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FIBONACCI AND LUCAS NUMBERS AS TRIDIAGONAL MATRIX DETERMINANTS

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1. INTRODUCTION

There are many known connections between determinants of tridiagonal matrices and the Fibonacci and Lucas numbers. For example, Strang [5, 6] presents a family of tridiagonal matrices given by:

\[ M(n) = \begin{pmatrix}
3 & 1 & & & \\
1 & 3 & 1 & & \\
& 1 & 3 & \ddots & \\
& & \ddots & \ddots & 1 \\
& & & 1 & 3
\end{pmatrix}, \tag{1} \]

where \( M(n) \) is \( n \times n \). It is easy to show by induction that the determinants \( |M(k)| \) are the Fibonacci numbers \( F_{2k+2} \). Another example is the family of tridiagonal matrices given by:

\[ H(n) = \begin{pmatrix}
1 & i & & & \\
i & 1 & i & & \\
& i & 1 & \ddots & \\
& & \ddots & \ddots & i \\
& & & i & 1
\end{pmatrix}, \tag{2} \]
described in [2] and [3] (also in [5], but with 1 and –1 on the off-diagonals, instead of \(i\)). The determinants \(|H(k)|\) are all the Fibonacci numbers \(F_k\), starting with \(k = 2\). In a similar family of matrices [1], the \((1,1)\) element of \(H(n)\) is replaced with a 3. The determinants now generate the Lucas sequence \(L_k\), starting with \(k = 2\) (the Lucas sequence is defined by the second order recurrence \(L_1 = 1, L_2 = 3, L_{k+1} = L_k + L_{k-1}, k \geq 2\)).

In this article, we extend these results to construct families of tridiagonal matrices whose determinants generate any arbitrary linear subsequence \(F_{ak+\beta}\) or \(L_{ak+\beta}\), \(k = 1, 2, \ldots\) of the Fibonacci or Lucas numbers. We then choose a specific linear subsequence of the Fibonacci numbers and use it to derive the following factorization:

\[
F_{2mn} = F_{2m} \prod_{k=1}^{n-1} \left( L_{2m} - 2 \cos \frac{\pi k}{n} \right).
\]  (3)

This factorization is a generalization of one of the factorizations presented in [3]:

\[
F_{2n} = \prod_{k=1}^{n-1} \left( 3 - 2 \cos \frac{\pi k}{n} \right).
\]

In order to develop these results, we must first present a theorem describing the sequence of determinants for a general tridiagonal matrix. Let \(A(k)\) be a family of tridiagonal matrices, where

\[
A(k) = \begin{pmatrix}
  a_{1,1} & a_{1,2} & \cdots & a_{1,k} \\
  a_{2,1} & a_{2,2} & \cdots & \cdots & a_{2,k-1} \\
  \vdots & \ddots & \ddots & \ddots & \ddots \\
  a_{k-1,1} & \cdots & a_{k-1,k-1} & a_{k-1,k} \\
  a_{k,1} & \cdots & a_{k,k-1} & a_{k,k}
\end{pmatrix}.
\]

**Theorem 1:** The determinants \(|A(k)|\) can be described by the following recurrence relation:
\[ |A(1)| = a_{1,1} \]
\[ |A(2)| = a_{2,2}a_{1,1} - a_{2,1}a_{1,2} \]
\[ |A(k)| = a_{k,k} |A(k-1)| - a_{k,k-1}a_{k-1,k} |A(k-2)|, \quad k \geq 3. \]

**Proof:** The cases \( k = 1 \) and \( k = 2 \) are clear. Now

\[
|A(k)| = \det \begin{pmatrix}
    a_{1,1} & a_{1,2} & & \\
    a_{2,1} & a_{2,2} & & \\
    & & \ddots & \\
    & & & a_{k-3,k-3}
\end{pmatrix},
\]

By cofactor expansion on the last column and then the last row,

\[
|A(k)| = a_{k,k} |A(k-1)| - a_{k-1,k} |A(k-2)|.
\]

2. **FIBONACCI SUBSEQUENCES**

Using Theorem 1, we can generalize the families of tridiagonal matrices given by (1) and (2) to construct, for every linear subsequence of Fibonacci numbers, a family of tridiagonal matrices whose successive determinants are given by that subsequence.

**Theorem 2:** The symmetric tridiagonal family of matrices \( M_{\alpha,\beta}(k), \ k = 1,2,\ldots \) whose elements are given by:
\[ m_{1,1} = F_{a+\beta}, \quad m_{2,2} = \begin{bmatrix} F_{a+\beta} \\ F_{a+\beta} \end{bmatrix} \]

\[ m_{j,j} = L_{a}, \quad 3 \leq j \leq k, \]

\[ m_{1,2} = m_{2,1} = \sqrt{m_{2,2}F_{a+\beta} - F_{2a+\beta}} \]

\[ m_{j,j+1} = m_{j+1,j} = \sqrt{(-1)^{j}} \quad 2 \leq j < k, \]

with \( \alpha \in \mathbb{Z}^+ \) and \( \beta \in \mathbb{N} \), has successive determinants \( |M_{a,\beta}(k)| = F_{dk+\beta} \).

In order to prove Theorem 2, we must first present the following lemma:

**Lemma 1**: \( F_{k+n} = L_{n}F_{k} + (-1)^{n+1}F_{k-n} \) for \( n \geq 1 \).

**Proof**: We use the second principle of finite induction on \( n \) to prove this lemma:

Let \( n = 1 \). Then the lemma yields \( F_{k+1} = F_{k} + F_{k-1} \), which defines the Fibonacci sequence. Now assume that \( F_{k+n} = L_{n}F_{k} + (-1)^{n+1}F_{k-n} \) for \( n \leq N \). Then

\[
F_{k+N+1} = F_{k+N} + F_{k+N-1}
\]

\[
= L_{N}F_{k} + (-1)^{N+1}F_{k-N} + L_{N-1}F_{k} + (-1)^{N}F_{k-N+1}
\]

\[
= (L_{N} + L_{N-1})F_{k} + (-1)^{N+2}(F_{k-N+1} - F_{k-N})
\]

\[
= L_{N+1}F_{k} + (-1)^{N+2}F_{k-(N+1)} \quad \blacksquare
\]

Now, using Theorem 1 and Lemma 1, we can prove Theorem 2.

**Proof of Theorem 2**: We use the second principle of finite induction on \( k \) to prove this theorem:

\[
|M_{a,\beta}(1)| = |det F_{a+\beta}| = F_{a+\beta}.
\]

\[
|M_{a,\beta}(2)| = \det \left( \begin{bmatrix} F_{a+\beta} \\ m_{2,2}F_{a+\beta} - F_{2a+\beta} \end{bmatrix} \right) = F_{2a+\beta}.
\]
Now assume that \( M_{\alpha,\beta}(k) = F_{\alpha k + \beta} \) for \( 1 \leq k \leq N \). Then by Theorem 1,

\[
M_{\alpha,\beta}(k + 1) = m_{k,k} \left| M_{\alpha,\beta}(k) \right| - m_{k,k-1} m_{k-1,k} \left| M_{\alpha,\beta}(k - 1) \right|
\]

\[
= L_\alpha \left| M_{\alpha,\beta}(k) \right| - (-1)^\alpha \left| M_{\alpha,\beta}(k - 1) \right|
\]

\[
= L_\alpha F_{\alpha k + \beta} + (-1)^{\alpha + 1} F_{\alpha (k - 1) + \beta}
\]

\[
= F_{\alpha (k + 1) + \beta} \quad \text{(by Lemma 1)}
\]

\[
= F_{\alpha (k + 1) + \beta} \quad \blacksquare
\]

Another family of matrices that satisfies Theorem 2 can be found by choosing the negative root for all of the super-diagonal and sub-diagonal entries. With Theorem 2, we can now construct a family of tridiagonal matrices whose successive determinants form any linear subsequence of the Fibonacci numbers. For example, the determinants of:

\[
\begin{pmatrix}
1 & 0 \\
0 & 8 & 1 \\
1 & 17 & 1 \\
& \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\
& & & & 1 & 17 \\
\end{pmatrix}, \quad \begin{pmatrix}
8 & \sqrt{6} \\
\sqrt{6} & 5 & i \\
i & 4 & i \\
\ddots & \ddots & \ddots & \ddots & \ddots & i \\
i & 4
\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}
13 & -\sqrt{5} \\
-\sqrt{5} & 3 & -1 \\
-1 & 3 & -1 \\
-1 & 3 & \ddots & \ddots & \ddots & \ddots & \ddots \\
-1 & 3 \\
\end{pmatrix}
\]

are given by the Fibonacci subsequences \( F_{4k-2} \), \( F_{3k+3} \) and \( F_{2k+5} \).

3. LUCAS SUBSEQUENCES

We can also generalize the families of tridiagonal matrices given by (1) and (2) to show a similar result for linear subsequences of Lucas numbers. We state this result as the following theorem:
\textbf{Theorem 3}: The symmetric tridiagonal family of matrices $T_{\alpha,\beta}(k)$, $k = 1, 2, \ldots$ whose elements are given by:

$$
t_{1,1} = L_{\alpha+\beta}, \quad t_{2,2} = \left[\frac{L_{\alpha+\beta}}{L_{\alpha+\beta}}\right],
$$

$$
t_{j,j} = L_{\alpha}, \quad 3 \leq j \leq k,
$$

$$
t_{1,2} = t_{2,1} = \sqrt{t_{2,2} L_{\alpha+\beta} - L_{2\alpha+\beta}}
$$

$$
t_{j,j+1} = t_{j+1,j} = \sqrt{(-1)^{\alpha}}, \quad 2 \leq j < k,
$$

with $\alpha \in \mathbb{Z}^+$ and $\beta \in \mathbb{N}$, has successive determinants $|T_{\alpha,\beta}(k)| = L_{\alpha k + \beta}$.

Again we begin with a lemma; its proof imitates the proof of Lemma 1.

\textbf{Lemma 2}: $L_{k+n} = L_{n}L_{k} + (-1)^{n+1}L_{k-n}$ for $n \geq 1$.

\textbf{Proof of Theorem 3}: We use induction:

$$
|T_{\alpha,\beta}(1)| = \det L_{\alpha+\beta} = L_{\alpha+\beta}.
$$

$$
|T_{\alpha,\beta}(2)| = \det \left( L_{\alpha+\beta}, \sqrt{m_{2,2} L_{\alpha+\beta} - L_{2\alpha+\beta}}, \left[\frac{L_{\alpha+\beta}}{L_{\alpha+\beta}}\right] \right) = L_{2\alpha+\beta}.
$$

Now assume that $|T_{\alpha,\beta}(k)| = L_{\alpha k + \beta}$ for $1 \leq k \leq N$. Then by Theorem 1,

$$
|T_{\alpha,\beta}(k+1)| = t_{k,k} |T_{\alpha,\beta}(k) - t_{k,k-1} t_{k-1,k} |T_{\alpha,\beta}(k-1)|
$$

$$
= L_{\alpha} |T_{\alpha,\beta}(k)| - (-1)^{\alpha} |T_{\alpha,\beta}(k-1)|
$$

$$
= L_{\alpha} L_{\alpha k + \beta} + (-1)^{\alpha+1} L_{\alpha(k-1)+\beta}
$$

$$
= L_{\alpha \alpha k + \beta} \quad \text{(by Lemma 2)}
$$

$$
= L_{\alpha(k+1)+\beta} \quad \blacksquare
$$
With Theorem 3, we can now construct a family of tridiagonal matrices whose successive determinants form any linear subsequence of the Lucas numbers. For example, the determinants of:

\[
\begin{pmatrix}
3 & 0 & & \\
0 & 6 & -1 & \\
& -1 & 7 & -1 \\
& & & -1 & 7 \\
& & & & -1 & 7
\end{pmatrix}, \begin{pmatrix}
18 & \sqrt{14} & i & \\
\sqrt{14} & 5 & i & \\
i & 4 & i & \\
& i & 4 & i \\
& & i & 4
\end{pmatrix}, \text{ and } \begin{pmatrix}
29 & \sqrt{11} & & \\
\sqrt{11} & 3 & 1 & \\
3 & 1 & & \\
& & & 1
\end{pmatrix}
\]

are given by the Lucas subsequences \(L_{4k-2}, L_{3k+3}\) and \(L_{2k+5}\).

4. A FACTORIZATION OF THE FIBONACCI NUMBERS

In order to derive the factorization (3) given by \(F_{2mn} = F_{2m} \prod_{i=1}^{n-1} \left( L_{2m - 2 \cos \frac{\pi k}{n}} \right) \), we consider the symmetric tridiagonal matrices:

\[
B_m(n) = \begin{pmatrix}
L_{2m}F_{2m} & \sqrt{F_{2m}} & & \\
\sqrt{F_{2m}} & L_{2m} & 1 & \\
& 1 & L_{2m} & 1 \\
& & & \ddots & 1 \\
& & & & 1
\end{pmatrix}.
\]

By Lemma 1, \(F_{4m} = L_{2m}F_{2m}\), and \(\left[ F_{6m}/F_{4m} \right] = L_{2m} - \left( F_{2m}/F_{4m} \right) = L_{2m} \). Furthermore,

\[
\sqrt{F_{6m}/F_{4m}} = \sqrt{L_{2m}F_{4m} - F_{6m}} = \sqrt{F_{2m}}, \text{ so } B_m(n) = M_{2m,2m}(n) \text{ is a specific instance of the tridiagonal family of matrices described in Theorem 2. Therefore, by Theorem 2,}
\]

\[
\left| B_m(n) \right| = F_{2m(n+1)}.
\]

By using the property of determinants that \(|AB| = |A||B|\), and by defining \(e_j\) to be the \(j\)th column of the \(n \times n\) identity matrix \(I\), we have \(\left| B_m(n) \right| = F_{2m} \left| C_m(n) \right|\), where:
\[ C_m(n) = \left( I + \frac{1}{F_{2m}} e_1 e_1^T \right) B_m(n). \]

The determinant is the product of the eigenvalues. Therefore, let \( \lambda_k, k = 1, 2, \ldots, n \) be the eigenvalues of \( C_m(n) \) (with associated eigenvectors \( x_k \)), so \( |C_m(n)| = \prod_{k=1}^{n} \lambda_k \). Letting \( G_m(n) = C_m(n) - L_{2m}I \), we see that \( G_m(n)x_k = C_m(n)x_k - L_{2m}x_k = \lambda_k x_k - L_{2m}x_k = (\lambda_k - L_{2m})x_k \). Then \( \gamma_k = \lambda_k - L_{2m} \) are the eigenvalues of \( G_m(n) \).

An eigenvalue \( \gamma \) of \( G_m(n) \) is a root of the characteristic polynomial \( |G_m(n) - \gamma I| = 0 \). Note that \( |G_m(n) - \gamma I| = \left| \left[ I + \left( \frac{1}{\sqrt{F_{2m}}} - 1 \right) e_1 e_1^T \right] G_m(n) - \gamma I \left[ I + \left( \frac{1}{\sqrt{F_{2m}}} - 1 \right) e_1 e_1^T \right] \right| \), so \( \gamma \) is also a root of the polynomial:

\[
\begin{vmatrix}
-\gamma & 1 \\
1 & -\gamma & 1 \\
& 1 & -\gamma & 1 \\
& & \ddots & \ddots & \ddots & \ddots & 1 \\
& & & 1 & -\gamma
\end{vmatrix} = 0.
\]

This polynomial is a transformed Chebyshev polynomial of the second kind \([4]\), with roots \( \gamma_k = -2 \cos \frac{\pi k}{n+1} \). Therefore,

\[
F_{2m(n+1)} = |B_m(n)| = F_{2m} |C_m(n)| = F_{2m} \prod_{k=1}^{n} \lambda_k = F_{2m} \prod_{k=1}^{n} \left( L_{2m} - 2 \cos \frac{\pi k}{n+1} \right). \]

(3) follows by a simple change of variables.

**REFERENCES**


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