2000

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HOPF BIFURCATION SUBJECT TO A LARGE DELAY IN A LASER SYSTEM*

DIDIER PIEROUX†, THOMAS ERNEUX†, ATHANASIOS GAVRIELIDES‡, AND VASSILIOS KOVANIS†

Abstract. Hopf bifurcation theory for an oscillator subject to a weak feedback but a large delay is investigated for a specific laser system. The problem is motivated by semiconductor laser instabilities which are initiated by undesirable optical feedbacks. Most of these instabilities are starting from a single Hopf bifurcation. Because of the large delay, a delayed amplitude appears in the slow time bifurcation equation which generates new bifurcations to periodic and quasi-periodic states. We determine analytical expressions for all branches of periodic solutions and show the emergence of secondary bifurcation points from double Hopf bifurcation points. We study numerically different cases of bistability between steady, periodic, and quasi-periodic regimes. Finally, the validity of the Hopf bifurcation approximation is investigated numerically by comparing the bifurcation diagrams of the original laser equations and the slow time amplitude equation.

Key words. semiconductor laser instabilities, system of delay-differential equations, two-time solution, bifurcation to periodic and quasi-periodic oscillations

AMS subject classifications. 37G15, 34E13, 34K18, 34D15, 78A60, 78M35

1. Introduction. Semiconductor lasers are the technology of choice for many applications requiring a coherent light source because they are of relatively small size, they are mass-produced, and they are easy to operate. Applications of semiconductor lasers span a broad range of areas from optical communication to optical ranging and sensing. However, semiconductor lasers are extremely sensitive to optical feedback (OFB). A small amount of OFB may result from the reflection from an optical disk or from the end of an optical fiber. This feedback perturbs the normal output of the laser and generates dynamical instabilities. As a result, the laser line is observed to broaden dramatically to a width of several GHz. It is therefore of practical interest to understand how the stability of the laser depends on OFB. A simple model was formulated in 1980 by Lang and Kobayashi (LK) and consists of two nonlinear ordinary delay-differential equations for the complex electrical field $Y(t)$ and the carrier number $Z(t)$ [1]. Time $t$ is measured in units of the photon lifetime $\tau_p$. A term proportional to $Y(t-\tau)$ appears in the field equation and describes the effect of OFB after the reflection of light from a mirror. $\tau$ is proportional to the round-trip time from laser to mirror and back to laser. Computer simulations have shown that the LK delay-differential equation correctly describes the dominant effects observed experimentally. This includes the occurrence of mode hopping [2, 3], coexisting time-periodic attractors [4, 5, 6] and different forms of chaotic attractors called low-frequency fluctuations [7, 8, 9]. More recently, a numerical bifurcation study suggests that LK instabilities


*Received by the editors August 13, 1999; accepted for publication (in revised form) May 12, 2000; published electronically October 4, 2000. This research was supported by U.S. Air Force Office of Scientific Research grant AFOSR F49620-98-1-0400, National Science Foundation grant DMS-9973203, Fonds National de la Recherche Scientifique (Belgium), and InterUniversity Attraction Pole of the Belgian government.

http://www.siam.org/journals/siap/61-3/36013.html

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may result from cascading bifurcations starting from a Hopf bifurcation [10]. Ideally, LK equations model an experimental set-up consisting of a laser exposed to weak reflection of light from a mirror located at a sufficiently large distance to avoid multiple reflections (one to two meters). This implies a round-trip time \( \tau' \) from laser to mirror and back to laser of the order of 1 ns, which is relatively large compared to the photon lifetime \( \tau_p \sim 1 \text{ ps} \) (i.e., a dimensionless delay \( \tau \equiv \tau'/\tau_p = O(10^3) \)). This large delay \( \tau \) is responsible for the laser instabilities, even for low feedback rates, and motivates our mathematical interest for the Hopf bifurcation subject to a strong delay.

Periodic solutions of delay-differential equations and their behavior for large delay have been studied mathematically for equations of the form

\[
u' = f(u, u(t-\tau), \lambda),
\]

where \( u \in \mathbb{R} \). Applications appear in the biological sciences [14, 15], chemical problems [16], and nonlinear optics [17, 18]. Most of the solutions computed numerically correspond to square-wave oscillations exhibiting a period close to \( \tau \) as \( \tau \to \infty \). The condition for the bifurcation to these square-wave periodic regimes has been examined mathematically in [11, 12]. By contrast to these square-wave oscillations, the oscillatory solution of LK equations for low feedback is nearly harmonic in time [19, 20], and the frequency of the oscillations is \( O(1) \) as \( \tau \to \infty \). The effect of a large \( \tau \) appears through a secondary bifurcation to quasi-periodic oscillations which exhibits a low frequency proportional to \( \tau^{-1} \). This typical two-time behavior of the solution of the LK problem for low feedback rates but large delays motivates the interest for Hopf bifurcation problems of the form

\[
u' = f(u, \epsilon^2 u(t-\tau), \lambda),
\]

where \( u \in \mathbb{R}^n (n \geq 2) \).

In (2), \( \epsilon \) and \( \tau \) represent a small parameter and a large parameter, respectively. General Hopf bifurcation theories for systems of delay-differential equations are difficult (see [13]) and we shall take advantage of the small parameter \( \epsilon \). Specifically, we assume that the \( \epsilon = 0 \) problem admits a Hopf bifurcation at \( \lambda = \lambda_0 \) from \( u = 0 \) to a stable time-periodic state. Our objective is to investigate this bifurcation for small \( \epsilon \) and progressively larger values of \( \tau \). To this end, we may apply a two-time perturbation method introducing a fast time \( t \) and a slow time \( \nu \equiv \epsilon^2 t \) [26, 27]. The two independent time variables imply that \( u(t) = u(t, \nu) \) and \( u(t-\tau) = u(t-\tau, \nu - \epsilon^2 \tau) \). If \( \tau = O(1) \), \( u(t-\tau, \nu - \epsilon^2 \tau) \) simplifies as

\[
u(t-\tau, \nu - \epsilon^2 \tau) = u(t-\tau, \nu) + O(\epsilon^2)
\]

and the effect of the delay on the slow time \( \nu \) will not appear in the leading order amplitude equation.

On the other hand, if \( \tau = \epsilon^{-2} \tau_0 \), where \( \tau_0 = O(1) \), we cannot expand \( u(t-\tau, \nu - \epsilon^2 \tau) \) for small \( \epsilon \), and the slow time delay given by \( \epsilon^2 \tau = \tau_0 \) will appear explicitly in the bifurcation equation through a delayed amplitude. As we shall demonstrate, this slow time delay is responsible for new, isolated branches of periodic solutions that are absent if \( \tau = O(1) \).

Although it is simple to find the scaling of \( \tau(\epsilon) \) leading to a different Hopf problem, a general theory for a class of delay-differential equations remains difficult. Algebraic transcendental equations appear as soon as we try to find time-dependent solutions of the slow time amplitude equation and limit our chances for analytical results.
In addition, our bifurcation equation may now admit multiple branches of solutions with gradually increasing amplitudes, and we need to check the numerical validity of Hopf local approximation. The difficulty of analyzing delay-differential equations and the multiplicity of solutions of the bifurcation equation motivate the study of a simple problem and a comparison of numerical bifurcation diagrams obtained from the original equations and from the slow-time amplitude equation.

Specifically, we shall consider a semiconductor laser subject to a delayed feedback controlled optoelectronically [21]. The problem is mathematically simpler than the LK problem because the phase of the laser field does not play a role and the dynamical variables reduce to the intensity of the laser field and the electronic carrier density. The laser equations are formulated in dimensionless form in section 2 and the slow-time amplitude equation is derived in section 3. In section 4, we determine its steady state solutions and formulate stability conditions which we investigate numerically. We conclude that multiple branches of periodic and quasi-periodic states are possible and are connected at double Hopf bifurcation points. In section 5, we evaluate the validity of our Hopf bifurcation analysis by comparing numerical bifurcation diagrams. We also identify different forms of bistability between steady, periodic, and quasi-periodic regimes which are of particular interest for experiments. Finally, section 6 discusses the physical relevance of our results for semiconductor lasers instabilities.

2. Formulation. The semiconductor laser system and its optoelectronic feedback is sketched in Figure 1. Part of the laser output is detected with a high-speed photodetector. The detector photocurrent component is selected with a $T$ bias, amplified and added to the DC pump current [21]. In dimensionless form, the laser rate equations modeling this system are given by [22]

\[
\frac{dY}{dt} = (1 + i\alpha)ZY,
\]

\[
T \frac{dZ}{dt} = P + \eta |Y(t - \tau)|^2 - Z - (1 + 2Z) |Y|^2,
\]

where $Y$ and $Z$ represent the complex electrical field and the electronic carrier density. Time $t$ is measured in units of the photon lifetime $\tau_p (t \equiv t'/\tau_p)$. $\alpha$ is defined as the linewidth enhancement factor; $T \equiv \tau_n/\tau_p$ is a ratio of two time constants, where $\tau_n$ is defined as the carrier lifetime; $\tau \equiv \tau'/\tau_p$ is the dimensionless delay of the feedback, where $\tau'$ is the cavity transit time; and $P$ is the pump parameter above threshold. These equations are the traditional semiconductor laser rate equations with $P$ now replaced by $P + \eta |Y(t - \tau)|^2$, which models the effect of the DC coupled optoelectronic feedback. Typical values of the fixed parameters are $\alpha \sim 3 - 6$, $\tau_p \sim 2$ ps, $\tau_n \sim 2$ ns, and $\tau \sim 1 - 10$ ns, which mean large $O(10^3)$ values, for both $T$ and $\tau$. We shall take advantage of these two large parameters in order to determine an approximation of the solution of (4) and (5). To this end, it is mathematically convenient to reformulate (4) and (5) in terms of new dependent and independent variables. First, we introduce the decomposition $Y = \sqrt{I} \exp(i\psi)$ into (4) and (5) and obtain the following equations for $I$, $Z$, and $\psi$:

\[
I' = 2ZI,
\]

\[
TZ' = P + \eta I(t - \tau) - Z - (1 + 2Z)I,
\]
Fig. 1. Semiconductor laser subject to an optoelectronic feedback. The figure illustrates the optoelectronic device used by Saboureau, Foing, and Schanne [21]. The feedback operates on the pump of the laser by using part of the output light which is injected into a photodetector connected to the pump. The delay of the feedback is controlled by changing the length of the optical path.

\( \psi' = \alpha Z, \)  

where prime means differentiation with respect to \( t \). Our dynamical problem now reduces to the first two equations for \( I \) and \( Z \). Knowing \( Z \), we obtain \( \psi \) by integrating (8). Second, the large \( T \) parameter which multiplies the left-hand side of (7) can be removed by introducing the new variables \( x, y, \) and \( s \) defined as

\[
I = P(1 + y), \quad Z = \frac{\omega}{2}x, \quad s = \omega t,
\]

where

\[
\omega \equiv \sqrt{2P/T}
\]

is known as the laser relaxation oscillation frequency. Equations (6) and (7) then become

\[
x' = -y - \epsilon^2 x \left( 1 + \frac{2P}{1+2Py} \right) + \eta [1 + y(s - \theta)],
\]

\[
y' = (1+y)x,
\]

where prime now means differentiation with respect to \( s \). The new parameters \( \epsilon \) and \( \theta \) are defined by

\[
\epsilon \equiv \sqrt{\frac{\omega^2(1+2P)}{2P}} \quad \text{and} \quad \theta \equiv \omega \tau.
\]
\( \epsilon \) is a small parameter because \( T \) is typically an \( O(10^3) \) large quantity. From (10), we note that \( \omega = O(T^{-1/2}) \), and using the definition of \( \epsilon \) in (13), we find that \( \epsilon = O(T^{-1/4}) \). The delay \( \tau \) is typically large in the experiments (\( \tau = O(T) \)). Since \( \omega = O(T^{-1/2}) \), the definition of \( \theta \) in (13) implies that \( \theta = O(T^{-1/2}) \). The bifurcation parameter is the feedback rate \( \eta \) which is typically small. In the next section, we analyze (11) and (12) in the limit \( \epsilon \to 0 \) assuming

\[
\eta = \epsilon^2 C \quad \text{and} \quad \theta = \epsilon^{-2} \Theta,
\]

where \( C \) and \( \Theta \) are \( O(1) \).

Note that (11) and (12) are equivalent to the equations studied in [22, 23] for a variety of different optoelectronic devices. In [22, 23], the objective was to determine strongly pulsating periodic solutions for moderate values of the delay. Here, we determine periodic solutions which are nearly harmonic for large values of the delay. Recent attempts to investigate the large delay limit for similar laser equations are proposed in [24, 25]. Simplified amplitude equations were derived but their solutions were not compared to the numerical solutions of the original laser equations. As explained previously, one of the objectives of this paper is to investigate the numerical validity of all our analytical approximations.

### 3. Hopf bifurcation

In this section, we determine a slow time amplitude equation by a two-time analysis of (11), (12), and (14) which is based on the limit \( \epsilon \) small. The perturbation analysis is quite simple and we summarize the main results. Specifically, we seek a two-time solution of the form

\[
x(s, \nu, \epsilon) = \epsilon x_1(s, \nu) + \epsilon^2 x_2(s, \nu) + \cdots,
\]

\[
y(s, \nu, \epsilon) = \epsilon y_1(s, \nu) + \epsilon^2 y_2(s, \nu) + \cdots,
\]

where \( \nu \) is a slow time variable defined by

\[
\nu = \epsilon^2 s.
\]

Inserting (14)–(16) into (11) and (12), using the chain rule \( d/ds = \partial/\partial s + \epsilon^2 \partial/\partial \nu \), and noting that \( y(s - \theta) = y(s - \theta, \nu - \Theta) \), lead to a sequence of problems for the unknown coefficients in (15) and (16). The solution of each problem is easily determined. The leading approximation is given by

\[
x = \epsilon (A(\nu) \exp(is) + \text{c.c.}) + O(\epsilon^2),
\]

\[
y = -\epsilon (iA(\nu) \exp(is) + \text{c.c.}) + O(\epsilon^2),
\]

where c.c. means complex conjugate, and the slow-time amplitude \( A \) satisfies the following ordinary differential equation:

\[
\frac{dA}{d\nu} = \frac{1}{2} \left[ iCA - \frac{i}{3} A^2 A^* - A - iCA(\nu - \Theta) \exp(-i\theta) \right].
\]

The main difference between (19) and the equation that results when \( \theta = O(1) \) is the presence of the delayed amplitude \( A(\nu - \Theta) \) instead of \( A(\nu) \). It is instructive to review the significance of each term in the right-hand side of (19). The first two terms in the right-hand side of (19) represent the linear and nonlinear correction to the frequency and could be anticipated from a study of the laser equations (11) and (12) without
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Table 1

Laser parameters and their typical values. \( T = 1000 \), \( P = 0.5 \), and \( \tau = 1000 \) to 2000.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
<th>Value</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega )</td>
<td>( \sqrt{\frac{2P}{T}} )</td>
<td>0.03</td>
<td>relax. oscill. frequency</td>
</tr>
<tr>
<td>( \epsilon^2 )</td>
<td>( \omega \frac{\sqrt{2P}}{T} )</td>
<td>0.06</td>
<td>damping rate</td>
</tr>
<tr>
<td>( \theta )</td>
<td>( \omega \tau = \sqrt{\frac{2P}{T}} \tau )</td>
<td>31.62 to 63.25</td>
<td>scaled delay</td>
</tr>
<tr>
<td>( \Theta )</td>
<td>( \epsilon^2 \theta = \frac{\sqrt{2P}}{T} \tau )</td>
<td>2 to 4</td>
<td>scaled delay</td>
</tr>
<tr>
<td>( C )</td>
<td>( \eta / \epsilon^2 )</td>
<td>-1.6 to 1.6</td>
<td>scaled feedback rate</td>
</tr>
</tbody>
</table>

damping (\( \epsilon = 0 \)) and without feedback (\( \eta = 0 \)). The third term in the right-hand side of (19) comes from the coefficient multiplying \( \epsilon^2 \) in (11). It represents the physical damping of the laser relaxation oscillations. Finally, the last term in the right-hand side of (11) is the contribution of the feedback. Note that the coefficient of the cubic term in \( A \) is purely imaginary. Nonlinear damping necessary for bifurcation appears through the delayed feedback term in (19).

Equation (19) is the starting point of our bifurcation analysis. For clarity, the different parameters introduced in the last two sections and their values are shown in Table 1.

4. Analysis of the slow time amplitude equation. In this section, we determine the steady state solutions of (19) and investigate their linear stability properties. The validity of (19) as an approximation of the original laser equations (4) and (5) will be investigated numerically in section 5. The main advantage of (19) is the fact that periodic and quasi-periodic solutions of the original laser equations now correspond to steady and periodic solutions for the amplitude \( |A| \). It is mathematically convenient to introduce the decomposition \( A = R \exp(i\phi) \) into (19). We then obtain the following equations for \( R \) and \( \phi \):

\[
R' = \frac{1}{2} \left[ -R + CR(\nu - \Theta) \sin(-\theta + \phi(\nu - \Theta) - \phi) \right],
\]

\[
R\phi' = \frac{1}{2} \left[ CR - CR(\nu - \Theta) \cos(-\theta + \phi(\nu - \Theta) - \phi) - \frac{1}{3} R^3 \right],
\]

where prime means differentiation with respect to \( \nu \). Before we analyze these equations in detail, it is interesting to review the solution of the regular problem when \( \theta = O(1) \). Equivalently, we solve (20) and (21) with \( \Theta = 0 \). A periodic solution of the original laser equations corresponds to \( R = \text{const} \) and \( \phi = \phi(0) + B\nu \). From (20) and (21) with \( \Theta = 0 \), we find that

\[
C = C_H^0 \equiv -\frac{1}{\sin(\theta)}
\]

and

\[
R = \sqrt{6(B_0 - B)},
\]

where \( B_0 \) is defined by

\[
B_0 \equiv -\frac{1}{2} \tan \left( \frac{\theta}{2} \right)
\]
and $B < B_0$ is arbitrary. $C = C_H^0$ is a Hopf bifurcation point, and the bifurcation is vertical at this order of the perturbation analysis in $\epsilon$. A higher order analysis will lead to the direction of bifurcation but we do not need this result (see Appendix in [22]).

Our main observation is that the Hopf bifurcation is unique if the delay $\theta$ is $O(1)$. As we shall now demonstrate, increasing $\theta$ will bend the vertical bifurcation branch (22)-(24) and produce new (bifurcation and isolated) branches of periodic solutions.

4.1. Periodic solutions. We consider (20) and (21) with $\Theta \neq 0$ and seek a solution of the form $R = \text{cst}$ and $\phi = \phi(0) + B\nu$. This solution corresponds to a periodic state of the original laser equations which exhibits a frequency equal to $1 + \epsilon^2 B$. The exact solution for $R^2 = R^2(C)$ and $B = B(C)$ can be formulated in parametric form as

\begin{equation}
C = \frac{1}{\sin((\theta + B\Theta)/2)},
\end{equation}

\begin{equation}
R^2 = 3[-2B - \tan((\theta + B\Theta)/2)] > 0,
\end{equation}

where the frequency $B$ is the parameter. Changing $B$ from $-\infty$ to $+\infty$, we obtain $C$ and $R$ using (25) and (26). See Figure 2, which shows that there exist several branches of solutions. The limit as $\Theta \to 0$ is discussed in Appendix A.

From (25) and (26) with $R = 0$, we find that the Hopf bifurcation points $(B, C) = (B_H, C_H)$ satisfy (25) and

\begin{equation}
2B + \tan((\theta + B\Theta)/2) = 0.
\end{equation}

Thus, we determine $B = B_H$ by solving (27) and then obtain $C = C_H$ using (25). A different and more physically interesting way to investigate the roots of (27) as a function of $\theta$ is to study the implicit expression given by

\begin{equation}
\theta = \frac{2}{1+BE^2} \left[m\pi - \arctan(2B)\right],
\end{equation}

where $m = 0, 1, 2, \ldots$. Having $\theta = \theta(B)$, we determine $C = C_H$ by (25). Thus, we obtain $\theta$ and $C_H$ by changing parameter $B$ continuously from $-\infty$ to $\infty$. See Figure 3. For each $m$, we have a line of bifurcation points which corresponds to a different branch of periodic solutions.

The direction of bifurcation can be determined from an analysis of (25) and (26) for $(B, C)$ close to $(B_H, C_H)$. We find that the bifurcation is defined for $C > C_H$ if

\begin{equation}
\cos(\theta + B\Theta) < 0
\end{equation}

or, equivalently, for $C < C_H$ if $\cos(\theta + B_H\Theta) > 0$. The direction of bifurcation changes if $\cos(\theta + B_H\Theta) = 0$, which implies, using (25), that $C_H = 1$. Since $C_H = 1$ at the minimum of the curve $C_H = C_H(\theta)$, the change of direction occurs exactly at that point. The successive minima are located at $\theta = \theta_m$. If $C_H > 0$, we have verified that the bifurcation is supercritical ($C > C_H$) if $\theta < \theta_m$ and is subcritical ($C < C_H$) if $\theta > \theta_m$.

Figure 3(a) shows that distinct Hopf bifurcation lines may intersect. These points correspond to degenerate Hopf bifurcation points which exhibit two distinct periodic modes (double Hopf bifurcation points). From these points, the solution is in first approximation a linear combination of the two periodic modes, and multiple branching
Fig. 2. Bifurcation diagram of the periodic solutions. (a) and (b) represent the amplitude $R$ and the frequency $B$ as functions of the feedback rate $C$, respectively. Circles denote Hopf bifurcation points. The values of the parameters are $P = 0.5$, $T = 1000$ and $\tau = 2000$ (implying $\theta \simeq 63.25$, and $\Theta = 4$).

of solutions is possible at these points. We anticipate that in addition to the two pure mode bifurcations, mixed mode bifurcations are possible. We verify this numerically and analytically in the next subsection. This mixed mode regime is typically quasi-periodic exhibiting the two pure mode frequencies.

By contrast to the Hopf points, the limit points $(B, C) = (B_L, C_L)$ lead to isolated branches of periodic solutions. They satisfy the condition $\sin(\theta + B_L \Theta) = \pm 1$ and they are located at

\begin{equation}
C_L = \pm 1 \quad \text{and} \quad B_L = \frac{1}{\Theta} \left(\mp (1 + 4n)\pi/2 - \theta\right),
\end{equation}

where $n = 0, 1, 2, \ldots$. 

4.2. Quasi-periodic bifurcations. We next investigate the stability of the periodic states given by (25) and (26). A detailed stability analysis of these solutions is difficult because the characteristic equation is transcendental. But we may examine the case of small eigenvalues and the case of purely imaginary eigenvalues.

From (20) and (21), we find the linearized equations for the deviations $r$ and $\Phi$. They are given by

\begin{align}
\dot{r} &= \frac{1}{2} \left[ -r - Cr(\nu - \Theta) \sin(\theta + B\Theta) + CR \cos(\theta + B\Theta) \left( \Phi(\nu - \Theta) - \Phi \right) \right], \\
\dot{\Phi} &= \frac{1}{2} \left[ \frac{C}{\pi} (r - r(\nu - \Theta)) \cos(\theta + B\Theta) - C \sin(\theta + B\Theta) \left( \Phi(\nu - \Theta) - \Phi \right) - \frac{3}{4} Rr \right].
\end{align}
From (31) and (32) we determine the following characteristic equation for the growth rate $\sigma$:

$$
4\sigma^2 + C^2 \left( \exp(-\sigma\Theta) - 1 \right)^2 + (\exp(-\sigma\Theta) - 1) C \left[ 4\sigma \sin(\theta + B\Theta) + \cos(\theta + B\Theta) \frac{2}{3} R^2 \right] = 0.
$$

Assuming $\sigma$ small, we obtain an expression for $\sigma$ of the form

$$
\sigma \simeq \beta \cos(\theta + B\Theta) R^2,
$$

where the coefficient $\beta$ is always positive. If all the remaining eigenvalues have a negative real part, (34) implies that the Hopf bifurcation branch is stable near its bifurcation point $C = C_H$ (limit $R \to 0$, $\cos(\theta + B\Theta)$ fixed) provided that (29) is satisfied, i.e., if the bifurcation is supercritical. The expression (34) also implies the local stability of one of the two branches of periodic solutions that emerge from a limit point $C = C_L$ (recall that $\cos(\theta + B\Theta) = 0$ at $C = C_L$ and consider the limit $|\cos(\theta + B\Theta)| \to 0$, $R$ fixed of (34)).

We next examine the conditions for a Hopf bifurcation from the $R = \text{cst}$ solution meaning a secondary bifurcation to quasi-periodic oscillations for the laser intensity. Substituting $\sigma = i\mu$ into the characteristic equation, we obtain two conditions for the critical feedback rate $C$ and frequency $\mu$. They are given by

$$
-4\mu^2 + C^2 \left( (\cos(\mu\Theta) - 1)^2 - \sin^2(\mu\Theta) \right)
+(\cos(\mu\Theta) - 1) C \cos(\theta + B\Theta) \frac{2}{3} R^2
- \sin(\mu\Theta) 4\mu = 0
$$

and

$$
-2C^2(\cos(\mu\Theta) - 1) \sin(\mu\Theta)
-4\mu(\cos(\mu\Theta) - 1)
- \sin(\mu\Theta) C \cos(\theta + B\Theta) \frac{2}{3} R^2 = 0,
$$

where $R$ and $B$ are related to $C$ by (25) and (26). The solution of these equations has been determined numerically. See Figure 4. We find successive lines of quasi-periodic bifurcation points that all emerge from the double Hopf bifurcation points. These points are denoted by $P_1$ and $P_4$ in Figure 4. In Appendix B, we show analytically that these lines emerge from the double Hopf bifurcation points with the new frequency $\mu = B_2 - B_1$. $B_1$ and $B_2$ denote the two Hopf frequencies at the double Hopf point. This is consistent with a bifurcation theory near a multiple eigenvalue since we expect a solution of the form

$$
A \sim R_1 \exp(i(B_1\nu + \phi_1)) + R_2 \exp(i(B_2\nu + \phi_2))
$$

near this point. At the end of Appendix B, we show that the secondary bifurcating solution has indeed the form of (37). A detailed analysis of the solutions near the double Hopf bifurcation point is investigated elsewhere using numerical continuation methods [28]. In the next section, we investigate numerically the bifurcation possibilities suggested by Figure 4 and emphasize cases of coexisting stable states (bistability).
Primary and secondary Hopf bifurcation lines. The Hopf bifurcation lines $C = C_H$ (solid lines) and the quasi-periodic bifurcation line $C = C_{QP}$ (dashed lines) are represented as functions of the scaled delay $\theta$. The points $P_1$ and $P_4$ are double Hopf bifurcation points. The point $P_2$ corresponds to a change of direction of the Hopf bifurcation point (supercritical for $\theta < \theta_{P_2}$ and subcritical for $\theta > \theta_{P_2}$). At the point $P_3$, a Hopf bifurcation point and a quasi-periodic bifurcation point may have the same value of the bifurcation parameter $C$ but they exhibit different amplitudes. The values of the fixed parameters are $P = 0.5$, $T = 1000$.

5. Numerical bifurcation diagrams. The objectives of this section are twofold. First, we investigate numerically several cases of coexisting (steady or not) attractors. These possibilities are of physical interest because they can be found experimentally. Second, we examine the validity of our approximations by comparing the numerical bifurcation diagram of the original laser equations and the bifurcation diagram obtained from the amplitude equation.

We first determine the bifurcation diagram of the solutions of (19) for different values of $\theta$. The main advantage of this equation is that steady and periodic states correspond to periodic and quasi-periodic states of the original laser equations. The simplicity of (19) allowed us to have analytical expressions for all the steady state branches. We shall limit our analysis to the range $60 < \theta < 70$ and consider $C > 0$. We follow the predictions of Figure 4, and our results are summarized by the three qualitative diagrams shown in Figure 5 and by the detailed diagram given in Figure 6. In Figure 5, we concentrate on the gradual change of the first Hopf bifurcation branch as the delay $\theta$ is progressively increased. Figure 6 examines a case of two coexisting branches of periodic solutions. All our diagrams are based on the numerical determination of the stable solutions of the slow time amplitude equations (20) and (21) as well as the exact analytical expressions of the branches of periodic
solutions given by (25) and (26). In Figure 6, we determine the stable solutions from the amplitude equations as well as from the original laser equations. Four key points in Figure 4 mark qualitative changes of the bifurcation diagrams. As described in section 4.2, $P_1$ and $P_4$ are double Hopf bifurcation points where lines of secondary bifurcation points appear. $P_2$ refers to the change of direction of the Hopf bifurcation and was analyzed in section 4.1. $P_3$ verifies the condition $C_H = C_{QP}$ but corresponds to distinct bifurcation points (different amplitudes of the solution).

The simplest case occurs if $\theta_{P_1} < \theta < \theta_{P_2}$ (Figure 5(a)). A supercritical Hopf bifurcation is followed by a secondary bifurcation to quasi-periodic oscillations. If $\theta_{P_2} < \theta < \theta_{P_3}$ (Figure 5(b)), the sequence of bifurcations is similar to the sequence shown in Figure 5(a) except that the Hopf bifurcation is subcritical and allows the coexistence of a stable steady state and a stable periodic regime. This coexistence becomes richer if $\theta_{P_3} < \theta < \theta_{P_4}$ (Figure 5(c)): in addition to coexisting steady and periodic states, we may now have coexisting stable steady and quasi-periodic regimes. If $\theta > \theta_{P_4}$, new branches of periodic and quasi-periodic solutions appear.

In Figure 6, we compare the bifurcation diagrams of the periodic and quasi-periodic states obtained from the full laser equations (11) and (12) (Figure 6(a)) and from the slow time amplitude equations (20) and (21) (Figure 6(b)). In Figure 6(b), we observe a stable Hopf bifurcation branch that emerges from zero at $\eta \sim 0.076$. This bifurcation is followed by a secondary bifurcation to a branch of stable quasi-periodic oscillations at $\eta \sim 0.086$. Near $\eta \sim 0.1$, we note a tertiary bifurcation to a more com-
Fig. 6. Numerical bifurcation diagrams. (a) and (b) represent the bifurcation diagram of the periodic and quasi-periodic solutions obtained numerically from the original laser equations (11) and (12) and from the amplitude equations (20) and (21), respectively (we represent \(\max(y) = 2\epsilon R\) in (b)). The two main branches of periodic solutions and their bifurcations have been obtained by changing \(\eta\) by successive steps first forward and then backward. The \(P_1\) branch emerges from a Hopf bifurcation point of the basic steady state. The \(P_2\) branch emerges from a limit point. The dotted lines in (b) have been obtained using the analytical expressions (25) and (26) for the (stable and unstable) periodic solutions. For the periodic solutions \(P_1\) and \(P_2\), we show the maximum of \(y(t)\) as a function of \(\eta\). For the quasi-periodic solutions \(QP_1\) and \(QP_2\), we show the smallest and largest values of the maximum of \(y(t)\) as functions of \(\eta\). Diamonds denote bifurcations from the two-frequency quasi-periodic solution to a three-frequency quasi-periodic solution (not followed). A better agreement between the two figures is observed if we take into account the \(O(\epsilon^2)\) correction term for \(\max(y)\) in (b). In (b), the lower branch for \(QP_1\) reaches zero and then increases. This point is not a bifurcation point but marks a change of behavior of the phase. The values of the parameters are the same as in Figure 2.
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This branch then exhibits a secondary bifurcation to a quasi-periodic state at \( \eta \sim 0.068 \), which is itself followed by a tertiary bifurcation to more complex quasi-periodic oscillations (\( \eta \sim 0.082 \)). We compare this bifurcation diagram to the diagram obtained by solving directly the laser equations (Figure 6(a)). The agreement is excellent for the low amplitude branches. The agreement becomes more qualitative for the higher amplitude branches of periodic solutions, as we may expect, but the same sequence of bifurcation transitions is observed in both diagrams. Note that \( \epsilon = 0.17 \) for our parameter values. We have verified that the numerical agreement between diagrams improves for lower \( \epsilon \) (higher \( T \)).

6. Discussion. The main objective of this paper was to evaluate the validity of Hopf bifurcation theory for a system of two delay-differential equations modeling semiconductor lasers subject to a weak optical feedback but exhibiting a strong delay. To this end, we have compared the numerical bifurcation diagrams obtained from the original laser equations and from the slow time amplitude equations. In addition, we have determined useful analytical expressions for branches of periodic solutions and formulated conditions for their bifurcations. These expressions considerably helped the numerical study. Previous authors [24, 25] have tried to take advantage of this large delay in order to derive simplified laser equations but they never analyze the asymptotic validity of their approximations.

An important result of our bifurcation analysis is the observation that a strong delay generates multiple branches of periodic and quasi-periodic oscillations even for low feedback. These multiple Hopf bifurcations and isolated branches of solutions are predicted by our amplitude equation. Because the derivation of this amplitude equation is relatively easy, Hopf local theory was a useful guide for our numerical study of the laser equations.

The multiple branching of solutions does not necessarily lead to a chaotic laser. We have shown that different forms of bistable responses are possible near the first instability of the laser. These cases are of practical interest for semiconductor laser experiments which are mainly based on the determination of power spectra. Jumps between steady, periodic, or quasi-periodic regimes can be observed by the changes of frequencies.

Secondary bifurcation to quasi-periodic oscillations is intimately connected to the fact that the laser problem admits successive primary Hopf bifurcations as the delay is increased. We have found that the bifurcation to stable quasi-periodic oscillations is generated by the interaction of pairs of primary Hopf bifurcations. Other cases of secondary bifurcation leading to the coexistence of two stable periodic solutions have been discussed for the LK problem [4, 5, 29].

Appendix A. The small delay limit (\( \Theta \to 0 \)). In this appendix, we examine the behavior of the Hopf bifurcation and limit points for small \( \Theta \) using (25) and (28). As \( \Theta \to 0 \), all Hopf bifurcation branches move to higher values of \( B \) (\( B = O(\Theta^{-1}) \)) except one, which converges to the vertical branch (22). More precisely, we find that

\[
R \simeq \sqrt{\frac{6}{\Theta} \left( \frac{C - C_H}{C_H \cot(\theta)} \right)} \quad \text{and} \quad B \simeq -\frac{1}{6} R^2 + B_0,
\]

where \( B_0 \) is defined by (24) and the new bifurcation point \( C_H \simeq C_H^0 \) is shifted as

\[
C_H(\Theta) \equiv C_H^0 \left(1 - \Theta B_0 \cot(\theta)\right).
\]
Thus the bifurcation becomes vertical as \( \Theta \to 0 \) \( (C - C_H^0 = O(\Theta) \) if \( R = O(1) \)). All the other Hopf bifurcation branches move to infinity as \( \Theta \to 0 \).

\[ C_H \simeq \frac{1}{\Theta} ((1 + 2m)\pi - \theta), \]

\[ R = \sqrt{6} (C - C_H(m)). \]

**Appendix B. Double Hopf bifurcation points.** For critical values of the delay \( \theta \) (points \( P_1 \) and \( P_4 \) in Figure 4), a Hopf bifurcation point may exhibit two periodic modes with the distinct frequencies \( B = B_1 \) and \( B = B_2 \). From (25), we find the simple condition \( \sin(\theta + B_1\Theta) = \sin(\theta + B_2\Theta) \), which implies that either

\[ (B_2 - B_1)\Theta = 2m\pi, \]

where \( m = 1, 2, \ldots \), or

\[ 2\theta + (B_2 + B_1)\Theta = \pi + 4n\pi, \]

where \( n = 0, 1, 2, \ldots \). We first examine case (42). From (27), we know that \( B_1 \) and \( B_2 \) verify the equations

\[ 2B_1 + \tan((\theta + B_1\Theta)/2) = 0, \]

\[ 2B_2 + \tan((\theta + B_2\Theta)/2) = 0. \]

Subtracting (44) and (45), we obtain

\[ 2(B_1 - B_2) + \frac{\sin((B_1 - B_2)\Theta/2)}{\cos((\theta + B_1\Theta)/2) \cos((\theta + B_2\Theta)/2)} = 0. \]

Using now (42), we find that (46) cannot be satisfied. We next consider case (43). We may rewrite the denominator in (46) as a sum of two cosine functions and, using (43), we obtain

\[ (B_1 - B_2) + \tan((B_1 - B_2)\Theta/2) = 0. \]

Equation (47) is an equation for the difference between the two frequencies. Similarly, we obtain an equation for the sum of the two frequencies by adding (44) and (45).

Using (43) again, we find

\[ \frac{\pi + 4n\pi - 2\theta}{\Theta} + \frac{1}{\cos((B_1 - B_2)\Theta/2)} = 0. \]

Introducing \( z \equiv (B_1 - B_2)\Theta/2 \) and eliminating \( \theta \) and \( \Theta = \beta\theta \) using (47), we obtain an equation for \( z \) only:

\[ \beta [ (\pi + 4n\pi) \sin(z) - 2z ] + 4z \cos(z) = 0. \]

Having \( z \) from (49), we determine \( \theta \) or \( \Theta = \theta/\xi \) by using (47):

\[ \Theta = \frac{-2z}{\tan(z)}. \]
For example, with \( P = 0.5 \) and \( T = 1000 \), we find \( z \approx 2.28 \), \( \theta \approx 61.4 \) (equivalently, \( \Theta \approx 3.88 \)), which gives \( B_1 - B_2 \approx 1.17 \). This point corresponds to point \( P_1 \) in Figure 4.

We next consider conditions (35) and (36) for the secondary bifurcation point and show that the frequency of the quasi-periodic oscillations is \( \mu = (B_1 - B_2) \) at the double Hopf bifurcation point. To this end, we consider (35) and (36) evaluated at this point \((R = 0 \text{ and } C = C_H)\):

\[
\mu^2 + C_H^2 \sin^2(\frac{\mu \Theta}{2}) \cos(\mu \Theta) + \sin(\mu \Theta) \mu = 0
\]

and

\[
\sin^2(\frac{\mu \Theta}{2}) (C_H^2 \sin(\mu \Theta) + 2\mu) = 0.
\]

Equation (52) is satisfied if \( \sin(\frac{\mu \Theta}{2}) = 0 \), but from (51), we then find \( \mu = 0 \), which is not possible. The second possibility for satisfying (52) is to have

\[
C_H^2 \sin(\mu \Theta) + 2\mu = 0.
\]

With (53), we eliminate \( C_H \) in (51) and obtain, after many simplifications, and for \( \mu \neq 0 \), the equation for \( \mu \),

\[
\mu + \frac{\sin(\mu \Theta/2)}{\cos(\mu \Theta/2)} = 0,
\]

which we recognize as (47) for \( B_1 - B_2 \). We conclude that \( \mu = B_1 - B_2 \). The solution of the linearized problem is then of the form \( A \sim (R_1 + c \exp(i\mu \nu)) \exp[i (B_1 \nu + d \exp(i\mu \nu))] \), which we may rewrite as

\[
A \sim R_1 \exp(iB_1 \nu) + (c + idR_1) \exp(iB_2 \nu)
\]

for small \( c \) and \( d \). Thus the secondary bifurcating solution is close to a linear combination of the two periodic modes at the double Hopf bifurcation point.

REFERENCES


