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Applications of Renewal Theory to Pattern Analysis

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APPLICATIONS OF RENEWAL THEORY TO PATTERN ANALYSIS

by

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Abstract

In this thesis, renewal theory is used to analyze patterns of outcomes for discrete random variables. We start by introducing the concept of a renewal process and by giving some examples. It is then shown that we may obtain the distribution of the number of renewals by time t and compute its expectation, the renewal function. We then proceed to state and illustrate the basic limit theorems for renewal processes and make use of Wald's equation to give a proof of the Elementary Renewal Theorem. The concept of a delayed renewal process is introduced for the purpose of studying the patterns mentioned above. The mean and variance of the number of trials necessary to obtain the first occurrence of the desired pattern are found as well as how these quantities change in cases where a pattern may overlap itself. In addition to this, we explore the expected number of trials between consecutive occurrences of the pattern. If our attention is restricted to patterns from a given finite set, the winning pattern may be defined to be the one which occurs first and the game ends once a winner has been declared. We compute the probability that a given pattern is the winner as well as the expected length of the game. In concluding our discussion, we explore the expected waiting time until we obtain a complete run of distinct or increasing values.

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Contents

1. Introduction

A renewal process is a counting process where the interarrival times, the time between successive events, are independent and identically distributed (iid) random variables with an arbitrary distribution. One well known example of this is the Poisson process, whose interarrival times are iid exponential random variables. The general nature of a renewal process, allowing for any interarrival distribution, can be useful in the sense that any properties or definitions found to apply to a renewal process can then be used in a wide variety of cases. It is this reasoning that makes it valuable for pattern analysis.

1.1 Definition of a Renewal Process

Definition 1.1 *A counting process, $\{N(t),\ t \geq 0\}$, is a renewal process if $\{X_1, X_2, \dots\}$, the sequence of nonnegative random variables of interarrival times, are independent and identically distributed. [3]*

In other words, a renewal process is a counting process where the times between events, the interarrival times, all have the same distribution but are independent of each other. The occurrence of an event is referred to as a renewal.

Example 1.1 Suppose when a light bulb burns out, it is immediately replaced with another whose lifetime is iid to the previous. Assume there is an infinite supply of these light bulbs.

Allow us to define the following quantities.
X_n represents the lifetime of light bulb n; X_n is the n^{th} the interarrival time. A renewal is the replacing of a light bulb; the arrival of an event. Then $\{N(t),\ t \geq 0\}$ is a renewal process and $N(t)$ represents the number of light bulbs that have burned out by time t; the number of renewals by time t. [3]

■

For general cases, we will use the following notation. The interarrival distribution of $\{X_1, X_2, \dots\}$ will be denoted by F, the mean time between successive renewals is denoted by μ, and T_n is the time that the n^{th} renewal takes place.

Then, we have the following two equations that describe the simple relationship between the

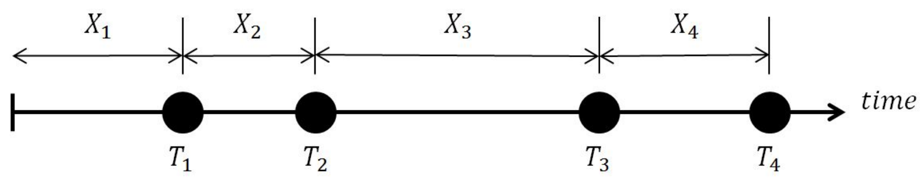

Figure 1: relationship between interarrival times and renewals

interarrival times, $\{X_1, X_2, \ldots\}$, and the times of renewals, T_n. [3]

$$T_0 = 0$$
$$T_n = \sum_{i=1}^{n} X_i, \qquad n \geq 1$$

1.2 Assumptions and Consequences

For the sake of eliminating trivial cases, the assumptions are made that no two arrivals may occur at the same time and that the interarrival distribution, F, is such that it's X_n are not identically zero. That is,

$$F(0) = P(X_n = 0) < 1.$$

Then, by extension,

$$\mu = E(X_n) > 0, \qquad n \geq 1$$

due to the fact that the X_n are not identically zero and since they are representative of times, they must be nonnegative. [3]

Considering the previous two equations, can an infinite number of renewals occur in a finite amount of time? Can a finite amount of renewals occur in an infinite amount of time?

Let us answer the first question.

$$N(t) = max\{n : T_n \leq t\} \qquad \text{(eq. 1.1)}$$

We can justify this equation in the following way. Since $N(t)$ is the count of renewals by time t, any time a renewal event occurs and as long as it's $T_n < t$, it will contribute to the count of $N(t)$. Say for example a renewal takes place at times $T_1 = 3$, $T_2 = 7$, $T_3 = 9$, and $T_4 = 15$. Then $N(14)$ will represent the renewals from times T_1, T_2, and T_3, so $N(14) = 3$. [3]

We can say

$$\frac{T_n}{n} \to \mu \quad \text{as} \quad n \to \infty$$

by the strong law of large numbers, which tells us that the average of the interarrival times will almost surely (with probability 1) converge on the expected value of the interarrival times. Because $\mu > 0$ and $n \to \infty$, then T_n must also be going to infinity. This tells us that only a finite number of T_n can be less than or equal to a finite t.

It can now be stated that for a finite value of t, $N(t)$ must be also be finite.

Is it possible for a finite number of renewals to occur in an infinite amount of time?

$N(t)$ is finite for finite t, but if we allow $t \to \infty$, then $N(\infty)$ represents the total amount of renewals that have ever occurred in the process and it follows that

$$N(\infty) = \lim_{t\to\infty} N(t) = \infty.$$

The only way to have a finite number of total renewals is if some of the interarrival times, X_n, were infinite. In fact, this can be shown below.

$$\begin{aligned} P(N(\infty) < \infty) &= P(\text{one or more interarrival time was infinite}) \\ &= P(X_n = \infty \text{ for some n}) \\ &= P(\cup_{i=1}^{\infty}(X_n = \infty)) \end{aligned}$$

Which must be less than or equal to the probability that every X_n is infinite.

$$\leq P(X_1 = \infty,\ X_2 = \infty,\ \ldots,\ X_n = \infty)$$

Because interarrival times are iid, this is the sum of their independent probabilities.

$$\begin{aligned} &\leq \sum_{i=1}^{\infty} P(X_n = \infty) \\ &= 0 \end{aligned}$$

Therefore, the probability of the total number of renewals being finite is zero here. So for an infinite t, $N(t)$ must also be infinite. [3]

2. Distribution of a Renewal Process

2.1 Obtaining the Distribution of N(t)

First noting an important relationship between the number of renewals and the time of each renewal, we can actually find the distribution of $N(t)$, at least in theory.

$$N(t) \geq n \quad \text{iff} \quad T_n \leq t \tag{eq. 2.1}$$

It is a simple relationship. If the time of the n^{th} renewal occurs at time t, or $T_n = t$, then $N(t) = n$. However, what if the time of the n^{th} renewal falls short of t? In this case, there is the possibility of another renewal occurring before time t. Say one more renewal does occur in that period, then $T_{n+1} < t$ and so $N(t) = n + 1$. If two renewals take place in that period, then $T_{n+1} < t$ and $T_{n+2} < t$, so $N(t) = n + 2$. By this reasoning, the previous equation is validated.

Using this relationship, we have,

$$\begin{aligned} P(N(t) = n) &= P(N(t) \geq n) - P(N(t) \geq n + 1) \\ &= P(T_n \leq t) - P(T_n \leq t) \\ &= F_n(t) - F_{n+1}(t). \end{aligned} \tag{eq. 2.2}$$

Remembering the definition of T_n, $T_n = \sum_{i-1}^{n} X_i$, we know that because the interarrival times are iid with distribution F, then the distribution of T_n is the n-fold convolution of F with itself, denoted by F_n. [3]

Example 2.1 A renewal process has geometric interarrival times. Find an expression for $P(N(t) = n)$.

To start, the distribution of T_n must be negative binomial since it is the sum of iid geometric random variables.

Recall that the pmf of the negative binomial is,

$$P(T_n = t) = \begin{cases} \binom{t-1}{n-1} p^n (1-p)^{t-n} & t \geq n \\ 0 & t < n \end{cases}$$

Because we are working with discrete random variables, $n \leq t$. This is because for each time unit, a renewal may or may not occur, but two renewals may not occur at the same time. So, at most, one renewal may take place per unit of time. This is useful because we can disregard the zero portion of the negative binomial pmf.

From here, (*eq.* 2.2) can be manipulated.

$$
\begin{aligned}
P(N(t)=n) &= F_n(t) - F_{n+1}(t) \\
&= P(T_n \le t) - P(T_{n+1} \le t) \\
&= \sum_{i=n}^{t} P(T_i = t) - \sum_{i=n+1}^{t} P(T_{i+1} = t) \\
&= \sum_{i=n}^{t} \binom{i-1}{n-1} p^n (1-p)^{i-n} - \sum_{i=n+1}^{t} \binom{i-1}{n} p^{n+1} (1-p)^{i-n-1} \\
&= p^n \left[\sum_{i=n}^{t} \binom{i-1}{n-1} (1-p)^{i-n} - p \sum_{i=n+1}^{t} \binom{i-1}{n} (1-p)^{i-n-1} \right] \\
&= p^n \left[\sum_{i=n}^{t} \binom{i-1}{n-1} (1-p)^{i-n} - (1-(1-p)) \sum_{i=n+1}^{t} \binom{i-1}{n} (1-p)^{i-n-1} \right] \\
&= p^n \left[\sum_{i=n}^{t} \binom{i-1}{n-1} (1-p)^{i-n} - \sum_{i=n+1}^{t} \binom{i-1}{n} (1-p)^{i-n-1} + \sum_{i=n+1}^{t} \binom{i-1}{n} (1-p)^{i-n} \right]
\end{aligned}
$$

Pulling out the $i = n$ term of the first sum allows for the first and third sums to be combined by use of the fact that $\binom{i-1}{n-1} + \binom{i-1}{n} = \binom{i}{n}$.

$$
\begin{aligned}
&= p^n \left[1 + \sum_{i=n+1}^{t} \binom{i}{n} (1-p)^{i-n} - \sum_{i=n+1}^{t} \binom{i-1}{n} (1-p)^{i-n-1} \right] \\
&= p^n \left[1 + \sum_{i=n+1}^{t} \binom{i}{n} (1-p)^{i-n} - \sum_{i=n}^{t-1} \binom{i}{n} (1-p)^{i-n} \right] \\
&= p^n \left[1 + \sum_{i=n+1}^{t} \binom{i}{n} (1-p)^{i-n} - \left(1 + \sum_{i=n+1}^{t-1} \binom{i}{n} (1-p)^{i-n} \right) \right] \\
&= p^n \left[\sum_{i=n+1}^{t} \binom{i}{n} (1-p)^{i-n} - \sum_{i=n+1}^{t-1} \binom{i}{n} (1-p)^{i-n} \right] \\
&= p^n \left[\binom{t}{n} (1-p)^{t-n} + \sum_{i=n+1}^{t-1} \binom{i}{n} (1-p)^{i-n} - \sum_{i=n+1}^{t-1} \binom{i}{n} (1-p)^{i-n} \right] \\
&= \binom{t}{n} p^n (1-p)^{t-n}
\end{aligned}
$$

The calculation ends with the binomial distribution, which makes sense here. $N(t)$ counts the number of renewals, so if we think of a renewal as a successful trial, then $N(t)$ refers to the number of successful outcomes by time t. [3]

■

A second method for finding the distribution of $N(t)$ is to condition on the value of T_n.

$$P(N(t) = n) = \int_0^\infty P(N(t) = n | T_n = x)\ f_{T_n}(x)\ dx$$

Because the only way for there to be exactly n events by time t is if the $n + 1^{st}$ renewal took place after t, we know that the length of X_{n+1} must be greater than $t - x$. We can simplify this in the following manner. [3]

$$= \int_0^t P(X_{n+1} > t - x | T_n = x)\ f_{T_n}(x)\ dx$$

Then since X_{n+1} and T_n are independent of eachother, the condition may be dropped, giving,

$$\begin{aligned}
&= \int_0^t P(X_{n+1} > t - x)\ f_{T_n}(x) dx \\
&= \int_0^t [1 - P(X_{n+1} \le t - x)]\ f_{T_n}(x)\ dx \\
&= \int_0^t [1 - F(t - x)]\ f_{T_n}(x)\ dx
\end{aligned}$$

Example 2.2 Suppose a renewal process' interarrival times follow the exponential distribution. Then the distribution of T_n is the convolution of n exponential random variables, the gamma distribution. Find $P(N(t) = n)$. [3]

Recall that the cumulative distribution function, cdf, of an exponential random variable is $F(x) = 1 - e^{-\lambda x}$ and the pdf of the gamma distribution is $f(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{n-1}}{(n-1)!}$.

$$\begin{aligned}
P(N(t) = n) &= \int_0^t [1 - F(t - x)]\ f_{T_n}(x)\ dx \\
&= \int_0^t \left[1 - (1 - e^{-\lambda(t-x)})\right]\ \frac{\lambda e^{-\lambda x}(\lambda x)^{n-1}}{(n-1)!}\ dx \\
&= \int_0^t e^{-\lambda(t-x)}\ \frac{\lambda e^{-\lambda x}(\lambda x)^{n-1}}{(n-1)!}\ dx \\
&= \int_0^t e^{-\lambda t} e^{\lambda x}\ \frac{\lambda e^{-\lambda x}(\lambda x)^{n-1}}{(n-1)!}\ dx \\
&= \frac{e^{-\lambda t}\lambda^n}{(n-1)!} \int_0^t e^{\lambda x}\ e^{-\lambda x} x^{n-1}\ dx \\
&= \frac{e^{-\lambda t}\lambda^n}{(n-1)!} \int_0^t x^{n-1}\ dx \\
&= \frac{e^{-\lambda t}\lambda^n}{(n-1)!} \left(\frac{t^n}{n}\right) \\
&= \frac{e^{-\lambda t}(\lambda t)^n}{n!}
\end{aligned}$$

■

2.2 Computing the Renewal Function

Definition 2.1 *The function which represents the expected value of $N(t)$, is the renewal function, $m(t)$. That is,*

$$m(t) = E(N(t)).$$

The renewal function is unique to its specific renewal process' interarrival distribution. [3]

The renewal function for discrete distributions can be calculated by use of (*eq.* 2.1).

$$\begin{aligned} m(t) &= E(N(t)) \\ &= \sum_{n=1}^{\infty} P(N(t) \geq n) \\ &= \sum_{n=1}^{\infty} P(T_n \leq n) \\ &= \sum_{n=1}^{\infty} F_n(t) \end{aligned}$$

By conditioning on the time of the first renewal, an integral equation for $m(t)$ can be made for continuous cases with probability density function $f(x)$. *

$$m(t) = E(N(t)) = \int_0^{\infty} E(N(t)|X_1 = x) f(x)\, dx \qquad \text{(eq. 2.3)}$$

If $x < t$, then,

$$\begin{aligned} E(N(t)|X_1 = x) &= E(1 + N(t-x)) \\ &= 1 + E(N(t-x)) \end{aligned}$$

because, probabilistically speaking, the renewal process starts over after each renewal occurrence. So $N(t)$ is increased by one and there is still the possibility of another renewal occurring in the remaining time before t.

On the other hand, if $x > t$, then no renewals will have occurred by time t, and then we know that

$E(N(t)|X_1 = x) = 0$. In this case, ($eq.$ 2.3) can be used as follows.

$$\begin{aligned} m(t) &= \int_0^t E(N(t)|X_1 = x) f(x)\, dx \\ &= \int_0^t [1 + E(N(t-x))] f(x)\, dx \\ &= \int_0^t [1 + m(t-x)] f(x)\, dx \\ &= \int_0^t f(x)\, dx + \int_0^t [m(t-x) f(x)\, dx \\ &= F(t) + \int_0^t m(t-x) f(x)\, dx \end{aligned} \qquad \text{(eq. 2.4)}$$

The previous equation is known as the renewal equation and, in some cases, it is possible to obtain the renewal function by solving it.

The following example will demonstrate the most simple case where this equation is solvable for $m(t)$. [3]

Example 2.3 Suppose $X_n \sim Uni(0,1)$ and assume $t \leq 1$, find an expression for $m(t)$.

Beginning with ($eq.$ 2.4),

$$\begin{aligned} m(t) &= F(t) + \int_0^t m(t-x) f(x)\, dx \\ &= \int_0^t f(x)\, dx + \int_0^t m(t-x) f(x)\, dx \\ &= \int_0^t 1\, dx + \int_0^t m(t-x)(1)\, dx \\ &= t + \int_0^t m(t-x)\, dx \end{aligned}$$

Differentiating both sides and applying the fundamental theorem of Calculus, we have,

$$m'(t) = t + m(t)$$

Making the replacement $g(t) = t + m(t)$, then $g'(t) = m'(t)$, and so the following equation may be solved by typical means.

$$g'(t) = g(t)$$

Then $g(t) = ce^t$ for some constant c. In terms of the original function $m(t)$, the generic solution is $m(t) = ce^t - 1$. Applying that $m(0) = 0$, which is intuitive, gives $c = 1$, and therefore, [3]

$$m(t) = e^t - 1, \quad 0 \leq t \leq 1.$$

■

3. Limit Theorems

3.1 Rate of Renewals

It was already been shown that $\lim_{t\to\infty} N(t) = \infty$. Now we will explore the rate at which this happens. Specifically, what can be said about $\lim_{t\to\infty} \frac{N(t)}{t}$?

If $T_{N(t)}$ represents the time of the most recent renewal before or at time t, then $T_{N(t)+1}$ denotes the time of the first renewal following t. Trivially, it can be stated then that [3]

$$T_{N(t)} \leq t < T_{N(t)+1}. \qquad \text{(eq. 3.1)}$$

Proposition 3.1 Almost surely, [1]

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu}.$$

Proof. The proof begins by making use of the simple relationship shown in (*eq.* 3.1).

$$\begin{aligned}
&T_{N(t)} \leq t < T_{N(t)+1} \\
&\frac{T_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{T_{N(t)+1}}{N(t)} \\
&\frac{T_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \left(\frac{T_{N(t)+1}}{N(t)+1}\right)\left(\frac{N(t)+1}{N(t)}\right) \\
&\lim_{t\to\infty} \frac{T_{N(t)}}{N(t)} \leq \lim_{t\to\infty} \frac{t}{N(t)} < \lim_{t\to\infty} \left(\frac{T_{N(t)+1}}{N(t)+1}\right) \lim_{t\to\infty} \left(\frac{N(t)+1}{N(t)}\right)
\end{aligned}$$

We know that $\lim_{t\to\infty} \left(\frac{N(t)+1}{N(t)}\right) = 1$, so we may simplify this to,

$$\lim_{t\to\infty} \frac{T_{N(t)}}{N(t)} \leq \lim_{t\to\infty} \frac{t}{N(t)} < \lim_{t\to\infty} \left(\frac{T_{N(t)+1}}{N(t)+1}\right)$$

From here, consider $\frac{T_{N(t)}}{N(t)}$. Because $T_{N(t)}$ is the sum of $N(t)$ variables, this quotient is an average and, therefore, will follow the strong law of large numbers. It follows that

$$\lim_{N(t)\to\infty} \frac{T_{N(t)}}{N(t)} = \lim_{t\to\infty} \frac{T_{N(t)}}{N(t)} = \mu$$

since, as shown previously, $N(\infty) = \infty$. Simplifying the inequality with this information gives the following:

$$\mu \leq \lim_{t\to\infty} \frac{t}{N(t)} \leq \mu.$$

In the line above, the simplification of the other side of the inequality was simple since it also must follow the same strong law of large numbers logic. Therefore, courtesy of the Squeeze Theorem,

$$\lim_{t\to\infty} \frac{t}{N(t)} = \mu$$

Finally, this gives us [3]

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu}.$$

■

Definition 3.1 *Since the average number of renewals per time unit is $\frac{1}{\mu}$, this quantity is the rate of a renewal process.*

Example 3.1 Suppose a bus departs from a bus-stop every 25-35 minutes, uniformly. At what rate do departures occur?

Here $X_n \sim Uni(25, 35)$, so we have $\mu = \frac{a+b}{2} = \frac{25+35}{2} = 30$. Then the rate is

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu} = \frac{1}{30}.$$

On average, a bus leaves the stop every half hour.

■

Example 3.2 Suppose the same bus has to undergo a brief safety check before it's doors are opened at a stop. This check will take 0 minutes if the bus driver forgets and up to 2 minutes if he remembers. Assume the length of the check is distributed uniformly. Now at what rate do bus departures take place?

Let X_1 represent the time it takes the bus to arrive at the stop and let X_2 denote the duration of the safety check. Then $X_1 \sim Uni(25, 35)$ and $X_2 \sim Uni(0, 2)$. The sum of $X_1 + X_2$ has mean $\mu = \mu_1 + \mu_2$ where $\mu = \frac{25+35}{2} + \frac{0+2}{2} = 30 + 1 = 31$ and the rate is

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu} = \frac{1}{31}.$$

On average, a bus departs every 31 minutes.

■

Example 3.3 Suppose each trial of an iid sequence will always result in some outcome i with probability p_i, $i = 1, 2, \ldots, n$. A renewal takes place each time the same outcome is observed n times in a row, after which, the game starts over. The outcome which caused this renewal is considered the winner. What is the probability that outcome i wins and what is the expected number of trials the game will last?

First, we can look at a simpler case by computing the expected number of coin tosses necessary to get a run of n successive heads. We are assuming each toss is independent and has a probability, p, of landing on heads.

Let T be the number of tosses needed to obtain n heads in a row, and let S denote the time of the first tails. Then, by conditioning on the time of the first tails, we obtain our starting equation and the rest follows.

$$E(T) = \sum_{k=1}^{\infty} E(T|S=k)\ P(S=k)$$

Because the probability that the first tails occurs at time k is geometric, we can simplify this to,

$$= \sum_{k=1}^{\infty} E(T|S=k)\ (1-p)p^{k-1}$$

Consider the possible values of $E(T|S=k)$. If $k > n$ and k is the first time a tails occurred, then there must have already been a run of at least n heads, so $E(T|S=k) = n$ if $k > n$. On the other hand, if $k < n$, then the occurrence of a tails at time k ruins the run of heads and we essentially start over again, therefore, $E(T|S=k) = k + E(T)$ when $k < n$. Using this information, we may break up the above summation.

$$\begin{aligned}
&= \sum_{k=1}^{n} (k + E(t))\ (1-p)p^{k-1} + \sum_{k=n+1}^{\infty} n\ (1-p)p^{k=1} \\
&= \sum_{k=1}^{n} (k + E(t))\ (1-p)p^{k-1} + n\ (1-p) \sum_{k=n+1}^{\infty} p^{k=1} \\
&= \sum_{k=1}^{n} (k + E(t))\ (1-p)p^{-1} + n\ (1-p) \left[\frac{p^n}{1-p}\right] \\
&= np^n + \sum_{k=1}^{n} \left[(1-p)p^{k-1}(k + E(T))\right]
\end{aligned}$$

$$\begin{aligned}
&= np^n + (1-p)\sum_{k=1}^{n} kp^{k-1} + (1-p)E(T)\sum_{k=1}^{n} p^{k-1} \\
&= np^n + (1-p)\left[\frac{1-p^n}{(1-p)^2} - \frac{np^n}{(1-p)}\right] + (1-p)E(T)\left[\frac{p^n-1}{(p-1)}\right] \\
&= np^n + \frac{1-p^n}{(1-p)} - np^n + \frac{(1-p)(p^n-1)}{(p-1)}E(T) \\
&= \frac{1-p^n}{(1-p)} - (p^n-1)E(T) \\
E(T) + (p^n-1)E(T) &= \frac{1-p^n}{(1-p)} \\
[1+(p^n-1)]E(T) &= \frac{1-p^n}{(1-p)} \\
E(T) &= \frac{1-p^n}{p^n(1-p)} \qquad \text{(eq. 3.2)}
\end{aligned}$$

Now we are ready to tackle the original example.

Because a win by outcome i is a renewal, we can apply Proposition 3.1. So,

$$\text{the rate at which i wins} = \frac{1}{\mu} = \frac{1}{E(N_i)}$$

here $E(N_i)$ represents the expected number of trials that take place between successive wins by i. In other words, $E(N_i)$ is the expected wait time until i wins. In fact, we've already computed this quantity, (*eq.* 3.2), and can now apply it here.

$$\text{the rate at which i wins} = \frac{1}{\mu} = \frac{1}{E(N_i)} = \frac{p_i^k(1-p_i)}{1-p_i^k} \qquad \text{(eq. 3.3)}$$

The proportion of games won by i are as follows.

$$\text{proportion of games won by } i = \frac{\text{rate at which } i \text{ wins}}{\text{rate at which wins occur}} = \frac{\frac{p_i^k(1-p_i)}{1-p_i^k}}{\sum_{j=1}^{n} \frac{p_j^k(1-p_j)}{1-p_j^k}}$$

Then, by the strong law of large numbers, the long-run proportion of games that i will win will, with probability 1, be equal to the probability that i wins any given game. So,

$$P(i \text{ wins}) = \frac{\frac{p_i^k(1-p_i)}{1-p_i^k}}{\sum_{j=1}^{n} \frac{p_j^k(1-p_j)}{1-p_j^k}}.$$

Lastly, we can compute the expected length of a game. It follows that,

$$\begin{aligned}\text{rate at which games end} &= \text{rate at which wins occur}\\ &= \sum_{j=1}^{n} \frac{p_j^k(1-p_j)}{1-p_j^k}.\end{aligned}$$

Then, from Proposition 3.1, [3]

$$\begin{aligned}E(\text{duration of a game}) &= \frac{1}{\text{rate at which games end}}\\ &= \frac{1}{\sum_{j=1}^{n} \frac{p_j^k(1-p_j)}{1-p_j^k}}\end{aligned}$$

■

3.2 Stopping Time and Wald's Equation

Definition 3.2 *If we have a non-negative integer-valued random variable, N, we can call this a stopping time for a sequence of independent random variables, $X_1, X_2, \ldots$ only in the case that for all $n = 1, 2, \ldots,$ then $(N = n)$ is independent of $X_{n+1}, X_{n+2}, \ldots$. [3]*

In other words, a stopping time is the time at which a sought after event occurs, as seen in the next example, and it is no longer necessary to continue with trials.

Example 3.4 Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with only two potential outcomes, 1 or 0. Then we can say,

$$\begin{aligned}&P(X_i = 1) = p\\ &P(X_i = 0) = 1 - p = q, \text{for} p > 0.\end{aligned}$$

Suppose we are seeking the event where $(X_1 + X_2 + \cdots + X_n > k)$, then we have N being a stopping time for the sequence if we define it as $N = \min(n : X_1 + X_2 + \cdots + X_n > k)$. [3]

■

Theorem 3.1 (Wald's Equation) [3] If we have a sequence of iid random variables, $X_1, X_2, \ldots$ with some finite expectation, denoted $E(X)$, and if N is a stopping time for the sequence $X_1, X_2, \ldots,$ then assuming $E(N) < \infty$,

$$E\left(\sum_{n=1}^{N} X_n\right) = E(N)E(X)$$

Proof. Let I_n be an binary indicator variable such that $I_n = 1$ if $n \leq N$ and 0 if $n > N$, for $n = 1, 2, \ldots$. With this, we have the relationship

$$\sum_{n=1}^{N} X_n = \sum_{n=1}^{\infty} X_n I_n.$$

If we consider the expected values of these sums, it can be seen that

$$\begin{aligned} E\left(\sum_{n=1}^{\infty} X_n\right) &= E\left(\sum_{n=1}^{\infty} X_n I_n\right) \\ &= \sum_{n=1}^{\infty} E(X_n I_n) \\ &= \sum_{n=1}^{\infty} E(X_n) E(I_n) \\ &= E(X) E\left(\sum_{n=1}^{\infty} I_n\right) \\ &= E(X) E(N) \end{aligned}$$

where the above relies on the argument that that X_n and I_n are independent since the value of I_n relies solely on whether or not we have yet stopped. [3]

■

Proposition 3.2 Suppose we have the interarrival times of a renewal process given by $X_1, X_2, \ldots$, then,

$$E(X_1 + X_2 + \cdots + X_{N(t)+1}) = E(X)\ E(N(t) + 1).$$

Altering notation shows us another equivalent expression, $E(T_{N(t)+1}) = \mu(m(t) + 1)$. [2]

Theorem 3.2 (Central Limit Theorem for Renewal Processes) [3] For large values of t and for $X_i \sim F(\mu, \sigma^2)$, then $N(t) \overset{approx.}{\sim} Normal\left(\frac{t}{\mu}, \frac{t\sigma^2}{\mu^3}\right)$. Recall that F denotes the distribution of the interarrival times.

$$\lim_{t\to\infty} P\left(\frac{N(t) - \frac{t}{\mu}}{\sqrt{\frac{t\sigma^2}{\mu^3}}} < x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-x^2}{2}}\ dx$$

$$\lim_{t\to\infty} \frac{Var(N(t))}{t} = \frac{\sigma^2}{\mu^3} \tag{3.1}$$

3.3 Elementary Renewal Theorem

The theorems and properties developed in the previous section are the ideas on which the proof of the Elementary Renewal Theorem relies, which is why they were presented first. We are now ready to investigate this theorem and prove it.

Theorem 3.3 (Elementary Renewal Theorem) [1] $\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}$ or $\lim_{t\to\infty} \frac{E(N(t)}{t} = \frac{1}{E(T)}$

Proof. Since $T_{N(t)+1}$ is the time of the first renewal that occurs after a time t, we can determine that

$$T_{N(t)+1} = t + Y(t)$$

and

$$E(T_{N(t)+1}) = E(t + Y(t)) = t + E(Y(t))$$

where $Y(t)$ is the excess at the time t. We define the excess as the difference in time between t and the next renewal, or $Y(t) = T_{N(t)+2} - t$.

By use of Proposition 3.2, this can be rearranged to yield the following:

$$\begin{aligned} \mu(m(t)+1) &= t + E(Y(t)) \qquad \text{(eq. 3.4)} \\ m(t) &= \frac{t}{\mu} + \frac{E(Y(T))}{\mu} - 1 \\ \frac{m(t)}{t} &= \frac{1}{\mu} + \frac{E(Y(T))}{t\mu} - \frac{1}{t} \end{aligned}$$

The stipulation that $Y(t) \geq 0$ allows us to see that

$$\frac{m(t)}{t} \geq \frac{1}{\mu} - \frac{1}{t}$$

which, when $t \to \infty$, simplifies this. That is,

$$\lim_{t\to\infty} \frac{m(t)}{t} \geq \frac{1}{\mu}.$$

Assuming that there exists a finite B, an upper boundary on the values of X_i, it must satisfy the property that $P(X_i < B) = 1$. Since all $X_i < B$, it follows that $Y(t) < B$ and consequently, $E(Y(t)) < B$.

We then have

$$\frac{m(t)}{t} \leq \frac{1}{\mu} + \frac{B}{t\mu} - \frac{1}{t}$$

Now if we consider $t \to \infty$, it can be seen that the limit of $\frac{m(t)}{t}$ does not exceed $\frac{1}{\mu}$.

$$\lim_{t\to\infty} \frac{m(t)}{t} \leq \frac{1}{\mu}$$

With that statement, due to the Squeeze theorem, the theorem is now proven for the cases where we may bound the interarrival times. We can say,

$$\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}.$$

For the cases where the interarrival times are unbounded, we define $B > 0$ and choose $\{N_B(t),\ t \geq 0\}$ to be a renewal process with $\min(X_i, B),\ i \geq 1$ as the interarrival times. Since, for all i, $\min(X_i, B) \leq X_i$, it follows that $N_B(t) \geq N(t),\ \forall t$. It then further follows that since $N(t) \leq N_B(t)$, we also have $E(N(t)) \leq E(N_B(t))$, which yields

$$\lim_{t\to\infty} \frac{E(N(t))}{t} \leq \lim_{t\to\infty} \frac{E(N_B(t))}{t} = \frac{1}{E(\min(X_i, B))}$$

We can claim the previous equality due to the interarrival times of $N_B(t)$ being bounded.

By letting $B \to \infty$, we obtain that $\lim_{B\to\infty} E(\min(X_i, B)) = E(X_i) = \mu$ and then $\lim_{t\to\infty} \frac{m(t)}{t} \leq \frac{1}{\mu}$.

The theorem has been proven for the case of unbounded interarrival times as well.

Therefore, the theorem is proven in all cases. [3]

■

Example 3.5 Let $D \sim Uni(0,1)$ and set $X_k = \begin{cases} k & D \leq \frac{1}{k} \\ 0 & D \geq \frac{1}{k} \end{cases}$.

D will always be greater than 0, and for suitably large k, then X_k will be 0. With this, we conclude that X_k converges to zero almost surely as k goes to infinity. The above, if considering Proposition 3.1, implies that it's expected value should be infinite.

However, we instead have that

$$E(X_k) = k\ P(D \leq \frac{1}{k}) = k\frac{1}{k} = 1.$$

Therefore, each X_k will have an expected value of one even though the sequence is expected to converge to zero. This showcases the differences between Theorem 3.3 and Proposition 3.1. [3]

■

Example 3.6 Assume X_i to be iid random variables for $i \geq 1$. Here, rather than the time between events, let X_i denote the change in position from one event to the next. Then we can define R_n to be the total displacement at time n from the process' original position.

We assume the process started at position 0 and let $R_n = \sum_{i=1}^{n} X_i,\ n > 0$. The process $\{R_n,\ n \geq 0\}$ is a random walk process and can be considered a renewal process if you let a renewal occur each time the displacement reaches a new low. That is, if $R_n < \min(0, R_1, R_2, \ldots, R_{n-1})$, then a renewal has taken place at time n. Let us assume that the expected value of X_i is negative.

The strong law of large numbers gives us

$$\lim_{n\to\infty} \frac{R_n}{n} = E(X_i).$$

For this quotient's limit to be negative, we must have R_n approaching negative infinity since n is positive. Let the probability that the random walk is always negative after the first move be denoted by $\alpha = P(R_n < 0,\ n \geq 1)$.

The next renewal, or record displacement, after n occurs k units of time afterward if

$$\begin{aligned} X_{n+1} &\geq 0 \\ X_{n+1} + X_{n+2} &\geq 0 \\ &\vdots \\ X_{n+1} + \cdots + X_{n+k-1} &\geq 0 \\ X_{n+1} + \cdots + X_{n+k} &< 0 \end{aligned}$$

Since all X_i are iid, the preceding event is independent of the values of X_i and the probability of occurrence does not depend on n, which validates that this can be considered a renewal process. That means we can say *

$$\begin{aligned} P(\text{renewal at time } n) &= P(R_n < 0, R_n < R_1, \ldots, R_n < R_{n-1}) \\ &= P(X_1 + X_2 + \cdots + X_n < 0, X_2 + X_3 + \cdots + X_n < 0, \ldots, X_n < 0) \end{aligned}$$

There are replacements that can be made without loss of generality since $X_n, X_{n-1}, \ldots, X_1$ shares the joint probability distribution with $X_1, X_2, \ldots, X_n$. We can replace X_1 by X_n, X_2 by X_{n-1}, X_3 by X_{n-2} and so on until we replace X_n by X_1.

Looking back, we now have

$$\begin{aligned} P(\text{ renewal at time } n) &= P(X_n + \cdots + X_1 < 0, X_{n-1} + \cdots + X_1 < 0, \ldots, X_1 < 0) \\ &= P(R_n < 0, R_{n-1} < 0, \ldots, R_1 < 0) \end{aligned}$$

We can now see that $\alpha = \lim_{n \to \infty} P(\text{renewal at time } n) = P(R_n < 0,\ n \geq 1)$.

Invoking the Elementary Renewal Theorem implies that $\alpha = \frac{1}{E(T)}$ with $T = \min(n : R_n < 0)$ being the time between renewals. [3]

■

4. Pattern Analysis Applications

4.1 Delayed Renewal Process

Definition 4.1 *A delayed or general renewal process is a counting process with independent interarrival times given by $X_1, X_2, X_3, \ldots$ if the distribution of X_1 does not match the identical distribution of $X_2, X_3, \ldots$. However, all limiting theorems about $N(t)$ are still true for these processes. [2]*

Let $X_1, X_2, \ldots$ be independent such that $P(X_i = j) = p(j)$ for all non-negative j. We also let T denote the time of the first occurrence of a pattern $x_1, x_2, \ldots, x_r$. If there is a renewal that occurs at time n with $n \geq r$ and if $(X_{n-r+1}, \ldots, X_n) = (x_1, \ldots, x_r)$, then we have a delayed renewal process $\{N(n), n \geq 1\}$ where $N(n)$ represents the number of renewals that have occurred by the time n.

From here, if we consider μ and σ to represent the mean and standard deviation of the time between successive renewals, we have the following consequence of the Central Limit Theorem: [3]

$$\lim_{n\to\infty} \frac{E(N(n))}{n} = \frac{1}{\mu} \tag{4.1}$$

$$\lim_{n\to\infty} \frac{Var(N(n))}{n} = \frac{\sigma^2}{\mu^3}. \tag{4.2}$$

4.2 Mean and Variance of T

Next, we explore how renewal theory results we have been exploring can be used to compute both the mean and variance of T.

Here we will use the same binary indicator variable as seen previously. Let $I(i) = 1$ if there is a renewal occurring at time i and $i \geq r$ and let it be 0 otherwise. Also let $p = \Pi_{i=1}^{r} p(x_i)$. We now have

$$P(I(i) = 1) = P(i_1 = X_{i-r+1}, \ldots,\ i_r = X_i) = p$$

The equation above, in other words, says that the probability that an event occurs at time i is the same as the probability that the r^{th} and final element of our sought after pattern occurs at time i. So it is the probability that our pattern is made complete at time i.

This allows us to determine that $I(i)$ with $i \geq r$ to be Bernoulli random variables with a given parameter p because we either have a success at time i, the pattern is completed, or else it is a failure. As $N(n)$ is the count of renewals that have taken place by time n, we may define this as the following summation. [3]

$$N(n) = \sum_{i=r}^{n} I(i)$$

Which gives us

$$\begin{aligned} E(N(n)) &= \sum_{i=r}^{n} E(I(i)) \\ &= \sum_{i=r}^{n} (1p + 0(1-p)) \\ &= \sum_{i=r}^{n} p \\ &= (n-r+1)p \end{aligned}$$

If we divide this by n and subsequently take the limit as $n \to \infty$, the following can be seen:

$$\begin{aligned} E(N(n)) &= (n-r+1)p \\ \frac{E(N(n))}{n} &= \frac{(n-r+1)p}{n} \\ \lim_{n\to\infty} \frac{E(N(n))}{n} &= \lim_{n\to\infty} \left(p - \frac{rp}{n} + \frac{p}{n}\right) \end{aligned}$$

Therefore, we know that,

$$\lim_{n\to\infty} \frac{E(N(n))}{n} = p$$

Combining this with 4.1 yields

$$p = \frac{1}{\mu} \quad \Rightarrow \quad \mu = \frac{1}{p} \tag{4.3}$$

This shows that the mean time between successive occurrences of the pattern is given by $\frac{1}{p}$. We are also able to determine that

$$\begin{aligned} \frac{Var(N(n))}{n} &= \frac{1}{n} \sum_{i=r}^{n} Var(I(i)) + \frac{2}{n} \sum_{i=r}^{n-1} \sum_{n \geq j > i} Cov(I(i), I(j)) \\ &= \frac{n-r+1}{n} p(1-p) + \frac{2}{n} \sum_{i=r}^{n-1} \sum_{i<j\leq \min(i+r-1,n)} Cov(I(i), I(j)) \end{aligned}$$

The above is obtained as $I(i)$ and $I(j)$ are independent when $|i-j| \geq r$; that is, when they are at least r trials apart. This reasoning makes sense because the occurrence of renewals is independent and there must be at least a distance of r, the length of the pattern, between them for another renewal to even take place at time j.

Since the $Cov(I(i), I(j))$ depends on i and j through $|j-i|$, so we now have

$$\lim_{n\to\infty} \frac{Var(N(n))}{n} = p(1-p) + 2\sum_{j=1}^{r-1} Cov(I(r), I(r+j))$$

Combining this with equation 4.2, then we obtain the following expression for variance. [3]

$$\begin{aligned} \frac{\sigma^2}{\mu^3} &= p(1-p) + 2\sum_{j=1}^{r-1} Cov(I(r), I(r+j)) \\ \frac{\sigma^2}{\left(\frac{1}{p}\right)^3} = \sigma^2 p^3 &= p(1-p) + 2\sum_{j=1}^{r-1} Cov(I(r), I(r+j)) \\ \sigma^2 &= \frac{1}{p^2}(1-p) + 2\frac{1}{p^3}\sum_{j=1}^{r-1} Cov(I(r), I(r+j)) \end{aligned} \qquad \text{(eq. 4.1)}$$

4.3 Cases of Overlap

Definition 4.2 *(Overlap) Overlap is defined as the maximal number of values at the end of a pattern that could make up the beginning of the next occurrence. It is of size* $k : k > 0$ *if [3]*

$$k = \max(n < m : (i_{m-n+1}, \ldots, i_m) = (i_1, \ldots, i_n))$$

Example 4.1 Consider the following coin toss patterns and their respective overlaps.

HTT	HTT	overlap $k = 0$
$HT\underline{H}$	$\underline{H}TH$	overlap $k = 1$
$HHT\underline{HH}$	$\underline{HH}THH$	overlap $k = 2$
$HHTT\underline{HHT}$	$\underline{HHT}THHT$	overlap $k = 3$

■

Let's now consider the overlap cases.

The first case concerns a zero overlap. Let $(N(n),\ n \geq 1)$ be an ordinary renewal process and choose T to be distributed as an interarrival time between renewals with mean and variance μ and

σ^2 respectively. The following stems from equation 4.3:

$$E(T) = \mu = \frac{1}{p} \tag{4.4}$$

Since the minimum distance between two patterns is r, it can be seen that $I(r)I(r+j) = 0$ for $1 \leq j \leq r-1$, yielding

$$Cov(I(r), I(r+j)) = -E(I(r))E(I(r+j)) = -p^2 \text{ for } 1 \leq j \leq r-1$$

Now, if we invoke equation eq. 4.1, we are able to obtain [3]

$$\begin{aligned} Var(T) &= \sigma^2 \\ &= \frac{1}{p^2}(1-p) - 2\frac{1}{p^3}(r-1)p^2 \\ &= \frac{1}{p^2} - \frac{1}{p}(2r-1) \end{aligned} \tag{eq. 4.2}$$

An interesting property worth noting about T is as follows. If some time, n, has passed already and the pattern we seek has yet to occur, there is no reason to believe that it will take less time to continue trials and receive the pattern than it would to just completely start over.

In fact, T is classified as a new better than used (NBU) variable, which means (non-rigorously) that it tends to be faster to start trials over rather than continuing them if a pattern has not occurred by some time, n.

Because of this, the distribution of T may be considered approximately memoryless, and then it can be claimed that T is approximately exponentially distributed. Therefore, since $Var(T) \approx E^2(T)$ in the exponential distribution, then we should expect here that $\frac{Var(T)}{E^2(T)} \approx 1$ for large μ. [3]

Example 4.2 Let's say we are seeking a pattern such as $HTTH$ and that we are interested in finding how many coin tosses are required to come across this pattern. In this case, $r = 4$ and $p = \frac{1}{2^4} = \frac{1}{16}$ and we have zero overlap. Applying equations 4.4 and eq. 4.2 we can see that

$$\begin{aligned} E(T) &= \frac{1}{p} = 16 \\ Var(T) &= \frac{1}{p^2} - \frac{1}{p}(2r-1) = 16^2 - (7 * 16) = 144 \\ \frac{Var(T)}{E(T)^2} &= \frac{16}{144} = 0.111\overline{11} \end{aligned}$$

Another pattern we could seek is abcdefg where each letter has $p = \frac{1}{7}$ of occurring. Here $r = 7$ and $p = \frac{1}{823{,}543}$ (for the whole pattern) and we again have overlap zero. Using the same technique as above, we can calculate

$$\begin{aligned} E(T) &= 823,543 \\ Var(T) &= (823,543)^2 - 13(823,543) \approx 6.78 \times 10^{11} \\ \frac{Var(T)}{E(T)^2} &\approx \frac{6.78 \times 10^{11}}{823,543} \approx 0.99967 \end{aligned}$$

■

Now let's consider the second case; the case where overlap is of size k. We start by defining

$$T = T_{i_1,\ldots,i_k} + T^*$$

where $T_{i_1,\ldots,i_k}$ is the time until the pattern $i_1, \ldots, i_k$ is seen. We also consider T^* to be distributed as the interarrival time of the renewal process and to be the additional time needed to receive the pattern $i_1, \ldots, i_r$ when we start with $i_1, \ldots, i_k$. These random variables are independent of each other, so we can state the following [3]

$$E(T) = E(T_{i_1,\ldots,i_k}) + E(T^*) \tag{4.5}$$

$$Var(T) = Var(T_{i_1,\ldots,i_k}) + Var(T^*) \tag{4.6}$$

If we consider 4.3, then we see that

$$E(T^*) = \mu = \frac{1}{p} \tag{4.7}$$

It can be seen that $I(r)I(r+j) = 0$ since no two renewals occur within $r-k-1$ of each other if $1 \le j \le r-k-1$. Keeping this in mind and applying equation eq. 4.1, we can show that

$$\begin{aligned} Var(T^*) = \sigma^2 &= \frac{1}{p^2}(1-p) + \frac{2}{p^3}\left(\sum_{j=r-k}^{r-1} E\left[I(r)I(r+j)\right] - (r-1)p^2\right) \\ &= \frac{1}{p^2} - \frac{1}{p}(2r-1) + \frac{2}{p^3}\left(\sum_{j=r-k}^{r-1} E\left[I(r)I(r+j)\right]\right) \end{aligned} \tag{4.8}$$

We are able to calculate $E[I(r)I(r+j)]$ for the above equations by considering the specific pattern. For the rest of the calculation of 4.5 and 4.6, we need the mean and variance of $T_{i_1,\ldots,i_k}$ which can be obtained by the same method. [3]

Example 4.3 Let's say we want to determine how many trials are needed to find the pattern 00100 if the only possible outcomes are 0 and 1 and they each have the same probability of occurring.

For this case, we have $r = 5$, $p = \frac{1}{32}$ and an overlap of $k = 2$. We need to obtain $E[I(5)I(8)]$ and $E[I(5)I(9)]$ as equation 4.8 sums the expected values over $j = r - k$ to $r - 1$. We specifically have $j = 5 - 2 = 3$ and $r - 1 = 5 - 1 = 4$. Equation 4.8 then becomes $\sum_{j=3}^{4} E\left[I(r)I(r+j)\right] = E[I(5)I(8)] + E[I(5)I(9)]$. This results in

$$E\left[I(5)I(8)\right] = P(00100100) = \frac{1}{256}$$
$$E\left[I(5)I(9)\right] = P(001000100) = \frac{1}{512}$$

Again, invoking equations 4.7 and 4.8, we can see that

$$E(T^*) = 32$$
$$Var(T^*) = 32^2 - 9(32) + 2(32)^3(\frac{1}{256} + \frac{1}{512}) = 1120$$

We can also consider the pattern 00 with $r = 2$, $p = \frac{1}{4}$ and $k = 1$. This pattern yields

$$E\left[I(2)I(3)\right] = \frac{1}{8}$$
$$E(T_{00}) = E(T_0) + 4$$
$$Var(T_0) = Var(T_0) + 16 - 3(4) + 2\frac{64}{8} = Var(T_0) + 20$$

As we now need to know about the simple pattern 0, we repeat this process once more with $r = 1$ and $p = \frac{1}{2}$ to see that $E(T_0) = 2$ and $Var(T_0) = 2$. This culminates in the following results: [3]

$$E(T_0) = 2$$
$$Var(T_0) = 2$$

$$E(T_{00}) = E(T_H) + 4 = 6$$
$$Var(T_{00}) = Var(T_H) + 20 = 22$$

$$E(T) = E(T_{00}) + 32 = 38$$
$$Var(T) = Var(T_{00}) + 1120 = 1142$$

■

Example 4.4 Consider $P(X_n = i) = p_i$. If we are looking for the pattern 10232210, we have $p = p_1 p_0 p_2 p_3 p_2 p_2 p_1 p_0 = p_1^2 p_0^2 p_2^3 p_3$ with $r = 8$ and an overlap of $k = 2$. We must first find $E[I(8)I(14)]$ and $E[I(8)I(15)]$.

For $E[I(8)I(14)]$, we have it equal to $P(10232210232210) = p_1^3 p_0^3 p_2^6 p_3^2$. Finding $E[I(8)I(15)]$ is the easier of the two, simply because that pattern cannot exist, so it evaluates to zero.

Then applying equations 4.5 and 4.7, we see that

$$E(T) = E(T_{10}) + \frac{1}{p}$$

since $T_{i_1,\ldots,i_k} = T_{10}$. Next, if we consider equations 4.6 and 4.8, we determine the variance of T as

$$Var(T) = Var(T_{10}) + \frac{1}{p^2} - \frac{15}{p} + \frac{2}{p p_1 p_0}.$$

This pattern, 10, has no overlap, hence we do not need to use equation 4.5 to further break this down like was necessary in the previous example.

To finish this example, we see that $E(T_{10}) = \frac{1}{p_1 p_0}$ and $Var(T_{10}) = \frac{1}{(p_1 p_0)^2} - \frac{3}{p_1 p_0}$. When we combine our collected information, we end with [3]

$$E(T) = \frac{1}{p_1 p_0} + \frac{1}{p}$$
$$Var(T) = \frac{1}{(p_1 p_0)^2} - \frac{3}{p_1 p_0} + \frac{1}{p^2} - \frac{15}{p} + \frac{2}{p p_1 p_0}$$

■

4.4 Sets of Patterns

In the previous section, we considered cases consisting of only a single pattern. Let us now consider what occurs when we have multiple patterns. Say there are s number of patterns, denoted as $A(1), A(2), \ldots, A(s)$ and that care to find the mean time until one of the patterns occurs. We also would like to determine the probability mass function of the first pattern. Assuming that we may have an overlap, but that no $A(i)$ is completely contained in any $A(j)$ for $i \neq j$, we consider the non-trivial cases.

Let $T(i)$ denote the time it takes for $A(i)$ to first occur, with $i = 1, 2, \ldots, s$ and let $T(i, j)$ be the additional time required for $A(j), i \neq j$ to first be seen.

We can compute $E(T(i, j))$ by subtracting the time of overlap occurrence from the time of the second pattern, keeping in mind that $E(T_{i_1,\ldots,i_k}) = 0$ in cases where there is no overlap.

Now, it can be seen that [3]

$$T(i,j) = T(j) - T_{i_1,\ldots,i_k})$$
$$E(T(j)) = E(T(j)) - E(T_{i_1,\ldots,i_k})$$

Example 4.5 Consider the patterns $A(1) = dcbdab$ and $A(2) = abcbd$, then $T(1,2) = T(2) - T(1)$. Since the completion of the first pattern is also the first occurrence of the overlap, we may instead say that $T(1,2) = T(2) - T_{i_1,\ldots,i_k}$ and so $E(T(1,2)) = E(T(2)) - E(T_{ab})$.

■

If we are to assume that $E(T(i))$ and $E(T(i,j))$ have both been computed used the aforementioned methods, we have the definitions

$$M = \min_i T(i)$$
$$P(i) = P(A(i) \text{ is the first pattern to occur }) = P(M = T(i)), i = 1, \ldots, s$$

Note that we must find an equation for $E(M)$ since we wish to know the mean time until the first pattern. We may now determine the value of $E(T(j))$ as

$$\begin{aligned} E(T(j)) &= E(M) + E(T(j) - M \\ &= E(M) + \sum_{i:i\neq j} E(T(i,j))P(i) \ \ , \ j = 1, \ldots, s \end{aligned} \tag{4.9}$$

Since $E(X) = \sum_i^n E(X|A_i)P(A_i)$ for mutually exclusive and exhaustive A_i, allows us to state equation 4.9 as a fact.

This, in conjunction with $\sum_{i=1}^n P(i) = 1$, gives us $s+1$ equations with $s+1$ unknowns which may be solved to determine a value for $E(M)$ as well as all values of $P(i)$. [3]

Example 4.6 Say we flip a fair coin with the purpose of finding the pattern $A(1) = tthh$ and $A(2) = hhtt$. We can show that

$$E(T(1)) = 16$$
$$E(T(2)) = 16$$
$$E(T(1,2)) = E(T(2)) - E(T_{hh}) = 16 - (E(T_h) + 4) = 10$$
$$E(T(2,1)) = E(T(1)) - E(T_{ht}) = 16 - (E(T_h) + 4) = 10$$

Using this, we can construct a series of equations, which when solved yield the values of $P(1), P(2)$ and $E(M)$.

$$\begin{aligned} E(T(1)) = E(M) + E(T(2,1))P(2) \quad &\rightarrow \quad 16 = E(M) + 10P(2) \\ E(T(2)) = E(M) + E(T(1,2))P(1) \quad &\rightarrow \quad 16 = E(M) + 10P(1) \\ &\quad\;\; 1 = P(1) + P(2) \end{aligned}$$

Solving these equations gives us [3]

$$\begin{aligned} P(1) &= \frac{1}{2} \\ P(2) &= \frac{1}{2} \\ E(M) &= 11 \end{aligned}$$

■

A unique fact concerning this example is that each pattern may occur first with the same probability as each other. This happens even though the mean time for the first pattern exceeds that of the second.

If we consider the case of zero overlap, we have $E(T_{i_1,\ldots,i_k}) = 0$ as stated previously. Note that with this and with zero overlap, we have $E(Ti,j)) = E(T(j))$ which allows us to simplify 4.9 in the following manner

$$\begin{aligned} E(T(j)) &= E(M) + \sum_{i \neq j} E(T(i,j))P(i) \\ &= E(M) + \sum_{i \neq j} E(T(j))P(i) \\ &= E(M) + E(T(j)) \sum_{i \neq j} P(i) \\ &= E(M) + E(T(j))(1 - P(j)) \\ 0 &= E(M) - E(T(j))P(j) \\ P(j) &= \frac{E(M)}{E(T(j))} \end{aligned}$$

Summing both sides over all values of j gives us the equation

$$\sum_j P(j) = \sum_j \frac{E(M)}{E(T(j))}$$
$$1 = E(M) \sum_j \frac{1}{E(T(j))}$$
$$E(M) = \frac{1}{\sum_j \frac{1}{E(T(j))}} \tag{4.10}$$

This may be further manipulated to find a definition for $P(j)$.

$$P(j) = \frac{1}{E(T(j)) \sum_j \frac{1}{E(T(j))}} \tag{4.11}$$

Consider, for a moment, what a powerful tool these equations are. Simply by having the mean time until each pattern individually, we we are then able to compute the expected run time until the first renewal as well as the individual probabilities of each pattern occurring first. To illustrate this, let us consider a previous example. [3]

Example 4.7 Recall that we previously found

$$E(T) = \frac{1-p^n}{p^n(1-p)}$$

where T was the number of tosses needed to obtain a sequence of n heads. This example can be further generalized by instead defining $T(i)$ a the time it takes to obtain the first occurrence of pattern $A(i)$ which has probability p_i and length $n(i)$. Now, we have

$$E(T(i)) = \frac{1-p_i^{n(i)}}{p_i^{n(i)}(1-p_i)}$$

If we then call on equations 4.10 and 4.11, we now have the following new information, [3]

$$E(M) = \frac{1}{\sum_{j=1}^{s} \left[p_j^{n(j)}(1-p_j)/(1-p_j^{n(j)})\right]}$$
$$P(i) = \frac{p_i^{n(i)}(1-p_i)}{(1-p_i^{n(i)}) \sum_{j=1}^{s} \left[p_j^{n(j)}(1-p_j)/(1-p_j^{n(j)})\right]}$$

■

4.5 Renewal Runs

Let $X_i, i \geq 1$ be iid random variables that may take on any value $1, \ldots, m$ with equal probability. Suppose we observe these variables sequentially and choose T to denote the first occurrence of a run of m consecutive values that includes the values $1, \ldots, m$. That is,

$$T = \min(n : X_{n-m-1}, \ldots, X_n \text{ are distinct})$$

The value of $E(T)$ may be found by defining a renewal process with the first occurring at time T. At this occurrence, start anew and let the next renewal occur after there is another run of m consecutive distinct values. [3]

To illustrate this concept, let $X_i = \{1, 2, 3\}$. The value of m, in this case, is 3. If the trials were to be as such:

$$3, 1, 1, 3, 2^{\alpha}, 2, 1, 2, 1, 3^{\beta}, 1, 2, 2,$$

we would have renewals at time $T = 5$ (denoted by α) and at $T = 10$ (denoted by β).

Definition 4.3 *The sequence of m distinct values that make up a renewal is defined as a renewal run.*

Definition 4.4 *Suppose for a given renewal process that we receive a reward each time there is a renewal, which we identify as R_n for the n^{th} renewal, with $n \geq 1$. R_n is iid. We also have $R(t)$ defined as the sum of R_n over all n representing the total reward earned by a time t.*

It is important to note that if $E(R_n) < \infty$ and the expected value of the random variable which R_n is attributed to is also finite, say $E(X) < \infty$, then we can say that [3]

$$\lim_{t \to \infty} \frac{R(t)}{t} = \frac{E(R)}{E(X)}$$

$$\lim_{t \to \infty} \frac{E(R(t))}{t} = \frac{E(R)}{E(X)}$$

The idea of a renewal reward process can be transformed into a delayed renewal reward process by assigning the reward at any time $n \geq m$ as long as the previous m values were distinct. As an example of this, look back to our example of a renewal run, but we no longer start over at time $T = 5$ and $T = 9$. Using the same data,

$$3, 1, 1, 3, 2^{\gamma}, 2, 1, 2, 1, 3^{\gamma}, 1, 2^{\gamma}, 2,$$

we assign rewards at $T = 5, 10$, and 12, shown by γ.

If we have R_i as the reward earned at time i, applying the limit definitions given when renewal reward processes were defined, and if we let $R = \sum_{i=1}^{n} R_i$, we then have

$$\lim_{n\to\infty} \frac{E\left(\sum_{i=1}^{n} R_i\right)}{n} = \lim_{n\to\infty} \frac{E(R)}{n} = \frac{E(R)}{E(T)} \tag{4.12}$$

Now, consider $A(i)$ to be the first set of i values of a renewal run and $B(i)$ to be the first i values following this renewal run. We can now show the following for R. [3]

$$\begin{aligned}
E(R) &= 1 + \sum_{i=1}^{m-1} E(\text{ reward earned at time } i \text{ after renewal}) \\
&= 1 + \sum_{i=1}^{m-1} P(A_i = B_i) \\
&= 1 + \sum_{i=1}^{m-1} \frac{i!}{m^i} \\
&= \sum_{i=0}^{m-1} \frac{i!}{m^i}
\end{aligned} \tag{4.13}$$

Asserting that $i \geq m$, we can then say that $E(R_i) = P(X_{i-m-1}, \ldots, X_i \text{ are distinct }) = \frac{m!}{m^m}$, which when used with equation 4.12 yields

$$\begin{aligned}
\lim_{n\to\infty} \frac{E\left(\sum_{i=1}^{n} R_i\right)}{n} &= \lim_{n\to\infty} \frac{\sum_{i=1}^{n} E(R_i)}{n} \\
&= \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \frac{m!}{m^m}}{n} \\
&= \frac{m!}{m^m} \lim_{n\to\infty} \frac{\sum_{i=1}^{n} 1}{n} \\
&= \frac{m!}{m^m} \\
&= \frac{E(R)}{E(T)}
\end{aligned}$$

This can be used with equation 4.13 to derive a value of $E(T)$. [3]

$$\begin{aligned}
\frac{m!}{m^m} &= \frac{E(R)}{E(T)} \\
E(T) &= \frac{m^m}{m!} E(R) \\
E(T) &= \frac{m^m}{m!} \sum_{i=0}^{m-1} \frac{i!}{m^i}
\end{aligned}$$

Next, we give an example showing how this method allows us to have another way to compute the expected time until the appearance of a specified pattern.

Example 4.8 Say we want to find the pattern *tthtt* from a coin with $P(\text{heads}) = p$ and $P(\text{tails}) = 1 - p = q$. We need to find $E(T)$. Start the process over after the first renewal occurs. A reward of 1 is earned each time we come across the pattern in our trials. If we say that R is the reward earned between renewal start overs, we can then say that [3]

$$\begin{aligned} E(R) &= 1 + \sum_{i=1}^{4} E(\text{reward received } i \text{ units after the renewal}) \\ &= 1 + 0 + 0 + 1P(B_i = htt) + 1P(B_i = thtt) \\ &= 1 + pq^2 + pq^3 \end{aligned}$$

We also have that $E(R_i) = 1P(tthtt) = pq^4 = \frac{E(R)}{E(T)}$, which can be further manipulated to illustrate that

$$E(T) = \frac{E(R)}{E(R_i)}$$
$$E(T) = \frac{1 + pq^2 + pq^3}{pq^4}$$

■

Now, instead of seeking a run of distinct values, suppose we are seeking a run of some length of increasing values. Choose a sequence of iid continuous random variables, $X_1, X_2, \ldots$ and choose T to denote the first occurrence of a string of r consecutive increasing values; that is, $T = \min(n \geq r : X_{n-r-1} < X_{n-r-2} < \cdots < X_n)$. [3]

To find $E(T)$, we need to use a renewal process. Say the first renewal occurs at T_1, start over and consider the next renewal to occur when a string of r consecutive increasing values is seen. As an example, if $r = 3$ and we have the data values

$$21, 3, 12, 13^{\alpha}, 18, 21, 9, 11, 17^{\beta}, 16, 8, 15, 3, 6, 10^{\gamma}, 13,$$

we have renewals occurring at $T = 4, 9$ and 15 as marked.

If we invoke the Elementary Renewal Theorem, by letting $N(n)$ represent the number of renewals by time n, we have that

$$\lim_{n\to\infty} \frac{E(N(n))}{n} = \frac{1}{E(T)}$$

In order to compute $E(N(n))$, let us define a stochastic process such that S_k is representative of the state at a given time k and equals the number of consecutive increasing values at the time. That is to say, for $1 \leq j \leq k$, $S_k = j$ if $X_{k-j} > X_{k-j-1} < \cdots < X_{k-1} < X_k$.

If we assume $X_0 = \infty$ a renewal will occur at a time k if and only iff $S_k = ir$ for some specific $i \geq 1$. [3]

Example 4.9 Say $r = 3$ and we have X_i, $1 \leq i \leq 8$ such that $X_1 > X_2 > X_3 < X_4 < X_5 < X_6 < X_7 < X_8$. We can determine that $S_3 = 1$, $S_4 = 2$, $S_5 = 3$, $S_6 = 4$, $S_7 = 5$, and $S_8 = 6$. Renewals occur at $T = k = 5$ and $T = k = 8$.

■

However, if we have $k > j$, then we can assert that

$$\begin{aligned}
P(S_k = j) &= P(X_{k-j} > X_{k-j-1} < \cdots < X_{k-1} < X_k) \\
&= P(X_{k-j-1} < \cdots < X_{k-1} < X_k) - P(X_{k-j} < X_{k-j-1} < \cdots < X_{k-1} < X_k) \\
&= \frac{1}{j!} - \frac{1}{(j+1)!} \\
&= \frac{(j+1)! - j!}{j!(j+1)!} \\
&= \frac{j!(j+1) - j!}{j!(j+1)!} \\
&= \frac{j}{(j+1)!}
\end{aligned}$$

This equality comes about due to the fact that all possible orderings of the random variables are just as likely as each other since all X_i are iid.

Before we find a final expression for $E(T)$, first note the following fact regarding Cesàro means, stated without proof.

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sum_{k=1}^{n} a_k}{n} &= \lim_{k\to\infty} a_k \\
\lim_{n\to\infty} \frac{n}{\sum_{k=1}^{n} a_k} &= \lim_{k\to\infty} \frac{1}{a_k}
\end{aligned} \tag{4.14}$$

Now an expression for $E(T)$ may be found by first flipping the Elementary Renewal Theorem.

$$\begin{aligned}
E(T) &= \lim_{n\to\infty} \frac{n}{E(N(n))} \\
&= \frac{n}{\sum_{k=1}^{n} P(\text{renewal at time } k)}
\end{aligned}$$

Using equation 4.14, we continue our derivation

$$E(T) = \lim_{k\to\infty} \frac{1}{P(\text{renewal at time } k)}$$
$$= \lim_{k\to\infty} \frac{1}{\sum_{i=1}^{\infty} P(S_k = ir)}$$

Then, by our most recent calculation of $P(S_k = ir)$, we finally have [3]

$$E(T) = \lim_{k\to\infty} \frac{1}{\sum_{i=1}^{\infty} \frac{ir}{(ir+1)!}}$$

With this, we complete the description of the method that can be used to analyze runs of increasing values.

5. Conclusion

While information regarding patterns can be gathered on an individual basis, using the general theorems and properties of renewal processes allows for application on a wide variety of patterns with arbitrary interarrival times. Using this as the motivating factor, a renewal process was defined as a counting process whose interarrival distribution is arbitrary. We were then able to make statements about the relationship between the number of renewals and time as well as the expected number of renewals and time.

These ideas were then applied to patterns with overlap zero or k, sets of patterns, and renewal runs. For patterns with overlap (zero or k) and we found expressions for the expected time it will take to receive it as well as the variance of this wait time. The same information was found for sets of patterns, but we were able to extend this knowledge to find the individual probabilities that each pattern would occur first, thus being deemed the winner and constituting a renewal.

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