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Quadrefringence of optical vortices in a uniaxial crystal

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The aim of this paper is to theoretically and experimentally describe the propagation of obliquely incident light in a uniaxial crystal. We also find the condition under which the generated vortices in each of the four individual beams propagate independently without changing their structure and have different locations in all beams for any crystal lengths. © 2008 Optical Society of America

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1. INTRODUCTION

The predictions of a new branch of modern optics called singular optics [1] sometimes enable us to catch sight of utterly unexpected features of well-known physical phenomena. For example, Berry recently demonstrated the utterly unexpected features of well-known physical phenomena. For example, Berry recently demonstrated the utter- larity trace the development of the process of splitting an initial optical vortex into four vortices imprinted onto independent crystal-propagating beams and to bring to light the conditions under which the combined singular beam can emerge as four separate vortex beams.

The paper is organized as follows: In Section 2 we obtain the set of off-axis paraxial beams with “mismatched” optical vortices and deduce the field expressions for an incident off-axis fundamental vortex beam. In Section 3 we consider the intrinsic features of dislocation reactions in the field components via variations of the beam inclination to the crystal axis. Section 4 is devoted to the analysis of the indistinguishability limit that restricts the independent propagation of the individual beams that have splintered. Section 5 shows that the nonuniform distribution of polarization states in the vicinity of polarization singularities is an inherent property—even when strongly splintered beams weakly overlap.

2. SOLUTIONS TO THE PARAXIAL WAVE EQUATION

We consider the propagation of a light beam whose monochromatic electric field may be expressed as \( \mathbf{E}(r,t) = \mathbf{E}(r) \exp(-i\omega t) \), where \( r = \hat{x}x + \hat{y}y + \hat{z}z \). We are interested in the fundamental vortex beam whose complex amplitude at the plane \( z' = 0 \) inside the crystal has the form

\[
\mathbf{E}(x', y', z' = 0, \alpha_v) = \mathbf{A}(\alpha_v) [(x' - iy')/w_0] \exp(-r'_z^2/w_0^2)
\]

(1)

in the reference frame \( \{x', y', z'\} \) tilted in the \( (x, z) \) plane of the crystal optical axis at the angle \( \alpha_v \). Here \( r'_z^2 = r_z^2 + y'^2 \) and \( w_0 \) is the beam waist radius. The beam is assumed to propagate through a uniaxial crystal with a permittivity tensor written in a reference frame \( \{x, y, z\} \) of the crystal as
so that $\varepsilon_1 = \varepsilon_2 = \varepsilon$, $\varepsilon$ and $\varepsilon_3$ are real constants, and $\varepsilon_3 < \varepsilon$. Here the crystal $c$ axis is parallel to the $z$ axis of our system. As is well known [15], an oblique beam in a uniaxial crystal gets elliptically deformed. However, we will consider very small inclination angles $\alpha$ in the crystal such that $\alpha \ll 1$; $\sin \alpha = \alpha$, $\cos \alpha = 1 - \alpha^2/2$, and this geometrical deformation of the beam cross section does not manifest itself in an explicit form. In the crystal, the complex amplitude $E(x,y,z>0)$ obeys the wave equation

$$ \nabla^2 + k^2 \hat{\varepsilon} = \nabla (\nabla \cdot \hat{E}), $$

(3)

where $k = 2\pi/\lambda$ is the vacuum wavenumber and $\lambda$ is the vacuum wavelength.

We will restrict our attention to the paraxial approximation. We represent the beam as if its wavefront propagates along the $z$ axis with a wave vector of magnitude $k_o = k \sqrt{\varepsilon}$, while the transverse amplitude of the field is

$$ E_\perp(x,y,z,\alpha) = \tilde{E}_\perp(x,y,z,\alpha) \exp(-ik_z z), $$

(4)

where $\tilde{E}_\perp(x,y,z,\alpha)$ is a slowly varying amplitude having components in the transverse $xy$ plane. The inclination of the beam axis is taken into account with a shift of the origin along the $y$ axis in the imaginary region [16] at distance $y_0 = -\alpha z_0 y - \alpha z_0 \alpha$, where $z'_0 = k'_w /2$ and $k'$ is a wavenumber of the beam in the crystal. In the paraxial approximation, we come to the wave equation [11,17]

$$ \nabla^2 + 2ik_z \partial_z \tilde{E}_\perp = \frac{\Delta E}{\varepsilon_3} \nabla \cdot (\nabla \cdot \tilde{E}_\perp), $$

(5)

where $\nabla \cdot E_\perp = \partial_x E_\perp + \partial_y E_\perp$. Since we will not consider the $E_\perp$ component in the paper, hereafter we stop using the subscript (\perp) for the field.

The vector equation (5) can be reduced to a simple scalar form by means of two substitutions:

$$ \tilde{E}^{(o)} = (\hat{\varepsilon} \hat{k}_o - \hat{y} \hat{k}_o) \Psi_o $$

(6)

and

$$ \tilde{E}^{(e)} = \nabla \cdot \Psi_e, $$

(7)

where $\Psi_o(x,y,z,\alpha_o)$ and $\Psi_e(x,y,z,\alpha_e)$ are arbitrary scalar fields that must satisfy the paraxial conditions for $\tilde{E}^{(o)}$ and $\tilde{E}^{(e)}$. Equations (6) and (7) describe groups of ordinary and extraordinary beams with two values of wavenumbers $k' \rightarrow k_o$, $k_e$ and $z'_0 \rightarrow z_o$, $z_e$ and two values of angles $\alpha \rightarrow \alpha_o$, $\alpha_e$, whereas we assume that the waists radii of the ordinary $w_o$ and the extraordinary $w_e$ beams are the same: $w_o = w_e = w_0$. Notice that Eqs. (6) and (7) along with condition (1) represent single-valued functions. Thus, Eq. (5) transforms into two scalar equations:

$$ (\nabla^2 + 2ik_o \partial_z) \Psi_o = 0, $$

(8)

$$ (\nabla^2 + 2ik_e \partial_z) \Psi_e = 0, $$

(9)

with $k_o = (\varepsilon_3 /\sqrt{\varepsilon}) k_0$. In particular, the simplest solutions of these equations are

$$ \Psi_o = \sigma_o^{-1} \exp[- (x^2 + y_o^2) / w_o^2 \sigma_o] \exp(- \alpha_o^2 k_o z /2), $$

(10)

$$ \Psi_e = \sigma_e^{-1} \exp[- (x^2 + y_o^2) / w_o^2 \sigma_e] \exp(- \alpha_e^2 k_e z /2), $$

(11)

where $y_o = y + i \alpha o z$, $x_o = x + i \alpha e z$, $z_o = k_o w_o^2 /2$, $z_e = k_e w_e^2 /2$, $\alpha_o(z,k_o) = 1 - iz / z_o$, and $\alpha_e(z,k_e) = 1 - iz / z_e$. Here the beam waists, both of radial size $w_0$, coincide with the plane $z = 0$. Besides, with the help of Eq. (1) we find $\Psi_o(z=0) = \Psi_e(z=0) = \Psi(z=0)$ so that $\alpha_o k_o = \alpha_e k_e$.

Combining Eqs. (4), (6), (7), (10), and (11), we obtain particular solutions for the ordinary and extraordinary fundamental vortex beam fields in the circularly polarized basis: $\hat{e}_o = \hat{x} + i \hat{y}$, $\hat{e}_e = \hat{x} - i \hat{y}$ in the form

$$ E^{(o)} = \tilde{E}^{(o)} \exp(-ik_z z) = \left[ \hat{e}_o \frac{x - iy_o}{\sigma_o(z,k_o)} - \hat{e}_e \frac{x + iy_o}{\sigma_o(z,k_o)} \right] \frac{\Psi_o}{w_0} \times \exp(-ik_z z), $$

(12)

$$ E^{(e)} = \tilde{E}^{(e)} \exp(-ik_z z) = \left[ \hat{e}_o \frac{x - iy_o}{\sigma_e(z,k_e)} + \hat{e}_e \frac{x + iy_o}{\sigma_e(z,k_e)} \right] \frac{\Psi_e}{w_0} \times \exp(-ik_z z). $$

(13)

Clearly, each polarization component of the beams in Eqs. (12) and (13) contains a fundamental vortex. The right-handed polarized components have negative topological charges, $l^{(o)} = l^{(e)} = -1$, that are displaced along the $x$ axis to the point $x^{(o)} = x^{(e)} = -\Delta x_o$. The left-handed polarized components have positive charges, $l^{(o)} = l^{(e)} = +1$, and displacements $x^{(o)} = x^{(e)} = \Delta x_o$, where $\Delta x = |\alpha_o z_o|$. An intriguing property of Eqs. (12) and (13) is that the vortex trajectories are parallel to the $z$ axis, whereas the beam envelopes deviate from the $z$ axis by the angles $\alpha_o$ and $\alpha_e$, respectively.

Clearly, Eqs. (12) and (13) cannot satisfy Eq. (1) for any polarization states $A(a)$. Let us soften this requirement by requiring one of the circularly polarized components of the sum of Eqs. (12) and (13) to be zero at the plane $z=0$, say, a left-handed polarized component vanishes: $E_e(x,y,z=0)$. As a result we find

$$ E_1 = \tilde{E}^{(o)} + \tilde{E}^{(e)} = \hat{e}_o \frac{x - iy_o}{w_0} \left( \frac{\Psi_o}{\sigma_o} + \frac{\Psi_e}{\sigma_e} \right) - \hat{e}_e \frac{x + iy_o}{\sigma_e} \left( \frac{\Psi_o}{\sigma_o} - \frac{\Psi_e}{\sigma_e} \right). $$

(14)

In fact, it means that if we form the field $E_1(z=0) = \hat{e}_o(x - iy_o) \psi^{(0)}(z=0) /w_0$ at the plane $z=0$ of the crystal whose vortex is shifted relative to the origin, the energy flux will evolve at some angle to the $z$ axis while the optical vortex will propagate parallel to the $z$ axis.
In order to satisfy Eq. (1) let us find one more field lacking in phase singularities similar to that of a fundamental Gaussian beam in free space. Toward this end we make use of the following recipe [11]: \( G = \int G(x, y, z)\, dz \). After a little algebra, we come to the expression

\[
G = \hat{e}_o (\tilde{\Psi}_o + \tilde{\Psi}_e) - \hat{e}_e \left( \frac{x + iy}{r_o} \right)^2 \frac{w_0^2}{r_o^2} (\sigma_o \tilde{\Psi}_o - \sigma_e \tilde{\Psi}_e) \]

\[
+ (\tilde{\Psi}_o - \tilde{\Psi}_e) \]

(15)

where \( r_o^2 = x^2 + y^2 \). The above expression characterizes the evolution of a Gaussian beam in a crystal [3,11]. Notice that a left-handed component \( G_o \) of the oblique beam vanishes where \( \Psi_e^2 = 0 \) and fulfilling condition (1) for a right-handed polarized component \( E_o \), whereas a left-handed component vanishes \( E_e(z=0)=0 \):

\[
\tilde{E}_o = \left\{ \frac{x - i(y - \alpha_o z)}{w_0 \sigma_o} \tilde{\Psi}_o + \frac{x - i(y -\alpha_e z)}{w_0 \sigma_e} \tilde{\Psi}_e \right\},
\]

(16)

\[
\tilde{E}_e = \left\{ \frac{x + iy_o}{w_o} \Psi_e - \frac{x + iy_o}{w_o} \Psi_e \right\} + \alpha \left( \frac{x + iy_o}{r_o} \right)^2 \frac{w_0^2}{r_o^2} (\sigma_o \Psi_o - \sigma_e \Psi_e) \]

\[
+ (\tilde{\Psi}_o - \tilde{\Psi}_e) \]

(17)

The obtained expressions describe the oblique propagation of a fundamental vortex beam in a uniaxial crystal.

3. DISLOCATION REACTIONS IN THE FIELD COMPONENTS

A. Energy Flux

Equations (16) and (17) describe the intrinsic features of the phase and polarization singularities in the oblique beams in a crystal. Each component of the field comprises a superposition of two individual beams (the ordinary and extraordinary ones) transmitted along different directions defined by the angles \( \alpha_o \) and \( \alpha_e \) (see Fig. 1).

In addition, these individual beams have different structures of the wavefront defined by the wavenumbers \( k_o \) and \( k_e \). The immediate corollary of such field behavior is that the energy flux in each field component at any crystal section varies rapidly as a function of the beam parameters. Indeed, the \( z \) component \( P_z \) of the energy flux of the beam field in the paraxial case can be calculated as

\[
P_z \approx \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |E_o|^2 + |E_e|^2 \, dx \, dy = I_o + I_e.
\]

(18)

By using Eqs. (16) and (17), we find the dependency of the energy flux \( P_z \) in each field component on the angle \( \alpha_o \), shown in Fig. 2. When the distance between the two maxima in the interference pattern is larger than the waist radius of the beam and the dark interferential fringe is positioned at the center of the beam cross section, we observe the intensity minimum. The smaller the waist radius \( w_o \), the larger the number of interferential fringes positioned on the beam cross section. The intensity oscillations are smoothed (compare top and center

![Fig. 1. Sketched representation of the off-axis individual paraxial beams in the uniaxial anisotropic medium.](Image)

![Fig. 2. (Color online) Variations of the total beam intensities \( I_o \) and \( I_e \) for \( E_o \) and \( E_e \) components of a singular beam in a LiNbO3 crystal with \( n_o = \sqrt{\varepsilon} = 2.3 \), and \( n_e = \sqrt{\varepsilon} = 2.2 \).](Image)
of the radius \( w_o \) and the utmost accuracy of the angle variation \( \Delta \alpha_{in} \approx 0.03^\circ \) that enable us to plot the relatively smooth experimental curves \( I_1(\alpha_{in}) \) presented in Fig. 2. Notice that the total intensity \( I = I_1 + I_2 \) experiences no variations via the angle \( \alpha_{in} \), but its value depends on the beam waist \( w_o \).

Our preliminary measurements showed also that (1) the initial optical vortex splits in the crystal into four centered optical vortices imprinted in four individual beams; (2) right-handed \( E_+ \) and left-handed \( E_- \) components of the electric vector carry over oppositely charged optical vortices on the axis at very small angles \( \alpha_{in} \approx 0 \); and (3) after splitting the individual beams at the angles \( \alpha_{in} > 10^\circ \), all four centered vortices have the same topological charges. Thus, hereafter we will focus our attention on bringing to light a physical mechanism responsible for such a conversion process.

### B. Dislocation Reactions

While superposition of two oblique nonsingular beams results in the interference pattern in the form of ordinary fringes or rings, the interference of two oblique singular beams entails dislocation reactions: Chains of birth and annihilation events of optical vortices. In the general case, any variations of the angle \( \alpha_n \) cause deformations of the beam structure consisting of polarization singularities: C-lines and L-surfaces \([18]\). The L-surface represents a locus where the electric field is linearly polarized. The C-line forms a locus where one of the circularly polarized field components vanishes. Its amplitude is zero, while the phase is uncertain. The L-surfaces encircle the C-lines. In fact, the C-line represents a space trajectory of the optical vortex imprinted in the circularly polarized field component. In this section we will bring to light the intrinsic features of the dislocation reactions accompanying the deformation of the beam structure in one of the circularly polarized components.

First, let us consider transformations of phase singularities in the \( E_- \) component of the oblique beam [Eq. (17)] provided that only the \( E_- \) component is at the plane \( z = 0 \). We can imagine that the input face of the crystal coincides with the plane \( z = 0 \) and consider the demand \( E_-(z=0)=0 \), while the reflected beam is neglected as the boundary condition.

Let the right-handed circularly polarized beam bearing the vortex with a negative topological charge \( l = -1 \) fall on the crystal input perpendicular to its verge \( (\alpha_{in} = \alpha_o = 0) \). Equation (17) shows that the left-handed circularly polarized component \( E_- \) in the crystal carries over the on-axis optical vortex with a positive topological charge \( l = +1 \). Figure 4 illustrates the theoretically predicted and experimentally measured intensity distributions and interferential patterns of the \( E_- \) component. The single-branch spiral with a left-handed rotation for \( \alpha_{in} = 0.5^\circ \), a double-charged vortex is torn off the beam following the crystal optical axis, while a single-charged vortex with \( l = -1 \) follows the beam direction. In the experiment we observe the interference pattern in the form of a two-branch
spiral. As the angle $\alpha_{in}$ increases ($\alpha_{in} > 2^\circ$), the portion of the light intensity falling on this vortex essentially decreases so that both the $E_+$ and $E_-$ components of the oblique beam carry over the vortices near their axes with the same signs of the topological charge ($l = -1$) equal to that in the initial one at the crystal input. In order to understand the fine structure of the beam, we consider vortex trajectories, written as

$$\text{Re}[E_z(x,y,z,\alpha_o)] = 0, \quad \text{Im}[E_z(x,y,z,\alpha_o)] = 0. \quad (19)$$

The structure of the projections of these trajectories onto the plane $xy$ at the starting range $0 < \alpha_s < 1^\circ$ is shown in Fig. 5(a). The open and closed circles in the figure refer to the birth and annihilation events, respectively. The gray circle indicates the initial vortex. The sequence of the ciphers characterizes the succession of the birth and annihilation events.

Let us consider dislocation reactions in the $E_-$ component. In order to observe some vortex processes with better resolution than that in Fig. 4, we choose an initial singular beam with a relatively large waist radius. Figure 5 shows that no sooner had the angle $\alpha_o$ started to change (event 1) than the positively charged optical vortex ($l = +1$) slid off the beam axis. Then the topological dipole is born near the beam axis $x = 0, y = -\alpha_o x$ (event 2, open circle). The positively charged vortex of the dipole pair aspires to the initial vortex far from the $z$ axis (Fig. 5(a)), whereas the negatively charged one follows the beam forming the general branch of the trajectory. Two positively charged vortices draw together at some distance from the $z$ axis. Their structure looks like an entire double-charged vortex in the interferential experiment with a relatively small waist of the initial beam (see Fig. 4). However, the vortices do not flow together but drift very slowly along the x direction (Fig. 5(a)). Two dipole pairs are born again at the points 3' and 3'' almost simultaneously. Two oppositely charged vortices of these pairs annihilate at point 4 (closed circle), while the other two identically charged vortices scatter very quickly in opposite directions, forming one of the transverse branches of the complex trajectory. On the general branch at point 5, a new vortex pair is born. Two oppositely charged vortices annihilate at point 6, while the residuary negatively charged vortex continues to build the general trajectory branch. Later on, this succession of events is reproduced right up to the critical angle $\alpha_{cr}$, where the general branch is bisected (see Section 4). The dislocation reactions in the $E_-$ component occur just as in the $E_+$ component if the very starting range of the trajectory, is not taken into account. Variations of the beam waist $w_0$ at the crystal input entail the deformation of the trajectory but its intrinsic features are preserved. Notice that some basic properties of these complex vector processes were perceived on the example of a simple scalar model of composite vortices [19] comprising two centered singly charged singular beams whose axes are shifted parallel to each other and whose fields have some phase difference.

4. INDISTINGUISHABILITY LIMIT

As we have seen above, any variations of beam parameters entail birth and annihilation vortices. We observe a complex vortex mixture. However, the experiment showed that starting with some critical angle $\alpha_{cr}$ (or some crystal length $d_{cr}$), there are two stable vortices in each circularly polarized component that take no part in the dislocation reactions. Each of these vortices is a special mark of an individual beam in the structure of the combined beam that follows the individual beam while the parameters of the initial beam are changed. The event when four stable vortices appear in the beam component is a distinguishability limit for the beam parameters, starting with which of the individual beams can be observed separately from the other one [20]. As a whole, this limit characterizes quadrefringence of the initial vortex in a uniaxial crystal, splitting the initial vortex into four stable, centered ones.

However, in order to distinguish two beams in each circularly polarized component, it is first necessary to define which singular beam the individual vortex belongs to. An important role in this process is played by that portion of the vortex trajectory in the vicinity of the critical angle $\alpha_{cr}$ in Fig. 5(b). The major branches of the vortex trajectory are divided here and do not interlace any more. Notice that the total number of vortices to be observed simulta-
neously on all trajectory branches for $\alpha < \alpha_c$ can be varied from one to four. While transiting the threshold $\alpha = \alpha_c$, one optical vortex disappears. It is this critical angle that characterizes the beam splitting. The birth and annihilation events emerge at the side lobes of the beams, forming the transversal trajectories. The individual beams can be distinguished. The total number of vortices observed simultaneously beyond the angle $\alpha > \alpha_c$ is varied from two to four. The critical angle $\alpha = \alpha_c$ can be regarded as the indistinguishability limit for two refracted beams with the same polarizations.

We can estimate mathematically the value of the critical angle $\alpha_c$, based on Eq. (16) for the $E_x$ component. Indeed, the optical vortices in two individual beams [the first and the second terms in Eq. (16)] do not take part in dislocation reactions when the front intensity maximum of the first individual beam coincides with the back intensity maximum of the second individual one. By using Eq. (16), we come to the expressions for the coordinates of the intensity maximum $y_{\text{from}}^{(e)} = \alpha_c z + w_z / \sqrt{2}$ and $y_{\text{back}}^{(e)} = \alpha_c z - w_z / \sqrt{2}$, where $w_z = w_0 z + 1 + z^2 / z_o^2$ and $w_z = w_0 z + 1 + z^2 / z_o^2$. The maxima matching obey the condition $\Delta y = y_{\text{from}}^{(e)} - y_{\text{back}}^{(e)} = 0$ from whence we find for $z_o / z \ll 1$:

$$
\frac{\alpha_c^2 \Delta x^2}{w_0^2 2z_o^2} \approx 1.
$$

(20)

The last equation shows that we cannot distinguish two singular beams at any crystal length $z$ if the angle of the ordinary beam is less than $\alpha = \alpha_c = \sqrt{(2 \alpha_{\text{diff}}^2 - 1) / 2}$. The curve presented in Fig. 6(a) characterizes the indistinguishability limit for two singular beams. The magnitudes $(\alpha_c, w_0)$ located on the right of the curve are associated with two beams that can be distinguished in the experiment. The nondistinguishable fields are perceived as one composite beam. The variation of the angle $\alpha_c$ is accompanied by the appearance of the phase singularities: From one to four vortices at the beam cross section in the indistinguishability area and from two to four vortices in the distinguishability area. As we have said above (see Section 3), the total intensity of the field component $E_x(\alpha_c)$ oscillates sharply within the range $(0, \alpha_c)$. The oscillation is smoothed; i.e., the amplitude of the oscillations decreases, and the period increases beyond the critical angle $\alpha_c$. The intensity maxima correspond to the minimum number of vortices (one or two) in the beam. The vortices on the transverse trajectories very quickly leave the area of the steady observation (in the experiment). The angle interval between the two maxima is approximately equal to $\Delta \alpha = \alpha_c [1 - \Delta \alpha / \epsilon_s]$. For a typical case, $n_s = 2.3, n_d = 2.208$, and $\alpha_c = 1.8^\circ$, we have $\Delta \alpha = 0.15^\circ$. Thus, within the region $\Delta \alpha$ we can register one or two vortices in the indistinguishability range and two vortices in the distinguishability range. The rest of the vortices leave the observation area very quickly via slight variations of the angle $\alpha_c$.

Based on these findings, we designed an experiment to verify the indistinguishability limit. Unlike the setup in Fig. 3, here we employed three lenses with the focus distances $f=5$ cm, $f=12$ cm, and $f=20$ cm. The presence of the vortices in the beam and the signs of their topological charges were measured by means of interferometric methods. We measured the indistinguishability limit $\alpha_c$ with an accuracy of about $\Delta \alpha = 0.1^\circ$. It makes up $\alpha_c = 2.7^\circ$ inside the crystal for the beam waist at the crystal input $w_0 = 50$ mm, and the crystal length $z = 2$ cm. Notice that the diaphragm $D_1$ in Fig. 3 distorted to some extent the shape of the initial Gaussian envelope at the crystal input in the form of Airy rings. Such distortion increases when decreasing the diaphragm pupil and the beam waist. The experimental error also increases. Nevertheless, the results shown in Fig. 6(b) illustrate a good agreement with the theoretical predictions.

5. VORTEX QUADREFRINGENCE

Let us focus our attention on splitting a centered vortex at the plane $z = 0$ into four centered vortices bearing four individual beams in terms of polarization singularities [18]. The vector singularities of off-axis beams in the uniaxial crystal were considered recently [13,14] within a small range of angles for the KDP crystal and for a very large input beam waist ($w_0 = 0.7$ mm) on the basis of a reductive model of the beam propagation in the birefringent media when the initial beam is linearly polarized. This regime corresponds to the indistinguishability range (the refracted beams in each polarized component cannot be distinguished). They revealed a very complex structure of C-lines resembling the braided vortex trajectories. In addition, different C-line branches can be reconnected via variations of the angle $\alpha_{in}$ (or the phase difference between the beams with orthogonal linear polarizations). In the present section we will consider the pattern as a whole for a broad range of the angles $\alpha_{in}$.

As is well known [18], the vortex trajectories in each circular polarized component and the C-lines in a nonuniformly polarized paraxial beam are tightly connected with one another. We made use of such conformity and plotted C-lines without the calculation of Stokes parameters. The computer simulation of the evolution of C-lines via variations of the angle $\alpha_{in}$ is depicted in Fig. 7. The curves in the figure can be divided into two groups (thin and bold curves) associated with the vortex trajectories in the $E_x$ and $E_y$ field components. Additionally, inside the group we can speak about C-lines of the general and transverse branches. The lines of all branches are neither intersected nor reconnected, although reconnections emerge along each branch within the indistinguishability range. The

![Fig. 6. Curves $\alpha_c = f(\rho, z)$ outlining the indistinguishability range for two singular beams.](image)
exceptions are only for the starting sections of the newborn branches of the C-lines in the splintered beams near the angles $\alpha_{cr}^{(s)}$ and $\alpha_{cr}^{(c)}$ when the transverse and general branches are united. Each of the two lines of the two different groups (bold and dashed lines for Fig. 5 and dashed and dotted lines for Fig. 7, respectively) behind the critical angles for any crystal lengths. Indeed, let us consider in Eqs. (16) and (17) that the lines at the critical angles for any crystal lengths.

The important point in our consideration is that the branches in each braidlet do not interflow after splitting for any crystal lengths. Let us consider in Eqs. (16) and (17) that for any crystal lengths.

Finally, we come to the expressions

$$\tilde{E}_e \propto \frac{X-iY}{Z_o} \Psi_o + \frac{X-iY}{Z_e} \Psi_e,$$

$$\tilde{E}_c \propto \frac{(X+1/\alpha)-iY_0}{Z_o} \Psi_o - \frac{(X+1/\alpha)-iY_e}{Z_e} \Psi_e,$$

where $Z_o=z/z_o$, $X=x/w_0$, and $Y_0=(y-\alpha x)/w_0$, $Y_e=(y-\alpha x)/w_0$. First of all, we see that vortices have the same signs as the topological charges in both circularly polarized components. Besides, we find from Eqs. (21) that four nodal lines for the $E_e$ and $E_c$ components lie on the $Y_o=0$ and $Y_e=0$ planes for $\alpha Z \approx 1$. Coordinates of the vortices on the neighboring trajectories differ in the value

$$\Delta x = w_0/\alpha = \frac{\lambda}{\pi n_o \alpha_o} = \frac{\lambda}{\pi n_e \alpha_e},$$

where $\lambda$ stands for wavelength in free space. For example, for the initial beam with $w_0=50 \mu m$, $\alpha_o=5^\circ$, and $n_o=2.3$, the value of the trajectory splitting is $\Delta x \approx 1 \mu m$. It means that in the frames of our approximation, nearly centered vortices bearing four individual beams do not flow together after splitting for any crystal lengths.

Using the method presented in [10], we plotted the maps of polarization states for a sufficiently large angle $\alpha_o=5.3^\circ > \alpha_{cr}$, shown in Figs. 8(a) and 8(b). The individual beams are practically separated. The uniform field of the linear polarization covers most of the beam cross sections in a manner similar to that presented in the model in [13,14]. Only a thin strand of the nonuniform polarization states separates the individual beams. However, the pairs of singular points in the form of the lemon and the star are positioned inside a dark area of each beam, perturbing a usually uniform picture of the birefringent process [see Fig. 8(b)]. We examined this singular structure with the help of a computer simulation for a broad range of crystal lengths (right up to $z=20 \ cm$). We revealed that there are two C-points encircled by the pattern in the forms of the star and the lemon [18] in the vicinity of the minimum of each individual beam. They approach each other at a distance of about $1 \mu m$ without a polarization unfolding at the crystal length $z=20 \ cm$, which is in good agreement with our estimation presented above. Shown in Figs. (c) and (d) is the map of the polarization states and the characteristic integral lines obtained in the experiment. We employed the computer-processing technique for plotting the map described in [10,21]. This technique enabled us to measure all Stokes parameters using only four measurements instead of six. It sufficiently improved the plottings of the map of polarization states in Figs. (c) and (d). In addition, our CCD camera has the capability of 160 pixels/mm that permits us to resolve the polarization states in the map with an accuracy not worse than $1 \mu m$.

Figures 8(c) and 8(d) describe the composite beam for the angle $\alpha_{cr}=12^\circ$ (or $\alpha_{cr}=5.2^\circ$) at the crystal input. The nonuniform distribution of polarization states at the area of the dark spots in Fig. 8(d) contrasts sharply with the nearly uniform distribution of the linear polarization state stretched over the rest portion of the beam’s cross section in Fig. 8(c). Two singular points, the lemon and the star (corresponding to two optical vortices in the field components), are positioned at a distance of about $20 \mu m$. 

![Fig. 7](image1.png)

Fig. 7. (Color online) C-lines for the singular beam with $w_0=50 \mu m$ and $z=2 cm$.

![Fig. 8](image2.png)

Fig. 8. (Color online) Maps of polarization states and integral curves for directions of the azimuthal angle of the polarization ellipses for the angles $\alpha_o=5.3^\circ$ ($w_0=50 \mu m$, $z=2 \ cm$) positioned against a background of the intensity distributions: (a), (b), theory; (c), (d), experiment.
This value is larger by 1 order of magnitude than our theoretical estimation $\Delta x \approx 1 \mu m$. Such a disagreement is caused by the fact that the value $\bar{a}Z_{\alpha} \approx 0.36$ in our experiment does not correspond to the condition $\bar{a}Z_{\alpha} \approx 1$. At the same time, this experimental result is in good agreement with the computer simulation in Fig. 8(b).

As a whole, the initial vortex at the plane $z = 0$ splits at first into two nearly centered vortices in the orthogonal field components (two C-lines in Fig. 7 for a very small angle $\alpha_c$). Then, after critical angles $\alpha_c^{(+)}$ and $\alpha_c^{(-)}$, there appear four nearly centered C-lines associated with four optical vortices. It appears as if the initial vortex splits into four vortices; i.e., it experiences a quadrefringence.

6. CONCLUSIONS

We have found solutions to the paraxial wave equation in a uniaxial anisotropic medium in the form of oblique vortex beams. We have theoretically and experimentally analyzed the singular structure of the beams. We have plotted the vortex trajectories and mapped the polarization states for different propagation directions of the initial beam. We have discovered that a centered initial optical vortex at the crystal input splits into two vortices in each circularly polarized component. These vortices do not take part in any dislocation reactions, and their trajectories do not coincide in space for any crystal lengths. We called this process the vortex quadrefringence.

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