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Rectangle Visibility Numbers of Graphs

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by

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ABSTRACT

Very-Large Scale Integration (VLSI) is the problem of arranging components on the surface of a circuit board and developing the wired network between components. One methodology in VLSI is to treat the entire network as a graph, where the components correspond to vertices and the wired connections correspond to edges. We say that a graph G has a *rectangle visibility representation* if we can assign each vertex of G to a unique axis-aligned rectangle in the plane such that two vertices u and v are adjacent if and only if there exists an unobstructed horizontal or vertical channel of finite width between the two rectangles that correspond to u and v. If G has such a representation, then we say that G is a *rectangle visibility graph*.

Since it is likely that multiple components on a circuit board may represent the same electrical node, we may consider implementing this idea with rectangle visibility graphs. The *rectangle visibility number* of a graph G, denoted $r(G)$, is the minimum k such that G has a rectangle visibility representation in which each vertex of G corresponds to at most k rectangles. In this thesis, we prove results on rectangle visibility numbers of trees, complete graphs, complete bipartite graphs, and $(1, n)$-hilly graphs, which are graphs where there is no path of length 1 between vertices of degree n or more.

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I. Introduction

I.1 VLSI Design

When manufacturing computer chips, Very-Large-Scale Integration (VLSI) is the process of arranging and placing transistors on the surface of a small chip, as well as the wired network between components. Designing an optimal arrangement can substantially decrease a circuit's footprint, allowing for the production of smaller and cheaper electronics. The optimal arrangement of VLSI components, however, is an NP-hard problem. Since a circuit's functionality is largely dependent on its arrangement, VLSI optimization is a critical aspect of a circuit design.

Various algorithms, mathematical models, and methodologies for the VLSI problem have been proposed and studied in recent years ([2,5,6,14,15]). While the placement of components on a circuit board heavily impacts the total physical area required in manufacturing and overall performance of the circuit, it is only one of three subproblems in the VLSI problem. The other two subproblems include the partitioning of components, which splits the circuit into logical sub-circuits, and the routing of the circuit, which constitutes the placement of wires used to connect components and develop the network. This has motivated the study of techniques to address this circuit design problem from different angles.

I.2 Visibility Graphs

The VLSI component placement subproblem can be confronted from a graph theoretic approach by assigning components on a computer chip's surface to vertices, and the wires that make up the network to edges. A *visibility representation* of a graph G is an assignment of the vertices in G to objects in a plane where vertices u and v are adjacent in G if and only if there is an unobstructed straight-line channel between the objects that correspond to u and v. If this condition is satisfied, then we say that the objects corresponding to u and v "see" one another. A graph with a visibility representation is called a *visibility graph*. While visibility graphs are limited in their application to VLSI design due to the restriction of edges in the graph corresponding only to objects that see one another, they form a starting point for the consideration of a physical ciruit network as a graph.

In [17], Wismath first introduced *bar visibility graphs* by considering the assignment of vertices of a graph G to horizontal line segments of finite nonzero length called bars, where two vertices u and v

are adjacent if and only if there is there is an unobstructed, vertical straight-line channel between the bars that correspond to u and v (see Figure 1 (a) and (b)). Wismath also developed a unifying characterization of bar visibility graphs, which Tamassia and Tollis [16] extended by considering how the visibility representation of a graph relates to the visibility representations of its subgraphs. Since this initial investigation, the study of visibility graphs has expanded to include many variants. Typically, visibility graphs are denoted by the shape and structure of the objects to which vertices are assigned, as is the case for bar visibility graphs. Additional objects that have been analyzed include rectangles [3] (see Figure 1 (c) and (d)), arcs (used in arc-and-circle visibility graphs [13]), and points [12]. Another element of interest in visibility graphs is object uniformity. Examples of these have been studied with bars [7] and rectangles [9]. A *unit bar visibility graph* is one with a bar visibility representation whose bars are all of equal length. A *unit rectangle visibility graph* is one with a rectangle visibility representation whose rectangles are all unit squares.

Dean et al. [8] introduced an alternative type of visibility graph that is better suited for the landscape of VLSI design. A *bar k-visibility graph* is one that admits two bars $b(u)$ and $b(v)$ corresponding to vertices u and v respectively to see one another if and only if the straight-line channel between them intersects at most k bars. This removes the need for an uninterrupted channel required for two bars to see one another. In a circuit design, it is not unusual for connections to be made between components that are not next to one another in the layout. Wires can easily be made to bend around components or insert into deeper layers of the circuit board in order to make these connections, so lessening the restriction of visibility in this manner facilitates a more realistic approach to VLSI design.

I.3 Visibility Numbers

Electronic designs in VLSI often include components that are represented more than once on the surface of a circuit board. It is therefore intuitive to consider this approach in the study of visibility representations. We define a *t-visibility representation* of a graph G to be a visibility representation where a maximum of t objects are assigned to each vertex in G. Note that there exists some t for every graph such that a t-visibility representation exists. The *visibility number* of a graph G is the minimum t such that a t-visibility representation of G exists. The study of visibility numbers in the literature thus far has been limited to bar visibility graphs [1, 4, 11]. A *t-bar representation* of a graph G is a bar visibility representation in which a maximum of t bars are assigned to each vertex in G. The *bar visibility number* of a graph, denoted $b(G)$, is the minimum t such that a

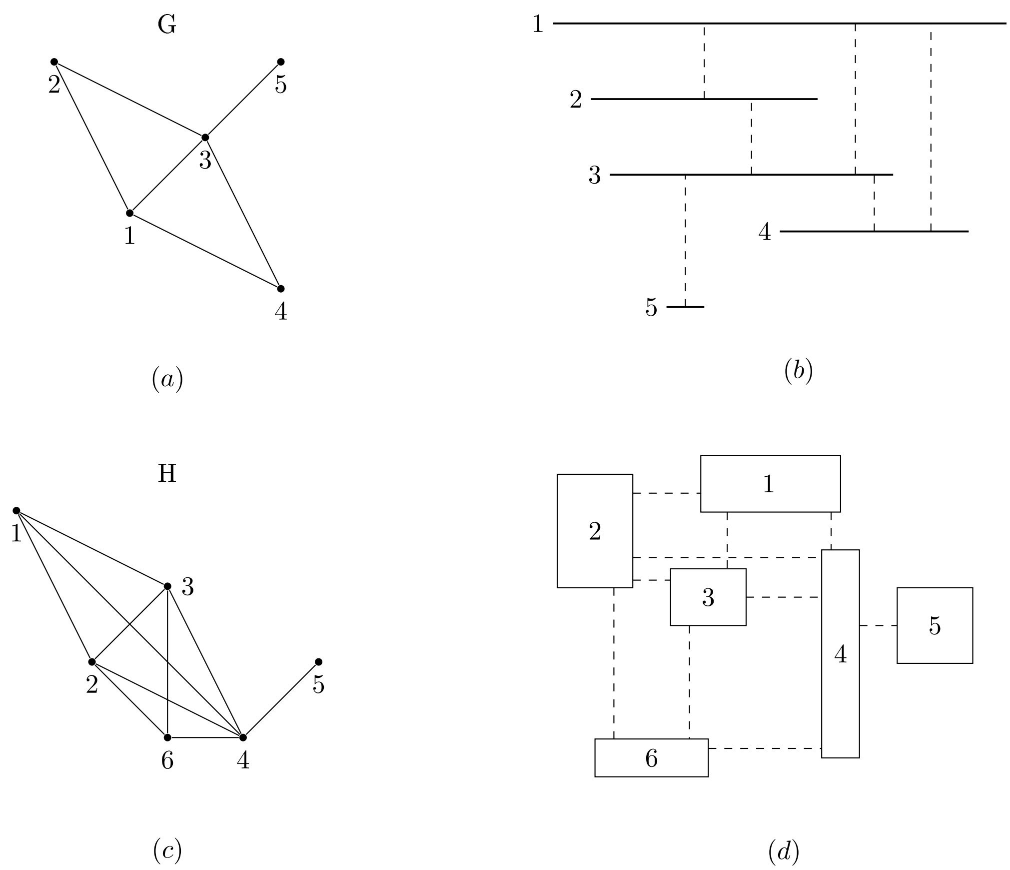

Figure 1: (a) A graph G (b) A bar visibility representation of G (c) A graph H (d) A rectangle visibility representation of H

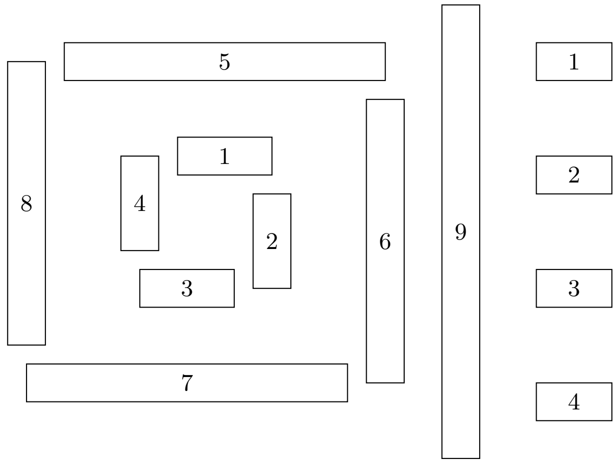

Figure 2: A 2-rectangle visibility representation of K_9

t-bar representation of G exists. Chang et al. [4] proved various results on bar visibility numbers including bounds on bar visibility numbers for planar graphs, complete bipartite graphs, complete graphs, and n-vertex graphs. Axenovich et al. [1] later studied bar visibility numbers of directed graphs. The *unit bar visibility number* of a graph, denoted $ub(G)$, is the bar visibility number of G under the condition that the bars are of equal length.

A *t-rectangle representation* of a graph G is a rectangle visibility representation in which a maximum of t rectangles are assigned to each vertex in G (see Figure 2). Similarly, the *rectangle visibility number* of a graph, denoted $r(G)$, is the minimum t such that a t-rectangle representation of G exists and the *unit rectangle visibility number* of a graph, denoted $ur(G)$, is the rectangle visibility number of G under the condition that the rectangles are unit squares. The purpose of this thesis is to provide an analysis of rectangle visibility numbers. A rectangle visibility graph can be thought of loosely as an extension of a bar visibility graph into two dimensions. Therefore, many of the results involving rectangle visibility numbers in this thesis are inspired by analogous results involving bar visibility numbers.

I.4 Overview of Thesis

In Section 2 we provide a brief history of visibility representations and visibility numbers. We start with the motivation and introduction of bar visibility graphs, which eventually led to the development of rectangle visibility graphs, and present several important results for these two classes of visibility graphs. We then move on to discuss advancements regarding bar visibility numbers and rectangle visibility numbers, as well as their unit length counterparts.

In Section 3 we present our main results. In Section 3.1, given a graph G, we determine a lower bound for the rectangle visibility number of G. In Section 3.2, given a tree T, we present an upper bound for the unit rectangle visibility number of G. We then show that this unit rectangle visibility number is equal to the unit rectangle arboricity of T by means of a general result that is applicable in n dimensions. In Section 3.3, we present an upper bound on the rectangle visibility number of a complete graph K_n. In Section 3.4, we present a lower bound on the rectangle visibility number of a complete bipartite graph $K_{m,n}$ and an upper bound when $m = n$. In Section 3.5, we introduce $(1, n)$-hilly graphs and show that if G is a $(1, 4)$-hilly graph, then $r(G) \leq 2$. We conclude by showing that, for all n, there exists a $(1, n)$-hilly graph whose rectangle visibility number can be bounded below.

In Section 4, we discuss possible extensions of the results in this thesis for future work.

I.5 Notation & Definitions

This section is devoted to defining terms that will be used throughout this thesis. A *graph* G is defined by two sets: the *vertex set* of G, denoted $V(G)$, and the *edge set* of G, denoted $E(G)$. Each edge is an unordered pair of vertices. A *simple graph* is one that contains no duplicate edges or self-loops. All graphs for the purpose of this thesis are simple graphs. An edge e is *incident* to a vertex v if it includes v. Two vertices u and v are *adjacent* if they are in the same edge. Similarly, u is a *neighbor* of v if u is adjacent to v. The *degree* of a vertex v in G, denoted $\deg v$, is the number of vertices that v is adjacent to. The *maximum degree of a graph* G, denoted $\Delta(G)$, is the maximum degree among all vertices in G. A *subgraph* H of G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An induced subgraph H of G is a subgraph where all edges between vertices in $V(H)$ are members of $E(H)$. Two vertex sets $V_1(G)$ and $V_2(G)$ are *disjoint* if there are no edges in $E(G)$ between vertices from $V_1(G)$ and $V_2(G)$. A *decomposition* of G is a partitioning of its edges. An edge $e = uv$ in a graph G can be *subdivided* into two edges by inserting a vertex w in between u and v so that e becomes $e_1 = uw$ and $e_2 = wv$. A *subdivision* of a graph G is a graph that is the result of subdividing edges in G. A *digraph* is a graph G where the each member of $E(G)$ is an arc that is defined by an ordered pair of vertices from $V(G)$.

The *path* on n vertices, denoted P_n, is the graph with $V(P_n) = \{v_1, v_2, ..., v_n\}$ and $E(P_n) = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n\}$. The *complete graph* on n vertices, denoted K_n, is the graph with every possible edge. The *cycle* on n vertices, denoted C_n, is the graph where $V(C_n) = \{v_1, v_2, ..., v_n\}$ and

$E(C_n) = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n, v_nv_1\}$. A cycle of length 3 is also called a *triangle*. A graph G is *triangle-free* if it contains no C_3 subgraphs.

A *walk* from v_1 to v_k is a union of paths $P_{n_1}, P_{n_2}, ..., P_{n_j}$ where P_{n_1} starts at v_1, P_{n_j} ends at v_k, and the starting vertex in P_{n_i} is the end vertex in $P_{n_{i-1}}$ for all i. A graph G is *connected* if there exists a path between any two vertices in $V(G)$. A graph G is *k-connected* if the removal of any $k-1$ vertices does not disconnect G. A *connected component* of G is an induced subgraph H where $V(H)$ and $V(G) - V(H)$ are disjoint vertex sets, and all vertices in $V(H)$ are connected. A *cut-vertex* is a vertex v such that G is connected but the graph G with v and all edges incident to v removed, denoted $G - v$, is not connected.

A *bipartite graph* G is one where we can partition the vertex set $V(G)$ into two sets $V_1(G)$ and $V_2(G)$ such that every edge in $E(G)$ contains one vertex from $V_1(G)$ and one from $V_2(G)$. The *complete bipartite graph*, denoted $K_{m,n}$, is the bipartite graph where $|V_1(G)| = m$, $|V_2(G)| = n$ that contains every possible edge.

A *tree* T is a connected graph that contains no cycle. Vertex v is a *leaf* of T if it only has one neighbor. The *distance* between vertices u and v is k if the shortest path between u and v has k edges. A *star* is a tree T such that a single vertex v is adjacent to all other vertices. A *caterpillar* C is a graph containing a path P where every vertex in C is either contained in P or adjacent to some vertex in P. A *forest* is a disjoint union of trees and a *caterpillar forest* is a disjoint union of caterpillars. A *spanning path* is one that includes all vertices of G. An *interval graph* G is one where each vertex can be assigned to an interval on the real number line such that two vertices u and v are adjacent if and only if the intervals corresponding to u and v intersect.

The following definitions consider the embedding of a graph G in the plane where vertices are represented by points and edges are represented by curves that connect vertices if and only if they are adjacent. Two edges *cross* one another if their curves overlap in the embedding of G. The graph G is a *planar graph* if there exists an embedding of G such that no two edges cross. The *exterior face* of a graph is the region of space that surrounds the graph.

II. Prior Work

Bar visibility graphs were the first visibility graphs to be studied. Wismath [17] began by observing that a bar visibility representation is planar, so bar visibility graphs form a subset of all planar graphs. His work included a characterization of bar visibility graphs.

Theorem 1. *(Wismath [17]) If G is a graph and G^+ is G with one extra vertex that is joined to all cut-vertices, then G is a bar visibility graph if and only if G^+ is planar.*

Tamassia and Tollis [16] differentiated between two types of visibility. A graph G has a *weak visibility representation* if the vertices of G can be mapped to objects in the plane such that if $v_1, v_2 \in G$ are adjacent, then the objects they correspond to see each other. In contrast, G has a *strong visibility representation* if the vertices of G can be mapped to objects in the plane such that $v_1, v_2 \in G$ are adjacent if and only if the objects they correspond to see each other. It follows that if G has a strong visibility representation, then it also has a weak visibility representation. In this thesis, we adapt the convention that a visibility representation of a graph is a strong visibility representation, unless otherwise specified. It was shown in [16] that any 2-connected planar graph has a weak visibility representation.

The logical next step in the study of visibility graphs was to investigate the assignment of vertices of a graph G to axis-aligned rectangles instead of bars, forming a rectangle visibility representation of G. Bose et al. [3] developed several important results for classes of graphs that have rectangle visibility representations. Here, a rectangle visibility representation is said to be *noncollinear* if no two rectangles have sides contained in the same (horizontal or vertical) line, and *collinear* if two rectangles are permitted to have sides contained in the same (horizontal or vertical) line.

Theorem 2. *(Bose et al. [3]) If a graph G can be decomposed into two caterpillar forests, then it has a noncollinear rectangle visibility representation.*

Bose et al. proved this result by giving a generalized construction of a rectangle visibility representation for a graph that can be decomposed into two caterpillar forests C_1 and C_2. This construction carefully lays out C_1 and C_2 as interval graphs using horizontal and vertical bars respectively such that the intersections of the projections of bars from C_1 and C_2 form nonintersecting rectangles. These rectangles form the rectangle visibility representation of G.

The following two results, also from [3], characterize rectangle visibility graphs with maximum degree 3 and 4.

Theorem 3. *(Bose et al. [3]) If a graph G has maximum vertex degree 3, then it has a noncollinear rectangle visibility representation.*

Theorem 4. *(Bose et al. [3]) If a graph G has maximum vertex degree 4, then it has a weak rectangle visibility representation.*

Since VLSI design often involves the layout of millions of transistors, it is likely that many of these components are uniform in size. To model this situation, we may require the bars in visibility representations to have uniform length. Graphs with such visibility representations are called unit bar visibility graphs. Dean and Veytsel [7] studied unit bar visibility graphs from various graph classes, including complete graphs, complete bipartite graphs, outerplanar graphs, and trees. Currently, there is no characterization of unit bar visibility graphs.

A graph that has a rectangle visibility representation composed entirely of unit squares is called a *unit rectangle visibility graph*. As with unit bar visibility graphs, these do not yet have a complete characterization, although Dean et al. [9] have made progress with classifying complete graphs, complete bipartite graphs, hypercube graphs, outerplanar graphs, and trees that have unit rectangle visibility representations.

The assignment of multiple components on a circuit board to the same electrical node is a common reality of VLSI design. Such nodes are often connected in deeper layers of the circuit board but may be located in completely different areas of the surface circuit layout. This idea has motivated the study of bar visibility numbers and rectangle visibility numbers. A *t-bar representation* of a graph G is a bar visibility representation of G where a maximum of t bars are assigned to each vertex in G. The *bar visibility number* of G, denoted $b(G)$, is the minimum t such that G has a t-bar visibility representation. A *t-rectangle representation* of a graph G is a rectangle visibility representation of G where a maximum of t rectangles are assigned to each vertex in G. Similarly, the *rectangle visibility number* of G, denoted $r(G)$, is the minimum t such that G has a t-rectangle visibility representation. Visibility numbers have only been studied thus far for bar visibility graphs [4], bar visibility directed graphs [1], and unit bar visibility graphs [11]. The following are general results from [4] that apply to any bar visibility graph.

Theorem 5. *(Chang et. al [4]) If G has n vertices, then $b(G) \leq \lceil \frac{n}{6} \rceil + 2$.*

Theorem 6. *(Chang et. al [4]) If G has n vertices and e edges, then $b(G) \geq \lceil \frac{e+6}{3n} \rceil$.*

These theorems bound the bar visibility number of a graph G. Other main results for bar visibility numbers have focused on bounds (lower and/or upper) for various graph classes, including complete graphs and complete bipartite graphs.

Theorem 7. *(Chang et. al [4]) If $n \geq 7$, then $b(K_n) = \lceil \frac{n}{6} \rceil$.*

Theorem 8. *(Chang et. al [4]) If $r = \left\lceil \frac{mn+4}{2m+2n} \right\rceil$, then $r \leq b(K_{m,n}) \leq r + 1$.*

Axenovich et al. [1] proved a lower bound for the bar visibility number for any digraph based on the number of vertices and arcs, and an upper bound based only on the number of vertices.

Theorem 9. *(Axenovich et al. [1]) If G is a digraph with n vertices, then $b(G) \leq \frac{n+10}{3}$.*

Theorem 10. *(Axenovich et al. [1]) If G is a digraph with n vertices and m arcs, then $b(G) \geq \frac{m+6}{3n}$. If G is triangle-free, then $b(G) \geq \frac{m+4}{2n}$.*

The *unit bar visibility number* of a graph G, denoted $ub(G)$, is the bar visibility number of G under the condition that all bars have the same length. As with bar visibility graphs, the main results have been the determination of bounds for specific graph classes. In particular, results for unit bar visibility numbers of trees have been determined. Although there is not yet a complete characterization for all unit bar visibility graphs, Dean and Veytsel [7] determined a characterization specifically for unit bar visibility trees.

Theorem 11. *(Dean and Veytsel [7]) A tree T is a unit bar visibility graph if and only if it is a subdivided caterpillar with maximum degree 3.*

Based on the above characterization, an upper and lower bound for $ub(T)$ that differ by one was determined by Gaub et al. [11]. In addition, a fast algorithm was developed to compute $ub(T)$ and provide the unit bar visibility representation for an input T.

Theorem 12. *(Gaub et al. [11]) If T is a tree, then* $\left\lceil \frac{\Delta(T)}{3} \right\rceil \leq ub(T) \leq \left\lceil \frac{\Delta(T)+1}{3} \right\rceil$.

The *unit rectangle visibility number* of a graph G, denoted $ur(G)$, is the rectangle visibility number of G where all rectangles are unit squares. A characterization of unit rectangle visibility trees exists.

Theorem 13. *(Dean et al. [9]) A tree T is a unit rectangle visibility graph if and only if it is the union of two subdivided caterpillar forests, each with maximum degree 3.*

III. Main Results

III.1 Preliminaries

Our first result is a general lower bound based on a result for bar visibility graphs due to Chang et al. [4]. For any graph G that has a rectangle visibility representation, the edge bound on planar graphs is used to establish a lower bound on its rectangle visibility number $r(G)$.

Theorem 14. *If G is a graph with n vertices and e edges, then $r(G) \geq \lceil \frac{e+12}{6n} \rceil$.*

Proof. Consider a t-rectangle representation of G and let N be the total number of rectangles that are used. This implies that $N \leq nt$. Draw an edge joining each pair of rectangles that see each other and color all horizontal edges red, and all vertical edges blue. Shrink each rectangle until it becomes a point and consider the graphs U and V that contain all red edges and all blue edges, respectively. Since U and V are simple planar graphs, the edge bound for planar graphs due to the Euler characteristic implies that G will have at most $6N - 12$ edges. Thus, $e \leq 6nt - 12$, so $t \geq \frac{e+12}{6n}$. Hence, $r(G) \geq \lceil \frac{e+12}{6n} \rceil$. □

Corollary 14.1. $r(K_n) \geq \lceil \frac{n-1}{12} + \frac{2}{n} \rceil$.

We also note provide a short lemma describing how to take a union of disjoint representations and place them in the plane such that there are no unwanted visibilities between these representations.

Lemma 15. *A disjoint union of rectangle visibility representations is a rectangle visibility representation.*

Proof. Let G_1 and G_2 be graphs with rectangle visibility representations R_1 and R_2. Place R_1 in the plane and then place R_2 such that all rectangles are completely to the left and above all rectangles in R_1. Thus, there will be no unwanted horizontal or vertical visibilities between R_1 and R_2. □

III.2 Trees

From Wismath [17], we get that b(T) = 1 for any tree T, so then by giving each bar in a representation of T a small width, we have a rectangle visibility representation of T. Therefore, $r(T) = 1$ for any tree. As a result, we will only consider using unit rectangles.

A tree that is a unit bar visibility graph is called a *unit bar visibility tree*, and a graph whose components are unit bar visibility trees is called a *unit bar visibility forest*. Likewise, a tree that is a unit rectangle visibility graph is called a *unit rectangle visibility tree*, and a graph whose components are unit rectangle visibility trees is called a *unit rectangle visibility forest*.

In [11], it is shown that the unit bar visibility number of a tree T is less than or equal to $\left\lceil \frac{\Delta(T)+1}{3} \right\rceil$. The following theorem uses a similar inductive argument to establish an upper bound for the unit rectangle visibility number of a tree T. For the following results, we will assume that the side length of any unit rectangle is 1 unit.

Theorem 16. *If T is a tree, then $ur(T) \leq \left\lceil \frac{\Delta(T)+1}{6} \right\rceil$*

Proof. We will use induction on the number of vertices in T. For the base case, assume that T is a star. Decompose $T = K_{1,n}$ into $\left\lfloor \frac{n}{6} \right\rfloor$ copies of $K_{1,6}$, and one copy of $K_{1,r}$ where $n \equiv r \pmod 6$ and $0 \leq r \leq 5$. We will say that a rectangle $r(v)$ in a representation is *receptive* if it is contained in an open channel width of 2 units in either the horizontal or vertical direction that does not intersect any other rectangles as well as an open channel width of 1 unit in the perpendicular direction. Construct a unit rectangle visibility representation of T such that every leaf corresponds to a receptive rectangle (see rectangles corresponding to vertex v and its receptive leaves w_1, w_2, w_3, and w_4 in Figure 3). This is possible due to the fact that each leaf rectangle of T sees only one other rectangle in either the horizontal or vertical direction, so the required open channels of width 1 and width 2 can be established.

Next, assume that T is not a star and consider some $v \in V(T)$ that has exactly one non-leaf neighbor. Let $L(v)$ be a set of neighbors of v that are leaves. Also, let T' be a decomposition of $T - L(v)$ into $m = \left\lceil \frac{\Delta(T')+1}{6} \right\rceil$ unit rectangle visibility trees, which exists due to the inductive hypothesis. If $|L(v)| = 6$, then add one copy of $K_{1,6}$ to the decomposition of T'. If $|L(v)| = 5$, then add one copy of $K_{1,5}$ to the decomposition of T'. If $|L(v)| = r$, where $1 \leq r \leq 4$, then delete the r leaves of v that make up $L(v)$. By induction, T' already has an m-rectangle representation where v is assigned to exactly one receptive rectangle. Add receptive rectangles for each of the deleted

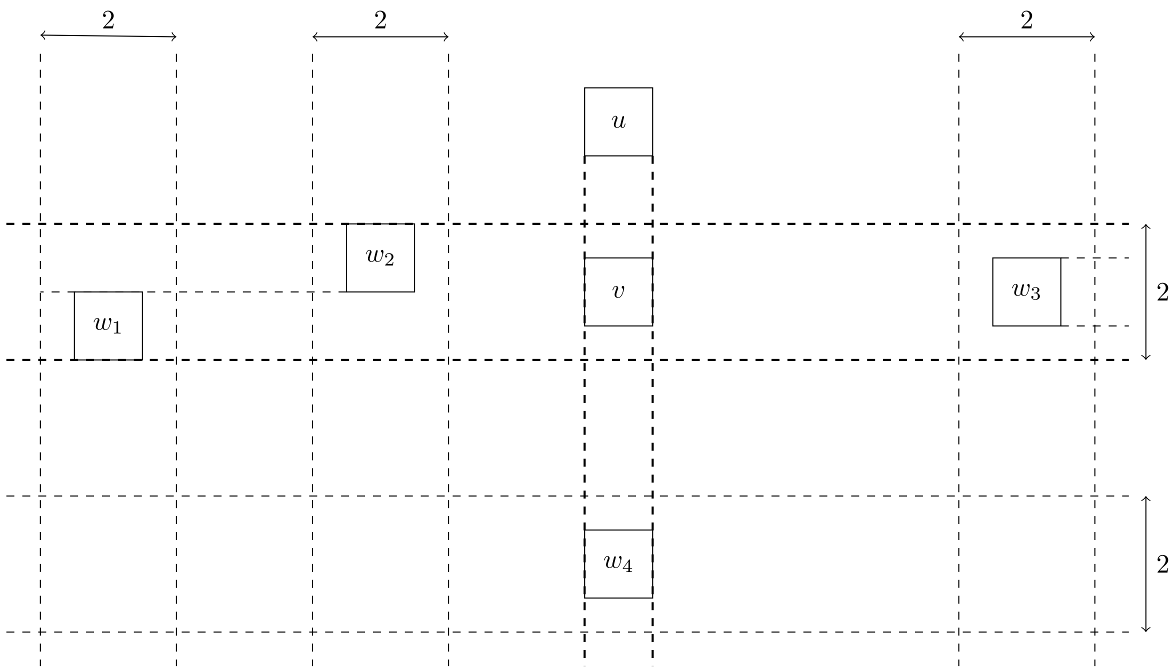

Figure 3: Arrangement of receptive rectangles corresponding to leaves w_1, w_2, w_3 and w_4 of v with u corresponding to the non-leaf neighbor of v

neighbors of v to the decomposition of T' as described in Figure 3. Note that if $r < 4$, leaves of v can be deleted in Figure 3 as necessary.

Since T is composed of finitely many vertices, the unit rectangle visibility representation will contain finitely many rectangles. Hence, we can place rectangles corresponding to leaves of v as far out in their respective directions as necessary to ensure that these rectangles are receptive. As a result, we have obtained the desired decomposition of T into $\left\lceil \frac{\Delta(T)+1}{6} \right\rceil$ unit rectangle visibility trees. $\square$

We define the *unit bar visibility arboricity* of a graph G, denoted $\Upsilon_{ub}(G)$, as the minimum number of unit bar visibility forests required to decompose G. The *unit rectangle visibility arboricity* of a graph G, denoted $\Upsilon_{ur}(G)$, is the minimum number of unit rectangle visibility forests required to decompose G. The next theorem relates the unit rectangle visibility arboricity of a tree T to its unit bar visibility arboricity.

Theorem 17. *If T is a tree, then $\Upsilon_{ur}(T) \geq \left\lceil \frac{\Upsilon_{ub}(T)}{2} \right\rceil$.*

Proof. If $\Upsilon_{ur}(T) = k$, then there exists a decomposition of T into k unit rectangle visibility forests. We can further decompose each of those k forests into 2 unit bar visibility forests by looking at horizontal and vertical visibilities separately. The union of these $2k$ unit bar visibility forests is also T. Hence, $\Upsilon_{ub}(T) \leq 2k$, so $\Upsilon_{ur}(T) \geq \left\lceil \frac{\Upsilon_{ub}(T)}{2} \right\rceil$. □

Our next result concludes that the unit rectangle visibility arboricity for a tree T is the same as its unit rectangle visibility number. By the above theorem, this result implies that we can relate the unit bar visibility arboricity of T directly to its unit rectangle visibility number.

In order to arrive at this result, we prove a result that applies to *n-hypercube visibility graphs*, which are visibility graphs with vertices corresponding to unit axis-aligned hypercubes of dimension n. A tree that is an n-hypercube visibility graph is called an *n-hypercube visibility tree*. An *n-hypercube visibility forest* is a graph whose components are unit n-hypercube visibility trees, and the *n-hypercube arboricity* of a graph G, denoted $\Upsilon_h^{(n)}(G)$ is the minimum number of n-hypercube visibility forests required to decompose G.

Finally, the n-hypercube visibility number of a graph G, denoted $h^{(n)}(G)$ is the minimum t such that a n-hypercube visibility representation exists where a maximum of t n-hypercubes are assigned to each vertex in G.

Theorem 18. *If T is a tree, then $\Upsilon_h^{(n)}(T) = h^{(n)}(T)$.*

Proof. If $\Upsilon_h^{(n)}(T) = k$ then there exists a decomposition of T into k n-hypercube visibility forests. Observe that a vertex may correspond to only one n-hypercube per forest, so there are at most k n-hypercubes corresponding to each vertex in T. Consequently, this decomposition is a k-n-hypercube visibility representation of T, so $\Upsilon_h^{(n)}(T) \leq h^{(n)}(T)$.

Let $h^{(n)}(T) = t$ and let R be a t-n-hypercube representation such that each component of R contains the minimum number of n-hypercube pairs $\{h, h'\}$ that correspond to the same vertex. Here, we define a *component* of R to be a set of n-hypercubes that corresponds to a component of the n-hypercube visibility graph with representation R (where the n-hypercubes are assigned to distinct vertices).

Assume that there exists some $v \in V(T)$ such that two n-hypercubes corresponding to v are in the same component. Call the constituent R_1. Draw in all lines of sight in R_1. Each line of sight has a nonzero Euclidean length and some lines of sight correspond to the same edge in T. Color a line of sight red if its Euclidean length is minimum over all lines of visibility corresponding to the same edge, breaking ties arbitrarily. This ensures that all red lines of sight in R_1 are between n-hypercube pairs corresponding to distinct vertex pairs. Furthermore, the red lines of sight induce

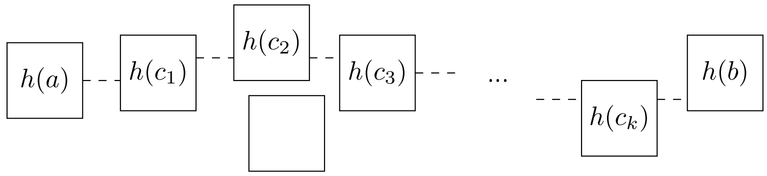

Figure 4: Representation of W with 2-hypercubes $h(a)$ and $h(b)$ using blockers $h(c_1)$,...,$h(c_k)$

a spanning forest of the n hypercubes in R_1. By Lemma 15, we can partition R_1 by shifting each component of the forest with red lines of sight so that each n-hypercube in each new component does not see n-hypercubes in any other component. Observe that this partitioning of R_1 retains all edges represented in R_1.

We now show that there are no extra edges that appear from this partitioning due to extra lines of sight. Suppose that n-hypercubes $h(a)$ and $h(b)$ correspond to distinct vertices a and b that are not adjacent in T, but $h(a)$ and $h(b)$ see each other and are in the same component, call it A, after applying the above partitioning of R_1.

This implies that there is a finite-width channel of visibility between $h(a)$ and $h(b)$ in A that does not exist before partitioning R_1. In order for this to be true, there must be a collection of n-hypercubes that intersect the channel of visibility between $h(a)$ and $h(b)$ before the partitioning of R_1 and collectively block this channel. We will refer to these as *blockers*. Observe that there exists a walk W between a and b that is represented by the blocking n-hypercubes $h(c_1), h(c_2), ..., h(c_k)$ (where $h(a)$ sees $h(c_1)$ and $h(c_k)$ sees $h(b)$) with lines of sight parallel to the channel between $h(a)$ and $h(b)$ (see Figure 4). Let the length of the channel between $h(a)$ and $h(b)$ be d. Then, the sum of the Euclidean lengths of the lines of sight corresponding to edges in W is equal to $d - k$.

Since $h(a)$ and $h(b)$ are in the same component after partitioning R_1, the path between a and b in T is represented in the visibility representation such that the lines of sight between n-hypercubes corresponding to vertices along this path are minimum in Euclidean length for each vertex pair. Let the path between a and b in T be of length m and call the n-hypercube representation of this path with minimum line of sight lengths P. Also, let P consist of hypercubes $h(p_1), h(p_2), ..., h(p_m)$. Consequently, $m \leq k$. Observe that the sum of these lines of sight cannot be less than d, the Euclidean length between $h(a)$ and $h(b)$, minus the widths of the m n-hypercubes corresponding to vertices along this path. Thus, this sum is greater than or equal to $d - m \geq d - k$.

Consider the case where $d - m > d - k$. In this case, the sum of the lengths of the lines of sight in W is greater than the sum of the lengths of the lines of sight in P. Then, by the pigeonhole principle,

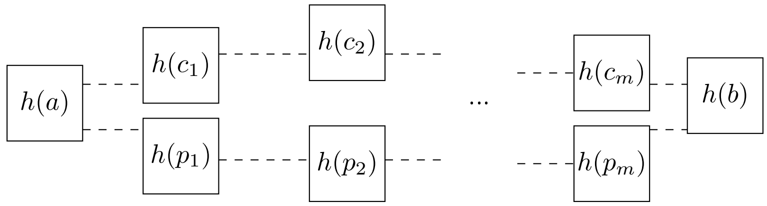

Figure 5: 2-hypercubes $h(a)$ and $h(b)$ showing the contradiction of the path between a and b being represented by both W and P where lines of sight between correponding n-hypercubes are of equal length

there is one line of sight between two n-hypercubes $h(u)$ and $h(v)$ in P whose length is greater than that of the line of sight between two n-hypercubes $h'(u)$ and $h'(v)$ in W that correspond to the edge between u and v. This contradicts the fact that all lines of sight between n-hypercubes in P between $h(a)$ and $h(b)$ are minimum.

Now consider the case where $d - m = d - k$. This is the case where $m = k$ and the walk between a and b is a path. We assume that P must have all lines of sight be a minimum for each n-hypercube pair. Since the sum of the lengths of lines of sight in W and P is the same, this is only satisfied when each line of sight in P between two n-hypercubes $h(u)$ and $h(v)$ is the same length as that of the corresponding line of sight between $h'(u)$ and $h'(v)$. Furthermore, since the lines of sight in W were required to be parallel to the channel between $h(a)$ and $h(b)$, this requires for lines of sight in P to also be parallel to this channel to retain equality of all corresponding lengths. But then each n-hypercube in W sees the n-hypercube in P corresponding to the same vertex in T, which is a contradiction (see Figure 5).

We conclude that it is not possible for $h(a)$ and $h(b)$ to be in the same component without a set of blockers that completely block the channel between the two n-hypercubes. This leads to the validity of the partitioning of R_1 as described above.

This successful partitioning of R_1 contradicts the minimality of R. Hence, $h^{(n)}(T) \geq \Upsilon_h^{(n)}(T)$, so $h^{(n)}(T) = \Upsilon_h^{(n)}(T)$. $\square$

The following corollary is the case from the previous theorem where $n = 2$, which is for unit rectangles.

Corollary 18.1. *If T is a tree, then $\Upsilon_{ur}(T) = ur(T)$.*

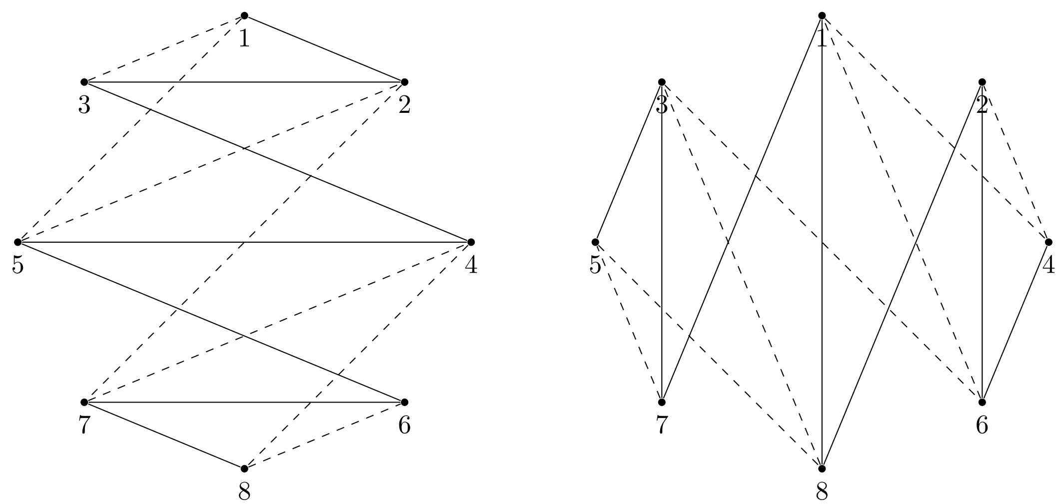

Figure 6: Zigzag spanning path decomposition of A_i where $m = 2$

III.3 Complete Graphs

An upper bound on the rectangle visibility number of a complete graph K_n can be determined using a methodology similar to a result in [4] for bar visibility graphs.

Theorem 19. $r(K_n) \leq \lceil \frac{n}{12} \rceil + 1$

Proof. Let $m = \lceil \frac{n}{12} \rceil$ and partition K_{12m} into three sets of $4m$ vertices each, call them A_1, A_2, and A_3. We can decompose a complete graph of size $4m$ into $2m$ spanning paths that correspond to rotations of a zigzag path formed when the vertices are placed in a circle. After forming these paths, pair up every j^{th} zigzag (where $j \in [2m]$ and j is odd) with every $(j+1)^{th}$ zigzag. This creates m pairs consisting of two spanning paths each that are pairwise edge-disjoint (see Figure 6).

Next, arrange each pair of zigzag paths from A_i as a bar visibility graph where the bars overlap in a "staircase" such that each bar has an unobstructed channel of visibility in both vertical and horizontal directions. Let one path in each pair be arranged with vertical channels of visibility and the other with horizontal channels of visibility. For each of m pairs, place rectangles in the plane that correspond to the intersection of these channels (see Figure 7). After applying this construction to all m pairs, we have m sets of $4m$ rectangles in the plane corresponding to the vertices of A_i such that each rectangle has a channel of visibility in all four directions. We will refer to these m

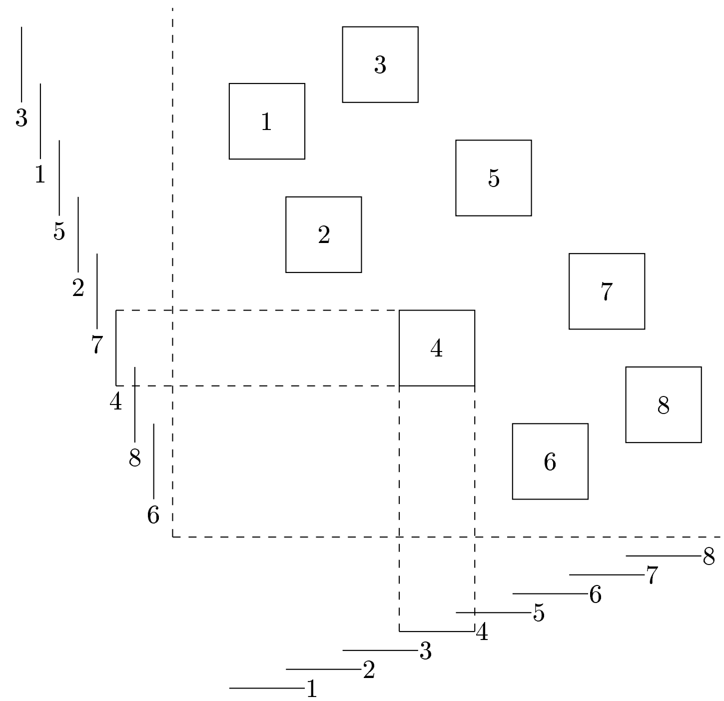

Figure 7: Arrangement of bar visibility graphs corresponding to a spanning path pair in Figure 6 with resulting module

rectangle visibility graphs as *modules*.

For each of m modules from A_i, place one long rectangle each above, below, to the left, and to the right of the $4m$ rectangles in the model, as shown in Figure 8. Assign these four rectangles to vertices in A_{i+1} where indices are taken modulo 3. This amounts to m pairwise edge-disjoint copies of the $4m$ rectangles corresponding to vertices in A_i, whose union covers the complete graph for A_i and all edges between A_i and A_{i+1}. Repeat for each A_i, such that we have $3m$ modules in total whose union is the complete graph K_{12m}.

Observe that each vertex of A_i is used in the m modules for A_i and then once more as one of the four rectangles surrounding one module from A_{i-1}. Hence, each vertex in K_{12m} is assigned to $m + 1 = \lceil \frac{n}{12} \rceil + 1$ rectangles. If n is not divisible by 12, then we delete unwanted rectangles from the $(m + 1)$-representation of K_{12m} to obtain an $(m + 1)$-representation of K_n. $\square$

The upper bound established above is not sharp. We can easily show that $r(K_{17})$ is actually lower than the suggested bound by means of a construction, which is of interest because the layout does

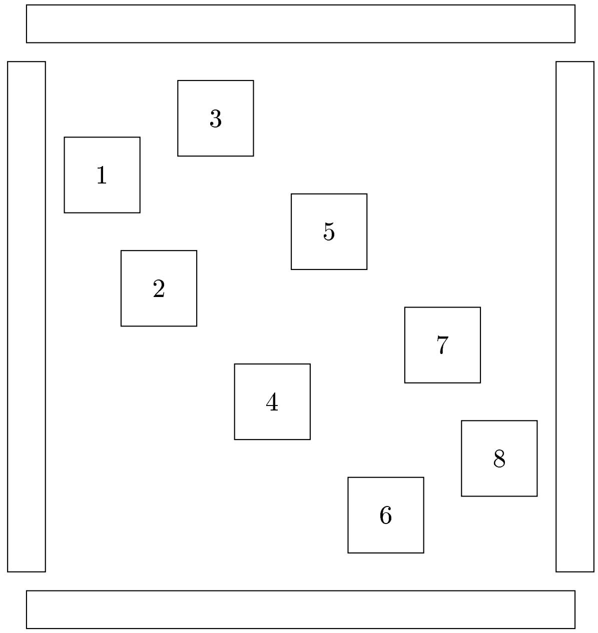

Figure 8: Arrangement of four outer rectangles from A_{i-1} surrounding the module in Figure 7

not suggest a pattern for larger complete graphs. Moreover, slight alterations of this construction do not provide one for K_{18}, so it is a standalone result.

Theorem 20. $r(K_{17}) = 2$.

Proof. Since there does not exist a 1-rectangle visibility representation for K_9, it follows that $r(K_9) \geq 2$, so $r(K_{17}) \geq 2$. Figure 9 shows a 2-rectangle representation of K_{17}.

□

III.4 Complete Bipartite Graphs

The following theorem is based on a similar result in [11] for unit bar visibility graphs and establishes a lower bound for the unit rectangle visibility number of any complete bipartite graph $K_{m,n}$. As with Theorem 14, planar graph edge bounds are utilized to the lower bound.

Theorem 21. *For $m \geq n \geq 2$, $r(K_{m,n}) \geq \left\lceil \frac{n}{4(m+n)}m + \frac{2}{m+n} \right\rceil$. If $n = m - o(m)$, then $r(K_{m,n}) \geq \left\lceil \frac{m}{8} - o(m) \right\rceil$.*

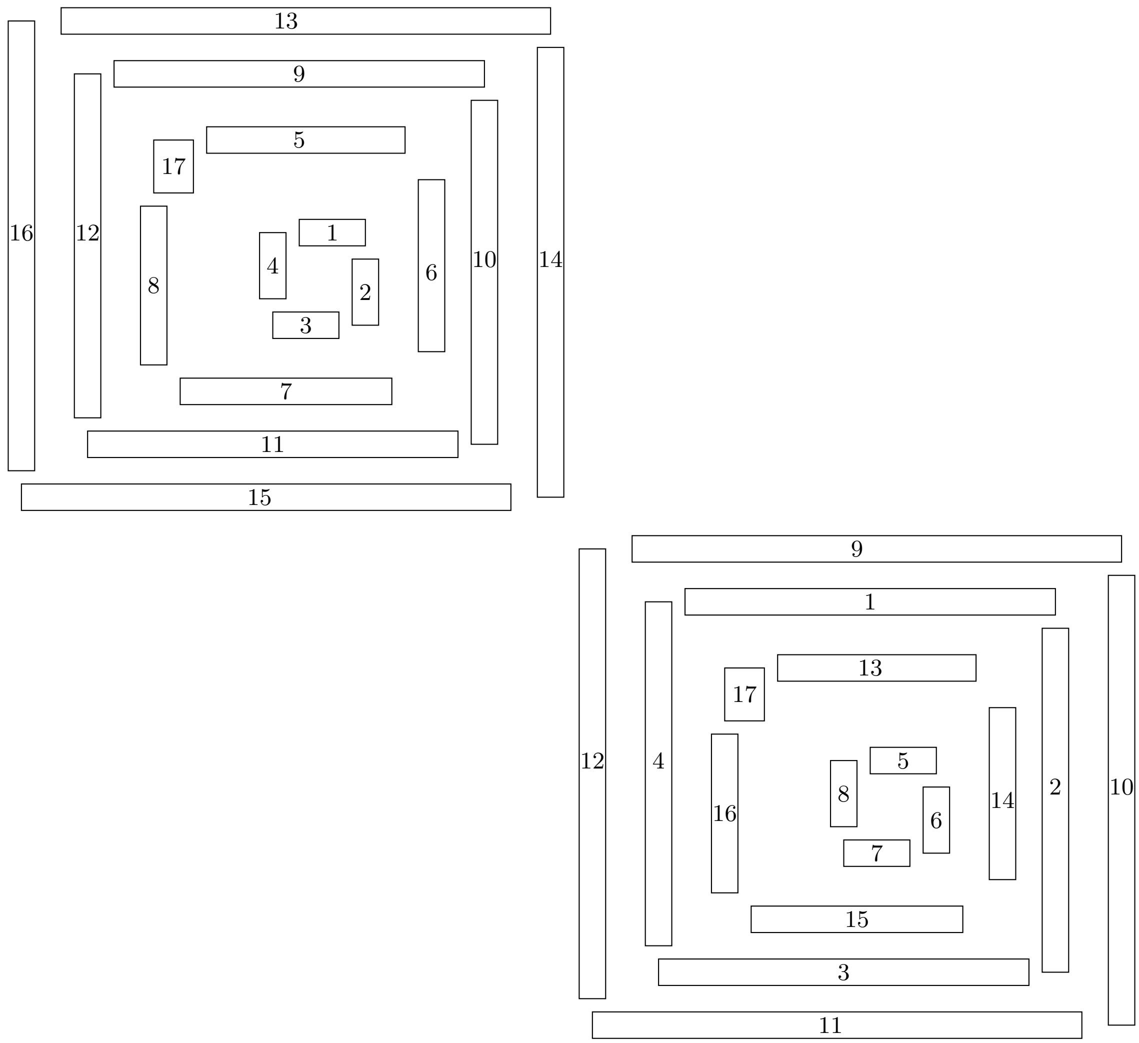

Figure 9: A 2-rectangle representation of K_{17}

Proof. Let $m \geq n \geq 2$. Consider a t-rectangle representation of $K_{m,n}$. Let N be the total number of rectangles that are used in the representation. Observe that $N \leq (m+n)t$. Draw an edge joining each pair of rectangles that see each other and then color all horizontal edges red, and all vertical edges blue. Consider subgraphs U and V such that U is the subgraph containing all red edges and V is the subgraph containing all blue edges. Observe that U and V are planar and edge-disjoint. Shrink each rectangle until it becomes a point in the plane while keeping all edges intact to retain the planarity of U and V. Since the t-rectangle representation of $K_{m,n}$ is bipartite, the planar edge disjoint subgraphs U and V will be bipartite as well. Each planar bipartite graph has at most $2N-4$ edges by the Euler characteristic, so $K_{m,n}$ has at most $4N-8$ edges. Therefore, $mn \leq 4(m+n)t - 8$, so $t \geq \frac{mn+8}{4(m+n)}$. Hence, $r(K_{m,n}) \geq \left\lceil \frac{n}{4(m+n)}m + \frac{2}{m+n} \right\rceil$.

If $n = m - o(m)$, then $t \geq \left\lceil \frac{m^2}{8m-o(m)} - \frac{o(m^2)}{8m-o(m)} + \frac{2}{2m-o(m)} \right\rceil = \left\lceil \frac{m}{8} - o(m) \right\rceil$. □

The following theorem is based on a similar result in [11] for unit bar visibility graphs and establishes a lower bound for the unit rectangle visibility number of any complete bipartite graph $K_{m,n}$.

Theorem 22. $r(K_{n,n}) \leq \left\lceil \frac{n}{8} \right\rceil + 2$.

Proof. We present a partial construction of $K_{n,n}$ and then show that the edges not in the partial rectangle visibility representation can be added without significantly increasing $r(K_{n,n})$.

Construction 1.

Let $K_{n,n}$ have partite sets X and Y. We define a *staircase* to be a set of n unit rectangles where the lower right corner of the i^{th} rectangle is fixed to the upper left corner of the $(i+1)^{th}$ rectangle, for all $i \in [n-1]$. Let the topmost rectangle in a staircase be the *starting rectangle*. Lay out $2(\left\lceil \frac{n}{8} \right\rceil + 1)$ staircases according to Figure 10 such that each staircase from top to bottom is shifted slightly to the left. We will number these staircases from 1 to $\left\lceil \frac{n}{8} \right\rceil + 1$ starting from the topmost staircase. Assign the rectangles in 1^{st} staircase vertices from X, starting with x_1 and ending at x_n. Assign the rectangles in the 2^{nd} staircase to vertices from Y starting with y_1 and ending at y_n. Assign the rectangles in the remaining odd numbered staircases with vertices from X where the index of the starting rectangle in a staircase increases by 5 modulo n from top to bottom. Likewise, assign the rectangles in the remaining numbered staircases to vertices from Y where the index of the starting rectangle in a staircase decreases by 3 modulo n from top to bottom. This construction is depicted in Figure 10. Let the i^{th} staircase be an X staircase such that the index of its starting rectangle is the same as that of the Y staircase directly below it. This occurs when $1 + 5i \equiv 1 - 3i \pmod{n}$.

The first time after the initial staircases where this occurs is when $8i \equiv \pmod{n}$, or equivalently $i = \lceil \frac{n}{8} \rceil$. Consequently, the indices of the starting rectangles in adjacent X and Y staircases are only the same for the first two staircases and the last two staircases.

We will first make an assumption that all rectangles in Construction 1 that correspond to vertices in X have degree 8 and show that the rectangles that have degree less than 8 can be considered in special cases. Consider some rectangle that corresponds to the vertex $x_j \in X$, where $j \in [1, n]$, and is located in the k^{th} X staircase (where k increases from top to bottom). Call this rectangle $r_k(x_j)$. Let the rectangles from the $(k-1)^{th}$ Y staircase that $r_k(x_j)$ sees be defined as $r_{k-1}(y_i), r_{k-1}(y_{i+1}), r_{k-1}(y_{i+2})$, and $r_{k-1}(y_{i+3})$, where addition of i is performed modulo n. Thus, the rectangles that $r(x_j)$ sees in the k^{th} Y staircase are $r_k(y_{i-4}), r_k(y_{i-3}), r_k(y_{i-2})$, and $r_k(y_{i-1})$. Now consider $r_{k+1}(x_j)$, the rectangle in the $(k+1)^{th}$ X staircase that corresponds to the vertex x_j. It follows that the rectangles $r_{k+1}(x_j)$ sees in the Y staircases above and below it are $r_k(y_{i-8}), r_k(y_{i-7}), r_k(y_{i-6}), r_k(y_{i-5})$ and $r_{k+1}(y_{i-12}), r_{k+1}(y_{i-11}), r_{k+1}(y_{i-10}), r_{k+1}(y_{i-9})$ respectively. We have now shown that the degree 8 rectangles corresponding to the same vertex in X see sets of eight rectangles corresponding to pairwise disjoint sets of vertices in Y. Note that since the indices of the starting rectangles for the first two staircases and last two staircases are the same, then the rectangles from X in these staircases see rectangles in that correspond to the same vertices from Y in these staircases. Since there are $\lceil \frac{n}{8} \rceil + 1$ rectangles corresponding to each vertex, then each rectangle corresponding to $x_i \in X$ sees $8 \lceil \frac{n}{8} \rceil + 8$ rectangles, n of which are distinct, and then up to 8 of which are repeated in the last set of staircases. Hence, all edges from X to Y are accounted for under the assumption that all rectangles are degree 8.

Now, consider rectangles that correspond to vertices in X that see less than eight other rectangles. Since we have just ensured that the first staircase, whose rectangles have maximum degree 4, does not lose any necessary visibilities by means of repetition in the last staircase, we only need to consider rectangles on both ends of all staircases. Observe in Construction 1 that the first two and last two rectangles in any given staircase lose a maximum of 3 possible visibilities. If we consider adding an additional rectangle corresponding to each vertex in X and Y in a separate area of the plane, we see that each rectangle then requires at most 3 visibilities. This corresponds to a subcubic graph, which has a rectangle visibility representation [3]. Hence, each vertex in $K_{n,n}$ corresponds to at most $(\lceil \frac{n}{8} \rceil + 1) + 1 = \lceil \frac{n}{8} \rceil + 2$ rectangles in this rectangle visibility representation. □

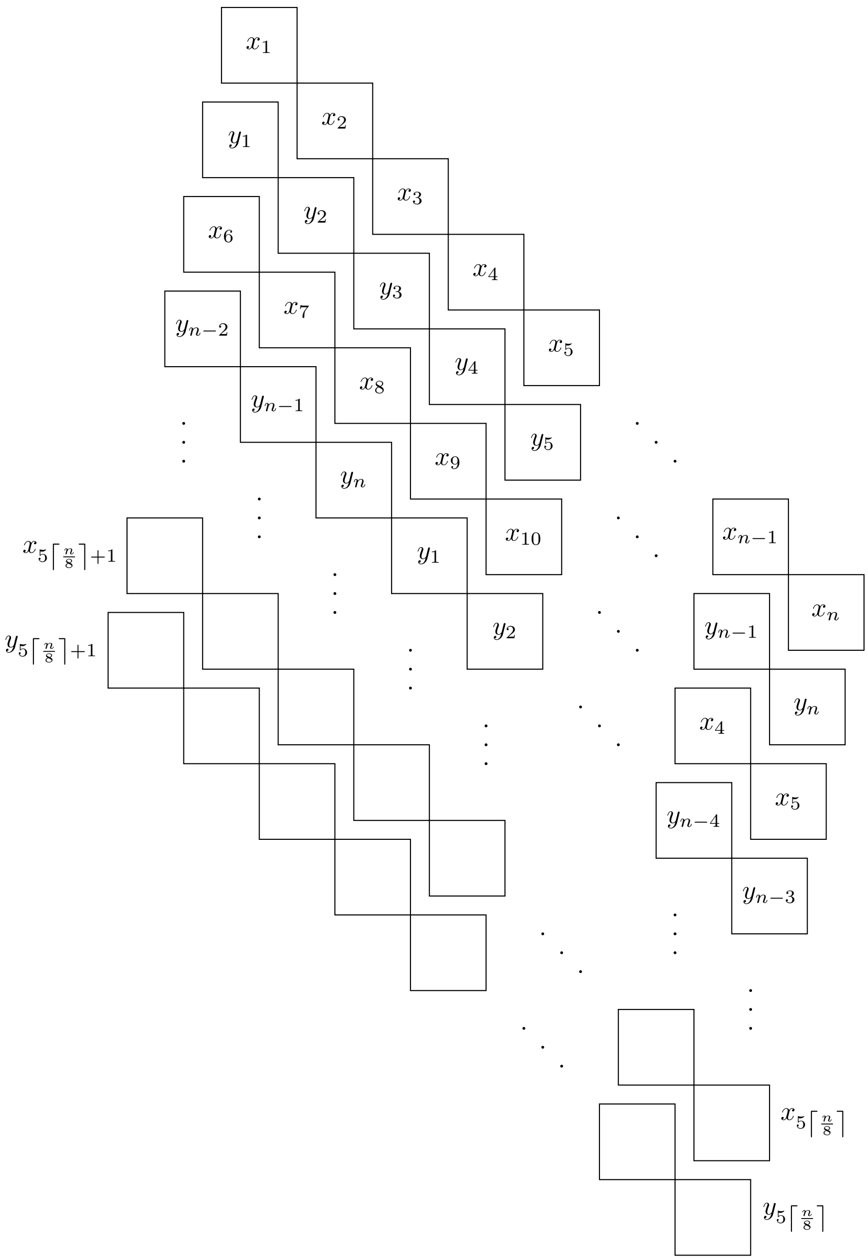

Figure 10: Arrangement of staircases and assignment of vertices in X and Y to rectangles outlined in Construction 1

III.5 (1,n)-Hilly Graphs

Bose et al. [3] define a graph G to be *k-hilly* if there is no path of length k or less between any two high degree vertices, where a vertex of *high degree* is one whose degree is 4 or more. Various results have been achieved with rectangle visibility representations involving 2-hilly graphs and 3-hilly graphs, but the literature is currently lacking concrete properties of 1-hilly graphs and their rectangle visibility representations.

First, we extend this definition to allow for the meaning of "high degree" to change. We define a graph to be *(k,n)-hilly* if there is no path of length k or less between two vertices of degree n or more.

Within the context of rectangle visibility graphs, we present an example of a $(1,4)$-hilly graph that does not even have a weak 1-rectangle visibility representation. The observation that not all $(1,4)$-hilly graphs are rectangle visibility graphs is made in [3], but a proof is not provided.

Theorem 23. *There exists a (1,4)-hilly graph that does not have a weak rectangle visibility representation.*

Proof. Let $K_{k,\binom{[k]}{3}}$ be the bipartite graph with partite sets containing k vertices and $\binom{k}{3}$ vertices where each vertex in the $\binom{k}{3}$ partite set is degree 3 and adjacent to a distinct combination of 3 vertices from the size k partite set. We will consider the construction of a rectangle visibility representation of $K_{k,\binom{[k]}{3}}$ as an example of a 1-hilly graph and show that if k is large enough, then this graph does not have a rectangle visibility representation. Let G be an arbitrary graph with a rectangle visibility representation of $k = (n-1)^6 + 1$ rectangles laid out on a set of xy axes, where $n \geq 33$. We will characterize each rectangle in G by the coordinate value of the top left corner, (x, y). We will also assign each pair of rectangles G with one of six relationships based on their positions relative to one another as described below.

Choose two rectangles $R_i, R_j \in G$. For the following two definitions, let the x-coordinate of R_i to be strictly less than that of R_j. We define these two rectangles to be *nested left* if the projection of R_i onto the y-axis is completely contained inside the projection of R_j onto the y-axis. Complete containment, in this case, is inclusive of endpoints. Similarly, we define these two rectangles to be *nested right* if the projection of R_j onto the y-axis is completely contained inside the projection of R_i onto the y-axis. For the following two definitions, let the y-coordinate of R_i be strictly less than that of R_j. We define these two rectangles to be *nested above* if the projection of R_j onto the x-axis

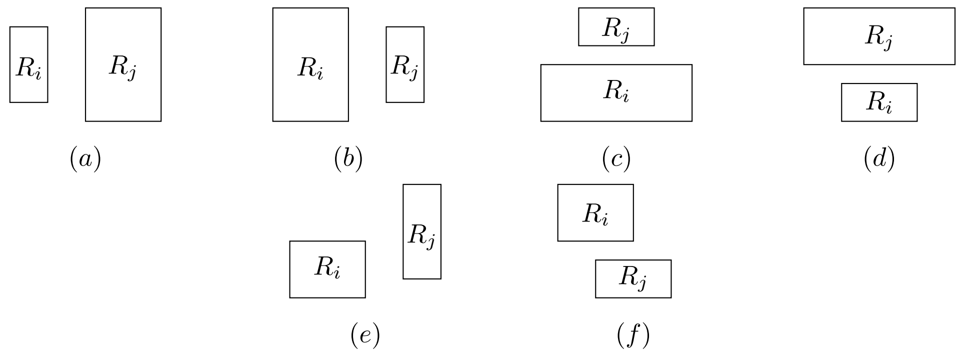

Figure 11: R_i and R_j (a) Nested left (b) Nested right (c) Nested above (d) Nested below (e) Ascending (f) Descending

is completely contained inside the projection of R_i onto the x-axis. Similarly, we define these two rectangles to be *nested below* if the projection of R_i onto the x-axis is completely contained inside the projection of R_j onto the x-axis. For the following two definitions, let the x-coordinate of R_i be strictly less than that of R_j. We define two rectangles as *ascending* if they are not nested in any direction (as defined above) and the y-coordinate of R_i is strictly less than that of R_j. In the final case, we define these two rectangles to be *descending* if they are not nested in any direction (as defined above) and the y-coordinate or R_i is strictly greater than that of R_j. Observe that if the y-coordinate of R_i and R_j is the same, then then these rectangles must be either nested left or nested right, so we need only consider inequality in the definitions of ascending and descending rectangles. Examples of these six relationships are shown in Figure 11.

List the rectangles in the representation of G by their x-coordinate in order of least to greatest and denote each rectangle by R_i where $i \in [k]$. In the case where rectangles have the same x coordinate value, we will list these by the y coordinate from greatest to least. Hence, the list of rectangles in G will read left to right as seen in the plane, except where rectangles have the same x coordinate, in which case the list will read top to bottom for these rectangles. Observe that each pair of rectangles must fall into one of the six aforementioned relationships. We then assign each rectangle pair (in order) with a 6-tuple coordinate, $(a_i, b_i, c_i, d_i, e_i, f_i)$ in which each of six relationships is assigned to one coordinate value. Let a_i be the length of a longest sublist where all rectangles are nested left ending at g_i. Similarly, let b_i, c_i, d_i, e_i, and f_i be the length of a longest sublist where all rectangles are nested right, nested above, nested below, ascending, or descending respectively that ends at g_i. Since there are $k = (n-1)^6 + 1$ rectangles in G, by the pigeonhole principle, there exists a monotone sublist with length n. Consider the rectangles that make up this sublist as an auxiliary graph H of G. Consequently, H will consist of n rectangles that are all either either nested in one direction, ascending, or descending. We will denote each rectangle by U_i where $i \in [n]$ where the assignment of U_i to all rectangles is done the same way as with rectangles in G previously.

Assume that all rectangles in H are nested left, which implies that the length of the left side of U_1 is less than that of all other rectangles in H. In the plane, the general shape of H would be rectangles that fan outward from left to right. We attempt to place $\binom{n-1}{2} = \frac{n^2-3n+2}{2}$ rectangles from the partite set of $\binom{k}{3}$ that see U_1 and two other distinct rectangles in H; this is required in the representation of $K_{k,\binom{[k]}{3}}$. We define a rectangle V that sees U_1 and two other distinct rectangles in H to be *parallel* if when considering the edges between the two rectangles that are not U_1, both are either horizontal or vertical. Similarly, we define a rectangle that sees U_1 and two other distinct rectangles in H to be *perpendicular* if when considering the edges between the two rectangles that are not U_1, one edge is horizontal and the other is vertical. Moreover, we denote rectangles to be located in the *upper region* when they are placed above any rectangle in H, and for rectangles to be located in the *lower region* when they are placed below any rectangle in H.

Consider placing rectangles in the $\binom{k}{3}$ partite set that are parallel. We will show that there is an upper bound on the number of rectangles that can be placed of this type. Let rectangles $U_i, U_j, U_k, U_l \in (H - U_1)$ such that $i < j < k < l$. Since H is an ordered set of rectangles, then we can define edge crossing according to [5]. Let $L(e)$ be the index of the left endpoint vertex of an edge e and $R(e)$ be the index of the right endpoint vertex of e. We say that two edges e and f *cross* if $L(e) < L(f) < R(e) < R(f)$. Furthermore, a *stack* is a set of edges such that no pair of edges cross. In general, a *k-stack* is a graph that whose edge set can be partitioned into k edge-disjoint stacks. Observe that if a parallel rectangle V is placed such that it sees U_i and U_k, then V will span across the entire length of U_j's band of visibility in the same direction. This makes it impossible to place a parallel rectangle W in the same region that shares two parallel edges in the same direction with U_j and U_l without blocking the band of visibility between V and U_k (see Figure 12).

We can therefore consider parallel rectangles V and W themselves as "edges" between rectangles in the ordered set H. Observe that if it were possible to place both V and W in this manner, then $L(V) < L(W) < R(V) < R(W)$ and therefore V and W would cross. Hence, we cannot allow edge crossings where edges are interpreted as parallel rectangles. As a result, the set of parallel rectangles that can be placed forms a 1-stack, which has a maximum of $2p - 3$ edges in a set of p vertices [10]. Note that the structure of H in Figure 12 illustrates that it is only possible to place parallel rectangles with horizontal edges that extend from the right. In addition, parallel rectangles in the upper region whose edges are vertical must be such that they extend from the bottom. It follows then that parallel rectangles in the lower region whose edges are vertical must

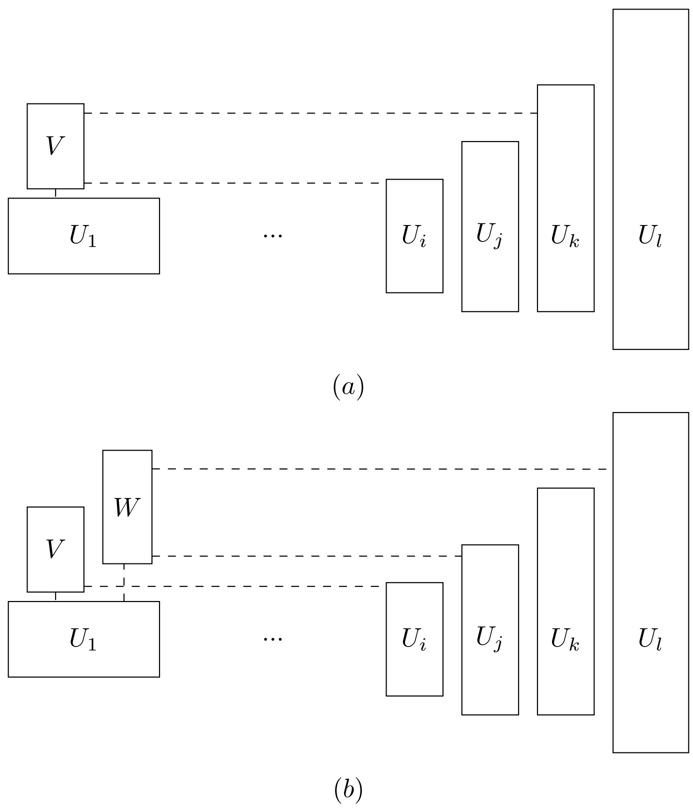

Figure 12: (a) Example of a parallel rectangle V with horizontal edges in the upper region that sees U_i, and U_k (b) Adding another parallel rectangle W with horizontal edges in the upper region that sees U_j and U_l but then blocks V's visibility to U_k

be such that they extend from the top. Consequently, we have a 4-stack in total to encompass all four possible combinations of parallel rectangles that can be placed. This leads to a maximum of $4[2(n-1)-3] = 8n-20$ rectangles that can be placed that see U_1 and share two parallel edges to distinct rectangles in $H - U_1$.

Next, consider placing rectangles in the $\binom{k}{3}$ partite set that are perpendicular. Similarly, we will show that there is an upper bound on the number of rectangles that can be placed of this type. Let rectangles $U_i, U_j, U_k, U_l \in (H - U_1)$ such that $i < j < k < l$. It should be noted that there are four distinct types of these rectangles that can be placed: horizontal edge on left with vertical edge on top, horizontal edge on right with vertical edge on top, horizontal edge on left with vertical edge on bottom, and horizontal edge on right with vertical edge on bottom. Since H is an ordered set of rectangles, then we can define edge nesting according to [5]. We say that an edge e is *nested* inside another edge f if $L(f) < L(e) < R(e) < R(f)$. Furthermore, a *queue* is a set of edges such that no that there are no nested edges. In general, a *k-queue* is a graph whose edge set can be partitioned into k edge-disjoint queues. Observe that if a perpendicular rectangle V is placed such that it sees U_i and U_l, then it becomes impossible to place a perpendicular rectangle W of the same type in the same region that shares two perpendicular edges in the same direction with U_j and U_k without blocking the band of visibility between V and U_i (see Figure 13).

We can therefore consider perpendicular rectangles V and W themselves as "edges" between rectangles in the ordered set H. Observe that if it were possible to place both V and W in this manner, then $L(V) < L(W) < R(W) < R(V)$ and therefore W would nest in V. Hence, we cannot allow edge nestings where edges are interpreted as perpendicular rectangles. As a result, the set of perpendicular rectangles of a single type that can be placed in the same region of the plane forms a 1-queue, which has a maximum of $2p - 3$ edges in a set of p vertices [10]. Figure 6 illustrates that it is not possible for a perpendicular rectangle to have its horizontal edge extend from the left and still see U_i, so only two types of perpendicular rectangles may be placed in each of the two regions. Consequently, we have a 4-queue that encompasses all four combinations of perpendicular rectangles that can be placed. This leads to a maximum of $4[2(n-1)-3] = 8n-20$ rectangles that can be placed that see U_1 and share two perpendicular edges to distinct rectangles in $H - U_1$.

We now observe that the above restrictions on parallel and perpendicular rectangles can be applied to nesting from any direction by rotating the plane until rectangles in H are all nested left. If all rectangles in H are ascending, then these restrictions still apply, although we note that in this case U_1 is always the leftmost rectangle in H, not necessarily the rectangle with the shortest left side as with the nested cases. Then, if all rectangles in H are descending, we can rotate the plane by 90

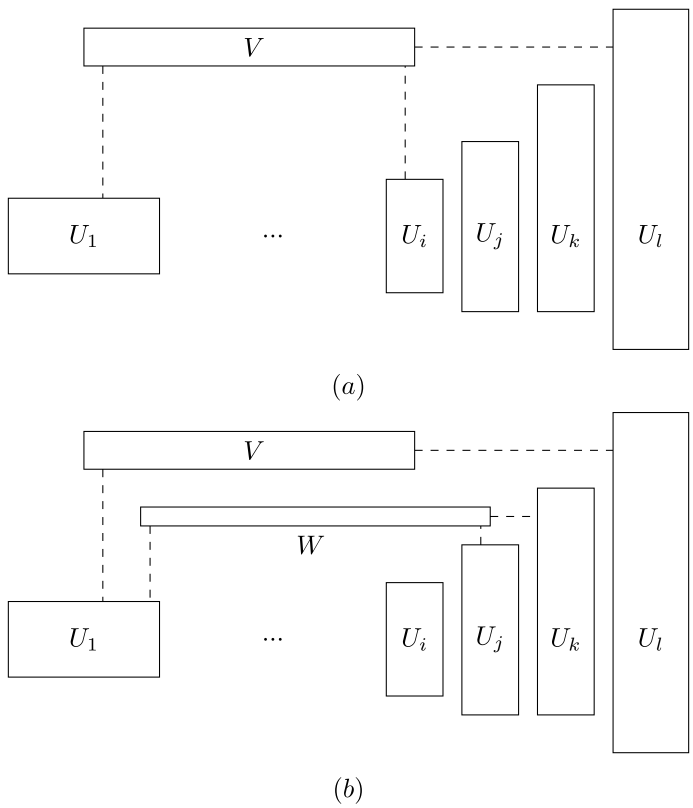

Figure 13: (a) Example of a perpendicular rectangle V in the upper region that sees U_i, and U_l (b) Adding another perpendicular rectangle W of the same type in the upper region that sees U_j and U_k but then blocks V's visibility to U_i

degrees to make them ascending.

We have therefore shown that in any case, we can place the required $\frac{n^2-3n+2}{2}$ rectangles as long as $\frac{n^2-3n+2}{2} \leq (8n-20)+(8n-20) = 16n-40$. This implies that we require $n^2 - 35n + 82 \leq 0$, but when $n \geq 33$, then $n^2 - 35n + 50 \geq 16$. This implies that there does not exist a rectangle visibility representation of a $K_{k,\binom{[k]}{3}}$ bipartite where $k = n^6 + 1$ and $n \geq 33$.

So, we have found a $(1,4)$-hilly graph that does not have a rectangle visibility representation. $\square$

Although not all $(1,4)$-hilly graphs have a 1-rectangle visibility representation, they do all have a 2-rectangle visibility representation.

Theorem 24. *If G is a $(1,4)$-hilly graph, then $r(G) \leq 2$.*

Proof. We will construct a 2-rectangle visibility representation of G as follows. Let H be the set of all rectangles that correspond to high degree vertices in G. Note that H consists of two rectangles per high degree vertex. Arrange rectangles in H in the plane as unit squares with unit length 1 as a descending staircase. We will show how to place rectangles corresponding to low degree vertices that are adjacent to high degree vertices for different cases.

Place all vertices in G that are adjacent to exactly one high degree vertex u as rectangles with dimensions $\frac{1}{m}$ by $\frac{1}{m}$ (where $m = \deg u$) completely inside the channel of visibility of one of the two rectangles corresponding to u as a descending staircase.

Place all vertices v in G that are adjacent to exactly two high degree vertices u and w as rectangles with dimensions $\frac{1}{m}$ by $\frac{1}{n}$ (where $m = \deg u$ and $n = \deg w$) completely inside the intersection of the channels of visibility of rectangles corresponding to u and w as a descending staircase.

Place all vertices v in G that are adjacent to three high degree vertices u, w, and x as rectangles with dimensions $\frac{1}{m}$ by $\frac{1}{n}$ (where $m = \deg u$ and $n = \deg w$) completely inside the intersection of the channels of visibility of rectangles corresponding to u and w as a descending staircase. For each of these vertices, place an additional rectangle of dimension $\frac{1}{p}$ by $\frac{1}{p}$ completely inside the channel of visibility of one of the two rectangles corresponding to x (see Figure 14) to form a descending stairacse.

Note that all high degree vertices and low degree vertices that are adjacent to three high degree vertices will be completely accounted for at this point in the rectangle visibility representation.

Observe that we can place at least one rectangle corresponding to low degree vertices that are

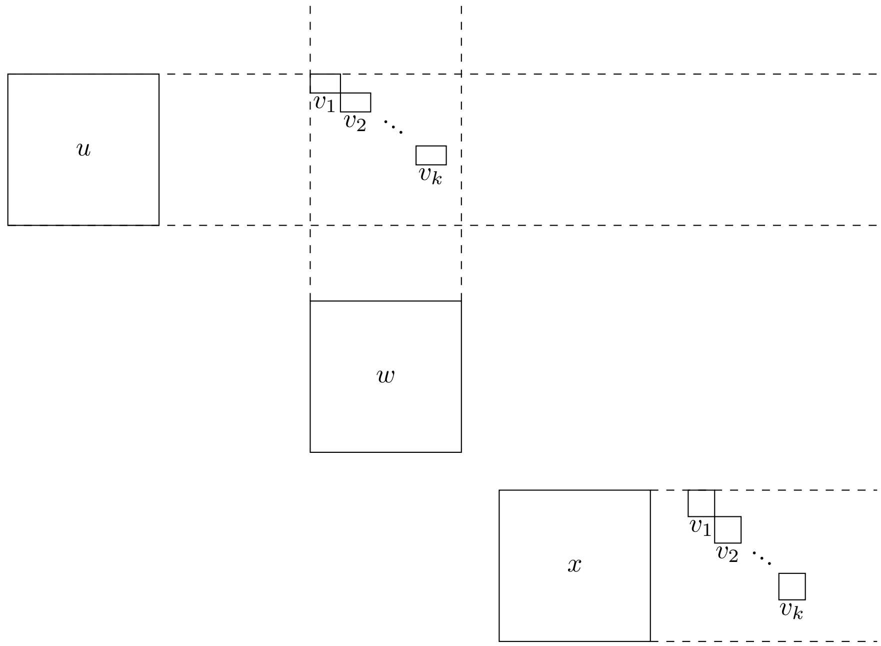

Figure 14: Placement of rectangles corresponding to vertices adjacent to high degree vertices u, w, and x $(v_1, ..., v_k)$

adjacent to at most two high degree vertices. This collection of vertices and edges forms a subcubic graph, which has a 1-rectangle visibility representation [3]. We can construct this representation according to Lemma 15. $\square$

Theorem 25. *There exists a* $(1, r+1)$*-hilly graph* G *such that* $r(G) \geq \left\lceil \frac{r-1}{8} + \frac{1}{r-1} \right\rceil$.

Proof. Let $k = 2r$ and let $G = K_{k,\binom{[k]}{r}}$, which is an example of a $(1, r+1)$-hilly graph. Next, let $k = 2r$. Observe that we can select $r-1$ vertices in the set of $\binom{[2r]}{r}$ that are all adjacent to a set of $r-1$ vertices in the partite set k. The r^{th} edge of all $r-1$ vertices from $\binom{[2r]}{r}$ can then be distributed among the remaining $r+1$ vertices in k. From this, we can conclude that there exists k such that $K_{r-1,r-1}$ is a subgraph of $K_{k,\binom{[k]}{r}}$ for all r.

Let H be a subgraph of G isomorphic to $K_{r-1,r-1}$ and remove all rectangles in G that do not correspond to vertices in H. Since there may have been rectangles in $G - H$ that were blocking potential visibilities between rectangles of vertices in H, the subgraph H may be a weak rectangle visibility graph in the plane once unnecessary rectangles are removed. We can still apply the edge bound stated in Theorem 21 for complete bipartite graphs because any extra visibilities in the weak representation of H will be between rectangles corresponding to vertices in the same partite set of $r-1$ vertices rather than between two vertices in different partite sets. Moreover, the visibility number of a graph must be no less than that of one of its subgraphs. Therefore, we have that $r(G) \geq r(H)$, then $r(G) \geq \left\lceil \frac{(r-1)^2+8}{8(r-1)} \right\rceil = \left\lceil \frac{r-1}{8} + \frac{1}{r-1} \right\rceil$.

$\square$

IV. Future Work

The following is a list of open problems related to the work presented in this thesis:

- Can we determine the maximum degree of a unit hyper n-cube tree for $n \geq 3$?
- Is $\Upsilon_{ur}(T) = \left\lceil \frac{\Upsilon_{ub}(T)}{2} \right\rceil$? We believe this to be true but have not yet been able to provide a proof.
- Is there a sharper bound on $r(K_n)$? We have seen that even small complete graphs such as $r(K_{17})$ are below the determined upper bound.
- Can we provide an upper bound for $r(K_{m,n})$, the complete graph where the partite sets are of different sizes?
- Can we analyze $r(G)$ where G is a (k, n)-hilly graph and $k \neq 1$?

One of the bigger questions that is not directly related to the work presented in this thesis is if there is a unifying characterization for rectangle visibility graphs and unit rectangle visibility graphs analagous to the characterization for bar visibility graphs given by Wismath [17]. In the framework of VLSI design, one might also consider future work to involve an analysis of visibility numbers for rectangle k-visibility graphs, where lines of sight may intersect at most k other rectangles. This looser definition of visibility is better for direct application to circuit design because it considers the possibilities of wired connections that travel through deeper layers of a circuit board before resurfacing or those that bend around components.

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