_2}(x) x^{l_1+l_2} e^{-x} dx = \begin{cases} 
0, & m_1 \neq m_2 \text{ and } l_1 = l_2 \\
\frac{(m+l)!}{m!}, & m_1 = m_2 \text{ and } l_1 = l_2
\end{cases}
\]  \hspace{1cm} (2.35)

Additionally, cosine functions are also orthogonal \([16]\), as in Eq. (2.36),

\[
\int_{-\pi}^{\pi} \cos(l_1x) \cos(l_2x) dx = \begin{cases} 
0, & l_1 \neq l_2 \\
1, & l_1 = l_2
\end{cases}
\]  \hspace{1cm} (2.36)

Given Eq. (2.35) and Eq. (2.36), it follows that the power coupling between two modes of electric fields \(E_{l_1,m_1}\) and \(E_{l_2,m_2}\) is

\[
\eta = \begin{cases} 
0, & m_1 \neq m_2 \text{ or } l_1 \neq l_2 \\
1, & m_1 = m_2 \text{ and } l_1 = l_2
\end{cases}
\]  \hspace{1cm} (2.37)

Modes of identical azimuthal and radial mode numbers completely overlap, and modes of different azimuthal mode number or radial mode number are orthogonal.
2.3.3 Intensity Profiles of some Gradient-index Modes

The intensity profile of a gradient-index multi-mode fiber modes is given by

\[ I_{l,m} = |E_{l,m}|^2. \quad (2.38) \]

Figure 2.6 shows the intensity plots of \( E_{01}, E_{04}, E_{32}, \) and \( E_{33} \) in a gradient-index fiber.
Figure 2.6: Intensity plots for four modes in gradient-index fiber.
Chapter 3

Transverse-Mode Phase Masks

The result of Fourier-transforming a signal that changes amplitude and phase in the near-field shows the change of amplitude and phase in the far-field. Figure 3.1 shows two images in the near-field and Figure 3.2 shows their corresponding amplitude and phase variations in the far-field.

The spectral amplitude and the spectral phase have been studied to see which preserves more information about the original signal. In some situations, only partial Fourier domain information is available or desired over the other to reconstruct the original signal. It has been shown that phase-only holograms, where the phase is recorded and the amplitude is assumed to be unity, the signal is reconstructed with more features of the original signal than the reconstructed signal from an amplitude-only hologram, where the amplitude is recored and the phase is assumed to be
zero [8][13]. Figure 3.3 shows this interesting finding. The phase of Image A is used in the reconstruction of Image B, while the phase of Image B is used in the reconstruction of Image A. We can see that the spectral phase carries more information about the original signal than the spectral amplitude. It follows that using a phase-only spatial light modulator, fiber modes can be reconstructed with minimal error.

Figure 3.1: In the near-field (a) Image A, (b) Image B.
Figure 3.2: In the far-field (a) Image A amplitude, (b) Image A phase, (c) Image B amplitude, (d) Image B phase.
Figure 3.3: Reconstructed Images (a) Amplitude of Image A with phase of Image B, (b) Amplitude of Image B with phase of Image A.
3.1 Construction of Fiber Modes using a Spatial Light Modulator

Consider a 4-f optical setup shown in Figure 3.4. A spatial light modulator (SLM) is placed in the Fourier plane of the first lens. The electric field $E_{si}$ incident upon the SLM is the Fourier transform of the signal originating from the input fiber and is given by [10]
\[ E_{si}(f_x, f_y) \propto FT\{E_i\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_i(x_1, y_1)e^{j2\pi(f_xx_1+f_yy_1)}dx_1dy_1 \]  \hspace{1cm} (3.1)

where

- \( E_i \) is the electric field from the input fiber
- \( f_x = \frac{x}{\lambda f} \) and \( f_y = \frac{y}{\lambda f} \) are the spatial frequencies
- \( (x, y) \) is the coordinate system in the Fourier plane
- \( \lambda \) is the signal wavelength
- \( f \) is the focal distance of the lens
- \( FT \) is the Fourier transform symbol

The incident field upon the second lens is the product of \( E_{si} \) and the transfer function of the SLM \( T(f_x, f_y) \). The second lens performs another Fourier transform of the incident signal and focuses the light into the transmission fiber. Thus, the electric field \( E_0 \) incident upon the transmission fiber is given by [10]

\[ E_0(x_2, y_2) \propto FT\{T(f_x, f_y)E_{si}(f_x, f_y)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T(f_x, f_y)E_{si}(f_x, f_y)e^{j2\pi(x_2f_x+y_2f_y)}df_xdf_y \]  \hspace{1cm} (3.2)
where \((x_2, y_2)\) is the coordinate system in the plane of the transmission fiber.

Although the field propagating out of the input fiber is usually considered to be Gaussian, it can be assumed to be an impulse function \(\delta(x_1, y_1)\) without much compromise [12]. With this assumption, it follows that \(E_{si} \propto FT\{\delta(x_1, y_1)\} = 1\). In turn, Eq. (3.2) becomes

\[
E_o \propto FT\{T(f_x, f_y)\} = \int_{-\infty}^{\infty} T(f_x, f_y)e^{j2\pi(x_2f_x+y_2f_y)}df_xdf_y. \tag{3.3}
\]

At the transmission fiber, to excite a specific mode, the incident electric field \(E_o(x_2, y_2)\) has to match or overlap with the mode to be excited. The quality of overlap is calculated with an overlap integral given by [11]

\[
\eta = \frac{\left| \int \int E_o(x_2, y_2)E_M^*(x_2, y_2)dx_2dy_2 \right|^2}{\int \int |E_o(x_2, y_2)|^2dx_2dy_2 \int \int |E_M(x_2, y_2)|^2dx_2dy_2} \tag{3.4}
\]

where \(\eta\) is the power coupling coefficient (PC) and \(E_M\) is the electric field of the mode to be excited.

The maximum value \(\eta\) takes on is 1 and this happens when \(E_o = E_M\), i.e., \(E_o\) and \(E_M\) completely overlap. It is then evident that in order to excite a given mode \(E_M\), the transfer function of the SLM has to be the inverse Fourier transform of \(E_M\).
\[ T(f_x, f_y) = FT^{-1}\{E_M(x_2, y_2)\} = \int_{-\infty}^{\infty} E_M(x_2, y_2)e^{-j2\pi(f_xx_2 + f_yy_2)}dx_2dy_2. \]

(3.5)

### 3.2 Unit Amplitude

As discussed earlier, the spectral phase of a signal carries more information about the original signal than the spectral amplitude. Using a phase-only spatial light modulator and assuming that the amplitude of the light field reaching the front end of the SLM is a unit over the transmissive region of the SLM, a desired mode is excited in the transmission fiber given that the transfer function of the SLM is equal to the spectral phase of the desired mode transform,

\[ \tilde{T}(f_x, f_y) = \frac{T(f_x, f_y)}{|T(f_x, f_y)|}. \]

(3.6)

#### 3.2.1 Results

This section shows the results of constructed gradient fiber modes using the setup shown in Figure 3.4. Figure 3.5 and Figure 3.6 shows how well modes are created and the cross-talk with other modes. Mode \( E_{0,1} \) is created with a power coupling of approximately 2.5x10\(^{-5} \) and cross-talks
with mode $E_{0,4}$ with a cross-talk coefficient of approximately $1.7 \times 10^{-5}$. The cross-talk coefficient is defined as the power coupling between a mode and another mode other than itself. Mode $E_{3,2}$ has a power coupling of approximately $8.8 \times 10^{-5}$, and a cross-talk coefficient with mode $E_{2,3}$ of approximately $0.2 \times 10^{-5}$.

![Power Coupling & Modal Cross-talk](image.png)

Figure 3.5: Power Coupling and Modal Cross-talk for mode $E_{0,1}$. 

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Figure 3.6: Power Coupling and Modal Cross-talk for mode $E_{3,2}$. 
As the results show, using a unit amplitude across the transmissive area of the SLM and using the phase signal of the target mode, the modes are created at the output fiber with very little power coupling. The following sections focus on improving this technique.

3.3 Optimal Aperture

3.3.1 Abbe’s Theory

Consider the 4-f optical system in Figure 3.7. Each point in the Fourier plane \((x, y)\) of the first lens corresponds to a spatial frequency and amplitude. Abbe’s theory states that to reconstruct the image of the original object, as many diffraction orders as possible must be recombined for higher quality image of the object [13]. These diffraction orders correspond to spatial frequencies. The second lens recombines the spatial frequencies to reconstruct the image in the object plane \((x_2, y_2)\). The spatial frequencies \((f_x, f_y)\) in the Fourier plane are related to \((x, y)\) system by Eq. (3.7) and Eq. (3.8) [4].
Figure 3.7: Abbe’s Theory of Imaging.

\[ f_x = \frac{x}{\lambda f} \]  \hspace{1cm} (3.7)

and

\[ f_y = \frac{y}{\lambda f} \]  \hspace{1cm} (3.8)

where \( \lambda \) is the light wavelength and \( f \) is the focal distance of the first and second lens.
3.3.2 Aperture Diameter Optimization

The assumption that the light field reaching the SLM has a constant amplitude across the transmissive area of the SLM requires that higher frequencies be filtered out for maximum power coupling. To understand this reasoning, let's consider the transfer function $T$ of the SLM to be the inverse Fourier transform of $E_M$ with both amplitude and phase. Here we assuming that the SLM is configured to modulate amplitude and phase. If $T$ is applied to the SLM, the resulting electric field $E_o$ at the input on the transmission fiber completely overlaps with $E_M$, i.e., $\eta = 1$. However, it is evident from Figure 3.8 that not all the spectral frequencies have the same amplitude; there is a maximum frequency passed which the amplitude becomes zero.
However, if the amplitude component of the transfer function is assumed to be constant across the spectrum, higher order frequencies have to be filtered out in order to achieve a high $\eta$, otherwise the resulting strong interference at high frequencies will cause the power coupling $\eta$ to be much less than 1. This is the case of Section 3.2. It follows that an aperture is required to block higher frequencies, and the resulting setup is shown in Figure 3.9. It is then imperative to determine the optimal aperture.
diameter that will result in maximum $\eta$.

Figure 3.9: An aperture is placed in front of SLM to block high frequencies.

We calculate the optimal aperture diameter by varying the maximum frequency allowed through the aperture and recording the resulting power coupling. Eq. (3.7) and Eq. (3.8) shows the relationship between $(x, y)$ coordinates in the Fourier plane and their corresponding spatial frequencies. Figure 3.10 and Figure 3.11 show the change of $\eta$ with respect to the size of the spatial frequency. We can see that there is an
optimal diameter at which the highest power coupling is reached for any given mode. For example, for mode $E_{0,4}$, the maximum frequency, for both $f_x$ and $f_y$, at which the highest power coupling of approximately 0.8 is 0.115$mm^{-1}$. Using Eq. (3.8) where the focal distance $f = 20cm$ and the wavelength $\lambda = 850nm$, we calculate the corresponding optimal diameter of the aperture to be approximately 0.04$mm$. 
Figure 3.10: Change of $\eta$ with respect to the highest allowed frequency for angular-independent modes ($E_{l=0,m}$).
Figure 3.11: Change of $\eta$ with respect to the highest allowed frequency for angular-independent modes ($E_{l\neq0,m}$).
With an optimal aperture in place, we have a new transfer function of the system optimal aperture-SLM, and is

\[ \tilde{T}(f_x, f_y) = \frac{T(f_x, f_y)}{|T(f_x, f_y)|} \delta \left( \frac{\sqrt{f_x^2 + f_y^2}}{f_{\text{max}}} \right) \] (3.9)

where

\[ \delta \left( \frac{\sqrt{f_x^2 + f_y^2}}{f_{\text{max}}} \right) = \begin{cases} 1, & \frac{\sqrt{f_x^2 + f_y^2}}{f_{\text{max}}} \leq 1 \\ 0, & \frac{\sqrt{f_x^2 + f_y^2}}{f_{\text{max}}} > 1 \end{cases} \] (3.10)

and \( f_{\text{max}} \) is the maximum frequency allowed through the aperture.

### 3.3.3 Results

As we can see in Figure 3.12 and Figure 3.13, the power coupling significantly increases for angular-independent and angular-dependent modes when using an aperture to block high frequencies in the far field.
Figure 3.12: Power Coupling and Modal Cross-talk for Mode $E_{0,1}$: Comparison between Unit Amplitude and Optimal Aperture Setups.
Figure 3.13: Power Coupling and Modal Cross-talk for Mode $E_{3,2}$: Comparison between Unit Amplitude and Optimal Aperture Setups.
3.4 Optimal Inner Aperture Block

This section investigates the impact of an aperture block at the center of the aperture for the purpose of increasing the power coupling. As it was shown in the previous section, the use of aperture is based on the fact that higher frequencies do not exist in the spectrum of the mode, i.e., higher frequencies have little or no amplitude. In Figure 3.14, far-field intensity profiles of four modes are shown. It is the property of Laguerre-based modes that they are invariant to the the Fourier transform [8]. As we can see in the figure, lower frequencies also do not exist in the spectrum of the modes, and as a result there is a minimum frequency under which the amplitude becomes zero. It is important to note that this is true for angular-dependent modes only, i.e., $E_{l\neq0,m}$. 

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It follows that by placing an aperture block to block lower frequencies will lead to an increase in power coupling. Additionally, the optimal size of the block that yields the highest increase in power coupling must be utilized. The process of determining the optimal diameter of the inner aperture block is similar to the one used to determine the optimal aperture diameter: by varying the minimum frequency allowed through the aperture (which is equivalent to varying the maximum frequency blocked by the inner aperture block), the power coupling is recorded, and the optimal inner aperture block diameter corresponds to the minimum allowed
frequency at which the power coupling is highest. Figure 3.15 shows the setup of the inner aperture block and Figure 3.16 shows the optimization process of the inner aperture block.

Figure 3.15: Setup with Inner Aperture Block: An inner aperture block is placed within the aperture.
Figure 3.16: Change of $\eta$ with respect to the lowest allowed frequency for angular-dependent and angular-independent modes.

With an optimal aperture and an inner aperture block in place, we have a new transfer function of the system optimal aperture-inner aperture block-SLM, and is
\[ \bar{T}(f_x, f_y) = \frac{T(f_x, f_y)}{|T(f_x, f_y)|} \cdot \delta \left( \sqrt{f_x^2 + f_y^2} \right) \cdot \gamma \left( \frac{f_{\text{min}}}{\sqrt{f_x^2 + f_y^2}} \right) \]  

(3.11)

where

\[
\gamma \left( \frac{f_{\text{min}}}{\sqrt{f_x^2 + f_y^2}} \right) = \begin{cases} 
1, & \frac{f_{\text{min}}}{\sqrt{f_x^2 + f_y^2}} \leq 1 \\
0, & \frac{f_{\text{min}}}{\sqrt{f_x^2 + f_y^2}} > 1 
\end{cases}
\]  

(3.12)

and \( f_{\text{min}} \) is the minimum frequency not blocked by the inner aperture block.

### 3.4.1 Results

Figure 3.17 and Figure 3.18 show the results of using an optimized inner aperture block. We can see that by utilizing an inner aperture block, the power coupling significantly increases in this way:

- As previously noted, only angular-dependent modes will experience a power coupling increase by the use of an inner aperture block.
- The percentage power coupling increase increases with increasing azimuthal mode number.
- The percentage power coupling increase decreases with increasing radial mode number.
Figure 3.17: Percentage Increase in Power Coupling: Increasing Azimuthal Mode Number.
Figure 3.18: Percentage Increase in Power Coupling: Increasing Radial Mode Number.
Chapter 4

Far-field Phase Noise

Thus far, in the creation of modes using far-field phase information, we have considered an noiseless SLM, i.e., the SLM transfer function is equal to the phase of the target mode, with no noise considered. However, in reality this noise exists and is due to the effect of opto-electrical components of the SLM. This chapter will explore the effect that the SLM noise has on the creation of modes.

4.1 SLM Noise

In this section, we consider the SLM noise $\phi(f_x, f_y)$ to be a White Gaussian noise of probability density function defined as

$$\xi = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{f_x^2 + f_y^2}{2\sigma^2}}. \quad (4.1)$$
where \( \sigma \) is the standard deviation of the noise.

With the SLM noise considered, the total transfer function becomes

\[
\tilde{T}(f_x, f_y) = \left( \phi(f_x, f_y) + \frac{T(f_x, f_y)}{|T(f_x, f_y)|} \right) \delta \left( \frac{\sqrt{f_x^2 + f_y^2}}{f_{\text{max}}} \right) \gamma \left( \frac{f_{\text{min}}}{\sqrt{f_x^2 + f_y^2}} \right)
\]

(4.2)

The signal to noise ration (SNR), which defines the level of the phase to the level of the noise, is defined as

\[
snr = 10 \log \frac{P(f_x, f_y)}{\phi(f_x, f_y)}, \quad (4.3)
\]

where

\[
P(f_x, f_y) = \frac{T(f_x, f_y)}{|T(f_x, f_y)|}
\]

(4.4)

### 4.2 Effect of SLM noise on creation of modes

Using the transfer function defined in Eq. 4.2, Figure 4.1 shows the creation of modes as a function of SNR. As we could have predicted, a higher SNR results in a more accurate creation of modes.
Figure 4.1: Effect of SLM noise of mode creation.

Figure 4.2 shows the impact of SNR on the power coupling. We see that a phase-to-noise ratio of approximately 30dB is sufficient to achieve maximum power coupling for modes $E_{0,1}, E_{0,4}, E_{4,1}, E_{4,4}, E_{8,1}, E_{8,4}, E_{12,1}, E_{12,4}$. It is also important to notice that the approximation of 30dB as the necessary phase-to-noise ratio to
reconstruct modes with maximum power coupling holds true independently of the azimuthal or radial mode number.

![Power Coupling vs. Phase/Noise Ratio](image)

Figure 4.2: Impact of SNR on Power Coupling.
Chapter 5

Concluding Remarks

This thesis investigated the creation of gradient-index fiber modes, first by studying the electromagnetic fields of light in step-index and gradient-index multi-mode fibers, and secondly by simulating the creations of modes using MATLAB utilizing different techniques with the goal to maximize the power coupling between the created mode and the desired mode.

We first reproduced a previous work by Stepniak et al. [8] and extended the work by introducing an inner aperture block in the far field, which resulted in an increase of power coupling of up to 23%. We examined this technique for angular-depend and angular-independent modes and found that this technique increases the power coupling for angular-dependent modes only. We also reported that the power coupling increases with increasing azimuthal mode number, and decreases with increasing radial...
Finally, we investigated the effect of SLM noise to the creation of modes. This is an important study because SLM noise is an inherent property that needs to be considered in the creation of modes. We reported that an approximate of 30dB of SNR is necessary to create modes with maximum possible power coupling.
Bibliography


Appendices
Appendix A

MATLAB Simulation of Step-Index Multi-Mode Fiber Modes

- Calculate the V number based on the fiber and light properties (fiber core radius, numerical aperture and light wavelength).
- Define eigenvalue \( u \) as a vector of maximum value \( V \).
- Calculate eigenvalue \( w \) based on Eq. 2.18.
- Define each term in Eq. 2.17 depending on the mode to be created.
- Graphical method
  - Plot each side of Eq. 2.17 with respect to \( u \) on the same plot.
Below is an example of a MATLAB simulation code and the resulting graphical solution for mode of azimuthal mode number 4.

- The intersections between the two plots are solutions to eigenvalue $u$. The first intersection is $u_{4,1}$, the second is $u_{4,2}$, and so on.

- Numerical method
  - Using MATLAB built-in functions, find the root of Eq. 2.19

- Using Eq. 2.17, calculate $w$

- Use Eq. 2.15 and Eq. 2.20, calculate the electric field of the mode and its intensity, respectively.

**Light Properties**

```matlab
lambda = .85;           %wavelength[microns]
k = 2*pi/lambda;        %wave number
```

**Fiber Properties**

```matlab
n1 = 1.46;              %refractive index of the fiber core
delta = 0.0178;         %normalized index difference
na = n1*sqrt(2*delta);  %numerical aperture
r = 62.5;               %radius[microns]
```
V Parameter

\[ v = 20; \frac{2\pi r n_0}{\lambda}; \]

Eigenvalues

\[ u = 0:.0001:v; \]
\[ w = \sqrt{v^2 - u^2}; \]

Bessel Functions for use in Eq. 2.16

\[ j_4 = \text{besselj}(4, u); \]
\[ j_3 = \text{besselj}(3, u); \]
\[ k_4 = \text{besselk}(4, w); \]
\[ k_3 = \text{besselk}(3, w); \]

Plotting for graphical solution

\[ \text{plot}(u, j_4./j_3, '-b', 'LineWidth', 1.1); \text{hold on}; \]
\[ \text{plot}(u, -(u./w).*(k_4./k_3), '-r', 'LineWidth', 1.1); \]
\[ \text{text}(0.5, -6, '\frac{-u}{w}\frac{K_4(w)}{K_3(w)}', 'interpreter', 'latex', 'Color', 'r', 'fontsize', 25); \]
\[ \text{ylabel}('\frac{J_4}{J_3}', 'interpreter', 'latex', 'fontsize', 25); \]
\[ \text{xlabel}('u', 'fontsize', 25); \]
\[ \text{axis}([0 v -30 30]) \]
\[ \text{set(gcf,'color','w');} \]
Figure A.1: Graphical solution to eigenvalue equation. The solution is the intersections of the two plots.

\[ \frac{J_4}{J_3} = \frac{-u \, K_4(w)}{w \, K_3(w)} \]